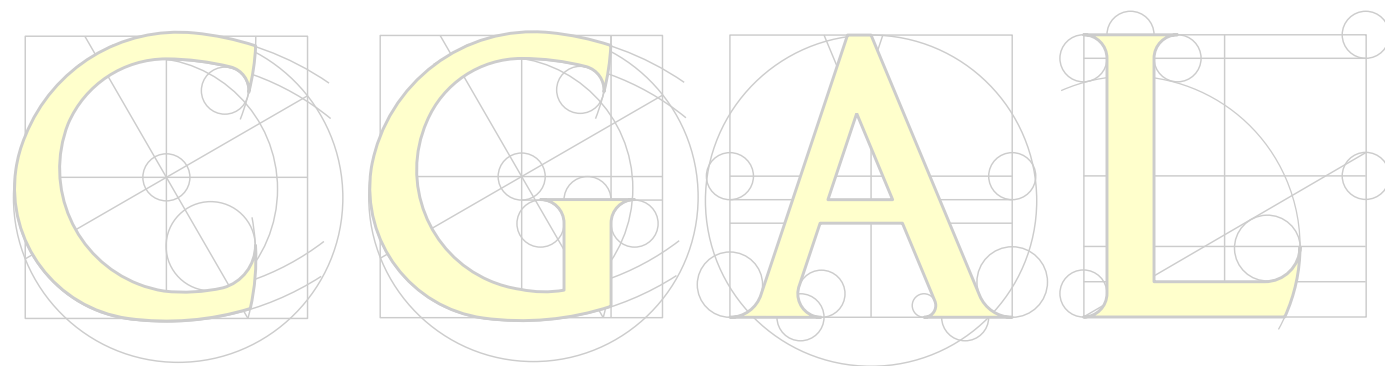
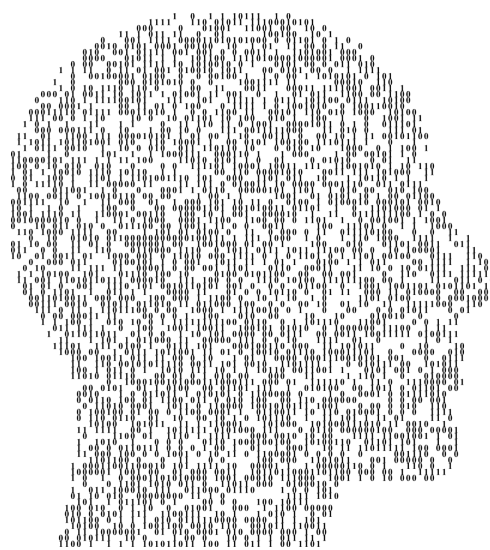


GEOMETRY & ALGEBRA IN THE ALGORITHMS LAB

Michael Hoffmann <hoffmann@inf.ethz.ch>

- I: Numbers: Representation & Computation
- II: Basic Geometric Computing using CGAL



Also check out the (incomplete) lecture notes on moodle.

RECOMMENDATIONS...

- ▶ Check out “A Short Introduction to C++ for the Algorithms Lab”: It is essential for some problems.
- ▶ Multiplicative constants matter: Asymptotics provide a baseline, but being 5 times slower does not cut it in most cases.
- ▶ Avoid “weird” optimizations: Think how to make your solution more efficient and avoid unnecessary computations. But only where it makes sense, also from a “theoretical” perspective.



PART I:

Numbers: Representation & Computation

OUTLINE

When working with numbers, choose an appropriate datatype for representation and computation.

- ▶ Does the input fit?
- ▶ Do the results of computations fit?
- ▶ Do not blindly trust limited precision arithmetic!
- ▶ Use exact algebraic computing where needed.

Here: Black box tool



GOALS

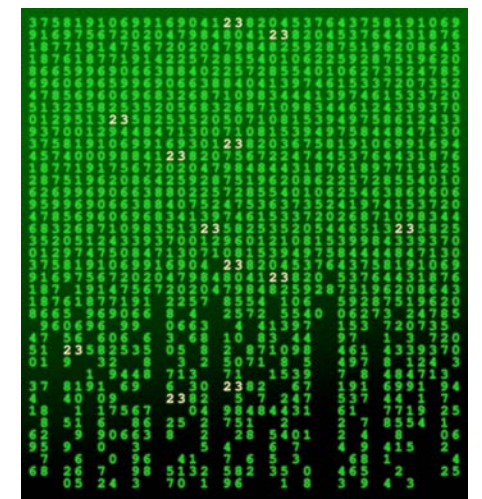
Awareness of discrepancy: infinite abstraction vs. finite computer.



- ▶ Consequences of using limited precision arithmetic for discrete decisions.
- ▶ How to ensure correctness.

Basic knowledge of limited precision arithmetic.

- ▶ How large is `int`, `long`, `double`, ...?
- ▶ How to bound results of a computation in terms of the input numbers.



DOES THE INPUT FIT?

Check Section 2.5 in “A Short Introduction to C++ for the Algorithms Lab”.

▶ If possible, read as an **int**

▶ else read as a **long**

32bit on Code Expert

64bit on Code Expert

Typical for 64-bit computers,
but not universally true....

Sanity check

```
#include <limits>
#include <stdexcept>
```

```
if (std::numeric_limits<int>::max() < 33554432.0)
    throw std::runtime_error("max(int) < 2^(25)");
```

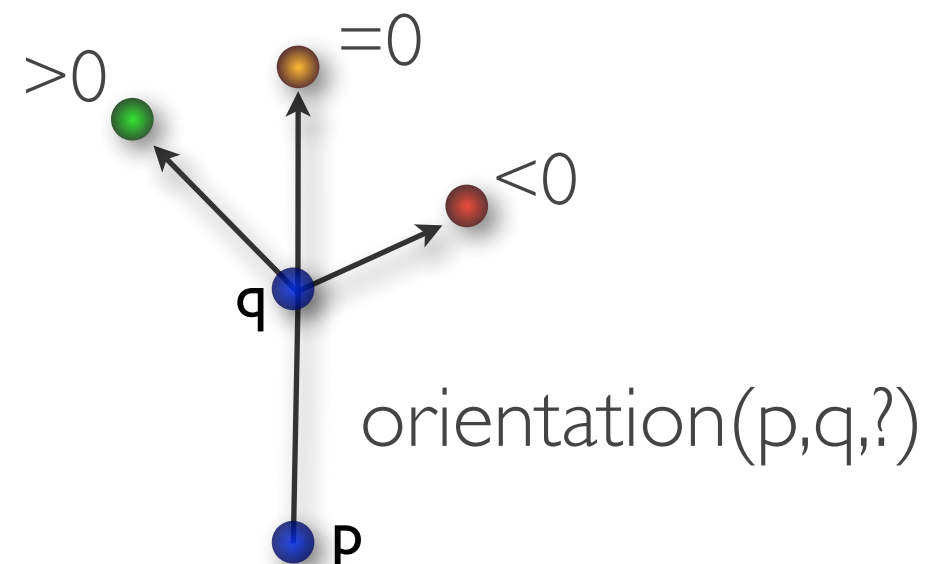
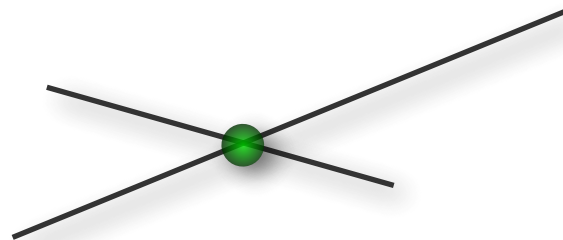
double literal for 2^{25}

COMPUTATIONS

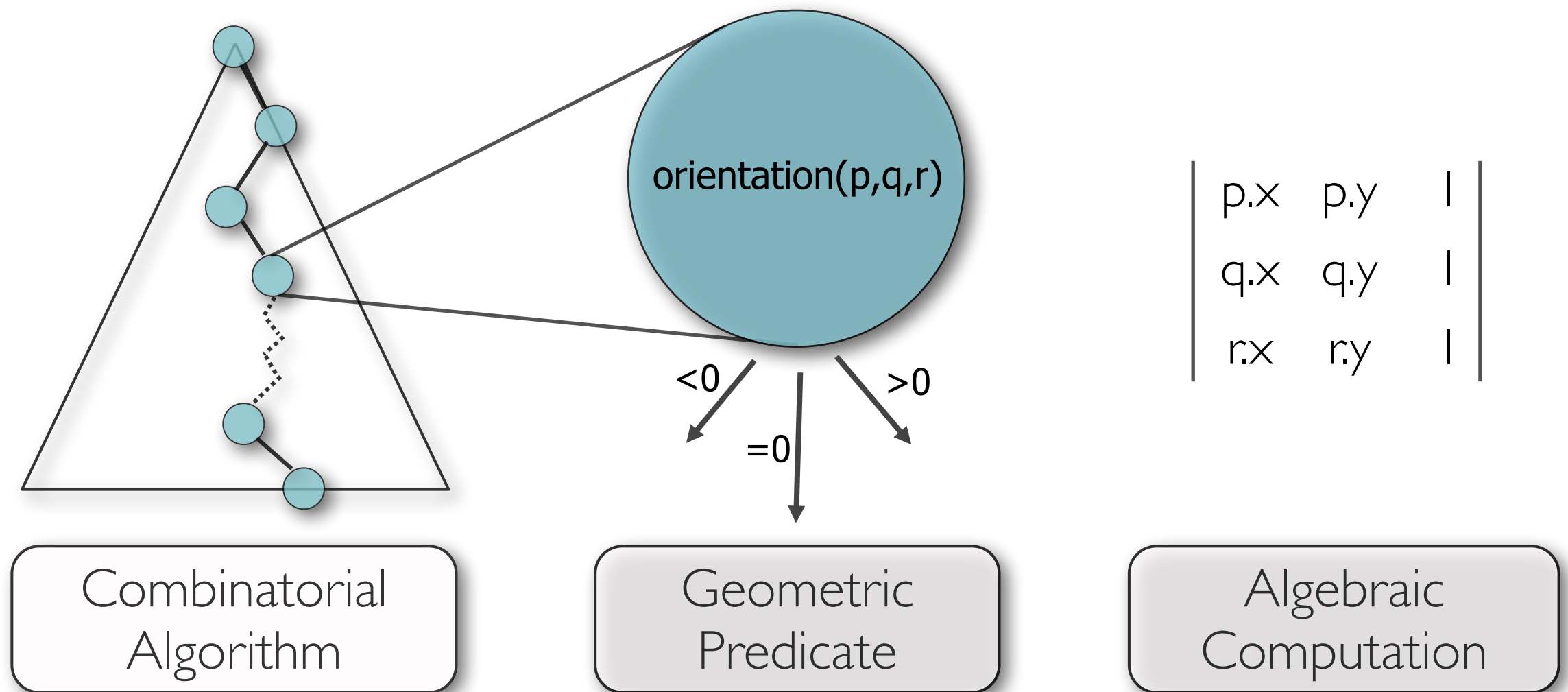
Some tasks require nontrivial computations, e.g.,

- ▶ computing Euclidean distances
- ▶ solving a linear system
- ▶ computing orientations of point triples

$$\begin{aligned}a_1 \cdot x + b_1 \cdot y &= c_1 \\a_2 \cdot x + b_2 \cdot y &= c_2\end{aligned}$$



LAYERS OF GEOMETRIC ALGORITHMS



Control flow depends on nontrivial algebraic computations.
How to do these efficiently and consistently?

ARITHMETIC

If computations are done using **limited precision** (floating point) arithmetic...

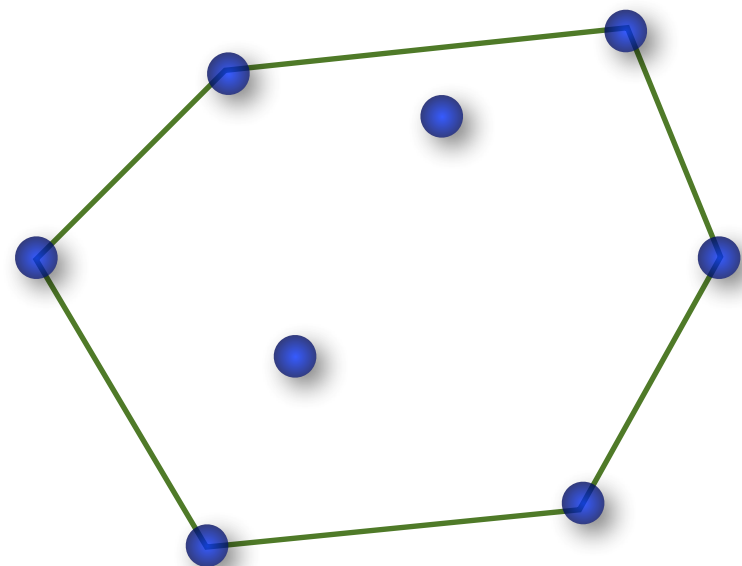
➔ Results may be **incorrect** due to roundoff.

Difference to numeric computing:

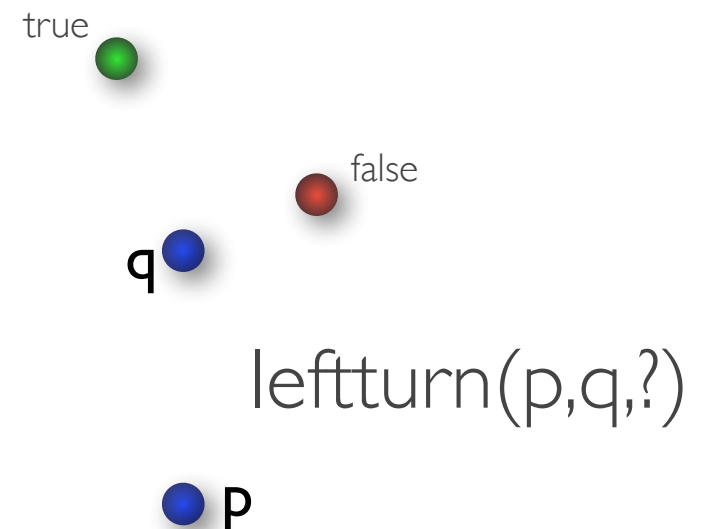
Results are interpreted combinatorially: yes or no.

Incorrect results often lead to a **complete failure** rather than to a reasonable approximation.

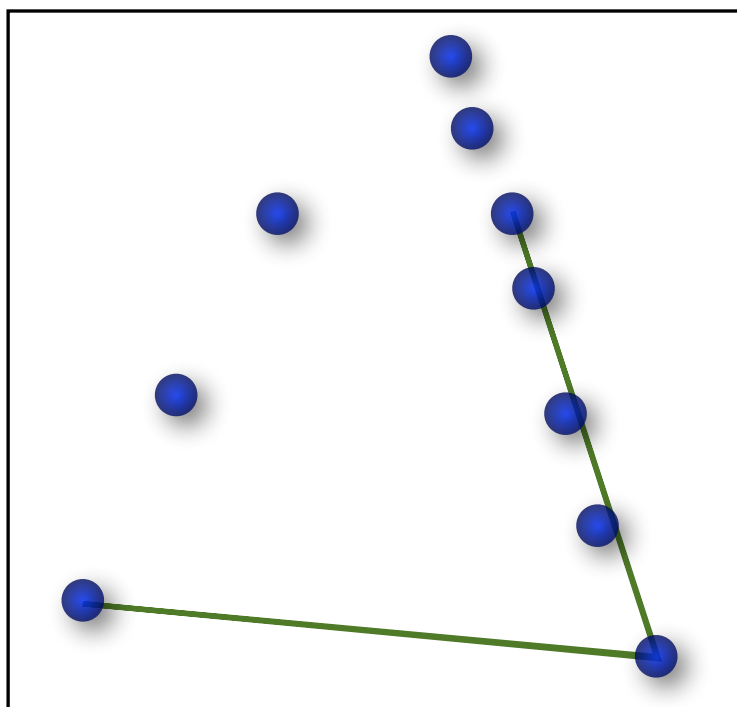
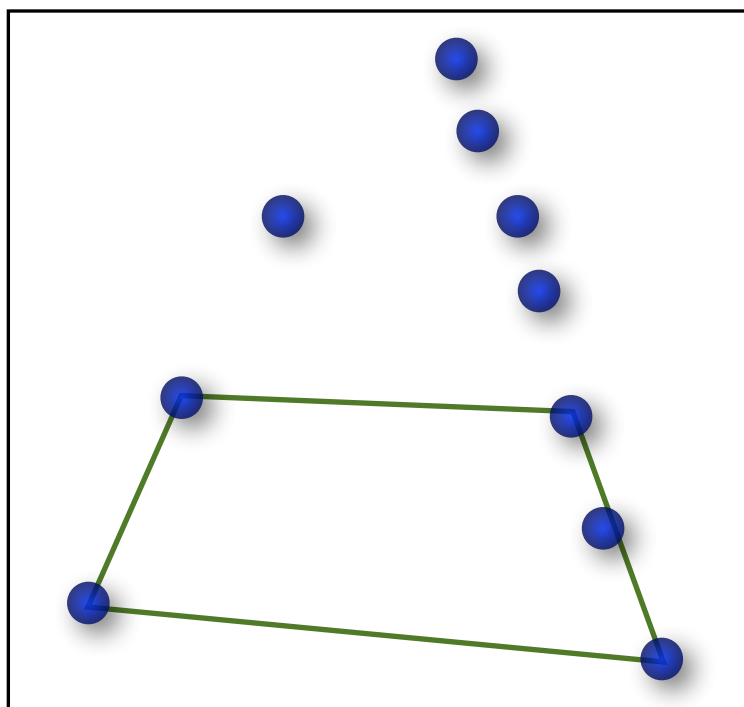
CONVEX HULL



Based on
orientation test.



Possible results with an unreliable orientation test:



ROUND OFF ERRORS?

```
#include <iostream>

int main()
{
    double x = 1.1;
    x -= 1;
    x -= 0.1;
    std::cout << x << "\n";
}
```

Output: 8.32667e-17

Why?

BUILTIN NUMBER TYPES

Type specifier	Standard	Code Expert	Min Integer	Max Integer
int	≥ 16 bits	32 bits	-2^{31}	$2^{31} - 1$
long	≥ 32 bits	64 bits	-2^{63}	$2^{63} - 1$
double	64 bits	64 bits	-2^{53}	2^{53}

IEEE 754 double precision

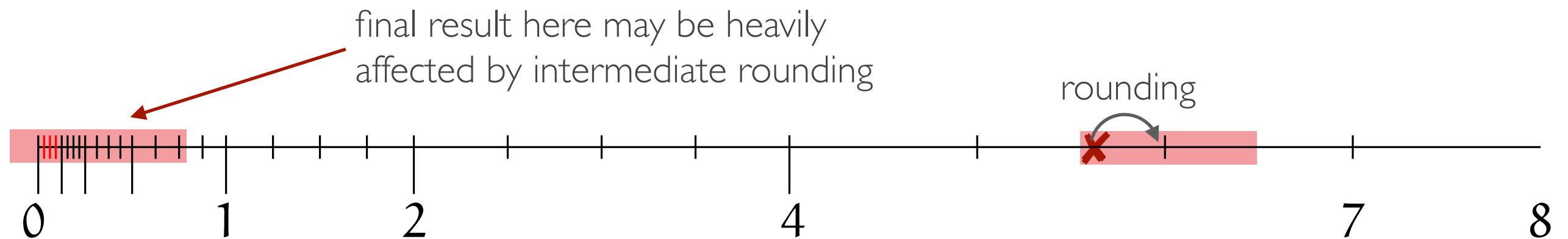
+/-	exponent	mantissa
1 bit	11 bits	53 bits

53 binary digits

Numbers $\pm m \cdot 2^x$, $1 \leq m < 2$, $-1022 \leq x \leq 1023$.

Results are rounded to nearest representable number.

FLOATING POINT NUMBERS



2bit m , 3bit x : $m \cdot 2^x$ with $m = (1.D_1D_2)_2$ and $-3 \leq x \leq 2$.

- ▶ Mantissa width \approx #ticks between 2^i and 2^{i+1}
- ▶ Uniform relative error
- ▶ But absolute error of large numbers can be large
- ▶ If intermediate results are large but the final result is small...

ORIENTATION

Leftturn(p, q, r) \Leftrightarrow

$$\begin{vmatrix} p.x & p.y & 1 \\ q.x & q.y & 1 \\ r.x & r.y & 1 \end{vmatrix} = \underbrace{(q.x-p.x)(r.y-p.y)}_{\text{maybe}} - \underbrace{(q.y-p.y)(r.x-p.x)}_{\text{may comparatively large}} > 0$$

maybe may comparatively large

.... even if the final result is very small.



Roundoff errors may lead to wrong results.

STRAIGHT LINES ?

$$\text{Orientation}(p, q, r) = \begin{vmatrix} p.x & p.y & 1 \\ q.x & q.y & 1 \\ r.x & r.y & 1 \end{vmatrix} = (q.x - p.x)(r.y - p.y) - (q.y - p.y)(r.x - p.x)$$

$$p = (0.5 + x \cdot u, 0.5 + y \cdot u)$$

$$q = (12, 12)$$

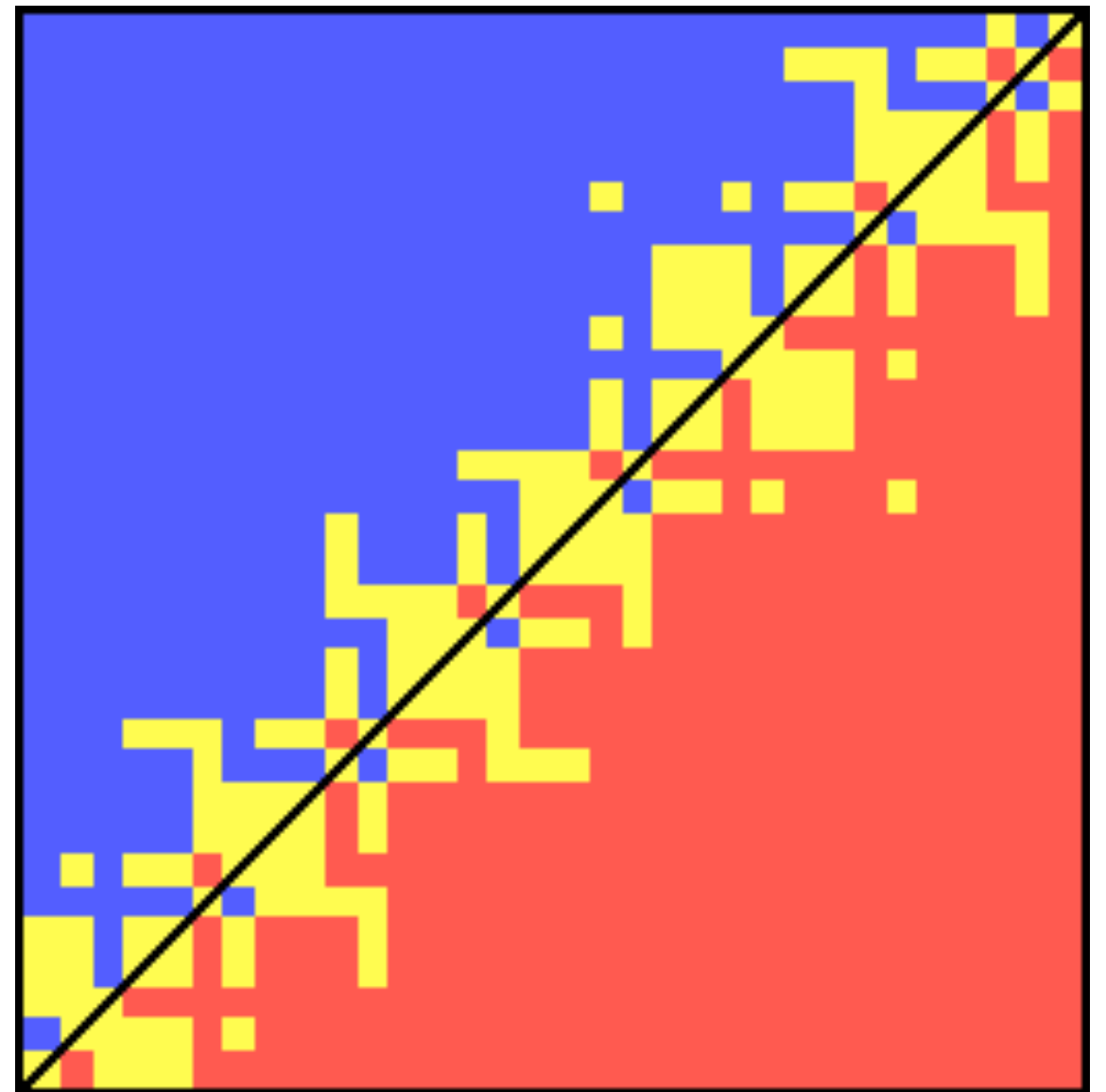
$$r = (24, 24)$$

$$0 \leq x, y < 256, \quad u = 2^{-53}$$

256x256 pixel image

red: <0, **yellow**: =0, **blue**: >0

evaluated with **double**



STRAIGHT LINES ?

$$\text{Orientation}(p, q, r) = \begin{vmatrix} p.x & p.y & 1 \\ q.x & q.y & 1 \\ r.x & r.y & 1 \end{vmatrix} = (q.x - p.x)(r.y - p.y) - (q.y - p.y)(r.x - p.x)$$

$$p = (0.5 + x \cdot u, 0.5 + y \cdot u)$$

$$q = (8.800000000000000000000007, \\ 8.800000000000000000000007)$$

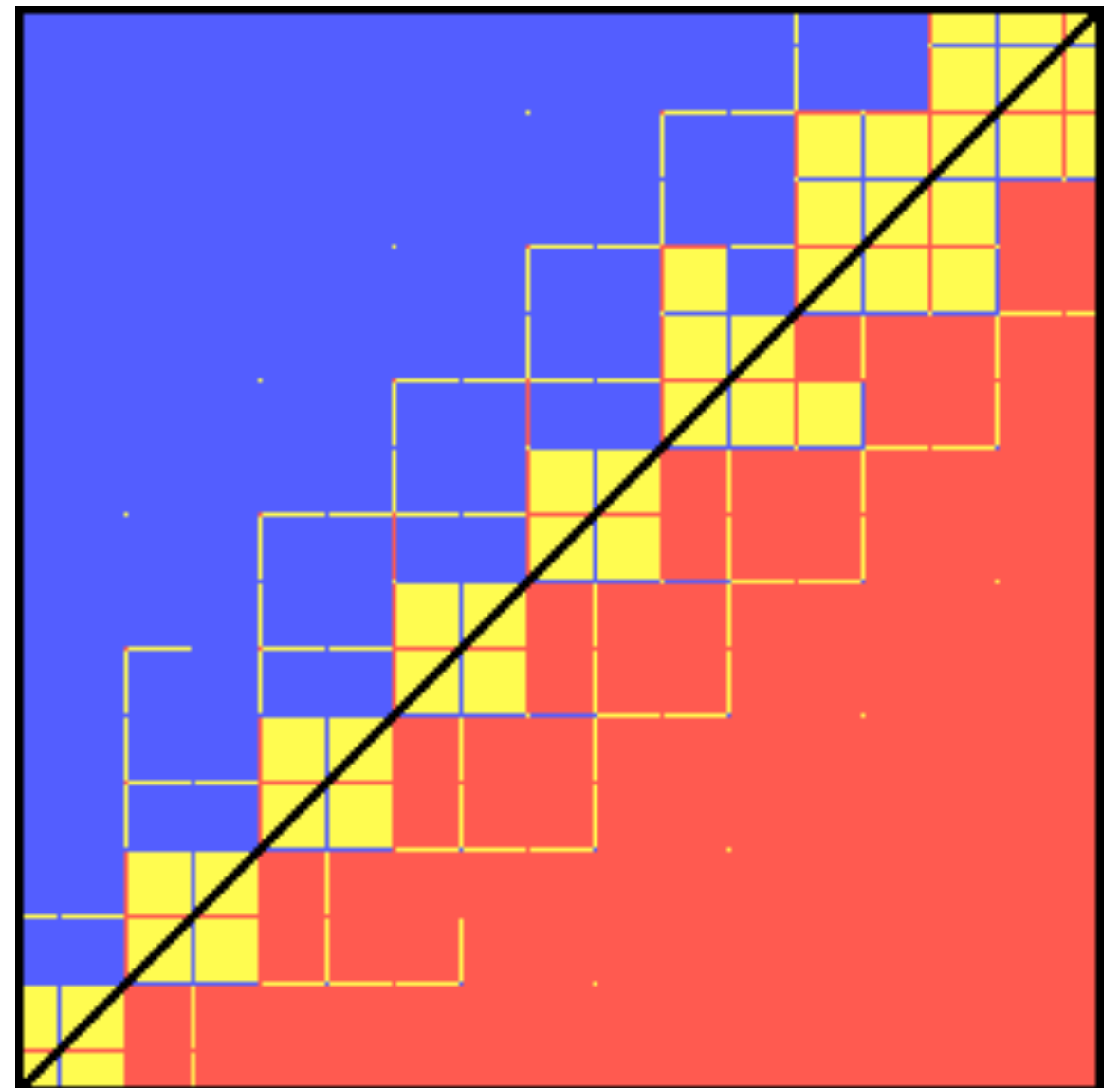
$$r = (12.1, 12.1)$$

$$0 \leq x, y < 256, u = 2^{-53}$$

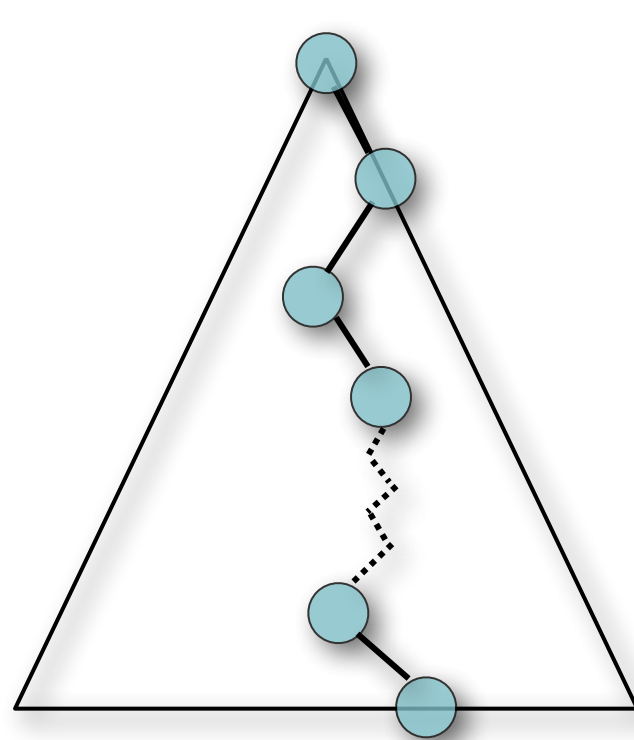
256x256 pixel image

red: <0, **yellow**: =0, **blue**: >0

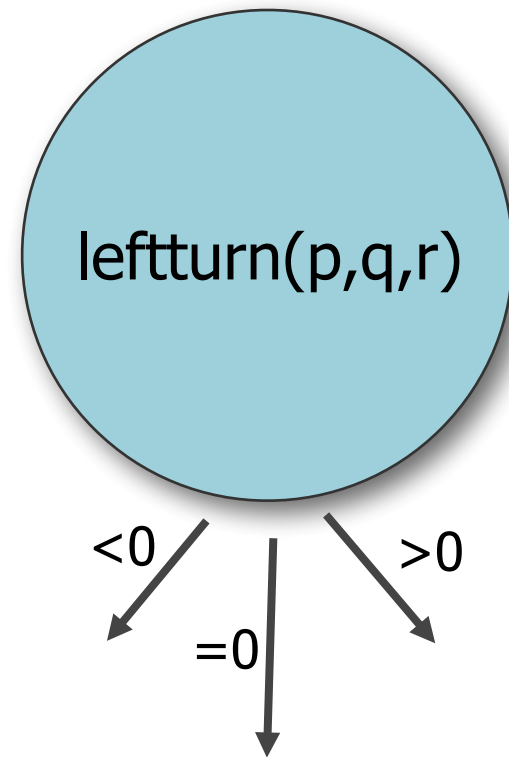
evaluated with **double**



EXACT COMPUTATION



Combinatorial
Algorithm



Geometric
Predicate

$$\begin{vmatrix} p.x & p.y & 1 \\ q.x & q.y & 1 \\ r.x & r.y & 1 \end{vmatrix}$$

Algebraic
Computation

Using tools from algebra, ensure that all predicates are computed correctly.



Correctness

SO, THAT'S IT? 😊

Not quite... 😞

Exact algebraic computation comes at a cost.

Usually, arithmetic operations are assumed to have **unit cost**.

If numbers grow, this assumption becomes **invalid**.



Use only as much algebra as needed.

HOW TO AVOID ROOTS

$$p = (p_x, p_y) \quad \bullet \quad \longleftrightarrow \quad \bullet \quad q = (q_x, q_y)$$

$$d(p, q) = \sqrt{(q_x - p_x)^2 + (q_y - p_y)^2}$$

For Euclidean distances, we need squareroots! (?)

Not necessarily! For comparison (e.g., to compute an MST) squared distances suffice.

Guideline #1: Avoid (square)roots where possible!

$$\text{For } x, y \geq 0 : \sqrt{x} < \sqrt{y} \iff x < y.$$

COMPUTING WITH FLOATING POINT NUMBERS

Guideline #1: Avoid (square)roots!

For $x, y \geq 0$: $\sqrt{x} < \sqrt{y} \iff x < y$.

Guideline #2: Avoid divisions!

For $b, d > 0$: $\frac{a}{b} < \frac{c}{d} \iff ad < bc$.

You saw in the 1.1-1-0.1 example how a division (here by 10) can create trouble.

Guideline #3: Estimate to check if loss of precision may occur! (See next slide...)

ON THE SIZE OF NUMBERS

Type specifier	Standard	Code Expert	Min Integer	Max Integer
int	≥ 16 bits	32 bits	-2^{31}	$2^{31} - 1$
long	≥ 32 bits	64 bits	-2^{63}	$2^{63} - 1$
double	64 bits	64 bits	-2^{53}	2^{53}

When computing with numbers, they may grow.

We can get easy upper bounds using:

$$\boxed{a \text{ bits}} \pm \boxed{b \text{ bits}} \leq \boxed{\max\{a,b\} + 1 \text{ bits}}$$

$$\boxed{a \text{ bits}} \cdot \boxed{b \text{ bits}} \leq \boxed{a+b \text{ bits}}$$

EX. EUCLIDEAN DISTANCES

$$p = (p_x, p_y) \quad \longleftrightarrow \quad q = (q_x, q_y)$$

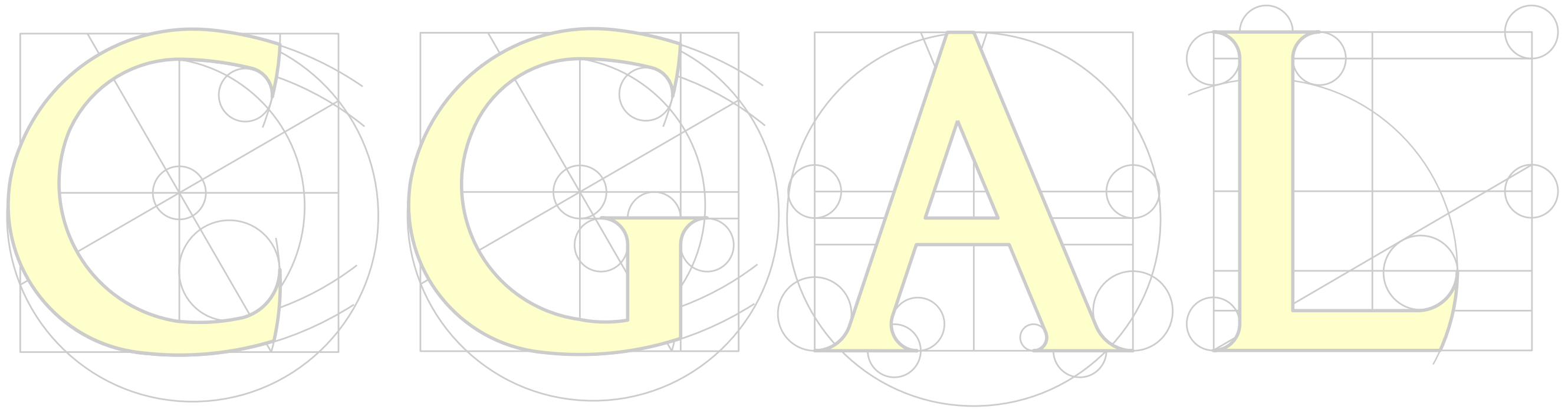
How large is d^2 for b -bit input coordinates?

$$d^2(p, q) = \underbrace{(q_x - p_x)^2}_{\substack{b+1 \\ 2b+2}} + \underbrace{(q_y - p_y)^2}_{\substack{b+1 \\ 2b+2}}$$

$2b+3$



For $b \leq 25$ **double** suffices.

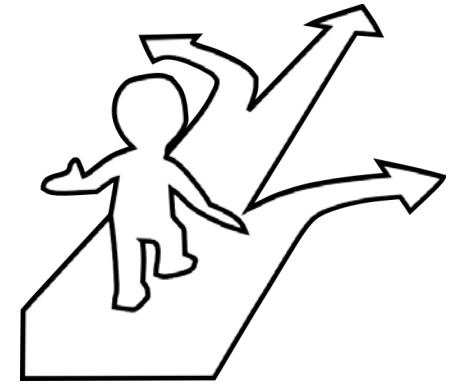


PART II:

Basic Geometric Computing using CGAL

GOALS

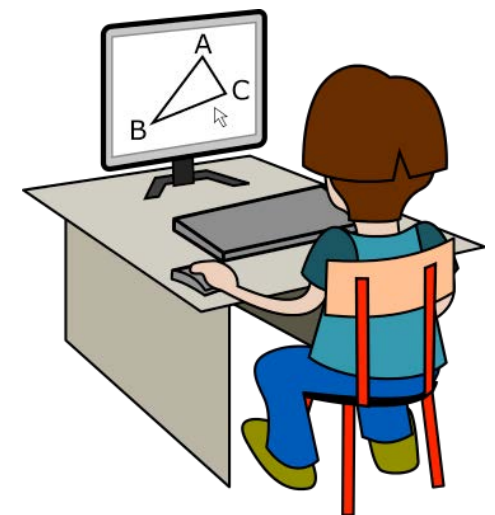
For a geometric algorithm, you are able to pick an adequate CGAL kernel.



- ▶ Are non-trivial constructions needed?
- ▶ Are exact roots needed?

You are able to do some basic geometric computations using CGAL.

- ▶ 2D kernel objects
- ▶ Intersections
- ▶ Minimum enclosing circles

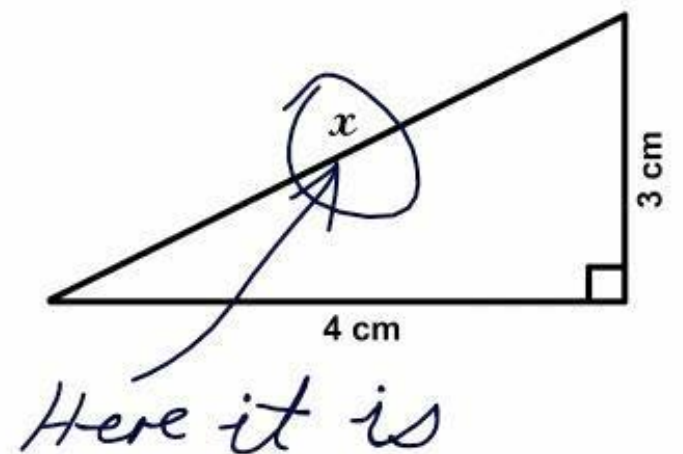


PREREQUISITES

You know basic Euclidean geometry (e.g., distance/area/volume, angles, Pythagoras, ...) and can apply this knowledge to describe and analyze problems, to design models and algorithms.

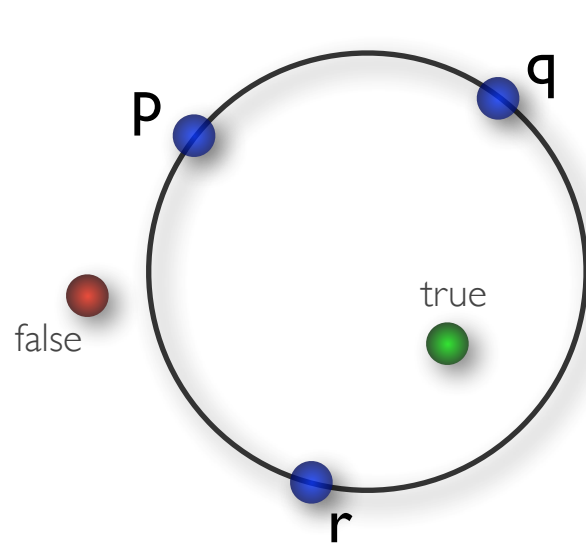
You know basic algorithmic techniques (e.g., D.P., binary search, sorting, union-find...). → You skillfully combine them with the geometric techniques discussed here.

3. Find x .

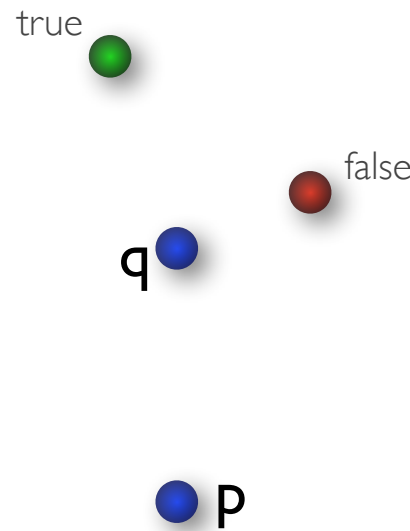


Ocular Trauma - by Wade Clarke ©2005

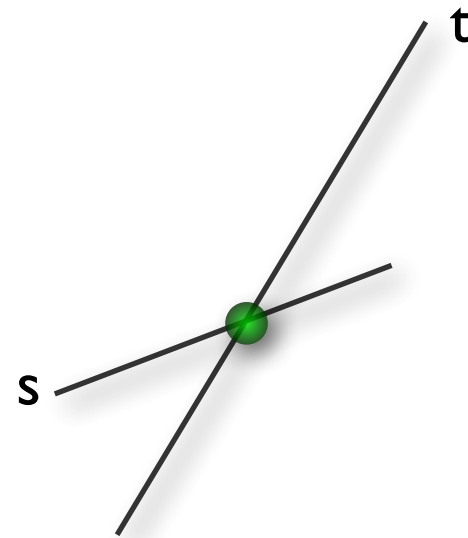
GEOMETRIC OPERATIONS



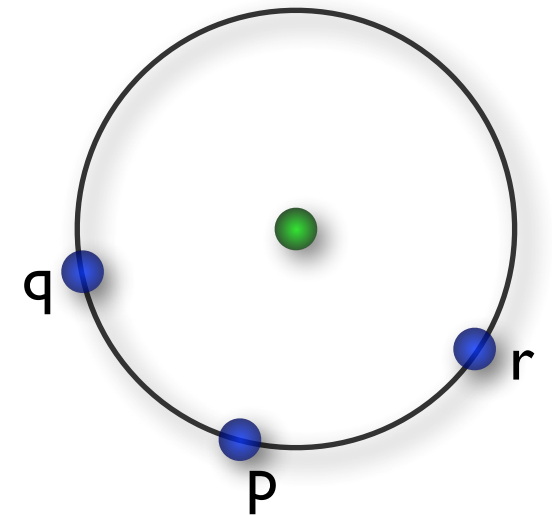
$\text{incircle}(p,q,r,?)$



$\text{leftturn}(p,q,?)$



$\text{intersection}(s,t)$



$\text{circumcenter}(p,q,r)$

Predicates

Result is constant size
(e.g., true/false)

Constructions

Result is not necessarily
constant size
(e.g., a real number or a
geometric object)

PREDICATES VS. CONSTRUCTIONS

If a program uses **predicates only**, numbers do not grow and it remains in the unit cost model.



Sometimes **exact constructions** are needed. Still, try to use as few as possible, and try to make do with elementary operations $+$, $-$, $*$, $/$.



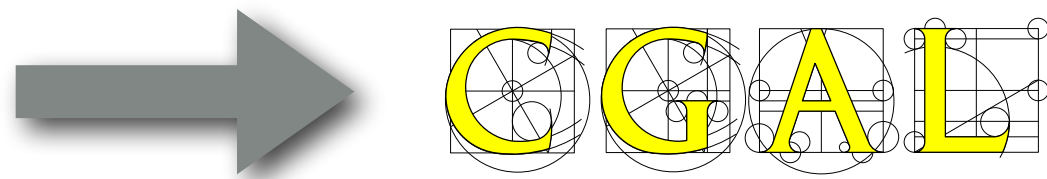
Sometimes you need also need **roots**.



KERNELS

Collection of geometric data types and operations.

There is no single true way to do geometric computing.



offers different kernels to serve various needs

You have to choose the right one for your particular case.

Predefined defaults: All three compute all predicates exactly.

► [CGAL::Exact_predicates_inexact_constructions_kernel](#)

Constructions use double.

“epic”

► [CGAL::Exact_predicates_exact_constructions_kernel](#)

Constructions use an exact number type supporting +,-,*,/.

“epec”

► [CGAL::Exact_predicates_exact_constructions_kernel_with_sqrt](#)

Constructions use an exact number type supporting +,-,*,/, and roots.

“sqrt”

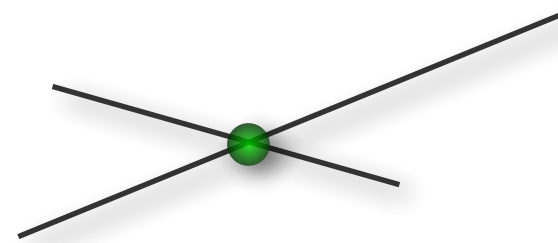
fast

slow

WHY ARE PREDICATES EASIER?

$$a_1 \cdot x + b_1 \cdot y = c_1$$

$$a_2 \cdot x + b_2 \cdot y = c_2$$



Testing for intersection/solution:

$$a_1 \cdot b_2 \neq b_1 \cdot a_2$$

Constructing intersection/solution:

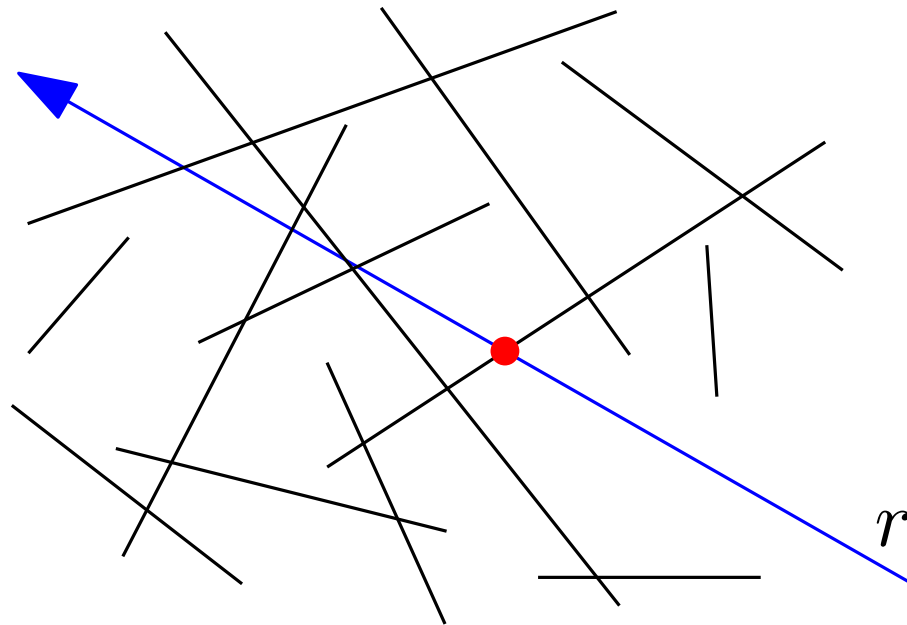
$$x = \frac{c_1 \cdot b_2 - c_2 \cdot b_1}{a_1 \cdot b_2 - a_2 \cdot b_1}$$

$$y = \frac{a_1 \cdot c_2 - a_2 \cdot c_1}{a_1 \cdot b_2 - a_2 \cdot b_1}$$



Three times as many computations.
Numbers grow by a factor of four.

EX. FIRSTHIT PROBLEM



Given: A set of n segments and a ray r in \mathbb{R}^2 .

Want: The first point along r on a segment (if any).

Hard to avoid constructing intersections
(needed for output). But can you reduce the
number of intersection constructions (e.g., in
favor of cheaper intersection tests)?



HELLO POINT

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
```

```
typedef CGAL::Exact_predicates_inexact_constructions_kernel K;
```

```
int main()
```

```
{
```

```
    K::Point_2 p(2,1), q(1,0), r(-1,-1);
```

```
    K::Line_2 l(p,q);
```

```
    K::FT d = CGAL::squared_distance(r,l);
```

```
    std::cout << d << "\n";
```

```
}
```

There is a bunch of hyperlinks here.
Click me to get to the CGAL manual.

Does this code use
constructions?
YES!

Output: 0.5

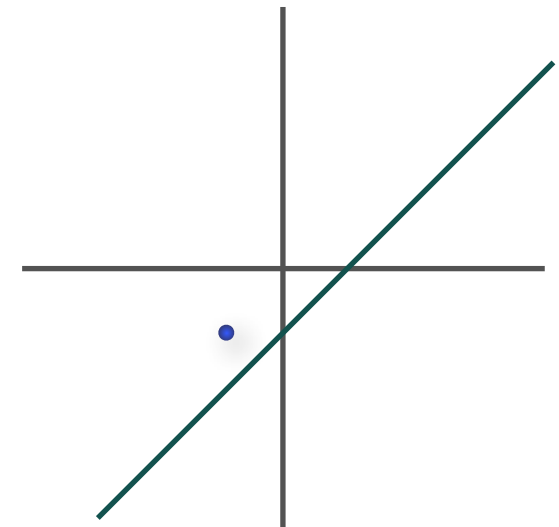
FT = field type

Here: double

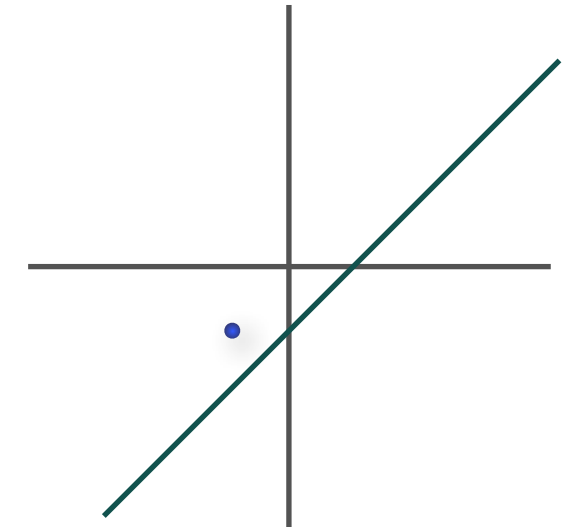
The number type used for the
underlying algebra. Supports all
field operations, i.e., +-*./.

Some (few) field types also support exact roots.

avoids square root computation



HELLO POINT



```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
```

```
typedef CGAL::Exact_predicates_inexact_constructions_kernel K;
```

```
int main()
```

```
{
```

```
    K::Point_2 p(2,1), q(1,0), r(-1,-1);
```

```
    K::Line_2 l(p,q);
```

```
    K::FT d = CGAL::squared_distance(r,l);
```

```
    std::cout << d << "\n";
```

```
}
```

For the small coordinates used here, things are probably fine. But in general...
this code is not safe!

Constructing a line from two points.
Trivial?

Depends on representation of lines... equation => **nontrivial construction**

Constructing a point from Cartesian **double** coordinates. All default kernels can do this exactly, by just storing the coordinates.
=> trivial construction, **no problem**

CGAL::Line_2<Kernel>

Definition

An object l of the data type `Line_2<Kernel>` is a directed straight line in the two-dimensional Euclidean plane \mathbb{E}^2 . It is defined by the set of points with Cartesian coordinates (x,y) that satisfy the equation $l : ax + by + c = 0$

The line splits \mathbb{E}^2 in a *positive* and a *negative* side. A point p with Cartesian coordinates (px, py) is on the positive side of l , iff $a px + b py + c > 0$, it is on the negative side of l , iff $a px + b py + c < 0$. The positive side is to the left of l .

Class

Also a **nontrivial construction**.
(Squared distance may be considerably larger than input coordinates, which may lead to overflow.)

HELLO POINT (EXACTLY)

```
#include <CGAL/Exact_predicates_exact_constructions_kernel_with_sqrt.h>
```

```
typedef CGAL::Exact_predicates_exact_constructions_kernel_with_sqrt K;
```

```
double floor_to_double(const K::FT& x)
```

```
{  
    double a = std::floor(CGAL::to_double(x));  
    while (a > x) a -= 1;  
    while (a+1 <= x) a += 1;  
    return a;  
}
```

Compute approximation of the
closest integer $\leq x$.
(Usually, this is ok. But there are
no guarantees...)

Compare to the **exact**
value to be sure.

(This assumes that x is somewhere within the range
of `double`, which will be the case in all our problems.)

```
int main()
```

```
{  
    K::Point_2 p(2,1), q(1,0), r(-1,-1);  
    K::Line_2 l(p,q);  
    K::FT d = CGAL::sqrt(CGAL::squared_distance(r,l));  
    std::cout << floor_to_double(d) << "\n";  
}
```

Compute squareroot exactly.

Output:

0

We need a precise specification for all output, in order to compare on the judge.

This is the recommended way to round down to an integer.

(The symmetric function `ceil_to_double(...)` to round up should be an easy exercise...)

TWO KERNELS IN ONE PROGRAM

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <CGAL/Exact_predicates_exact_constructions_kernel.h>
```

```
typedef CGAL::Exact_predicates_inexact_constructions_kernel IK;
```

```
typedef CGAL::Exact_predicates_exact_constructions_kernel EK;
```

```
int main()
```

```
{
```

```
    IK::Point_2 p(2,1), q(1,0), r(-1,-1);
```

```
    // do something that needs predicates only, e.g., ...
```

```
    std::cout << (CGAL::left_turn(p, q, r) ? "y" : "n") << "\n";
```

```
    // now we use non-trivial constructions...
```

```
    EK::Point_2 ep(p.x(), p.y()), eq(q.x(), q.y()), er(r.x(), r.y());
```

```
    EK::Circle_2 c(ep, eq, er);
```

```
    if (!c.has_on_boundary(ep))
```

```
        throw std::runtime_error("ep not on c");
```

```
}
```

This works because the coordinates of `IK::Point_2` are actually `double`.

It would not work the other way round, because the coordinates of `EK::Point_2` are of some elaborate number type.

We cannot just write `c(p, q, r)` because these are `IK::Point_2` and there is no general conversion between points from different kernels.

Output:

n

2D (LINEAR) KERNEL

▶ Point_2 

▶ Vector_2 

▶ Direction_2 

▶ Line_2 

▶ Ray_2 

▶ Segment_2 

▶ Triangle_2 

▶ Iso_rectangle_2 

▶ Circle_2 

Follow the links to see the manual.

2D KERNEL REPRESENTATIONS

- ▶ Point_2
 - ▶ Vector_2
 - ▶ Direction_2
 - ▶ Line_2
 - ▶ Ray_2
 - ▶ Segment_2
 - ▶ Triangle_2
 - ▶ Iso_rectangle_2
 - ▶ Circle_2
- } two FTs (Cartesian coordinates)
- three FTs (coefficients of line equation)
- } two points
- three points (corners)
- (two points, opposite corners)
- point and FT (center and squared radius)

So that you can tell which constructions are trivial...

2D KERNEL FUNCTIONALITY

See the  Manual: <http://www.cgal.org>

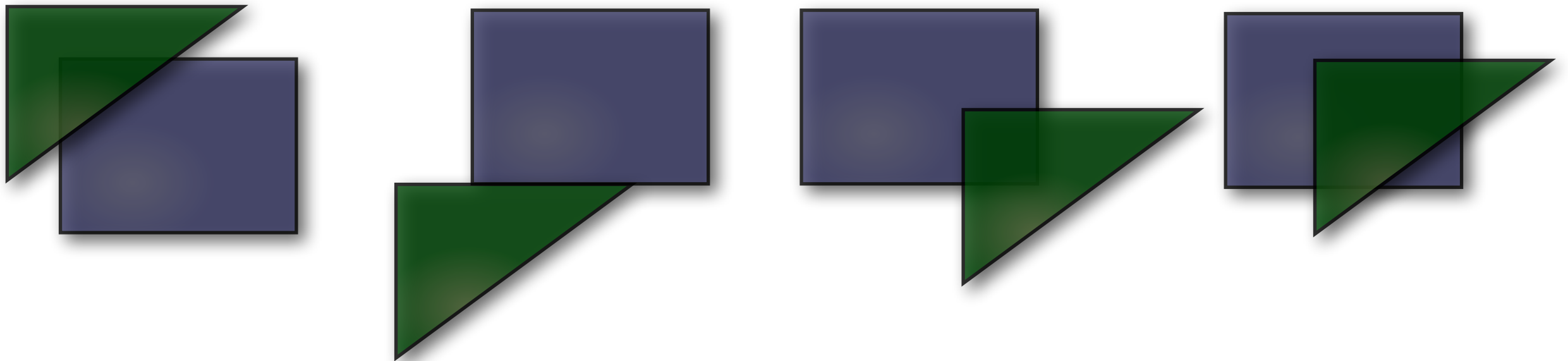
Most manual chapters have two parts:

- ▶ User Manual: general introduction and examples.
- ▶ Reference Manual: complete list of functionality.

Often one deals with several different interacting types and has to jump back and forth.

=> html is very convenient

INTERSECTIONS



Problem: We do not know the return type.

```
K::Iso_rectangle_2 r = ... ;  
K::Triangle_2 t = ... ;  
??? i = CGAL::intersection(r, t);
```

Solution: Use a generic wrapper class (based on boost::variant).
Test whether it contains an object of type T using boost::get<T>.

INTERSECTIONS

```
#include <CGAL/Exact_predicates_exact_constructions_kernel.h>
```

```
typedef CGAL::Exact_predicates_exact_constructions_kernel K;
```

```
typedef K::Point_2 P;
```

```
typedef K::Segment_2 S;
```

```
int main()
```

```
{
```

```
    P p[] = { P(0,0), P(2,0), P(1,0), P(3,0), P(.5,1), P(.5,-1) };
```

```
    S s[] = { S(p[0],p[1]), S(p[2],p[3]), S(p[4],p[5]) };
```

```
    for (int i = 0; i < 3; ++i)
```

```
        for (int j = i+1; j < 3; ++j)
```

```
            if (CGAL::do_intersect(s[i],s[j])) {
```

```
                auto o = CGAL::intersection(s[i],s[j]);
```

```
                if (const P* op = boost::get<P>(&*o))
```

```
                    std::cout << "point: " << *op << "\n";
```

```
                else if (const S* os = boost::get<S>(&*o))
```

```
                    std::cout << "segment: " << os->source() << " "
```

```
                        << os->target() << "\n";
```

```
                else // how could this be? -> error
```

```
                    throw std::runtime_error("strange segment intersection");
```

```
            } else
```

```
                std::cout << "no intersection\n";
```

```
}
```

Needs #include <type_traits>

The actual type is `std::result_of<K::Intersect_2(S,S)>::type`

Test for intersection (predicate)

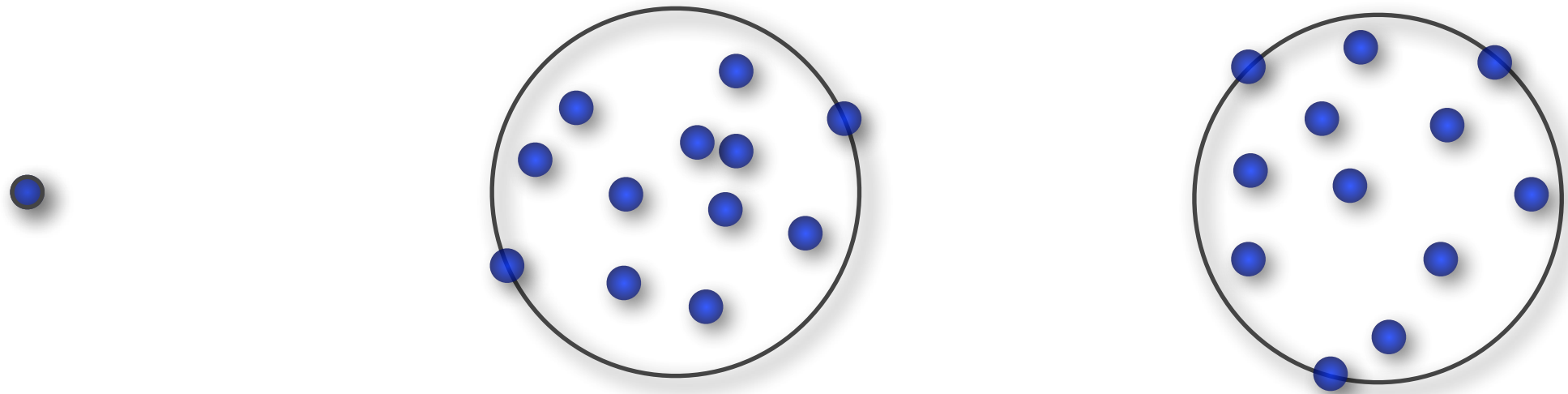
Construct intersection (construction :-)

Cast fails (=0) if o is not of type P.

Output:

```
segment: 1 0 2 0
point: 0.5 0
no intersection
```

MINIMUM ENCLOSING CIRCLE



Given: A set of n points in \mathbb{R}^2 .

Want: Their minimum enclosing circle.

► determined by ≤ 3 **support points** on its boundary

► can be computed in expected linear time using

`CGAL::Min_circle_2<Traits>`

(uses constructions internally \Rightarrow Epec*)

*Use this data structure, not
Min_sphere_of_spheres_d.*

MINIMUM ENCLOSING CIRCLE

```
#include <CGAL/Exact_predicates_exact_constructions_kernel.h>
#include <CGAL/Min_circle_2.h>
#include <CGAL/Min_circle_2_traits_2.h>
#include <vector>
```

Many data structures and algorithms have their own traits concept. It defines the geometric primitives needed, beyond those in the kernel.

```
// typedefs
```

```
typedef CGAL::Exact_predicates_exact_constructions_kernel K;
typedef CGAL::Min_circle_2_traits_2<K> Traits;
typedef CGAL::Min_circle_2<Traits> Min_circle;
```

```
int main()
{
```

```
    const int n = 100;
    std::vector<K::Point_2> P;
```

```
    for (int i = 0; i < n; ++i)
        P.push_back(K::Point_2((i % 2 == 0 ? i : -i), 0));
    // (0,0), (-1,0), (2,0), (-3,0), ...
```

Build from a range of points.

Attention! Constructions (circumcircle of three points) are used inside...

Randomize input order? Generally a good idea, unless input is known to be random, anyway.

```
    Min_circle mc(P.begin(), P.end(), true);
    Traits::Circle c = mc.circle();
    std::cout << c.center() << " " << c.squared_radius() << "\n";
```

Construct and return the circle.

Output:
-0.5 0 9702.25