Name: Nathan Flack

CSCE 531 Discrete Mathematics Fall 2018 Exam 1

Put your name on every page.

Your work must be your own.

The only permitted resources are:

- your personal notes,
- the course textbook, and
- the materials posted on the course Canvas site or linked directly from that site.

In particular,

- you may not use any other books or websites, and
- with the exception of the instructor, you may not communicate with another person in any way.

Tips:

- Read all questions prior to answering any, and budget your time accordingly.
- Leaving a question unanswered will result in zero points for that question.

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Multiple Choice – 3 points each

For each of the following, choose the **BEST** answer.

1. Given the statement $\forall x(x^2 + x - 2 \neq 0)$, where the domain consists of the **NONNEGATIVE** integers, which **ONE** of the following is **TRUE**?
   a. $x = -2$ is the only counterexample.
   b. $x = 1$ is the only counterexample.
   c. Both $x = -2$ and $x = 1$ are counterexamples.
   d. None of the above.

2. Which **ONE** of the following statements is **FALSE** if the domain of each variable consists of all **REAL** numbers?
   a. $\forall x \forall y \exists z(x = y + z)$ For all reals x and all reals y, there is a z such that $x = y + z$
   b. $\forall x \exists y \exists z(x = y + z)$ For all reals x there is a y and z such that $x = y + z$
   c. $\exists x \forall y \exists z(x = y + z)$ There is a x such that for every y there is a z where $x = y + z$
   d. $\exists x \exists y \forall z(x = y + z)$ there is a real x and y such that for every z $x = y + z$
   $z = x - y$

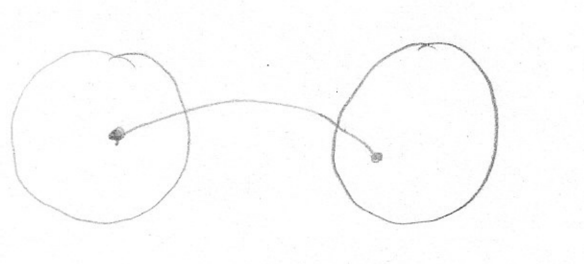
3. Which **ONE** of the rules of inference below is used in the following argument? "The Doctor is a Time Lord. If The Doctor is a Time Lord, then she can regenerate. Therefore, The Doctor can regenerate."
   a. Hypothetical syllogism
   b. Modus ponens
   c. Modus tollens
   d. None of the above.

   D is true
   $D \rightarrow R$
   R is true

4. For which **ONE** of the theorems below is the following proof a valid argument?
   Proof: Assume the premise of the theorem holds. Then $i = 2m$ for some $m \in \mathbb{Z}$. Now assume that $i + j$ is odd. Then $i + j = 2k + 1$ for some $k \in \mathbb{Z}$, so
   $$\begin{aligned} j &= j + (i - i) \\ &= (i + j) - i \\ &= (2k + 1) - 2m \\ &= 2(k - m) + 1, \end{aligned}$$
   which contradicts the premise.
   a. Premise: i is even and j is even. Conclusion: $i + j$ is even.
   b. Premise: i is even and j is odd. Conclusion: $i + j$ is odd.
   c. Premise: i is odd and j is even. Conclusion: $i + j$ is odd.
   d. Premise: i is odd and j is odd. Conclusion: $i + j$ is even.
   e. None of the above.

   if $i+j \neq$ odd
   then $i+j$ = even
   if i is even AND $i+j$ is even then j must be even

5. Which **ONE** of the following is a valid definition of a function with the domain $\mathbb{N}$ and the range $\mathbb{Z}^+$?
   $(0, 1, 2, ...)$ $(1, 2, 3, ...)$
   a. f assigns to each integer k the value of k mod 10.
   b. f assigns to each integer k the value of $k + 1$.
   c. f assigns to each nonnegative integer k the value of k mod 10.
   d. f assigns to each nonnegative integer k the value of $k + 1$.



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6. Suppose $f: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ is defined by $f(m,n) = m^2 - n$. Which **ONE** of the following is **TRUE**?
   a. $f(m,n)$ is both one-to-one and onto.
   b. $f(m,n)$ is neither one-to-one nor onto.
   c. $f(m,n)$ is not one-to-one but it is onto. $f(-2,2)$ and $f(2,2)$ both map to 2
   d. $f(m,n)$ is one-to-one but not onto.

7. Which **ONE** of the following is countably infinite?
   a. The set of blog posts that could be generated by an immortal monkey with an indestructible keyboard.
   ~~b.~~ The set of passwords that include at least one upper case letter, at least one lower case letter, at least one digit, and at least one special character, as well as having lengths of at least 8 and no more than 30.
   ~~c.~~ The set of real numbers between zero and $10^{-10^{100}}$. Uncountable
   ~~d.~~ The set of people who won the lottery without playing. 0

8. Consider the set F of functions mapping bit strings to Boolean values. Which **ONE** of the following is **TRUE**?
   A: {00, 01, 10, 00, 0, 1, 111, 000}
   B: {True, False}
   a. F is countably infinite
   b. F is empty
   c. F is finite
   d. F is uncountably infinite

9. Let $\mathbb{X} = \mathbb{R} - \mathbb{Z}$. Which **ONE** of the following is **TRUE**?
   R-Z = Real Numbers - Integers
   ~~a.~~ $\mathbb{Q} \subseteq \mathbb{X}$.
   ~~b.~~ $\mathbb{X}$ is countably infinite.
   ~~c.~~ $\mathbb{X}$ is finite.
   d. None of the above
   $\mathbb{X}$ is uncountably Infinite

10. Consider the cardinality of sets A, B, and C. If $(|A| \leq |B|) \wedge (|B| \leq |C|) \wedge (|C| \leq |A|)$, which **ONE** of the following is **TRUE**?
   $|A| \leq |B| \leq |C| \leq |A|$
   ~~a.~~ There cannot exist three sets that have these cardinality relationships.
   b. There exists a one-to-one function from A to B.
   ~~c.~~ All three sets are countable.
   d. None of the above
   The cardinality is equal for A, B, C.

11. Suppose set F is finite, set C is countably infinite, and set U is uncountably infinite. Which **ONE** of the following statements is **ALWAYS TRUE**?
   ~~a.~~ There exists a one-to-one correspondence between U and $\mathbb{Z}^+$.
   b. There exists a one-to-one correspondence between $C \cap U$ and $\mathbb{Z}^+$.
   ~~c.~~ There exists a one-to-one correspondence between $(C \cup F) - (C \cap F)$ and $\mathbb{Z}^+$.
   ~~d.~~ None of the above statements is always true.

12. Consider two countably infinite sets, J and K. Which **ONE** of the following is **FALSE**?
   a. There exists a function that maps J to $\mathbb{Z}^+$. ✓
   b. The exists a one-to-one correspondence between J and K. ✓
   ~~c.~~ Every function that maps J to K is invertible. ✓
   d. None of the above
   $J \leftrightarrow \mathbb{Z}^+$
   $K \leftrightarrow \mathbb{Z}^+$

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Short answer – Various point values

Present your work clearly and in an organized manner. If you need to do scratch work, do it elsewhere.

13. [6 pts] Refer to the rules of the island of knights and knaves described in Example 7 in Section 1.2. Suppose that you meet Anita, Boris, and Carmen. Anita says "I am a knave and Boris is a knight." Boris says "Exactly one of us is a knave." Determine whether each of the three people is a knight or a knave.

Let p = Anita is a Knigt, q = Boris is a knight, and r = Carmen is a Knight

I built a truth table. The only case where both hold is p = false, q = true, and r = true therefore:

p	q	r	-p	-q	-r	$-p \land q$	$-p \oplus -q \oplus -r$
0	1	1	1	0	0	1	1

$\boxed{\text{Anita = Knave, Boris = Knight, Carmen = Knight}}$

* See back for additional coments. *

14. [10 pts] Prove the following theorem: if n is a perfect square, then $n + 2$ is not a perfect square.
Hint: $i^2 - j^2 = (i + j)(i - j)$. Proof by Contraposition

p: "n is a perfect square" and q: n+2 is not a perfect square.

We are asked to prove $p \to q$. The contraposition is $-q \to -p$.
If $-q$ then $n+2$ is a perfect square. It follows that $n+2 = a^2$ where $a \in \mathbb{Z}$. Then $n = a^2 - 2$. Because $i^2 - j^2 = (i+j)(i-j)$, $n = a^2 - (\sqrt{2})^2 = (a+\sqrt{2})(a-\sqrt{2})$. If n is a perfect square $n = b^2$ where $b \in \mathbb{Z}$ therefore $b^2 = (a+\sqrt{2})(a-\sqrt{2})$ or $b \cdot b = (a+\sqrt{2})(a-\sqrt{2})$ (continued on back)

15. [10 pts] Explain the error in the following "proof":
"Theorem": In any set of $n \in \mathbb{Z}^+$ coffee mugs, all the mugs are the same size.
"Proof": Let $P(n)$ be the proposition that "in any set of n coffee mugs, all the mugs are the same size." Then $P(1)$ holds trivially. Now adopt the inductive hypothesis $P(k)$ for some $k \in \mathbb{Z}^+$, i.e. for any set of k coffee mugs, all the mugs have the same size. Next, let $C = \{m_0, m_1, m_2, \dots, m_{k-1}, m_k\}$ be a set of $k + 1$ coffee mugs. Also, define $C_0 = \{m_0, m_1, \dots, m_{k-1}\}$ and $C_k = \{m_1, m_2, \dots, m_{k-1}, m_k\}$, each of which is by construction a set of k coffee mugs. Thus, according to the inductive hypothesis, all the mugs in C_0 must have the same size, and similarly for C_k. Now choose any mug $m_i \in C_0 \cap C_k = \{m_1, m_2, \dots, m_{k-1}\}$. Then, because all the mugs in C_0 have the same size, so do m_0 and m_i, and because all the mugs in C_k have the same size, so do m_i and m_k. By transitivity, m_0 and m_k have the same size, and therefore so do all the mugs in C. We have shown that $P(k) \to P(k + 1)$, which completes the proof by induction.■

The proof did not have an Inductive Step that shows how $P(k) \to P(k+1)$ using the Inductive Hypothesis.
Because C_0 and C_K are not equal the Inductive Hypothesis cannot apply to both of them.
Assuming that C_K is true by the inductive hypothesis is a case of assuming the hypothesis (assuming m_K is the same size as m_0 through m_{K-1}). Must show that $\{m_0, m_1, \dots, m_K\}$ implies m_{K+1}.

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(14) $b \cdot b = (a+\sqrt{2})(b-\sqrt{2})$

Splitting this equation gives $b = a+\sqrt{2}$ and $b = a-\sqrt{2}$.
Since a is an integer and $1 \leq \sqrt{2} \leq 2$ then if b is an Integer we must add an Integer to a to get b. We know both 1 and 2 will not work.
Therefore b cannot be an Integer. This proves $\neg p$.
Therefore by contraposition the proof is complete.
Because $\neg q \rightarrow \neg p$ we also know $p \rightarrow q$. If n is a perfect square then n+2 is not a perfect square.

(13) continued: Anita = Knave
Boris = Knight
Carmen = Knight

However, since Anita said, "I am a Knave" she cannot be a Knave since she would be telling the truth, which only a Knight can do. Therefore, I believe the problem is unsolvable.

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16. [10 pts] Suppose that a restaurant offers gift certificates in denominations of \$5 and \$8. Use strong induction to prove that any integer value greater than \$32 can be formed.

This proposition [P(n)] holds for $32 \le n \le 36$, since

Basis Case:
- P(32): Gift cards of \$32 can be formed using four \$8 cards
- P(33): Gift cards of \$33 can be formed using five \$5 and One \$8 cards
- P(34): Gift cards of \$34 can be formed using two \$5 and three \$8 cards
- P(35): Gift cards of \$35 can be formed using seven \$5 cards.
- P(36): Gift cards of \$36 can be formed using four \$5 and two \$8 cards

I.H.: Now assume that for some $K \in \mathbb{Z}^+$ such that $K \ge 32$ and for all $j \in \mathbb{Z}^+$, $32 \le j \le K$, gift cards of j amount can be formed using \$5 and \$8 denominations. Then for $K+1 \ge 37$, we have $K-4 \ge 32$ and $K-4 \le K$. By the Inductive Hypothesis, $P(K-4)$ holds, it follows that gift certificates of $K-4$ amount can be formed using denominations of only \$5 and \$8. We can form $K+1$ amount by adding one more \$5 denomination to the current sum of $K-4$ $(K-4+5 = K+1)$. We have shown that $P(32)$, $P(33)$, $P(34)$, $P(35)$, and $P(36)$ is true and that $\bigwedge_{32 \le j \le K} P(j) \to P(K+1)$. By the principle of strong induction $P(n)$ is true for all $n \ge 32$. ■

17. [10 pts] Show how to use Fermat's Little Theorem and the Chinese Remainder Theorem to calculate $7^{3208} \bmod 2431 = 152$.

$\boxed{a^{p-1} \equiv 1 \pmod p}$ 2431 is

$2431 = 17 \cdot 13 \cdot 11$

$7^{3208} \bmod 17$: 3208 divided by 17-1
$3208 = 16 \cdot 200 + 8$
$7^{3208} = (7^{16})^{200} \cdot 7^8 = (1)^{200} \, 7^8 = 7^8$
$\boxed{7^{16} \equiv 1 \pmod{17}}$
$\equiv 7^8 \bmod 17 = \boxed{16}$

$7^{3208} \bmod 13$: 3208 divided by 12-1
$3208 = 12 \cdot 267 + 4$
$7^{3208} = (7^{12})^{267} \cdot 7^4 = (1)^{267} \cdot 7^4$
$\boxed{7^{12} \equiv 1 \bmod 13}$
$\equiv 7^4 \bmod 13 = \boxed{9}$

$7^{3208} \bmod 11$: 3208 divided by 11-1
$3208 = 10 \cdot 320 + 8$
$7^{3208} = (7^{10})^{320} \cdot 7^8 = (1)^{320} \cdot 7^8$
$\boxed{7^{10} \equiv 1 \bmod 11}$
$\equiv 7^8 \bmod 11 = (49)(49)(49)(49) \bmod 11 = 25 \cdot 25 \bmod 11 = 3 \cdot 3 \bmod 11$
$\equiv 9 \bmod 11 = \boxed{9}$

CRT: $x \equiv 16 \bmod 17$ $\quad x \equiv 9 \bmod 13$ $\quad x \equiv 9 \bmod 11$
$M_1 = 13 \cdot 11 = 143$ $\quad M_2 = 17 \cdot 11 = 187$ $\quad M_3 = 17 \cdot 13 = 221$
$a_1 = 16$ $\quad a_2 = 9$ $\quad a_3 = 9$

$y_1 \cdot 143 \equiv 1 \bmod 17$ $\quad y_1 = 5$
$y_2 \cdot 187 \equiv 1 \bmod 13$ $\quad y_2 = 8$
$y_3 \cdot 221 \equiv 1 \bmod 11$ $\quad y_3 = 1$

$x = a_1 M_1 y_1 + a_2 M_2 y_2 + a_3 M_3 y_3$
$x = 16 \cdot 143 \cdot 5 + 9 \cdot 187 \cdot 8 + 9 \cdot 221 \cdot 1$
$x = 11440 + 13464 + 1989 \pmod{2431} = 5014 \pmod{2431} = \boxed{152}$ ✓

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18. [10 pts] Consider a collection of m distinguishable n-sided dice where $n > m$. Assuming that m and n are both even, express the number of ways to roll the following.

a. Even numbers on all the dice. Even numbers on each die: $\frac{n}{2}$

$$\left(\frac{n}{2}\right)^m$$

b. An even number on exactly one die and odd numbers on the remaining dice.

$$\left(\frac{n}{2}\right)^m \cdot m!$$

c. An even number on at least one die.

$$\left(\frac{n}{2}\right)^1 \left(n^{m-1}\right) + \left(\frac{n}{2}\right)^m$$

d. Even numbers on exactly half the dice.

$$\left(\frac{n}{2}\right)^m \cdot m!$$

e. Distinct values on all the dice.

Same as choosing m-permutations of n elements without repetition

$$P(n,m) = \frac{n!}{(n-m)!}$$

19. [4 pts] Let A and B be finite sets.

a. How many distinct functions from A to B exist?

Must use all elements in A, but not in B

$\boxed{= |B|^{|A|}}$ If $|A| = 3$ and $|B| = 2$ then there are 2^3 distinct functions

b. How many of those functions are one-to-one?

If $|A| > |B|$ then there are no one-to-one functions.

If $|A| \leq |B|$ and $n = |A|$ and $m = |B|$ then there are $P(m,n) = \frac{m!}{(m-n)!}$ one-to-one functions.

20. [2 pts] At the end of Beggar's Night (a.k.a. Halloween) a boy makes his little sister an offer. He tells her that he has a total of n candy items, each of which is either a pack of gum, a lollipop, a chocolate bar, or a piece of taffy. If she can guess correctly how many of each type of item he has, he will give it all to her and do her chores for a year. Otherwise, she has to give him an item from her bag. How many possibilities does she have to choose from?

Order does not matter because she is just guessing the number of each candy. We have 3 moves and n "picks". Therefore, the answer is $C(n+4-1, n)$ or $\boxed{\binom{n+3}{n}}$

21. [2 pts] What is the term involving x^2 in $(3x + 2y)^9$?

$$(3x+2y)^9 = \sum_{j=0}^{9} \binom{9}{j} (3x)^{9-j} (2y)^j$$

For the x^2 term, $j = 7$ $(9-7=2)$

therefore, $\binom{9}{7} 3^2 x^2 2^7 y^7 = \boxed{\binom{9}{7} 9 \cdot 2^7 x^2 y^7}$