Dynamic Programming

'Those who do not remember the past are condemned to repeat it.'

First things first... **Problem of the Week: Deck of Cards** (simplified)

Input: n, k, and n non-negative integers $v_0, v_1, \ldots, v_{n-1}$

Output: a pair (i, j) such that $0 \le i \le j \le n-1$ and $\sum_{\ell=i}^{j} v_\ell = k$

Example: $n = 6$, $k = 7$

3	1	4	1	1	8

Solution: $i = 1$, $j = 4$

Problem of the Week: Deck of Cards (simplified)

Input: n, k, and n non-negative integers $v_0, v_1, \ldots, v_{n-1}$

Output: a pair (i, j) such that $0 \leq i \leq j \leq n-1$ and $\sum_{\ell=i}^{j} v_\ell = k$

- Test set 1: $n \leq 200 \longrightarrow$ time complexity $O(n^3)$ (e.g. just do it!)
- Test set 2: $n \leq 3000 \longrightarrow$ time complexity $O(n^2)$ (e.g. partial sums)
- Test set 3: $n \leq 100000 \longrightarrow$ time complexity $O(n \log n)$ (e.g. binary search)
- Test set 3: $n \leq 100000 \longrightarrow$ time complexity $O(n)$

How to solve Deck of Cards in $O(n)$?

Sliding Window

How to solve Deck of Cards in $O(n)$? **Sliding Window**!

Idea:

- Keep two pointers that keep track of the current interval (window)
- If the value of the window is too large: increase the left pointer
- If the value of the window is too small: increase the right pointer

Example: $n = 6$, $k = 7$

3	1	4	1	1	8

$i\ j$

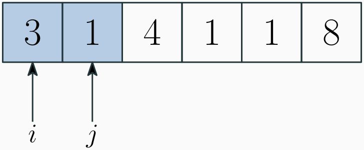
$v_0 = 3 < k \longrightarrow$ increase j

How to solve Deck of Cards in $O(n)$? **Sliding Window**!

Idea:

- Keep two pointers that keep track of the current interval (window)
- If the value of the window is too large: increase the left pointer
- If the value of the window is too small: increase the right pointer

Example: $n = 6$, $k = 7$



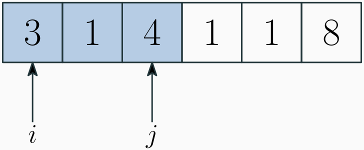
$$v_0 + v_1 = 4 < k \longrightarrow \text{increase } j$$

How to solve Deck of Cards in $O(n)$? **Sliding Window**!

Idea:

- Keep two pointers that keep track of the current interval (window)
- If the value of the window is too large: increase the left pointer
- If the value of the window is too small: increase the right pointer

Example: $n = 6$, $k = 7$



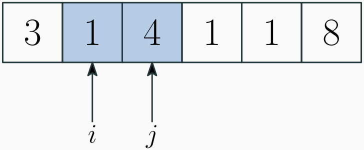
$$v_0 + v_1 + v_2 = 8 > k \longrightarrow \text{increase } i$$

How to solve Deck of Cards in $O(n)$? **Sliding Window**!

Idea:

- Keep two pointers that keep track of the current interval (window)
- If the value of the window is too large: increase the left pointer
- If the value of the window is too small: increase the right pointer

Example: $n = 6$, $k = 7$



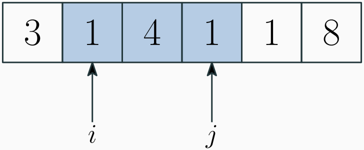
$$v_1 + v_2 = 5 < k \longrightarrow \text{increase } j$$

How to solve Deck of Cards in $O(n)$? **Sliding Window**!

Idea:

- Keep two pointers that keep track of the current interval (window)
- If the value of the window is too large: increase the left pointer
- If the value of the window is too small: increase the right pointer

Example: $n = 6$, $k = 7$



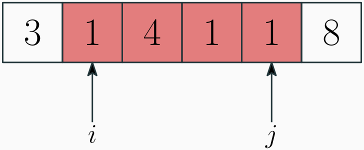
$$v_1 + v_2 + v_3 = 6 < k \longrightarrow \text{increase } j$$

How to solve Deck of Cards in $O(n)$? **Sliding Window**!

Idea:

- Keep two pointers that keep track of the current interval (window)
- If the value of the window is too large: increase the left pointer
- If the value of the window is too small: increase the right pointer

Example: $n = 6$, $k = 7$



$v_1 + v_2 + v_3 + v_4 = 7 = k \longrightarrow$ **YAY!** The solution is $i = 1$ and $j = 4$

How to solve Deck of Cards in $O(n)$? **Sliding Window**!

```
int i = 0, j = 0;
int val = v[0];

while (j < n) {
    if (val == k) break;

    if (val < k) {
        j++;
        if (j == n) break;
        val += v[j];
    } else {
        val -= v[i];
        i++;
        if (i > j) {
            if (i == n) break;
            j = i;
            val = v[i];
        }
    }
}
```

Sketch of the proof:

- at every point we increase either i or j by one, each can vary from 0 to n (so we terminate in $\leq 2n$ steps)
- assume j reaches the end of the target window before i reaches the start; then i keeps increasing until it hits the start of it
- assume i reaches the start of the target window before j reaches the end; then j keeps increasing until it hits the end of it

Exercise: Extend it to solve the real problem.

Trick/technique (Sliding window)

Some problems in which you need to find an **optimal interval** can be solved in linear time using a **sliding window** approach.

Let's get to the point!

- Most of you know Dynamic Programming (DP) (:
  - Solve a problem by reducing it to smaller subproblems of the same type
  - Describe the instance of the problem by a **state**, use smaller/previous states (subproblems) to solve the current state
- Many struggle to apply it :(
  - How to identify a DP problem?
  - How to tackle it?
  - How to implement it?
  - How to analyse its time complexity?
- Today we start from scratch
- Why DP? Runtime, runtime, and runtime again! From exponential to polynomial!

Outline for today:

- Three examples (Fibonacci, Rod Cutting, LIS)
- Elements of Dynamic Programming on examples
- Common pitfalls
- Tips & Tricks

First Example: Fibonacci Numbers

Definition: $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

Problem: compute F_n

Solution: transform the definition into a recursive algorithm

```
int F(int n) {
    if (n == 1 || n == 2) return 1;
    return F(n - 1) + F(n - 2);
}
```

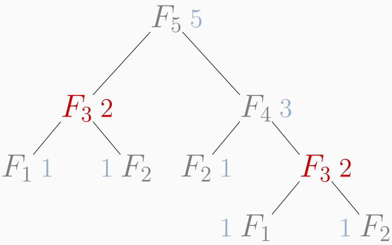
Time complexity: $\Theta(\varphi^n)$

Source of inefficiency?

???

```
int F(int n) {
    if (n == 1 || n == 2) return 1;
    return F(n - 1) + F(n - 2);
}
```

Example: F_5



Source of inefficiency?

Overlapping subproblems

$F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$

Idea: do not recompute, recall from memory

'Those who do not remember the past are condemned to repeat it.'

```
int F(int n) {
    if (n == 1 || n == 2) return 1;

    return F(n - 1) + F(n - 2);

}
```

```
vector<int> memo(n + 1, -1); // Memory

int F(int n) {
    if (n == 1 || n == 2) return 1;
    if (memo[n] == -1) // I do not remember it
        memo[n] = F(n - 1) + F(n - 2); // compute and remember it
    return memo[n]; // I remember it, recall
}
```

Time complexity: $\Theta(n)$

Memoization (or top-down DP) is **simple and powerful :)**
(not a typo, comes from *memo*)

Second Example: Rod Cutting

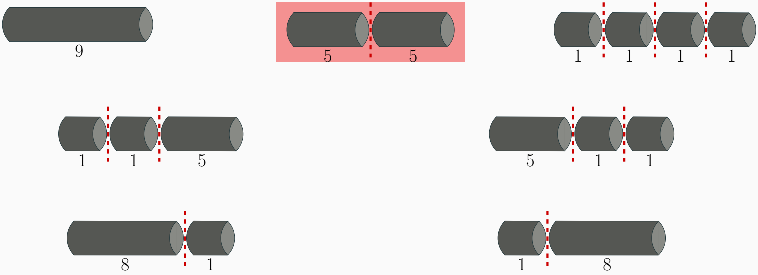
Input:

- a metal rod of length n
- values $p_1, \ldots, p_n$ s.t. p_i denotes the price for a rod of length i

Output: r_n, maximal possible revenue for a rod of length n (i.e. maximal sum of prices of pieces over all possible partitions)

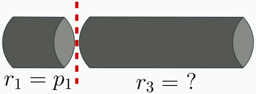
Example: $n = 4$

length i	1	2	3	4
price p_i	1	5	8	9



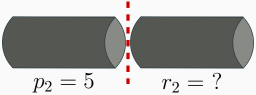
Recursive maximisation:

$$r_n = \max\{p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1\}$$



Reformulation: a piece containing the left end + a partition of the rest

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$



$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$

Recursive algorithm:

```
int r(vector<int> &p, int n) {
    if (n == 0) return 0;
    int res = -1;
    for (int i = 1; i <= n; i++) {
        res = max(res, p[i] + r(p, n - i));
    }
    return res;
}
```

Time complexity: $\Theta(2^n)$

Why? Overlapping subproblems. (Can you see them?)

How to be more efficient?

Dynamic Programming

Dynamic Programming

- Top-Down (recursion + memo)
- Bottom-Up (fill up a table)

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$

Top-Down DP

```
vector<int> memo(n + 1, -1);

int r(vector<int> &p, int n) {
    if (n == 0) return 0;
    if (memo[n] != -1) return memo[n];
    int res = -1;
    for (int i = 1; i <= n; i++) {
        res = max(res, p[i] + r(p, n - i));
    }
    memo[n] = res;
    return res;
}
```

Time complexity: $\Theta(n^2)$

Explanation: n function calls, i-th call takes $O(i)$ time. Total: $\sum_{i=1}^{n} O(i) = O(n^2)$

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$

Bottom-Up DP

```
vector<int> r(n + 1, -1);
r[0] = 0;
for (int i = 1; i <= n; i++) {
    for (int j = 1; j <= i; j++) {
        r[i] = max(r[i], p[j] + r[i - j]); // IMPORTANT: Current subproblem
                                           // relies only on the solution of
                                           // the smaller subproblems
    }
}
return r[n];
```

Time complexity: $\Theta(n^2)$

Reconstructing a Solution: What if we also want to know where to cut?

No problem!

```
vector<int> cut(n + 1, 0); // cut[i] stores where to optimally cut a rod of
                           // length i
vector<int> memo(n + 1, -1);

int r(vector<int> &p, int n) {
    if (n == 0) return 0;
    if (memo[n] != -1) return memo[n];
    int res = -1;
    for (int i = 1; i <= n; i++) {
        if (p[i] + r(p, n - i) > res) {
            res = p[i] + r(p, n - i);
            cut[n] = i; // We should cut at position i
        }
    }
    memo[n] = res;
    return res;
}
```

That was **easy**! Why is it so **difficult** in general?

Remebering is easy (right?)—apply memoization.

Deriving a recursive algorithm is the difficult part.

Usually problems are not given in a way that can straightforwardly be translated into a recursive definition. :(

Essential elements of a DP problem:

- Usually optimisation problems, i.e. maximise or minimise some quantity (but can also aim at computing a sum, a probability, or an expected value instead)
- Exhibit the optimal subproblem structure
- Overlapping subproblems

Essence of DP (with examples):

- Usually optimisation problems, i.e. maximise or minimise some quantity

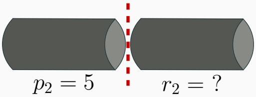
Example: Rod Cutting Problem

r_n, **maximal** possible revenue for a rod of length n

Essence of DP (with examples):

- Exhibit the optimal subproblem structure

Example: Rod Cutting Problem

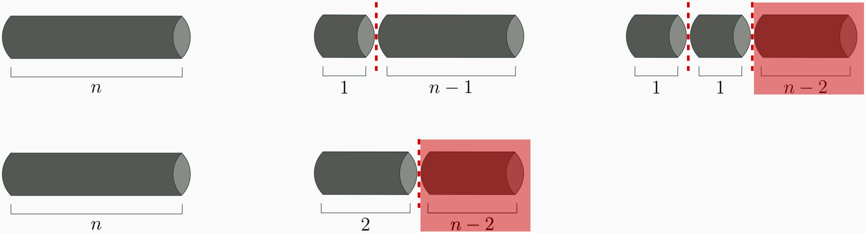


- Suppose the DP gods show you what the correct 'last' choice to make is
- Look at which subproblems arise once making this choice
- Show that the subproblems used in an optimal solution must themselves be optimal

Essence of DP (with examples):

- Overlapping subproblems

Example: Rod Cutting Problem



- Remember, do not recompute!
- *'Those who do not remember the past are condemned to repeat it.'*

Common Pitfalls

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$

```
vector<int> memo(n + 1, -1);

int r(vector<int> &p, int n) {
    if (n == 0) return 0;
    if (memo[n] != -1) return memo[n];
    int res = -1;
    for (int i = 1; i <= n + 1; i++) {
        res = max(res, p[i] + r(p, n - i));
    }
    memo[n] = res;
    return res;
}
```

This results in a **SEG FAULT**/**RUN ERROR**, can you see why?

Make sure that you stay 'within' the memo boundaries! Similarly, sometimes the memo table does not/cannot contain the base cases.

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$

```
vector<int> memo(n + 1, 0);

int r(vector<int> &p, int n) {
    if (n == 0) return 0;
    if (memo[n] != 0) return memo[n];
    int res = -1;
    for (int i = 1; i <= n; i++) {
        res = max(res, p[i] + r(p, n - i));
    }
    memo[n] = res;
    return res;
}
```

This results in a **TIMELIMIT**, can you see why?

Make sure that the **default** memo value is **not** a possible output!

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$

```
map<int, int> memo;

int r(vector<int> &p, int n) {
    if (n == 0) return 0;
    if (memo.find(n) != memo.end()) return memo[n];
    int res = -1;
    for (int i = 1; i <= n; i++) {
        res = max(res, p[i] + r(p, n - i));
    }
    memo[n] = res;
    return res;
}
```

This (possibly) results in a **TIMELIMIT**, can you see why?

`std::map` adds an $O(\log n)$ insert/find/access overhead.

Third Example: Longest Increasing Subsequence

Input: a sequence of n integers $a_1, \ldots, a_n$

Output: the length of a longest increasing subsequence (LIS)

Example:

$$2 \quad 4 \quad 3 \quad 7 \quad 4 \quad 5$$

Result: LIS $= 4$

Input: a sequence of n integers $a_1, \ldots, a_n$

Output: the length of a longest increasing subsequence (LIS)

First attempt: $f(i) :=$ 'length of the LIS in $a_1, \ldots, a_i$'.

- Base cases: $f(1) = 1$
- $f(i) =$???

Second attempt: $f(i) :=$ 'length of the LIS in $a_1, \ldots, a_i$ ending at a_i'.

- Base cases: $f(1) = 1$
- $f(i) = \max(\max_{j<i:a_j<a_i}(1 + f(j)), 1)$
- Answer is then $\max_{i \in \{1,\ldots,n\}} f(i)$, **not** $f(n)$!

We had to reformulate the problem s.t. it admits a recursive formulation, this is **difficult**! Important question: What should we compute?

Time complexity: $O(n^2)$

Explanation: n function calls (with memo), i-th call takes $O(i)$ time.

(**Exercise:** Can you do it in $O(n \log n)$?)

Tips & Tricks

Top-Down (memoization) vs Bottom-Up (iterative)

Usually both work and it boils down to a personal preference :)

- Simple to implement
- Easy to describe subproblems (by using a `std::map`)
- Computes only necessary subproblems
- Time complexity sometimes not so obvious
- Overhead of function calls

- More effort to code
- Subproblems must be described by integers
- Always computes all subproblems
- Time complexity obvious
- Saves some constant factors

How to determine the runtime?

Informally, a product of two factors: the overall number of subproblems and the number of possible choices for each subproblem

Important: Think before you code! No point coding a solution that turns out to be too slow. Compute the runtime *before* you program your solution, so that you don't waste your time.

Bottom-Up: Easy!

Top-Down: Sometimes harder to see immediately.

`std::map` or `std::vector`? **Always `std::vector`!**

Unless... the subproblem (state space) cannot be described by integers. Then use maps.

Remember! `std::map` has a insert/find/access overhead of $O(\log n)$.

DP – Summary

- Idea of DP: solve subproblems only once by storing solutions of subproblems
- Start by defining recurrence relation (on paper)
- Implement it. It will be correct but slow...
- Are there overlapping subproblems?
- Add memo (usually does the trick) or construct a DP table
- Practice deriving recurrence relations on paper for standard DP problems (e.g. Knapsack, SubsetSum, Coin Change, LCS, Edit Distance, LIS, etc.)

DP – How to come up with a recurrence relation

- What is an appropriate formulation of the problem that gives us subproblems we can work with? What will be the **function** we are trying to compute?
- What are the necessary and sufficient arguments of that function? That is, how do we describe a subproblem as succinctly as possible? What is our **state**?
- How can we use smaller subproblems to solve the problem? What is the **recurrence relation**?
- What will our **time complexity** be, after applying memoization? How many states are there, and how many possible choices per state? Is this fast enough?
- If you get stuck at any of these questions, consider changing your answer to one or more of the previous ones.

Runtime, runtime, and runtime again

- Practice determining the runtime of DP algorithms — it is not always easy
- At the exam, think about the runtime *before* you program your solution
- Plug into your time complexity the maximum value of n (and of any other relevant parameters) given in the problem statement
- Rule of thumb: 10 to 100 million operations per second
- Will your solution fit in the time limit? You can know the answer to this question before you even start programming!

That's all for today!