Power simulation for mixed-effects models

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Outline

1 Introduction mixed-effects models

Mixed-effects models

- Mixed-effects models are a class of statistical models that include fixed effects as well as random effects
- Fixed effects vs. random effects[1]
  - For fixed effects, only effects of the factor levels used in the present study are considered (manipulated conditions, e. g., assigned groups, but also sex, or other variables . . .)

    → Of interest is how these levels differ
  - For random effects, the factor levels considered in a study are regarded as a (random) sample from some population (e. g., words, raters, subjects, . . .)

    → Of interest are conlusions about the underlying population and its variation

[1] Some critical discussion on these definitions:
`http://andrewgelman.com/2005/01/25/why_i_dont_use/`

Linear mixed-effects model

- The linear mixed-effects model has the general form

$$\mathbf{y}_i = \mathbf{X}_i \boldsymbol{\beta} + \mathbf{Z}_i \boldsymbol{\upsilon}_i + \boldsymbol{\varepsilon}_i$$

with fixed effects $\boldsymbol{\beta}$, random effects $\boldsymbol{\upsilon}_i$, and the design matrices $\mathbf{X}_i$ and $\mathbf{Z}_i$ and the assumptions

$$\boldsymbol{\upsilon}_i \overset{iid}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\upsilon), \qquad \boldsymbol{\varepsilon}_i \overset{iid}{\sim} N(\mathbf{0}, \sigma^2 \mathbf{I}_{n_i})$$

- This implies for the marginal covariance matrix

$$Cov(\mathbf{y}_i) = \boldsymbol{\Sigma}_i = \mathbf{Z}_i \boldsymbol{\Sigma}_\upsilon \mathbf{Z}_i' + \sigma^2 \mathbf{I}_{n_i}$$

Linear mixed-effects model

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} 1 & x_{11} & x_{12} & \dots & x_{1p} \\ 1 & x_{21} & x_{22} & \dots & x_{2p} \\ 1 & x_{31} & x_{32} & \dots & x_{3p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{N1} & x_{N2} & \dots & x_{Np} \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix} + \begin{pmatrix} z_{10} & z_{11} & \dots & z_{1q} & \dots \\ z_{20} & z_{21} & \dots & z_{2q} & \dots \\ z_{30} & z_{31} & \dots & z_{3q} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{N0} & z_{N1} & \dots & z_{Nq} & \dots \end{pmatrix} \cdot \begin{pmatrix} \upsilon_{10} \\ \vdots \\ \upsilon_{1q} \\ \upsilon_{20} \\ \vdots \\ \upsilon_{Nq} \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \vdots \\ \varepsilon_N \end{pmatrix}$$

2 Longitudinal data analysis

Longitudinal data

- Consist of repeated measurements on the same subject taken over time
- Are a frequent use case for mixed-effects models
- Contain time as a predictor: time trends within and between subjects are of interest

```
library(lme4)
data(sleepstudy)
?sleepstudy
str(sleepstudy)
summary(sleepstudy)
head(sleepstudy)
```

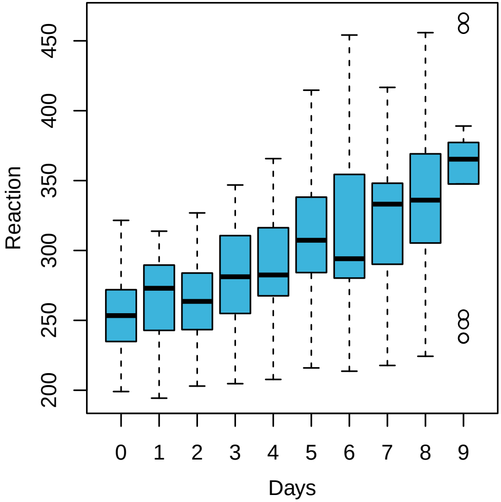
Sleep study

- Average reaction time per day for subjects in a sleep deprivation study
- On day 0, the subjects had their normal amount of sleep
- Starting that night they were restricted to 3 hours of sleep per night
- Observations represent the average reaction time on a series of tests given each day to each subject

A data frame with 180 observations on the following 3 variables

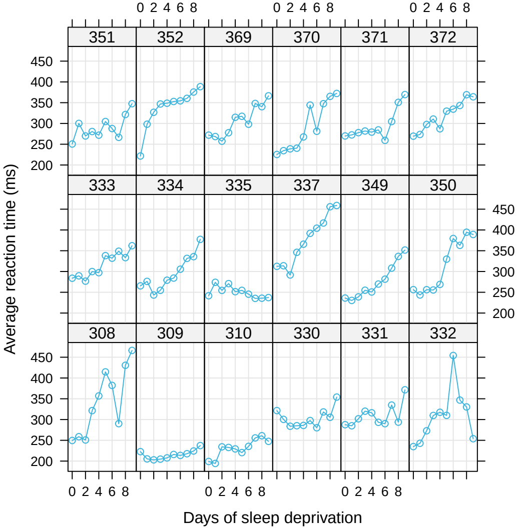
`Reaction`	Average reaction time (ms)
`Days`	Number of days of sleep deprivation
`Subject`	Subject number on which the observation was made

Visualization of data




```
boxplot(Reaction ~ Days, sleepstudy)
```

Visualization of individual data




```r
library(lattice)

xyplot(Reaction ~ Days | Subject,
  data = sleepstudy,
  type = c("g", "b"),
  xlab = "Days of sleep deprivation",
  ylab = "Average reaction time (ms)",
  aspect = "xy")
```

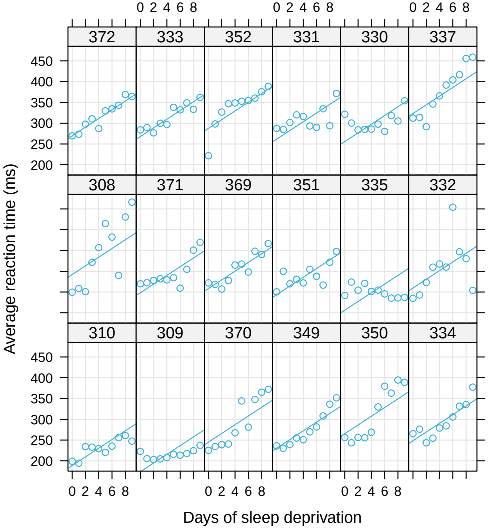
Random intercept model

- The random intercept model adds a random intercept for each subject

$$y_{ij} = \beta_0 + \beta_1\, Days_{ij} + \upsilon_{0i} + \varepsilon_i$$

with $\upsilon_{0i} \overset{iid}{\sim} N(0, \sigma_\upsilon^2)$, $\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$

- The slope is identical for each subject (and the population)



Random slope model

- The random slope model adds a random intercept and a random slope for each subject

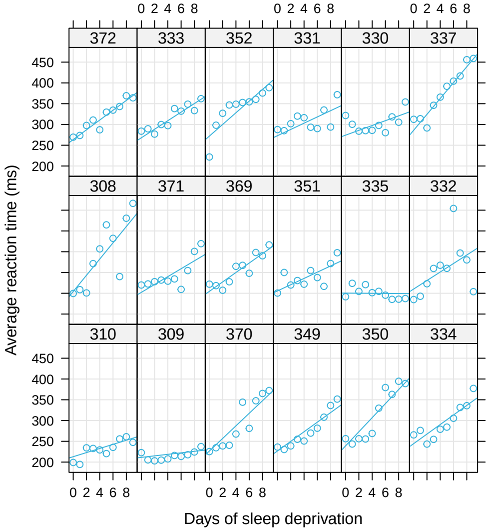
$$y_{ij} = \beta_0 + \beta_1 \, Days_{ij} + \upsilon_{0i} + \upsilon_{1i} \, Days_{ij} + \varepsilon_{ij}$$

with

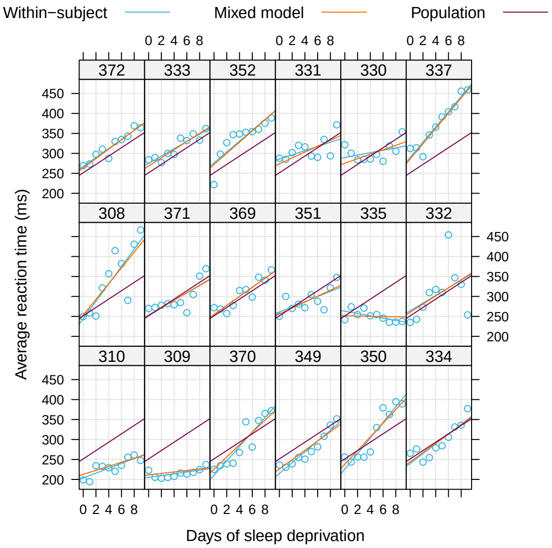
$$\begin{pmatrix} \upsilon_{0i} \\ \upsilon_{1i} \end{pmatrix} \overset{iid}{\sim} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{\Sigma}_\upsilon = \begin{pmatrix} \sigma^2_{\upsilon_0} & \sigma_{\upsilon_0 \upsilon_1} \\ \sigma_{\upsilon_0 \upsilon_1} & \sigma^2_{\upsilon_1} \end{pmatrix} \right)$$

$$\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2_\varepsilon)$$

- Individual slopes for each subject



Partial pooling



- Within-subject regression line shows regression line fitted to data for each individual
- Population regression line shows fixed effects for mixed-effects model
- Mixed model regression line shows individual regression lines as predicted by mixed-effects models

3 Crossed random effects

Crossed random effects

- In many experiments in psychology the reaction of each subject ($j = 1, \ldots, N$) to a complete set of stimuli or items ($k = 1, \ldots, K$) is measured

$$y_{ijk} = \beta_0 + \beta_i x_i + \upsilon_{0j} + \eta_{0k} + \varepsilon_{ijk}$$

with $\varepsilon_{ijk} \overset{iid}{\sim} N(0, \sigma^2)$, $\upsilon_{0j} \overset{iid}{\sim} N(0, \sigma^2_\upsilon)$, and $\eta_{0k} \overset{iid}{\sim} N(0, \sigma^2_\eta)$

- Data are completely crossed: all subjects are presented with all items

		Subject				
		1	2	3	...	20
	1	1	1	1	...	1
	2	1	1	1	...	1
Item	3	1	1	1	...	1
	⋮	⋮	⋮	⋮	⋮	⋮
	10	1	1	1	...	1

Lexical decision task (Baayen, Davidson, & Bates, 2008)

- Assume an example data set with three participants s1, s2 and s3 who each saw three items w1, w2, w3 in a priming lexical decision task under both short and long stimulus onset asynchrony (SOA) conditions
- The data are generated by the following model with random intercepts for subject and item, and random slopes for subject

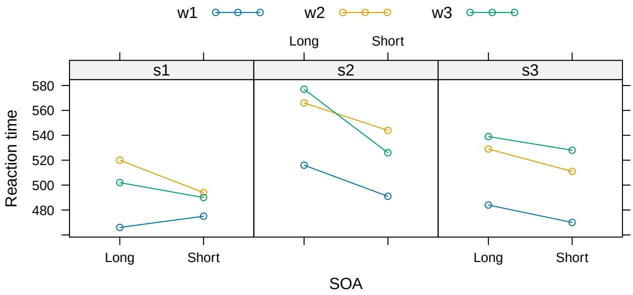
$$y_{ijk} = \beta_0 + \beta_1 SOA_k + \eta_{0j} + \upsilon_{0i} + \upsilon_{1i} SOA_k + \varepsilon_{ijk}$$

with $\boldsymbol{\upsilon} \sim N\left(\mathbf{0}, \boldsymbol{\Sigma}_\upsilon = \begin{pmatrix} \sigma^2_{\upsilon_0} & \sigma_{\upsilon_0\upsilon_1} \\ \sigma_{\upsilon_0\upsilon_1} & \sigma^2_{\upsilon_1} \end{pmatrix}\right)$, $\eta_{0j} \sim N(0, \sigma^2_\eta)$, $\varepsilon_{ijk} \sim N(0, \sigma^2_\varepsilon)$, all i.i.d.

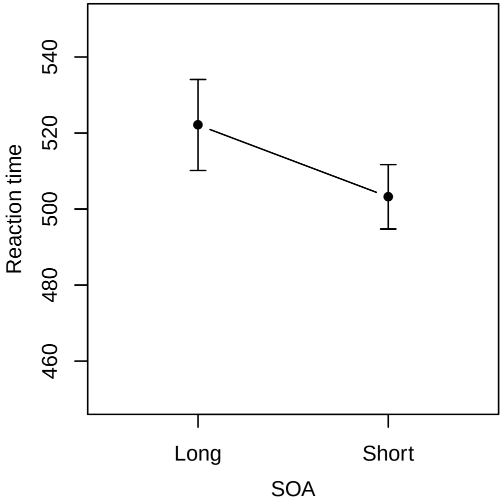
Structure of the data set

Subj	Item	SOA	RT
s1	w1	Long	466
s1	w2	Long	520
s1	w3	Long	502
s1	w1	Short	475
s1	w2	Short	494
s1	w3	Short	490
s2	w1	Long	516
s2	w2	Long	566
s2	w3	Long	577
s2	w1	Short	491
s2	w2	Short	544
s2	w3	Short	526
s3	w1	Long	484
s3	w2	Long	529
s3	w3	Long	539
s3	w1	Short	470
s3	w2	Short	511
s3	w3	Short	528

- When we collect data, we might get a data set like this
- We fit a model to the data to separate the structural and the stochastical parts



Aggregated data



Structure of the data set

Subj	Item	SOA	RT	Fixed		Random			Res
				Int	SOA	ItemInt	SubInt	SubSOA	
s1	w1	Long	466	522.2	0	−28.3	−26.2	0	−2.0
s1	w2	Long	520	522.2	0	14.2	−26.2	0	9.8
s1	w3	Long	502	522.2	0	14.1	−26.2	0	−8.2
s1	w1	Short	475	522.2	−19	−28.3	−26.2	11	15.4
s1	w2	Short	494	522.2	−19	14.2	−26.2	11	−8.4
s1	w3	Short	490	522.2	−19	14.1	−26.2	11	−11.9
s2	w1	Long	516	522.2	0	−28.3	29.7	0	−7.4
s2	w2	Long	566	522.2	0	14.2	29.7	0	0.1
s2	w3	Long	577	522.2	0	14.1	29.7	0	11.5
s2	w1	Short	491	522.2	−19	−28.3	29.7	−12.5	−1.5
s2	w2	Short	544	522.2	−19	14.2	29.7	−12.5	8.9
s2	w3	Short	526	522.2	−19	14.1	29.7	−12.5	−8.2
s3	w1	Long	484	522.2	0	−28.3	−3.5	0	−6.3
s3	w2	Long	529	522.2	0	14.2	−3.5	0	−3.5
s3	w3	Long	539	522.2	0	14.1	−3.5	0	6.0
s3	w1	Short	470	522.2	−19	−28.3	−3.5	1.5	−2.9
s3	w2	Short	511	522.2	−19	14.2	−3.5	1.5	−4.6
s3	w3	Short	528	522.2	−19	14.1	−3.5	1.5	13.2
						$\sigma^2_{\eta 0}$	$\sigma^2_{\upsilon 0}$	$\sigma^2_{\upsilon 1}$	σ^2_{ε}
							$\sigma_{\upsilon 0 \upsilon 1}$		

True values

- We assume the following true parameters for a data simulation

Parameter	Model
β_0	522.22
β_1	-19.00
σ_η	21.00
σ_{υ_0}	24.00
σ_{υ_1}	7.00
$\rho_{\upsilon_0 \upsilon_1}$	-0.70
σ_ε	9.00

$$y_{ijk} = \beta_0 + \beta_1 SOA_k + \eta_{0j} + \upsilon_{0i} + \upsilon_{1i} SOA + \varepsilon_{ijk}$$

with $\boldsymbol{\upsilon} \sim N\left(\mathbf{0}, \boldsymbol{\Sigma}_\upsilon = \begin{pmatrix} \sigma^2_{\upsilon_0} & \sigma_{\upsilon_0 \upsilon_1} \\ \sigma_{\upsilon_0 \upsilon_1} & \sigma^2_{\upsilon_1} \end{pmatrix}\right)$, $\eta_{0j} \sim N(0, \sigma^2_\eta)$, $\varepsilon_{ijk} \sim N(0, \sigma^2_\varepsilon)$

Matrix notation

For this simple example the model looks like this in matrix notation

$$
\begin{pmatrix} y_{111}\\ y_{121}\\ y_{131}\\ y_{112}\\ y_{122}\\ y_{132}\\ y_{211}\\ y_{221}\\ y_{231}\\ y_{212}\\ y_{222}\\ y_{232}\\ y_{311}\\ y_{321}\\ y_{331}\\ y_{312}\\ y_{322}\\ y_{332} \end{pmatrix}
=
\begin{pmatrix} 1 & 0\\ 1 & 0\\ 1 & 0\\ 1 & 1\\ 1 & 1\\ 1 & 1\\ 1 & 0\\ 1 & 0\\ 1 & 0\\ 1 & 1\\ 1 & 1\\ 1 & 1\\ 1 & 0\\ 1 & 0\\ 1 & 0\\ 1 & 1\\ 1 & 1\\ 1 & 1 \end{pmatrix}
\cdot \begin{pmatrix} \beta_0\\ \beta_1 \end{pmatrix} +
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0\\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0\\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0\\
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1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1\\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1\\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}
\cdot
\begin{pmatrix} \eta_{01}\\ \eta_{02}\\ \eta_{03}\\ \upsilon_{01}\\ \upsilon_{02}\\ \upsilon_{03}\\ \upsilon_{11}\\ \upsilon_{12}\\ \upsilon_{13} \end{pmatrix}
+
\begin{pmatrix} \varepsilon_{111}\\ \varepsilon_{121}\\ \varepsilon_{131}\\ \varepsilon_{112}\\ \varepsilon_{122}\\ \varepsilon_{132}\\ \varepsilon_{211}\\ \varepsilon_{221}\\ \varepsilon_{231}\\ \varepsilon_{212}\\ \varepsilon_{222}\\ \varepsilon_{232}\\ \varepsilon_{311}\\ \varepsilon_{321}\\ \varepsilon_{331}\\ \varepsilon_{312}\\ \varepsilon_{322}\\ \varepsilon_{332} \end{pmatrix}
$$

References

Baayen, R. H., Davidson, D. J., & Bates, D. M. (2008). Mixed-effects modeling with crossed random effects for subjects and items. *Journal of memory and language*, *59*(4), 390–412. doi: 10.1016/j.jml.2007.12.005

Bates, D. (2010). *lme4: Mixed-effects modeling with R (book draft).* Retrieved from `https://lme4.r-forge.r-project.org/`

Bates, D., Mächler, M., Bolker, B., & Walker, S. (2015). Fitting linear mixed-effects models using lme4. *Journal of Statistical Software*, *67*(1), 1–48. Retrieved from `https://CRAN.R-project.org/package=lme4/vignettes/lmer.pdf` doi: 10.18637/jss.v067.i01