Growth curve models

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Polynomial regression

- A lot of times the assumption of a linear time trend is too simple
- Change is not happening unbraked linearly but flattens out
- Quadratic regression model

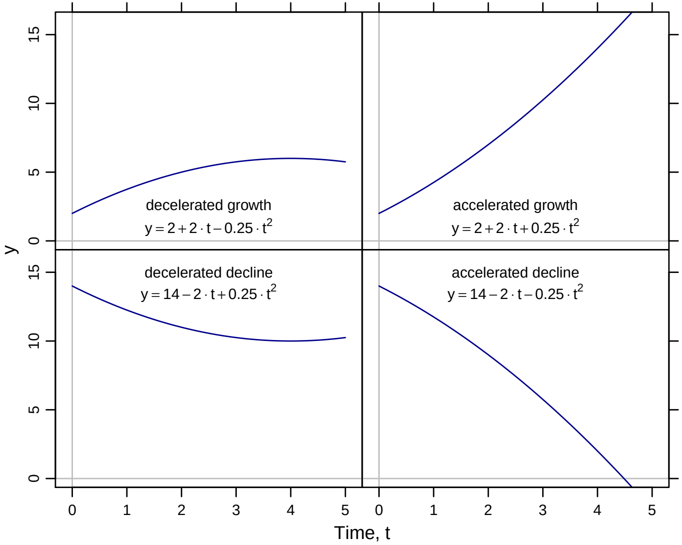
$$\begin{aligned} y_{ij} &= b_{0i} + b_{1i}\, t_{ij} + b_{2i}\, t_{ij}^2 + \varepsilon_{ij} \\ &= b_{0i} + (b_{1i} + b_{2i}\, t_{ij}) t_{ij} + \varepsilon_{ij} \end{aligned}$$

- The linear change depends on time t

$$\frac{\partial y}{\partial t} = b_{1i} + 2 b_{2i}\, t$$

- The intercept $t = -b_{1i}/(2b_{2i})$ is the point in time when a positive (negative) trend becomes negative (postive)

Quadratic time trends



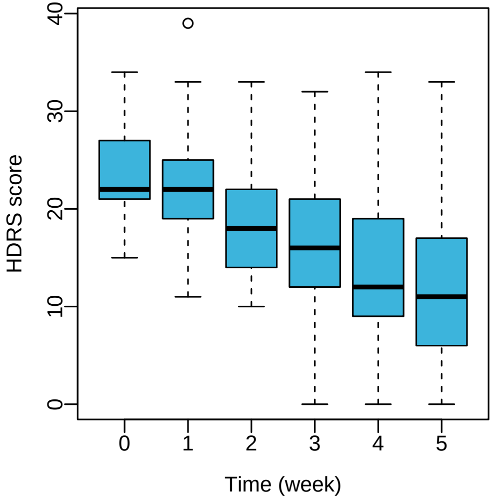
Outline

1 Depression and Imipramin

Depression and Imipramin (Reisby et al., 1977)

- Reisby et al. (1977) studied the effect of Imipramin on 66 inpatients treated for depression
- Depression was measured with the Hamilton depression rating scale (HDRS)
- Additionally, the concentration of Imipramin and its metabolite Desipramin was measured in their blood plasma
- Patients were classified into endogenous and non-endogenous depressed
- Depression was measured weekly for 6 time points; the effect of the antidepressant was observed starting at week 2 for four weeks

Descriptive statistics



HDRS score

t	W0	W1	W2	W3	W4	W5
M	23.44	21.84	18.31	16.42	13.62	11.95
SD	4.53	4.70	5.49	6.42	6.97	7.22
n	61	63	65	65	63	58

Empirical correlation matrix of HDRS score

	W0	W1	W2	W3	W4	W5
Week 0	1	.49	.41	.33	.23	.18
Week 1	.49	1	.49	.41	.31	.22
Week 2	.41	.49	1	.74	.67	.46
Week 3	.33	.41	.74	1	.82	.57
Week 4	.23	.31	.67	.82	1	.65
Week 5	.18	.22	.46	.57	.65	1

Model with quadratic trend

- Model with quadratic individual and group trend

$$y_{ij} = \beta_0 + \beta_1\, t_{ij} + \beta_2\, t_{ij}^2 + \upsilon_{0i} + \upsilon_1\, t_{ij} + \upsilon_2\, t_{ij}^2 + \varepsilon_{ij}$$

with

$$\begin{pmatrix} \upsilon_{0i} \\ \upsilon_{1i} \\ \upsilon_{2i} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2_{\upsilon_0} & \sigma_{\upsilon_0\upsilon_1} & \sigma_{\upsilon_0\upsilon_2} \\ \sigma_{\upsilon_0\upsilon_1} & \sigma^2_{\upsilon_1} & \sigma_{\upsilon_1\upsilon_2} \\ \sigma_{\upsilon_0\upsilon_2} & \sigma_{\upsilon_1\upsilon_2} & \sigma^2_{\upsilon_2} \end{pmatrix} \right) \text{ i.i.d.}$$

$$\varepsilon_i \sim N(\mathbf{0},\, \sigma^2 \mathbf{I}_{n_i}) \text{ i.i.d.}$$

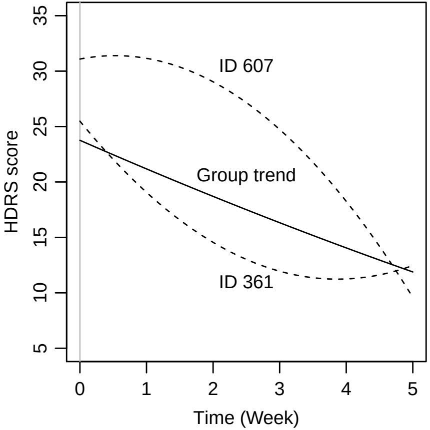
Depression and Imipramin

```r
dat      <- read.table("data/reisby.dat", header = TRUE)
dat$id   <- factor(dat$id)
dat$diag <- factor(dat$diag, levels = c("nonen", "endog"))
dat      <- na.omit(dat)     # drop missing values

# random intercept model
lme1 <- lmer(hamd ~ week + (1 | id), dat, REML = FALSE)

# random slope model
lme2 <- lmer(hamd ~ week + (week | id), dat, REML = FALSE)

# model with quadratic time trend
lme3 <- lmer(hamd ~ week + I(week^2) + (week + I(week^2) | id), dat,
  REML = FALSE)
```

ML estimates of parameters

```
...
Random effects:
 Groups   Name        Variance Std.Dev. Corr
 id       (Intercept) 10.4402  3.2311
          week         6.6381  2.5764   -0.11
          I(week^2)    0.1937  0.4402   -0.08 -0.83
 Residual             10.5160  3.2428
Number of obs: 375, groups:  id, 66

Fixed effects:
            Estimate Std. Error t value
(Intercept) 23.76025    0.55206  43.039
week        -2.63258    0.47900  -5.496
I(week^2)    0.05148    0.08835   0.583
...
```

Model predictions



- Averaged over persons an approximately linear trend is obtained, $\hat{\beta}_1 = -2.63$, $\hat{\beta}_2 = 0.05$
- Some of the predicted individual trends are strongly nonlinear

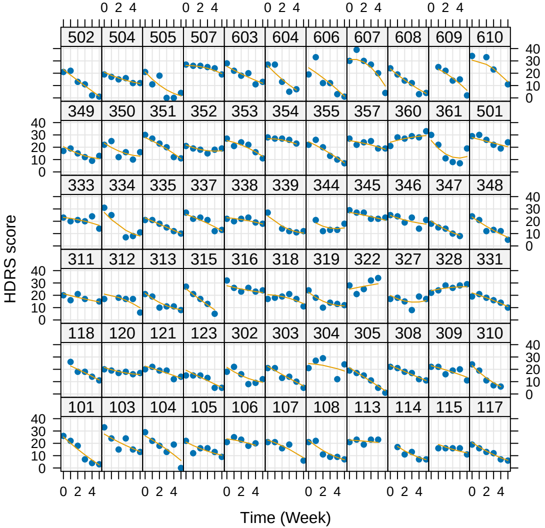
- Test against a model without individual quadratic trends

$$\mathrm{H}_0\colon\ \sigma^2_{\upsilon_2} = \sigma_{\upsilon_0\upsilon_2} = \sigma_{\upsilon_1\upsilon_2} = 0 \qquad G^2(3) = 10.98,\ p = .012$$

Model predictions

```r
# model without random quadratic time effect
lme3.0 <- lmer(hamd ~ week + I(week^2) + (week | id), dat,
  REML = FALSE)

# model without fixed quadratic time effect
lme3.1 <- lmer(hamd ~ week + (week + I(week^2) | id), dat,
  REML = FALSE)

# LRTs
anova(lme3.0, lme3)
anova(lme3.1, lme3)
```

Visualization of model predictions




```
xyplot(
  hamd + predict(lme3)
     ~ week | id,
  data = dat,
  type = c("p", "l", "g"),
  pch = 16,
  distribute.type = TRUE,
  ylab = "HDRS score",
  xlab = "Time (Week)")
```

Implied marginal covariance matrix

Predicted

$$\mathbf{Z}_i\hat{\boldsymbol{\Sigma}}_\upsilon\mathbf{Z}_i' + \hat{\sigma}^2\mathbf{I}_{n_i} = \begin{pmatrix} 20.96 & 9.41 & 8.16 & 6.68 & 4.98 & 3.06 \\ 9.41 & 23.86 & 15.57 & 16.08 & 14.88 & 11.97 \\ 8.16 & 15.57 & 31.07 & 23.11 & 23.26 & 20.98 \\ 6.68 & 16.08 & 23.11 & 38.31 & 30.12 & 30.09 \\ 4.98 & 14.88 & 23.26 & 30.12 & 45.98 & 39.29 \\ 3.06 & 11.97 & 20.98 & 30.09 & 39.29 & 59.11 \end{pmatrix}$$

Observed

$$\widehat{Cov}(\mathbf{y}_i) = \begin{pmatrix} 20.55 & 10.11 & 10.14 & 10.09 & 7.19 & 6.28 \\ 10.11 & 22.07 & 12.28 & 12.55 & 10.26 & 7.72 \\ 10.14 & 12.28 & 30.09 & 25.13 & 24.63 & 18.38 \\ 10.09 & 12.55 & 25.13 & 41.15 & 37.34 & 23.99 \\ 7.19 & 10.26 & 24.63 & 37.34 & 48.59 & 30.51 \\ 6.28 & 7.72 & 18.38 & 23.99 & 30.51 & 52.12 \end{pmatrix}$$

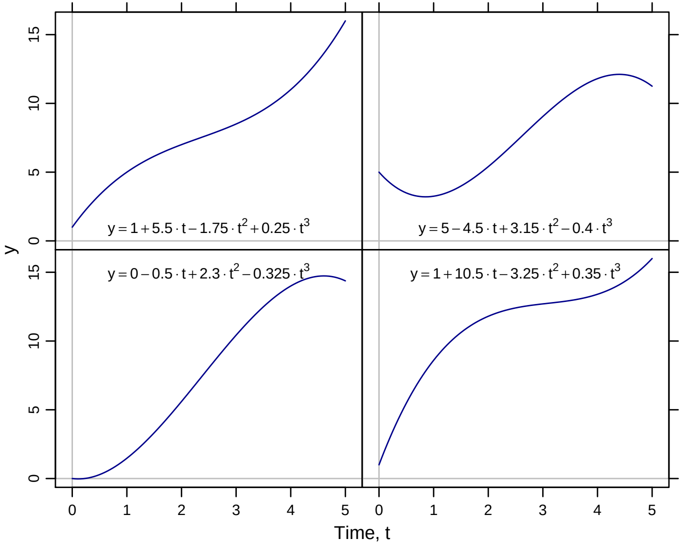
Centering variables

- If multiples of the time variables (t, t^2, t^3, etc.) are entered into the regression equation, multicollinearity can become a problem
- For example, $t = 0, 1, 2, 3$ and $t^2 = 0, 1, 4, 9$ correlate almost perfectly
- By centering the variables, this problem can be diminished: $(t - \bar{t}) = -1.5, -0.5, 0.5, 1.5$ and $(t - \bar{t})^2 = 2.25, 0.25, 0.25, 2.25$ are uncorrelated
- By centering variables the interpretation of the intercept in a linear model changes:
  - Uncentered intercepts represent the difference to the first time point $(t = 0)$
  - Centered intercepts represent the difference after half of the time

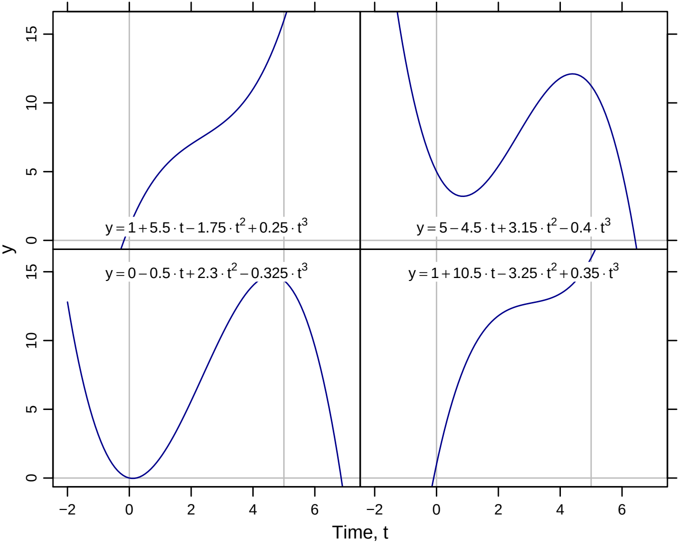
Higher-order polynomials

- Nonlinear time trends can be modelled in a flexibel and parsimonious way by using higher-order polynomials
- For example, saddle or reversal points in a time trend can be described
- Polynomials have the advantage that the regression model stays linear in its parameters
- They have the disadvantage that extrapolated values can quickly be outside of a range that can still be interpreted in a meaningful way

Cubic time trends



Polynomial regression: Extrapolation



2 Vocal charades

Example: Growth curve model

This example from linguistics is taken from Winter and Wieling (2016)

- Participants play a game of ‘vocal charades’
- At each round, a participant has to vocalize a meaning to the partner (e.g., ‘ugly’) without using language (e.g., through grunting or hissing)
- The partner has to guess the meaning of the vocalization
- This game is played repeatedly with the finding that over time, a dyad converges on a set of nonlinguistic vocalizations that assure a high degree of intelligibility between the two participants in the dyad

Example: Growth curve model

- Initially, participants may be struggling with the task and explore very different kinds of vocalizations
- Over time, they may converge on a more stable set of iconic vocalizations, that is vocalizations that resemble the intended referent (e.g., a high-pitched sound for 'attractive' and a low-pitched sound for 'ugly')
- Finally, after even more time, the dyad may conventionalize to idiosyncratic patterns that deviate from iconicity and become increasingly arbitrary

Example: Growth curve model

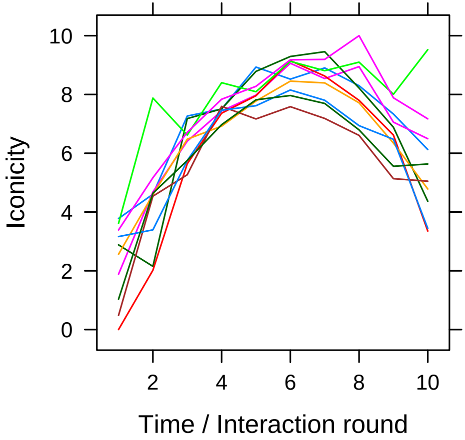
- 100 observations of three variables (simulated data set)

Variable	Description
`dyad`	different pairs of subjects playing the vocal charades game
`t`	sequential rounds for which the vocal charades game was played
`iconicity`	iconicity measure

- For better interpretation, `t` will be centered

```
dat$t_c <- dat$t - mean(dat$t)
```

Visualization of data




```
xyplot(
  iconicity ~ t, dat,
  groups = dyad,
  type = "l",
  xlab = "Time/Interaction round",
  ylab = "Iconicity")
```

Mixed-effects model with quadratic trend

- We will now consider a model with uncorrelated random effects

$$y_{ij} = \beta_0 + \beta_1 t_{ij} + \beta_2 t_{ij}^2 + \upsilon_{0i} + \upsilon_{1i} t_{ij} + \upsilon_{2i} t_{ij}^2 + \varepsilon_{ij}$$

with

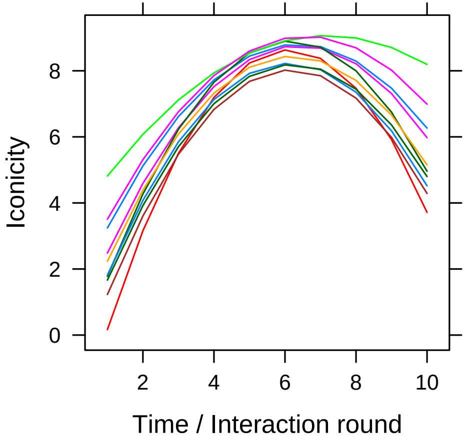
$$\begin{pmatrix} \upsilon_{0i} \\ \upsilon_{1i} \\ \upsilon_{2i} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{\upsilon_0}^2 & 0 & 0 \\ 0 & \sigma_{\upsilon_1}^2 & 0 \\ 0 & 0 & \sigma_{\upsilon_2}^2 \end{pmatrix} \right) \text{ i.i.d.}$$

$$\varepsilon_i \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{n_i}) \text{ i.i.d.}$$

- This model is fitted by

```
gcm1 <- lmer(iconicity ~ t_c + I(t_c^2) +
  (1 | dyad) + (0 + t_c | dyad) + (0 + I(t_c^2) | dyad),
  data = dat, REML = F)
```

Visualization of model predictions




```
xyplot(
  predict(gcm1) ~ t, dat,
  groups = dyad,
  type = "l",
  xlab = "Time/Interaction round",
  ylab = "Iconicity")
```

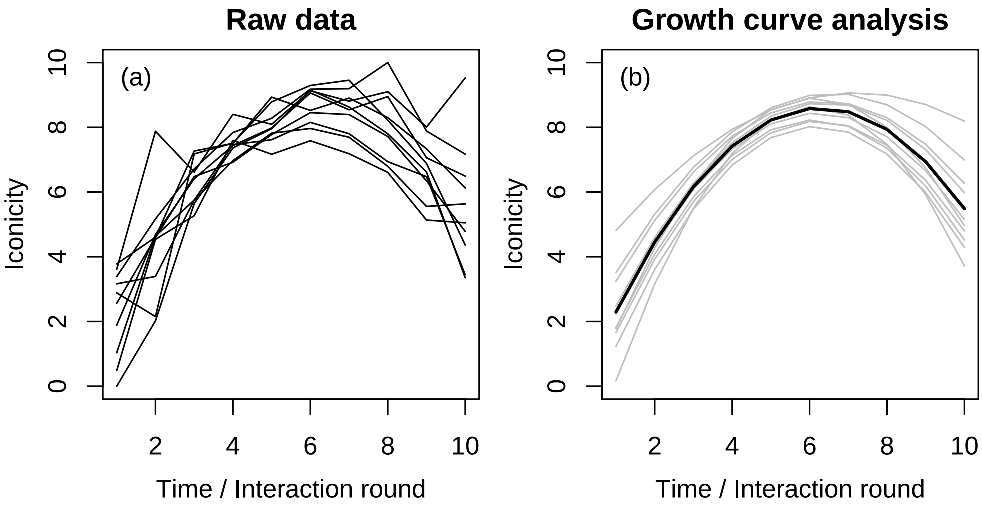
ML estimates of parameters

```
...
Random effects:
 Groups   Name        Variance Std.Dev.
 dyad     (Intercept) 0.152845 0.39095
 dyad.1   t_c         0.002595 0.05094
 dyad.2   I(t_c^2)    0.003504 0.05920
 Residual             0.429691 0.65551
Number of obs: 100, groups:  dyad, 10

Fixed effects:
            Estimate Std. Error t value
(Intercept)  8.45761    0.15849   53.36
t_c          0.35475    0.02793   12.70
I(t_c^2)    -0.22558    0.02078  -10.86
...
```

Interpretation of results

- There are now two slopes, one for the effect of linear time (t_c, $\beta_1 = 0.35$) and one for the effect of quadratic time (t_c^2, $\beta_2 = -0.23$), both of which are allowed to differ by dyad ($\sigma_{\upsilon_{1i}} = 0.05$ and $\sigma_{\upsilon_{2i}} = 0.06$)
- The negative value for the quadratic term indicates the inverse U-shape
- The point of reversal is $t = \bar{t} + \frac{-\hat{\beta}_1}{2 * \hat{\beta}_2} = 5.5 + \frac{-.35}{2*(-.22)} = 6.29$
- The model assumes that the random intercept and slopes are all uncorrelated

Plots from the article



References

Reisby, N., Gram, L. F., Bech, P., Nagy, A., Petersen, G. O., Ortmann, J., Ibsen, I., Dencker, S. J., Jacobsen, O., Krautwald, O., Sondergaard, I., & Christiansen, J. (1977).Imipramine: Clinical effects and pharmacokinetic variability. *Psychopharmacology*, *54*, 263–272.

Winter, B., & Wieling, M. (2016).How to analyze linguistic change using mixed models, growth curve analysis and generalized additive modeling. *Journal of Language Evolution*, *1*(1), 7–18. https://doi.org/10.1093/jole/lzv003