Dynamic Programming

'Those who do not remember the past are condemned to repeat it.'

First things first... **Problem of the Week: Deck of Cards** (simplified)

Input: n, k, and n non-negative integers $v_0, v_1, \ldots, v_{n-1}$

Output: a pair (i, j) such that $0 \le i \le j \le n - 1$ and $\sum_{\ell=i}^{j} v_\ell = k$

Example: $n = 6$, $k = 7$

3	1	4	1	1	8

Solution: $i = 1$, $j = 4$

Problem of the Week: Deck of Cards (simplified)

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Output: a pair (i, j) such that $0 \le i \le j \le n-1$ and $\sum_{\ell=i}^{j} v_\ell = k$

- ▶ Test set 1: $n \le 200 \longrightarrow$ time complexity $O(n^3)$ (e.g. just do it!)
- ▶ Test set 2: $n \le 3000 \longrightarrow$ time complexity $O(n^2)$ (e.g. partial sums)
- ▶ Test set 3: $n \le 100000 \longrightarrow$ time complexity $O(n \log n)$ (e.g. binary search)
- ▶ Test set 3: $n \le 100000 \longrightarrow$ time complexity $O(n)$

How to solve Deck of Cards in $O(n)$?

Sliding Window

How to solve Deck of Cards in $O(n)$? **Sliding Window!**

Idea:

▶ Keep two pointers that keep track of the current interval (window)
▶ If the value of the window is too large: increase the left pointer
▶ If the value of the window is too small: increase the right pointer

Example: $n = 6$, $k = 7$

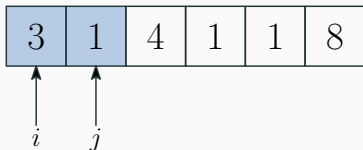| 3	1	4	1	1	8

$i\,j$

$v_0 = 3 < k \longrightarrow$ increase j

How to solve Deck of Cards in $O(n)$? **Sliding Window!**

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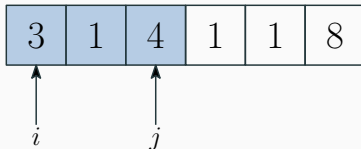
Example: $n = 6$, $k = 7$

| 3 | 1 | 4 | 1 | 1 | 8 |

i j

$v_0 + v_1 = 4 < k \longrightarrow$ increase j

How to solve Deck of Cards in $O(n)$? **Sliding Window**!

Idea:

▶ Keep two pointers that keep track of the current interval (window)
▶ If the value of the window is too large: increase the left pointer
▶ If the value of the window is too small: increase the right pointer

Example: $n = 6$, $k = 7$

$$v_0 + v_1 + v_2 = 8 > k \longrightarrow \text{increase } i$$

How to solve Deck of Cards in $O(n)$? **Sliding Window**!

Idea:

- ▶ Keep two pointers that keep track of the current interval (window)
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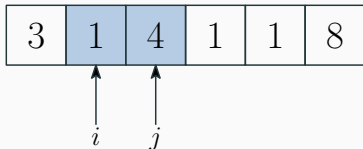
Example: $n = 6$, $k = 7$

3	1	4	1	1	8

$\qquad\quad i \qquad j$

$v_1 + v_2 = 5 < k \longrightarrow$ increase j

How to solve Deck of Cards in $O(n)$? **Sliding Window!**

Idea:

- Keep two pointers that keep track of the current interval (window)
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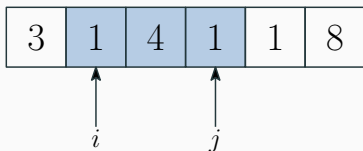
Example: $n = 6$, $k = 7$

3	1	4	1	1	8

$$\underset{i}{\uparrow} \qquad \underset{j}{\uparrow}$$

$$v_1 + v_2 + v_3 = 6 < k \longrightarrow \text{increase } j$$

How to solve Deck of Cards in $O(n)$? **Sliding Window**!

Idea:

- ▶ Keep two pointers that keep track of the current interval (window)
- ▶ If the value of the window is too large: increase the left pointer
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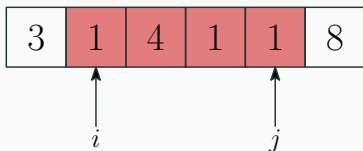
Example: $n = 6, k = 7$

3	1	4	1	1	8

i points to the second cell, j points to the fifth cell.

$v_1 + v_2 + v_3 + v_4 = 7 = k \longrightarrow$ **YAY!** The solution is $i = 1$ and $j = 4$

How to solve Deck of Cards in $O(n)$? **Sliding Window**!

```
int i = 0, j = 0;
int val = v[0];

while (j < n) {
    if (val == k) break;

    if (val < k) {
        j++;
        val += v[j];
    } else {
        val -= v[i];
        i++;
        if (i > j) {
            j = i;
            val = v[i];
        }
    }
}
```

Sketch of the proof:

► at every point we increase either i or j by one (so we eventually terminate)

► assume j reaches the end of the target window before i reaches the start; then i keeps increasing until it hits the start of it

► assume i reaches the start of the target window before j reaches the end; then j keeps increasing until it hits the end of it

Exercise: Extend it to solve the real problem.

Trick/technique (Sliding window)

Some problems in which you need to find an optimal interval can be solved in linear time using a **sliding window** approach.

Let's get to the point!

- ▶ Most of you know Dynamic Programming (DP) (:
- ▶ Many struggle to apply it :(
    - ▶ How to identify a DP problem?
    - ▶ How to tackle it?
    - ▶ How to implement it?
- ▶ Today we start from scratch

Outline for today:

- ► Three examples (Fibonacci, Rod Cutting, LSI)
- ► Elements of Dynamic Programming on examples
- ► Common pitfalls
- ► Tips & Tricks

First Example: Fibonacci Numbers

Definition: $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

Problem: compute F_n

Solution: transform the definition into a recursive algorithm

```
int F(int n) {
    if (n == 1 || n == 2) return 1;
    return F(n - 1) + F(n - 2);
}
```

Time complexity: $O(\varphi^n)$

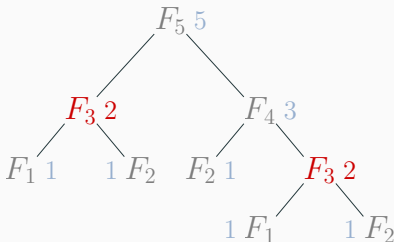Source of inefficiency?

???

```
int F(int n) {
    if (n == 1 || n == 2) return 1;
    return F(n - 1) + F(n - 2);
}
```

Example: F_5

Source of inefficiency?

Overlapping subproblems

$F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$

Idea: do not recompute, recall from memory

'Those who do not remember the past are condemned to repeat it.'

```cpp
vector<int> memo(n + 1, -1); // Memory
```

```cpp
int F(int n) {
    if (n == 1 || n == 2) return 1;

    return F(n - 1) + F(n - 2);

}
```

```cpp
int F(int n) {
    if (n == 1 || n == 2) return 1;
    if (memo[n] == -1) // I do not remember it
        memo[n] = F(n - 1) + F(n - 2); // compute and remember it
    return memo[n]; // I remember it, recall
}
```

Time complexity: $O(n)$

Memoization (or top-down DP) is **simple and powerful :)**
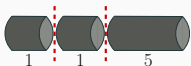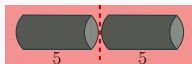(not a typo, comes from *memo*)

Second Example: Rod Cutting

Input:

- a metal rod of length n
- values $p_1, \ldots, p_n$ s.t. p_i denotes the price for a rod of length i

Output: r_n, maximal possible revenue obtained by cutting up a rod of length n and selling the pieces (i.e. maximal sum of prices of pieces over all possible partitions)
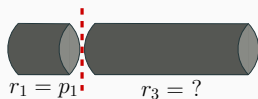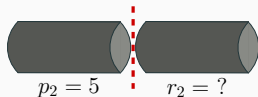
Example: $n = 4$

length i	1	2	3	4
price p_i	1	5	8	9

Recursive maximisation:

$$r_n = \max\left\{p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1\right\}$$

$r_1 = p_1$ $r_3 = ?$

Reformulation: a piece containing the left end + a partition of the rest

$$r_n = \max_{1 \le i \le n}(p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$

$p_2 = 5$ $r_2 = ?$

$$r_n = \max_{1 \leq i \leq n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$

Recursive algorithm:

```cpp
int r(vector<int> &p, int n) {
    if (n == 0) return 0;
    int res = -1;
    for (int i = 1; i <= n; i++) {
        res = max(res, p[i] + r(p, n - i));
    }
    return res;
}
```

Time complexity: $O(2^n)$

Why? Overlapping subproblems. (Can you see them?)

How to be more efficient?

Dynamic Programming

Dynamic Programming

- ▶ Top-Down (recursion + memo)
- ▶ Bottom-Up (fill up a table)

$$r_n = \max_{1 \leq i \leq n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$

Top-Down DP

```cpp
vector<int> memo(n + 1, -1);

int r(vector<int> &p, int n) {
    if (n == 0) return 0;
    if (memo[n] != -1) return memo[n];
    int res = -1;
    for (int i = 1; i <= n; i++) {
        res = max(res, p[i] + r(p, n - i));
    }
    memo[n] = res;
    return res;
}
```

Time complexity: $O(n^2)$

Explanation: n function calls, i-th call takes $O(i)$ time. Total: $\sum_{i=1}^{n} O(i) = O(n^2)$

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$

Bottom-Up DP

```cpp
vector<int> r(n + 1, -1);
r[0] = 0;
for (int i = 1; i <= n; i++) {
    for (int j = 1; j <= i; j++) {
        r[i] = max(r[i], p[j] + r[i - j]); // IMPORTANT: Current subproblem
                                           // relies only on the solution of
                                           // the smaller subproblems
    }
}
return r[n];
```

Time complexity: $O(n^2)$

Reconstructing a solution: What if we also want to know where to cut?

No problem!

```cpp
vector<int> cut(n + 1, 0); // cut[i] stores where to optimally cut a rod of
                           // lenght i
vector<int> memo(n + 1, -1);

int r(vector<int> &p, int n) {
    if (n == 0) return 0;
    if (memo[n] != -1) return memo[n];
    int res = -1;
    for (int i = 1; i <= n; i++) {
        if (p[i] + r(p, n - i) > res) {
            res = p[i] + r(p, n - i);
            cut[n] = i; // We should cut at position i
        }
    }
    memo[n] = res;
    return res;
}
```

That was **easy**! Why is it so **difficult** in general?

Remebering is easy (right?)—apply memoization.

Deriving a recursive algorithm is the difficult part.

Usually problems are not given in a way that can straightforwardly be translated into a recursive definition. :(

Essential elements of a DP problem:

- ▶ Usually optimisation problems, i.e. maximise or minimise some quantity
- ▶ Exhibit the optimal subproblem structure
- ▶ Overlapping subproblems

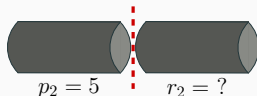▶ Usually optimisation problems, i.e. maximise or minimise some quantity

Example: Rod Cutting Problem

r_n, **maximal** possible revenue obtained by cutting up a rod of length n

Essence of DP (with examples):

▶ Exhibit the optimal subproblem structure
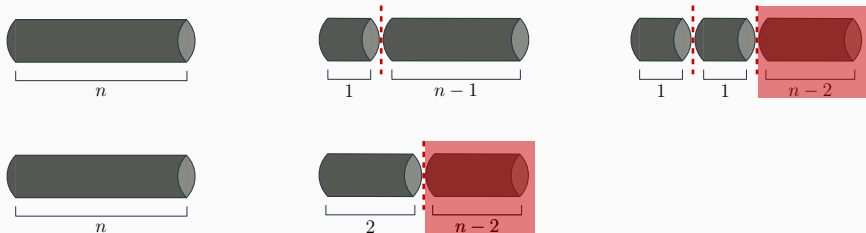
Example: Rod Cutting Problem

$$p_2 = 5 \qquad r_2 = ?$$

▶ Suppose the DP gods show you what is the 'last' choice to make
▶ Look at which subproblems arise once making this choice
▶ Show that the subproblems used in an optimal solution must themselves be solved optimally

Essence of DP (with examples):

▶ Overlapping subproblems

Example: Rod Cutting Problem

▶ Remember, do not recompute!

▶ *'Those who do not remember the past are condemned to repeat it.'*

Common Pitfalls

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$

```cpp
vector<int> memo(n + 1, -1);

int r(vector<int> &p, int n) {
    if (n == 0) return 0;
    if (memo[n] != -1) return memo[n];
    int res = -1;
    for (int i = 1; i <= n + 1; i++) {
        res = max(res, p[i] + r(p, n - i));
    }
    memo[n] = res;
    return res;
}
```

This results in a **SEG FAULT**/**RUN ERROR**, can you see why?

Make sure that you stay 'within' the memo boundaries! Similarly, sometimes the memo table does not/cannot contain the base cases.

$$r_n = \max_{1 \leq i \leq n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$

```cpp
vector<int> memo(n + 1, 0);
```

```cpp
int r(vector<int> &p, int n) {
    if (n == 0) return 0;
    if (memo[n] != 0) return memo[n];
    int res = -1;
    for (int i = 1; i <= n; i++) {
        res = max(res, p[i] + r(p, n - i));
    }
    memo[n] = res;
    return res;
}
```

This results in a **TIMELIMIT**, can you see why?

Make sure that the **default** memo value is **not** a possible output!

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$

```
map<int, int> memo;
```

```
int r(vector<int> &p, int n) {
    if (n == 0) return 0;
    if (memo.find(n) != memo.end()) return memo[n];
    int res = -1;
    for (int i = 1; i <= n; i++) {
        res = max(res, p[i] + r(p, n - i));
    }
    memo[n] = res;
    return res;
}
```

This (possibly) results in a **TIMELIMIT**, can you see why?

`std::map` adds an $O(\log n)$ insert/find/access overhead.

Third Example: Longest Increasing Subsequence

Input: a sequence of n integers $a_1, \ldots, a_n$

Output: the length of a longest increasing subsequence (LIS)

Example:

$$2 \qquad 4 \qquad 3 \qquad 7 \qquad 4 \qquad 5$$

Result: LIS $= 4$

Input: a sequence of n integers $a_1, \ldots, a_n$

Output: the length of a longest increasing subsequence (LIS)

First attempt: $f(i) :=$ 'length of the LIS in $a_1, \ldots, a_i$'

- Base cases: $f(1) = 1$
- $f(i) = ???$

Second attempt: $f(i) :=$ 'length of the LIS in $a_1, \ldots, a_i$ ending at a_i'

- Base cases: $f(1) = 1$
- $f(i) = \max_{j < i : a_j < a_i} (1 + f(j))$

We had to reformulate the problem s.t. it admits a recursive formulation, this is **difficult**!

Time complexity: $O(n^2)$

Explanation: n function calls (with memo), i-th call takes $O(i)$ time.

(**Exercise:** Can you do it in $O(n \log n)$?)

Tips & Tricks

Top-Down (memoization) vs Bottom-Up (iterative)

Usually both work and it boils down to a personal preference :)

- Simple to implement
- Easy to describe subproblems (by using a `std::map`)
- Computes only necessary subproblems
- Time complexity sometimes not so obvious
- Overhead of function calls

- More effort to code
- Subproblems must be described by integers
- Always computes all subproblems
- Time complexity obvious
- Saves some constant factors

How to determine the runtime?

Informally, a product of two factors: the overall number of subproblems and how many choices for each subproblem

Bottom-Up: Easy!

Top-Down: Sometimes harder to see immediately.

`std::map` or `std::vector`? Always **`std::vector`!**

Unless... the subproblem (state space) cannot be described by integers. Then use maps.

Remember! `std::map` has a insert/find/access overhead of $O(\log n)$.

DP – Summary

- ▶ Idea of DP: solve subproblems only once by storing solutions of subproblems
- ▶ Start by defining recurrence relation (on paper)
- ▶ Implement it. It will be correct but slow...
- ▶ Are there overlapping subproblems?
- ▶ Add memo (usually does the trick) or construct a DP table
- ▶ Practice deriving recurrent relations on paper for standard DP problems (e.g. Knapsack, SubsetSum, Coin Change, LCS, Edit Distance, LIS, etc.)

That's all for today!