

# Simple and multiple linear regression

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# Outline

- ① Basic concepts
- ② Assumptions
- ③ Multiple linear regression

# What is regression?

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Set of statistical processes for estimating the relationships between a dependent variable (often called the 'outcome variable') and one or more independent variables (often called 'predictors', 'covariates', or 'features')

[https://en.wikipedia.org/wiki/Regression\\_analysis](https://en.wikipedia.org/wiki/Regression_analysis)

# What is regression?

Set of statistical processes for estimating the relationships between a dependent variable (often called the 'outcome variable') and one or more independent variables (often called 'predictors', 'covariates', or 'features')

[https://en.wikipedia.org/wiki/Regression\\_analysis](https://en.wikipedia.org/wiki/Regression_analysis)

- Predict an outcome variable
- Compare predictions for different groups
- "Find the line that most closely fits the data"
- Continuous outcome  $y$

## 1 Basic concepts

# Simple linear regression

- For the pairs

$$(x_1, y_1), \dots, (x_n, y_n),$$

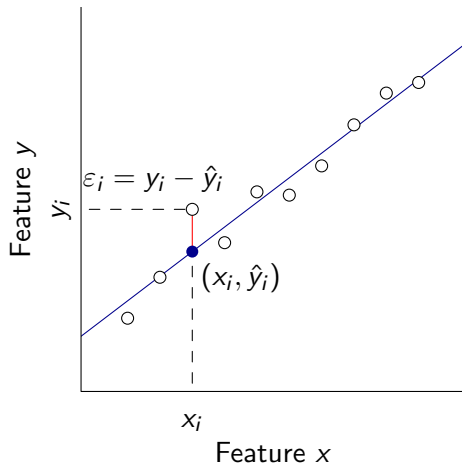
we get the stochastic model

$$y_i = \beta_0 + \beta_1 \cdot x_i + \varepsilon_i$$

$$\varepsilon_i \sim N(0, \sigma^2) \text{ i.i.d.}$$

for all  $i = 1, \dots, n$

- Errors are independent identically distributed (i.i.d.)



## Simple linear regression

- From the properties of the error variables, we conclude

$$E(y_i) = E(\beta_0 + \beta_1 \cdot x_i + \varepsilon_i) = \beta_0 + \beta_1 \cdot x_i = \bar{y}$$

and

$$\text{Var}(y_i) = \text{Var}(\beta_0 + \beta_1 \cdot x_i + \varepsilon_i) = \sigma^2$$

- For a given  $x_i$ , the stochastical independence of  $\varepsilon_i$  transfers to  $y_i$



## Parameter estimation

- The parameters  $\beta_0$  and  $\beta_1$  are estimated with the method of least squares
- Hereby, the sum of squares of the residuals is minimized

$$\sum_{i=1}^n \varepsilon_i^2 = \sum_{i=1}^n (y_i - \hat{y}_i)^2 = \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2 = \min$$

- The minimum is obtained by setting the partial derivatives for  $\beta_0$  and  $\beta_1$  to 0

$$\frac{\partial (\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2)}{\partial \beta_0} \quad \frac{\partial (\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2)}{\partial \beta_1}$$

- Solving these equations results in

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \quad \text{and} \quad \hat{\beta}_1 = \frac{\sigma_{xy}}{\sigma_x^2}$$

where  $\sigma_{xy}$  is the covariance between  $x$  and  $y$

## Correlation coefficient

- The correlation coefficient  $r$  is defined as the standardized covariance

$$r = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

- Hence, we get

$$\hat{\beta}_1 = \frac{\sigma_y}{\sigma_x} r$$

- When  $x$  and  $y$  are  $z$  standardized with  $\bar{x} = \bar{y} = 0$  and  $\sigma_x = \sigma_y = 1$ , we get

$$\hat{\beta}_0 = 0 \quad \text{and} \quad \beta_1 = r$$

## Determination coefficient

- With the assumptions made so far, it can be shown that the variance of the residuals

$$\sigma_{\varepsilon}^2 = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i)^2$$

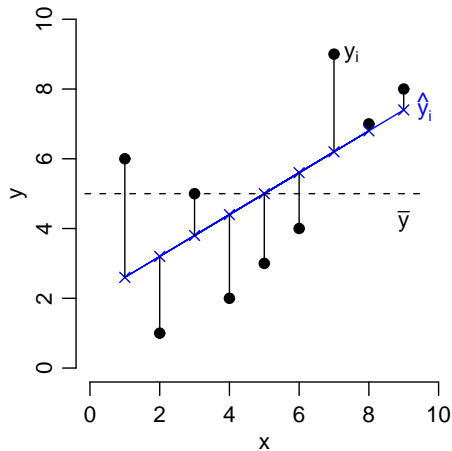
can be rewritten as

$$\sigma_{\varepsilon}^2 = (1 - r^2) \sigma_y^2$$

- The factor  $(1 - r^2)$  determines the proportion of the variance of  $y$  that cannot be explained by the regression of  $x$
- Hence, the determination coefficient  $r^2$  gives the proportion of the variance of  $y$  explained by  $x$

$$r^2 = \frac{\sigma_{\hat{y}}^2}{\sigma_y^2}$$

# Variance decomposition



$$\sigma_y^2 = \sigma_{\hat{y}}^2 + \sigma_e^2$$

$$\frac{1}{n} \sum_{i=1}^n (y_i - \bar{y})^2 =$$

$$\frac{1}{n} \sum_{i=1}^n (\hat{y}_i - \bar{y})^2 + \frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i)^2$$

## Exercise

- Simulate a data set based on a simple regression model with

$$\beta_0 = 0.2$$

$$\beta_1 = 0.3$$

$$\sigma = 0.5$$

$$x \in [1, 20] \text{ in steps of } 1$$

- What functions in *R* do we need?

## Simulate data set

```
x      <- 1:20
n      <- length(x)
a      <- 0.2
b      <- 0.3
sigma  <- 0.5
y      <- 0.2 + 0.3 * x + rnorm(n, sd = sigma)

dat <- data.frame(x, y)

# clean up workspace
rm(x, y)

# plot data
plot(y ~ x, data = dat)
```

## Fit regression model

```
lm1 <- lm(y ~ x, data = dat)
summary(lm1)

mean(resid(lm1))
sd(resid(lm1))
hist(resid(lm1), breaks = 15)

# plot data
plot(y ~ x, data = dat)
abline(lm1)
```

## Re-cover parameters

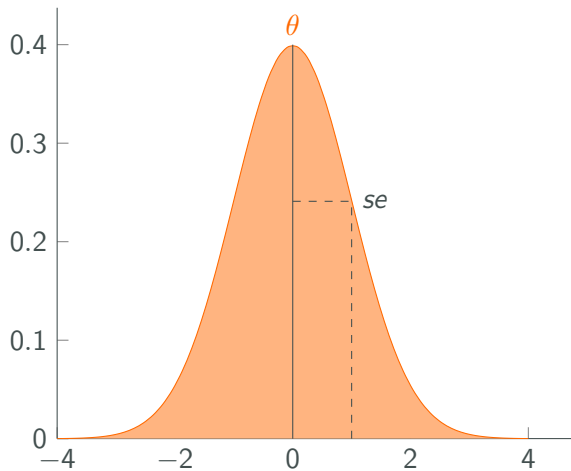
```
pars <- replicate(2000, {  
  ysim <- 0.2 + 0.3 * x + rnorm(n, sd = sigma)  
  lm1 <- lm(ysim ~ x, data = dat)  
  c(coef(lm1), sigma(lm1))  
})
```

```
rowMeans(pars)  
# standard errors  
apply(pars, 1, sd)
```

```
hist(pars[1, ])  
hist(pars[2, ])  
hist(pars[3, ])
```



## Sample distribution



## Exercise

- Simulate data with the parameters from slide 11
- Do not assume that we have one subject per value for  $x$ , but more than one subject
- Simulate data for  $n = 40$  and  $n = 100$   
Hint: Use `sample(x, n, replace = TRUE)`
- Re-cover your parameters as done on slide 14
- What happens to your standard errors?

## Confidence intervals

- We get the  $(1 - \alpha)$  confidence intervals for the estimates with

$$\left[ \hat{\beta}_0 - \hat{\sigma}_{\hat{\beta}_0} t_{1-\beta_0/2}(n-2), \hat{\beta}_0 + \hat{\sigma}_{\hat{\beta}_0} t_{1-\beta_0/2}(n-2) \right]$$

and

$$\left[ \hat{\beta}_1 - \hat{\sigma}_{\hat{\beta}_1} t_{1-\alpha/2}(n-2), \hat{\beta}_1 + \hat{\sigma}_{\hat{\beta}_1} t_{1-\alpha/2}(n-2) \right]$$

- For  $n > 30$  the  $t$  quantiles of the  $t(n-2)$  distribution can be replaced by quantiles of the  $N(0, 1)$  distribution
- For a sufficient sample size, even when the normality assumption is violated, the least square estimators are approximately  $t$  or normally distributed

# Hypothesis tests

## Wald test

- The estimates for  $\beta_0$  and  $\beta_1$  are unbiased, sufficient, consistent, and efficient
- The normality assumption implies

$$\hat{\beta}_0 \sim N(\beta_0, \sigma_{\hat{\beta}_0}^2) \quad \text{and} \quad \hat{\beta}_1 \sim N(\beta_1, \sigma_{\hat{\beta}_1}^2)$$

and with that for some hypothetical value  $\gamma_0$

$$T_{\beta_0} = \frac{\hat{\beta}_0 - \gamma_0}{\hat{\sigma}_{\hat{\beta}_0}} \sim t(n-2) \quad \text{und} \quad T_{\beta_1} = \frac{\hat{\beta}_1 - \gamma_0}{\hat{\sigma}_{\hat{\beta}_1}} \sim t(n-2)$$

with estimates  $\hat{\sigma}_{\hat{\beta}_0}^2$  and  $\hat{\sigma}_{\hat{\beta}_1}^2$  for the variances

- We are usually interested in the hypotheses  $\beta_0 = 0$  and  $\beta_1 = 0$ , hence  $\gamma_0 = 0$  and

$$T_{\beta_0} = \frac{\hat{\beta}_0}{\hat{\sigma}_{\hat{\beta}_0}} \sim t(n-2) \quad \text{und} \quad T_{\beta_1} = \frac{\hat{\beta}_1}{\hat{\sigma}_{\hat{\beta}_1}} \sim t(n-2)$$

## Overall $F$ test

- For simple linear regression, we can construct an equivalent test for testing  $\beta_1 = 0$  using variance decomposition
- This test can conceptionally be considered to test, if predictor  $x$  explains a significant proportion of the variance of  $y$

$$F = \frac{\frac{\sum_{i=1}^n (\hat{y}_i - \bar{y})^2}{1}}{\frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{n - 2}} = \frac{R^2}{1 - R^2} (n - 2)$$

- It can be shown that  $F = T^2 - \gamma_0$  for  $\gamma_0 = 0$  with  $F \sim F(1, n - 2)$
- With this more general test, we can also test the assumption that any variance of  $y$  is explained by the predictors in a multiple regression

## ② Assumptions

## Assumptions

- We have the stochastic model  $y_i = \beta_0 + \beta_1 \cdot x_i + \varepsilon_i$  with  $\varepsilon_i \sim N(0, \sigma^2)$  i.i.d.
- The error variables  $\varepsilon_i$  are considered to be non-observable and comprise influence that cannot be controlled and is unsystematic (or random)
- Hence, it makes sense to assume

$$E(\varepsilon_i) = 0, \text{ for all } i = 1, \dots, n$$

and

$$\text{Var}(\varepsilon_i) = \sigma^2, \text{ for all } i = 1, \dots, n$$

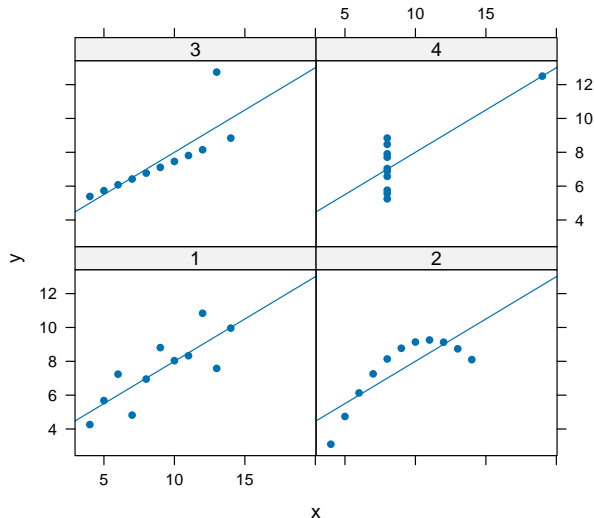
- From this follows

$$E(y_i) = E(\beta_0 + \beta_1 \cdot x_i + \varepsilon_i) = \beta_0 + \beta_1 \cdot x_i$$

and

$$\text{Var}(y_i) = \text{Var}(\beta_0 + \beta_1 \cdot x_i + \varepsilon_i) = \sigma^2$$

# Assumptions



- Four data sets by Anscombe (1973) with the same traditional statistical properties (mean, variance, correlation, regression line, etc.)
- Available in R with `data(anscombe)`



# Assumptions

```
data(anscombe)

lm1 <- lm(y1 ~ x1, anscombe)
lm2 <- lm(y2 ~ x2, anscombe)
lm3 <- lm(y3 ~ x3, anscombe)
lm4 <- lm(y4 ~ x4, anscombe)

rbind(coef(lm1), coef(lm2), coef(lm3), coef(lm4))

par(mfrow = c(2, 2))
plot(lm1)
plot(lm2)
plot(lm3)
plot(lm4)
```

## Exercise

- Create two vectors  $x$  and  $y$  with 100 observations each and  $X \sim N(1, 1)$  and  $Y \sim N(2, 1)$
- Create a data frame with variables `id`, `group` and `score`.  $X$  and  $Y$  are your score values
- Conduct a  $t$  test assuming that  $X$  and  $Y$  are independent having the same variances
- Then use the function `aov()` to compute an analysis of variance for these data
- Use then function `lm()` for a linear regression with predictor `group` and dependent variable `score`
- Compare your results

## Extending simple linear regression

Additional predictors  $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \varepsilon$

Nonlinear models  $\log y = \beta_0 + \beta_1 \log x + \varepsilon$

Nonadditive models  $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + \varepsilon$

Generalized linear models  $g(E(y)) = \beta_0 + \beta_1 x$

Mixed-effects models  $y = \beta_0 + \beta_1 x_1 + \beta_2 \textit{time} + v_0 + v_1 \textit{time} + \varepsilon$

...

### ③ Multiple linear regression

## Multiple linear regression

- Empirical observations consist of tuples for each observation unit

$$(y_i, x_{i1}, \dots, x_{ip}) \text{ with } i = 1, \dots, n$$

and we get the stochastical model

$$y_i = \beta_0 + \beta_1 \cdot x_{i1} + \dots + \beta_p \cdot x_{ip} + \varepsilon_i$$

$$\varepsilon_i \sim N(0, \sigma^2) \text{ i.i.d.}$$

which transfers to

$$y_i \sim N(\mu_i, \sigma^2) \text{ with } \mu_i = \beta_0 + \beta_1 \cdot x_{i1} + \dots + \beta_p \cdot x_{ip}$$

- The criterion variable  $y$  is always a metric variable, whereas the predictor variables  $x_1, \dots, x_p$  can be either metric or categorical variables, or both

## Overall $F$ test

- Hypotheses

$$H_0: \beta_1 = \dots = \beta_p = 0$$

$$H_1: \beta_j \neq 0 \text{ for at least one } j \in \{1, \dots, p\}$$

- Test statistic

$$F = \frac{R^2}{1 - R^2} \cdot \frac{n - p - 1}{p}$$

- Distribution of test statistic assuming  $H_0$  is true

$$F \sim F(p, n - p - 1)$$

- Rejection region

$$F > F_{1-\alpha}(p, n - p - 1)$$

## Incremental $F$ test

- We have two nested models  $M_1$  and  $M_0$ , meaning that  $M_0$  is a special case of  $M_1$  where some parameters  $\beta_{M_0,j} = 0$  and  $\beta_{M_1,j} \neq 0$  and want to test if

$$R_1^2 > R_0^2$$

- Test statistic

$$F = \frac{R_1^2 - R_0^2}{1 - R_1^2} \cdot \frac{n - q_1}{q_1 - q_0}$$

- Distribution of test statistic assuming  $H_0$  is true

$$F \sim F(q_1 - q_0, n - q_1)$$

- Rejection region

$$F > F_{1-\alpha}(q_1 - q_0, n - q_1)$$

## Likelihood ratio test

- The incremental  $F$  test is a special case of the more general likelihood ratio test

$$G^2 = 2 \log \frac{L_1}{L_0}$$

where  $L_1$  is the likelihood of the more general model and  $L_0$  the likelihood of the smaller model

- The models need to be nested
- The test statistic  $G^2$  is  $\chi^2$  distributed

$$G^2 \sim \chi^2(q_1 - q_0)$$

where  $q_1$  and  $q_0$  are the number of parameters for the bigger and the smaller model, respectively



## Example: Multiple linear regression

- We are fitting the following model

$$y_{ij} = \beta_0 + \beta_1 \cdot x_i + \beta_2 \cdot z_j + \varepsilon_{ij}$$

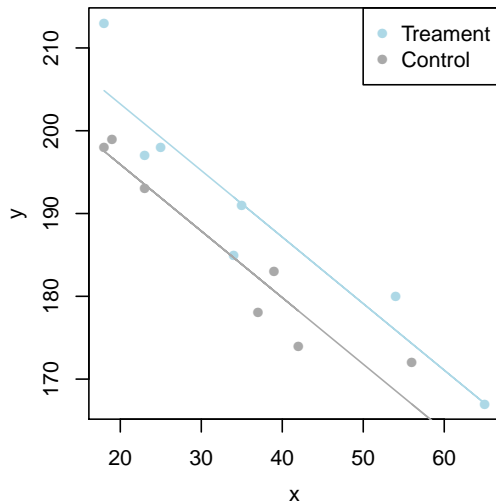
with  $i = 1 \dots N$  and  $j = 1, 2$  for two groups

- This means that we have one dummy variable for  $z$  which takes the values 0 and 1
- Hence, we get the two models

$$y_{i1} = \beta_0 + \beta_1 \cdot x_i + \beta_2 \cdot 0 + \varepsilon_{ij} = \beta_0 + \beta_1 \cdot x_i + \varepsilon_{ij}$$

$$y_{i2} = \beta_0 + \beta_1 \cdot x_i + \beta_2 \cdot 1 + \varepsilon_{ij} = (\beta_0 + \beta_2) + \beta_1 \cdot x_i + \varepsilon_{ij}$$

## Example: Multiple linear regression



## Example: Multiple linear regression

```
dat <- data.frame(  
  x = c(18, 23, 25, 35, 65, 54, 34, 56, 72, 19, 23, 42, 18, 39, 37),  
  y = c(213, 197, 198, 191, 167, 180, 185, 172, 153, 199, 193, 174,  
        198, 183, 178),  
  z = rep(c("treatment", "control"), c(7, 8))  
)  
  
aggregate(y ~ z, dat, mean)
```

## Example: Multiple linear regression

- We can now use the parameters to calculate adjusted means for the two groups
- The observed means are  $\bar{x}_{treat} = 190.14$  and  $\bar{x}_{contr} = 181.25$
- The adjusted means correspond to

$$\bar{x}_{contr} = \beta_0$$

$$\bar{x}_{treat} = \beta_0 + \beta_2$$

These are the means for a value of  $x = 0$  which should have a meaningful interpretation

- Hence, it might be indicated to center  $x$

## Example: Multiple linear regression

```
dat$xc <- dat$x - mean(dat$x)

lm2 <- lm(y ~ xc + z, dat)
summary(lm2)

# adjusted means
coef(lm2)[1]
coef(lm2)[1] + coef(lm2)[3]
```

## Exercise

- The data set `cars` contains speed and stopping distances of 50 cars
- Estimate the regression model

$$dist_i = \beta_0 + \beta_1 speed_i + \varepsilon_i$$

- How much variance of the stopping distances is explained by speed?
- Look at the residuals of the model. Are there any systematic deviances?
- Now estimate the model

$$dist_i = \beta_0 + \beta_1 speed_i + \beta_2 speed_i^2 + \varepsilon_i$$

Hint: Use `I(speed^2)` in the model formula in R

- Which model fits the data better?

## References

Anscombe, F. J. (1973). Graphs in statistical analysis. *The American Statistician*, 27(1), 17–21.

Gelman, A., Hill, J., & Vehtari, A. (2020). *Regression and other stories*. Cambridge University Press.