Algolab 2020
Winter Games

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Use them all!

Observations

- $2 \leq n \leq 90'000$ — probably not looking for $O(n^2)$
- no overlap — only "close neighbors" matter
- maximize radius — optimization problem?

Key ideas

- for a fixed cannon, the nearest neighbor determines an upper bound on the operation range
- only the closest cannon pair matters

Use them all! – Solution

Try them all!

- Compute $\binom{n}{2}$ pairwise distances $\rightarrow \Theta(n^2)$

Check only the distance to the nearest neighbor for each cannon

- Compute the Delaunay triangulation $\rightarrow \Theta(n \log n)$
- Iterate over the edges of the triangulation $\rightarrow \Theta(n)$

Implementation detail

- Squared distances fit into `double`:

$$d^2 = d_x^2 + d_y^2 \leq (2^{25})^2 + (2^{25})^2 = 2^{51}$$

Downhill course

Observations

- $1 \leq n \leq 5'000$ probably $O(n^2)$ is fine
- no overlap 2 options for every cannon (on/off)
- each cannon has at most 2 neighbors graph problem?

Key ideas

- graph problem: cannons are vertices, put an edge whenever two ranges overlap
- find maximum independent set
  - in general NP-complete
  - bipartite graphs König's Theorem, Matching
  - special cases trivial

Downhill course – Solution

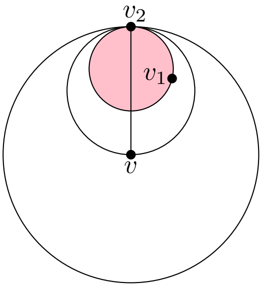
Construct a graph G to model the dependencies: vertices are cannons and there is an edge between two vertices if the respective operation ranges overlap

- Compute $\binom{n}{2}$ pairwise distances $\rightarrow \Theta(n^2)$

- It suffices to consider for every vertex v
  the nearest neighbor v_1 $\rightarrow \Theta(n \log n)$
  the second nearest neighbor v_2 $\rightarrow \Theta(n \log n)$

Lemma

Let v_1 be a nearest neighbor and v_2 be a second nearest neighbor of v. Then at least one of vv_2 or v_1v_2 is an edge of the Delaunay triangulation.



Each vertex v_1 can be the nearest neighbor of only a constant number of other vertices.

Downhill course – Solution

Construct a graph G to model the dependencies: vertices are cannons and there is an edge between two vertices if the respective operation ranges overlap

- Compute $\binom{n}{2}$ pairwise distances $\rightarrow \Theta(n^2)$

- It suffices to consider for every vertex v
  the nearest neighbor v_1 $\rightarrow \Theta(n \log n)$
  the second nearest neighbor v_2 $\rightarrow \Theta(n \log n)$

- maximal degree 2

$\Rightarrow$ the graph G must be a disjoint union of paths and cycles

$\Rightarrow$ greedy/ad-hoc solution for every component $\rightarrow \Theta(n)$

Software update

Observations

- $2 \leq n \leq 50$ tiny...
- minimal radius need solution of Exercise 2
- no overlap radii are bounded from above
- maximize sum of the radii optimization problem?

Key ideas

- optimization problem find a larger radius for each cannon
  - the objective is linear in the radii
  - lower bounds for the radii
  - implicit upper bounds: no overlap

Software update – Solution

Linear program with n variables and $n + \binom{n}{2}$ constraints:

- *Variables*: Operation range (radius) of every snow cannon

$$r_i \geq \left\lfloor \frac{\texttt{closest\_pair\_dist}}{2} \right\rfloor, \quad i = 1, \ldots, n$$

- *Constraints*: Operation ranges are not overlapping

$$r_i + r_j \leq \texttt{dist}(i, j), \quad 1 \leq i, j \leq n$$

- *Objective*: Maximize the sum of the radii

$$\max \sum_i r_i$$

Implementation details

- **Input type**: Exact type with sqrt