Introduction to

Quantitative Finance

in Python

CAMBRIDGE SPARK

imc

Part I

- Observing Prices
- Pricing Theory
- Derivatives
- Black-Scholes model

Part I

- Observing Prices
- Pricing Theory
- Derivatives
- Black-Scholes model

Part II

- The binomial model
- Advanced models

Some teaching, a lot of coding . . .

- lots of small code snippets to try,
- get a feel for how things can be coded,
- see things work,
- refer to content for help, modify at will.

I.I Introduction

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Key concepts:

- *asset*
- *derivative*
- *arbitrage*
- *law of one price*

Financial market = agents trading assets

Simple financial assets:

- stocks
- bonds
- commodities

`LA >>>` display financial quotes

Head to your notebook and ...

- retrieve quotes (eg. `AAPL`) over a time range
- display them using `matplotlib`

Financial market = agents trading assets

Simple financial assets:

- stocks
- bonds
- commodities

Composite assets or derivatives:

- contracts with a value depending upon that of another asset.

Forward = agreement to trade in the future

Characteristics:

- fixed future date T (*maturity*)
- fixed price K (*strike*)

Example:

*at $t = 0$, **S**am agrees to sell **L**aura 100 barrels of oil in $T = 2$ months for $K = £10k$.*

Forward = agreement to trade in the future

Characteristics:

- fixed future date T (*maturity*)
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Example:

> *at $t = 0$, **S**am agrees to sell **L**aura 100 barrels of oil in $T = 2$ months for $K = £10k$.*

Sam has a *short position*, Laura has a *long position*.

Arbitrage = make money without risk

Investment opportunity with

- no chance to lose money,
- a chance to make money.

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If there is such an opportunity, it will be exploited immediately (eg. high-frequency trading) $\rightarrow$ assume market is *arbitrage-free*.

Arbitrage = make money without risk

Example:

if the same asset can be bought at two different prices (e.g.: on two different markets) there is an arbitrage.

can you show it?

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Law of one price:

Identical assets must have the same price at any time.

Simplifying assumptions …

- fixed interest rate,
- no transaction fees,
- no bid-ask spread,
- no dividends,
- no cost of carry.

Recap…

Are these concepts clear?

- *asset*,
- *derivative*,
- *arbitrage*,
- *law of one price*.

I.II Observing prices

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Key concepts:

- *log-returns*
- *evolution of the mean and variance of the log-returns*
- *LogNormal model*

Log-returns = important process to model

Log of ratio of subsequent quotes:

$$r_i = \log \frac{S_i}{S_{i-1}}$$

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Log of ratio of subsequent quotes:

$$r_i = \log \frac{S_i}{S_{i-1}}$$

Note:

- scale-free $\rightarrow$ comparable
- $r_1, r_2, \ldots$ modeled as a sequence of random variables ie, a *random process*.

LA >>> analysing log-returns

Head to your notebook and…

- plot the log-returns
- plot the histogram of the log-returns
- fit a normal distribution

Model the log-returns as normally distributed

Observations suggest:

$$r_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$$

Model the log-returns as normally distributed

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Remarks:

- the fit is not perfect,
- how do μ_i, σ_i^2 vary if we increase the lag?

LA >>> mean and variance of the log-returns

Head to your notebook and…

- compute lagged log-returns for $\ell = 5$

$$r_i^\ell = \log \frac{S_i}{S_{i-\ell}}$$

- plot the corresponding histogram
- repeat this for varying ℓ and plot the evolution of the mean and variance with ℓ

Linear evolution of mean, var of log-returns

Observations suggest:

- mean and variance vary linearly with lag

The LogNormal model:

$$\log \frac{S_{t+\tau}}{S_t} \sim \mathcal{N}(\mu\tau, \sigma^2\tau)$$

Recap...

Are these concepts clear?

- *log-returns*,
- *LogNormal model*.

I.III Pricing Theory

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Key concepts:

- *discounting*
- *risk-neutral probability*
- *risk-neutral pricing*
- *replication pricing*

"The price of a financial asset is equal to its discounted expected payoff under the risk-neutral probability"

$$V_t \;=\; \Phi_r(t, T)\mathbb{E}_t^{\star}[V_T]$$

Discounting = taking time into account

At time t, invest X in bank with rate r:

- at $T > t$, you have $\Psi_r(t, T)X$
- inverse $\Phi_r(t, T) = \frac{1}{\Psi_r(t,T)}$ is the discounting factor

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Allows to compare future *cash-flows*:

- $(X_1, T_1) \to \Phi_r(0, T_1)X_1$
- $(X_2, T_2) \to \Phi_r(0, T_2)X_2$

Example:

You want to sell your house to a friend, he offers £200*k* today or £208*k* next year. The annual rate at your bank is 5%.

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What should you do?

the present value of getting £208k in one year is £208k/1.05 or £198.1k. You should take the money today.

Here, the discounting factor is $\Phi_r(0,1) = \frac{1}{1.05}$.

How much would you pay to play?

Consider a coin-flip game with rewards

- £10 if head,
- £0 if tail.

If you have to pay to play, what is the most you would agree to pay? (and why?)

Most agents are risk-averse

- As a seller, I'd like to set the price high but also low enough so that lots of people want to play.

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- Since most people are risk-averse, the price will reflect this and be lower than £5, say £3.

Most agents are risk-averse

- As a seller, I'd like to set the price high but also low enough so that lots of people want to play.
- Since most people are risk-averse, the price will reflect this and be lower than £5, say £3.
- *How can we interpret this price?*

Interpreting the price in a risk-neutral world

Interpretation 1:

- price $=$ (expected gain) $-$ (price of risk).

$$£3 = (£5) - (£2)$$

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Interpretation 1:

- price $=$ (expected gain) $-$ (price of risk).

$$£3 = (£5) - (£2)$$

- how to interpret the *price of risk*? (here £2)

Interpretation 2:

- price $=$ expected gain with *modified probabilities*.

$$£3 = £10 \times p_H^\star + £0 \times p_T^\star.$$

This artificial world with different probabilities is the risk-neutral world.

With the risk-neutral pdf, pricing is easy.

The risk-neutral pdf $p^\star$ is a *fictive* probability distribution that associates *different weights* to outcomes.

With it, you can price *any asset* depending on the same source of randomness and *guarantee there will not be an arbitrage*.

Example: introducing another game…

Same coin-flips but rewards (H:£6, T:£15).

- what should the price be?
- you can show that for any different price, there is an arbitrage (cf. notes)

So you think you can price…

We have seen the two key elements to pricing:

- using discounting to take time into account,
- using risk-neutral probabilities to price.

Together: $V_t \ = \ \Phi_r(t, T)\mathbb{E}_t^\star[V_T].$

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We have seen the two key elements to pricing:

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Together: $V_t \;=\; \Phi_r(t, T)\mathbb{E}_t^\star[V_T].$

Now what?

- how do you find the risk-neutral pdf?
- how do you compute the expected-value?

Another way: if you can replicate, you can price

If you can build a portfolio that *replicates* the cash-flows of a derivative, then, by the *law of one price*, the price of that derivative is equal to the value of the portfolio.

Replication in a simple market model…

Model the evolution over one time-step a stock, a bank account and a derivative as:

$$
\begin{array}{ccccccccc}
 & & uS_0 & & & & & & V_u \\
S_0 & \nearrow\!\!\!\!\searrow & & \qquad \Phi_r & \longrightarrow & 1 \qquad & V_0 & \nearrow\!\!\!\!\searrow & \\
 & & dS_0 & & & & & & V_d
\end{array}
$$

Replication in a simple market model…

Model the evolution over one time-step a stock, a bank account and a derivative as:

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\begin{tikzcd}[row sep=small]
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\end{tikzcd}
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\qquad
\begin{tikzcd}[row sep=small]
 & V_u \\
V_0 \arrow[ur] \arrow[dr] & \\
 & V_d
\end{tikzcd}
$$

Build a portfolio with αS_0 and $\beta\Phi_r$, it evolves as:

$$
\begin{tikzcd}[row sep=small]
 & \alpha u S_0 + \beta \\
\alpha S_0 + \beta\Phi_r \arrow[ur] \arrow[dr] & \\
 & \alpha d S_0 + \beta
\end{tikzcd}
$$

Replication in a simple market model…

$$
\begin{tikzcd}[column sep=small]
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\end{tikzcd}
\qquad\qquad
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 & V_u \\
V_0 \arrow[ur] \arrow[dr] & \\
 & V_d
\end{tikzcd}
$$

Replicating = determining $\alpha^\star$ and $\beta^\star$ such that

$$
\begin{cases}
\alpha^\star u S_0 + \beta^\star & = & V_u \\
\alpha^\star d S_0 + \beta^\star & = & V_d
\end{cases}
$$

Then, $V_0 = \alpha^\star S_0 + \beta^\star \Phi_r$.

Replication in a simple market model… 2

$$
\begin{tikzcd}[column sep=small]
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Then, $V_0 = \alpha^\star S_0 + \beta^\star \Phi_r$.

You could show the equivalence between this and the risk-neutral pricing (cf. notes).

Recap…

Are these concepts clear?

- *discounting*,
- *risk-neutral probabilities*,
- *risk-neutral pricing*,
- *replication pricing*.

I.IV Derivatives

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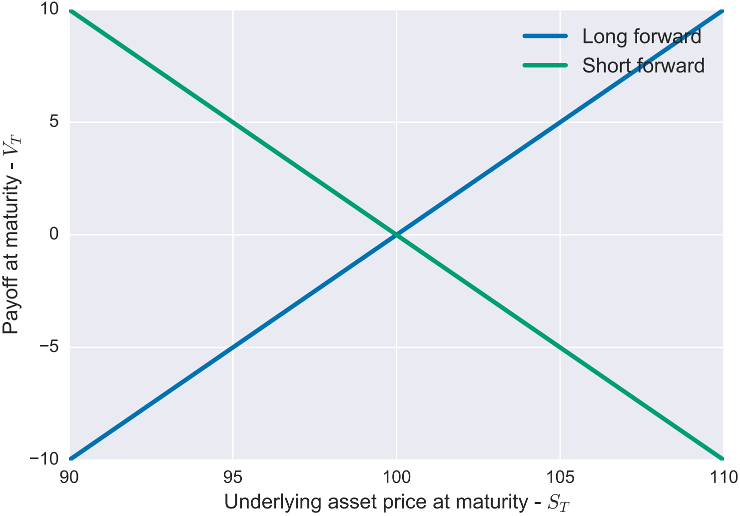
Key concepts:

- *payoff curve*
- *European & American options*
- *leverage*
- *put-call parity*

Payoff curves for the forward



Payoff = value of the exchange at maturity

For the long forward, the payoff at maturity is

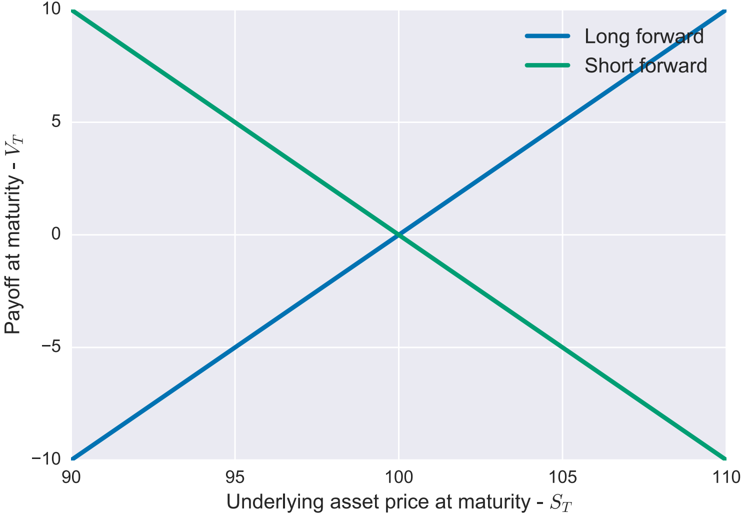
$$V_T^{\text{fwd}} \;=\; (S_T - K)$$

and the opposite for the short forward.

Recall that

- S_T is the price of the asset at maturity,
- K is the strike price of the forward.

Payoff curves for the forward



Characterising the long forward

- unlimited upside
- limited downside
- bullish

Characterising the long forward

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The payoff is also used to compute the price (on the board)

Options = right to buy/sell

Characterised by a strike price K and an expiry date T:

- Call: right to buy
- Put: right to sell

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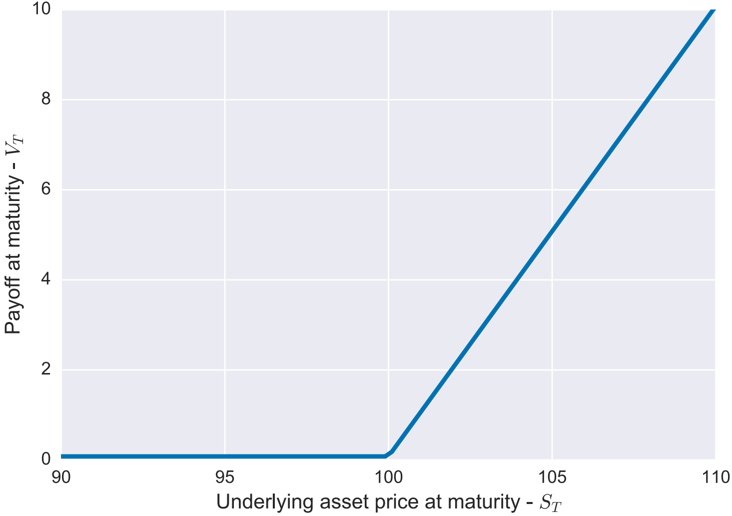
Call compared to forward, what changes?

Payoff curve for the call



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Call

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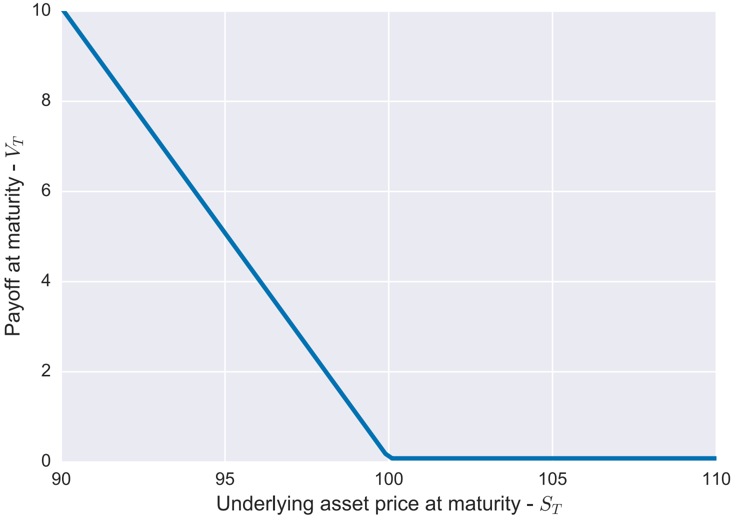
what about the put?

Payoff curve for the put



Characterising options

Call

- unlimited upside
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$$V_T^{\text{call}} = \max\{S_T - K, 0\}$$

what about the put?

- limited upside
- no downside
- bearish

$$V_T^{\text{put}} = \max\{K - S_T, 0\}$$

Characterising options

Call

- unlimited upside
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what about the put?

- limited upside
- no downside
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$$V_T^{\text{put}} = \max\{K - S_T, 0\}$$

You will see how to price those a bit further...

There are different options exercise style

- European: exercise is possible at (only at) T,

There are different options exercise style

- European: exercise is possible at (only at) T,
- American: exercise is possible at any time before T.

There are different options exercise style

- European: exercise is possible at (only at) T,
- American: exercise is possible at any time before T.

Nothing at all to do with geography! Bermudan options....

Key feature of options = leverage

Consider a call…

- buying a call is much *cheaper* than buying the underlying,
- you can therefore buy a large number of calls

Key feature of options = leverage

Consider a call…

- buying a call is much *cheaper* than buying the underlying,
- you can therefore buy a large number of calls
- if you choose to exercise, you then physically own a large number of units of the underlying and can sell it,
- this is called leverage.

Example: a golden call

Consider a call on an ounce of gold…

- the price of the call is at £15 and the strike at $K = £980$,
- the current (spot) price of the underlying is $S_t = £950$.

You have £10k to invest. Assume you invest everything in either the call or the underlying and that, at maturity, the spot price is at £1k.

What was the best choice?

Example: a golden call

Scenario 1: *investment in the underlying*

- with £10k you bought 10.5 ounces of gold,
- at maturity this is worth £10.5k
- profit: £500 or 5% return.

Example: a golden call

<u>Scenario 1</u>: *investment in the underlying*

- with £10k you bought 10.5 ounces of gold,
- at maturity this is worth £10.5k
- profit: £500 or 5% return.

<u>Scenario 2</u>: *investment in the call*

- with £10k you can buy more than 650 calls,
- at maturity, you exercise (since $S_T > K$) which gives you the right to buy 650 ounces of gold and sell them immediately making £20 per ounce.
- profit: $650 \times £20 = £13k$ or 30% return.

Example: a golden call… gone bad

Gold price *drops* to £900

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Scenario 1: *investment in the underlying*

- 10.5 ounces of gold now worth £9450.
- loss of £550 or -5.5%

Example: a golden call... gone bad

Gold price *drops* to £900

Scenario 1: *investment in the underlying*

- 10.5 ounces of gold now worth £9450.
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Scenario 2: *investment in the call*

- Call is out of the money, expires worthless
- you loose all your money -100%

Put-Call Parity = identity linking call and put

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Long call and short put with the same strike and expiry then:

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Long call and short put with the same strike and expiry then:

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Therefore, if you can price a call, you can also price a put:

$$V_t^{\text{put}} \;=\; \underbrace{\Phi_r(t,T)K - S_t}_{\text{price of forward}} + V_t^{\text{call}}.$$

Recap…

Are these concepts clear?

- *payoff curve*,
- *call and put options*,
- *European & American exercise*,
- *leverage*,
- *put-call parity*.

I.V Black-Scholes model

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Key concepts:

- *Black-Scholes formula*
- *Greeks* ($\Delta, \nu, \Theta, \Gamma$)
- *implied volatility*
- *volatility smile*

The LogNormal model in the risk-neutral world

LogNormal model:

$$\frac{S_{t+\tau}}{S_t} \sim \mathrm{LogNormal}(\mu\tau, \sigma^2\tau)$$

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$$\mathbb{E}\left[\frac{S_{t+\tau}}{S_t}\right] = \frac{\mathbb{E}[S_{t+\tau}]}{S_t} = \exp((\mu + 0.5\sigma^2)\tau)$$

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In the risk neutral world, we must have

$$\frac{\mathbb{E}^\star[S_{t+\tau}]}{S_t} = \exp(r\tau)$$

Modification: $\mu \to r - 0.5\sigma^2$

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In the risk neutral world, we must have

$$\frac{\mathbb{E}^\star[S_{t+\tau}]}{S_t} = \exp(r\tau)$$

Modification: $\mu \to r - 0.5\sigma^2$

Only the drift changes!

Pricing of European options: Black-Scholes

- Risk-neutral distribution: $\mathrm{LogNormal}((r - 0.5\sigma^2)\tau, \sigma^2\tau)$
- Payoff of a European call: $V_T^{\text{call}} = \max\{S_T - K, 0\}$

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- Risk-neutral price:

$$V_t^{\text{call}} = \Phi_r(t, T)\mathbb{E}_t^\star[V_T^{\text{call}}] = S_t F(d_1) - K\exp(-r\tau)F(d_2)$$

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with

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left(-\frac{1}{2}t^2\right) dt$$

$$d_{1,2} = \frac{\log(S_t/K) + (r \pm 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

and $\tau = (T - t)$.

LA >>> coding Black-Scholes

Head to your notebook and...

- write a function that computes the price of a European call
- test with $S_t = £105$, $K = £100$, $\tau = 2$ years, $r = 5\%$ per year, $\sigma = 15\%$ (the price should be £ ≈ 17.4)

$$V_t^{\text{call}} = S_t F(d_1) - K \exp(-r\tau) F(d_2)$$

and $d_{1,2} = (\log(S_t/K) + (r \pm 0.5\sigma^2)\tau)/\sigma\sqrt{\tau}$, use `norm.cdf` for F.

The Greeks: a way to quantify risk

Want to characterise variability of the value of the derivative with respect to:

- the price of the underlying $\rightarrow$ Delta (Δ),

The Greeks: a way to quantify risk

Want to characterise variability of the value of the derivative with respect to:

- the price of the underlying $\rightarrow$ Delta (Δ),
- the volatility of the underlying $\rightarrow$ Vega (ν),

The Greeks: a way to quantify risk

Want to characterise variability of the value of the derivative with respect to:

- the price of the underlying $\to$ Delta (Δ),
- the volatility of the underlying $\to$ Vega (ν),
- the time to expiry of the contract $\to$ Theta (Θ).

The *Delta*: variability with respect to price

Defined as:

$$\Delta = \frac{\partial V}{\partial S} \approx \frac{V(S + \Delta S) - V(S)}{\Delta S}$$

- For the European call, it can be computed exactly ($\Delta = F(d_1)$). Otherwise, can use approximations.

The *Delta*: variability with respect to price

Defined as:

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- For the European call, it can be computed exactly ($\Delta = F(d_1)$). Otherwise, can use approximations.
- It is a common strategy to attempt a Delta-neutral portfolio where $\Delta \approx 0$ at all times.

Using Black-Scholes to get the implied volatility

Current workflow:

$\boxed{\text{log-returns}}$

Using Black-Scholes to get the implied volatility

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$\boxed{\text{log-returns}} \rightarrow \boxed{\text{Volatility}}$

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Using Black-Scholes to get the implied volatility

Current workflow:

$$\boxed{\text{log-returns}} \longrightarrow \boxed{\text{Volatility}} \longleftrightarrow \boxed{\text{Prices}}$$

What happens if you *reverse* it?

- look at existing option prices,
- find the implied volatility σ_{IV} such that Black-Scholes returns the same prices.

Using Black-Scholes to get the implied volatility

Current workflow:

$\boxed{\text{log-returns}} \rightarrow \boxed{\text{Volatility}} \leftrightarrow \boxed{\text{Prices}}$

What happens if you *reverse* it?

- look at existing option prices,
- find the implied volatility σ_{IV} such that Black-Scholes returns the same prices.
- implied volatility comparable across funds

Using Black-Scholes to get the implied volatility

Current workflow:

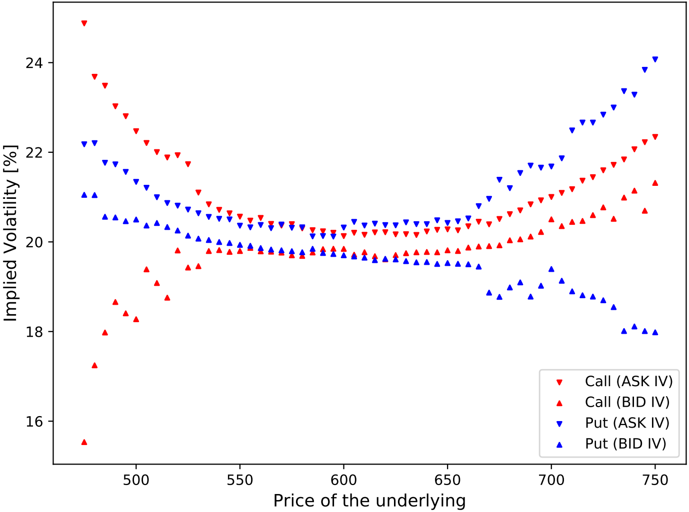
log-returns $\rightarrow$ **Volatility** $\leftrightarrow$ Prices

What happens if you *reverse* it?

- look at existing option prices,
- find the implied volatility σ_{IV} such that Black-Scholes returns the same prices.
- implied volatility comparable across funds
- we would expect $\sigma_{LR} \approx \sigma_{IV} \ldots$

A look at the implied volatility…

For each price you can compute the implied vol:



Volatility smile or limits of the LogNormal model

Under the LogNormal model, you should have expected a constant σ, this *does not hold*.

Workarounds?

- other model? $\rightarrow$ can do (e.g., Student-t based)

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Under the LogNormal model, you should have expected a constant σ, this *does not hold*.

Workarounds?

- other model? $\rightarrow$ can do (e.g., Student-t based)
- expressing σ as a function of K in Black-Scholes?
  $\rightarrow$ a common way around the issue.

Volatility smile or limits of the LogNormal model

Under the LogNormal model, you should have expected a constant σ, this *does not hold*.

Workarounds?

- other model? $\rightarrow$ can do (e.g., Student-t based)
- expressing σ as a function of K in Black-Scholes? $\rightarrow$ a common way around the issue.
    - further, express σ as a function of K *and* T: $\sigma(f(K, T))$

Recap...

Are these concepts clear?

- *Black-Scholes formula*,
- *Greeks* ($\Delta, \nu, \Theta, \Gamma$),
- *implied volatility*,
- *volatility smile*.

Part II – The binomial model

Part II – The binomial model

Key concepts:

- *recombining binomial tree (BT)*
- *calibrating a BT*
- *pricing with a BT*

Binomial tree = market as a branching process

$$
\begin{array}{ccc}
 & & uS_0 \\
 & \nearrow & \\
S_0 & & \\
 & \searrow & \\
 & & dS_0
\end{array}
$$

Binomial tree = market as a branching process

$$
\begin{tikzcd}
 & & u^2 S_0 \\
 & uS_0 \arrow[ur] \arrow[dr] & \\
S_0 \arrow[ur] \arrow[dr] & & udS_0 \\
 & dS_0 \arrow[ur] \arrow[dr] & \\
 & & d^2 S_0
\end{tikzcd}
$$

Binomial tree = market as a branching process

$$
\begin{tikzcd}
 & & u^2 S_0 \\
 & uS_0 \arrow[ur] \arrow[dr] & \\
S_0 \arrow[ur] \arrow[dr] & & udS_0 \\
 & dS_0 \arrow[ur] \arrow[dr] & \\
 & & d^2 S_0
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & & p^2 \\
 & p \arrow[ur] \arrow[dr] & \\
1 \arrow[ur] \arrow[dr] & & 2pq \\
 & q \arrow[ur] \arrow[dr] & \\
 & & q^2
\end{tikzcd}
$$

The last layer converges to a LogNormal distribution when the number of steps grows.

LA >>> convergence of BT to LogNormal

Head to your notebook and…

- compute the end probabilities and the end prices if
  - $p = 0.6, u = 1.03, d = 0.99, N = 100$
- compute the returns, plot the distribution,
- plot the log-returns and fit a normal distribution.

LA >>> hints...

- *k*th node at *N*th step:

$$\Pr(k, N) = \frac{N!}{k!(N-k)!} p^k q^{N-k}, \qquad S_T(k) = S_0 u^k d^{N-k}.$$

- use `scipy.misc.comb`

LA >>> hints…

- kth node at Nth step:

$$\Pr(k, N) = \frac{N!}{k!(N-k)!} p^k q^{N-k}, \qquad S_T(k) = S_0 u^k d^{N-k}.$$

- use `scipy.misc.comb`

Got all this already?

- can you return these quantities for *any* step?

Pricing with a BT? use risk-neutral pricing...

In the risk neutral-world, we must have

$$S_t = \Phi_r(t, t + \Delta t)\mathbb{E}_t^\star[S_{t+\Delta t}]$$

with $\Delta t = (T - t)/N$. This gives,

$$p^\star = \frac{\Phi_r^{-1}(t, t + \Delta t) - d}{u - d} = \frac{\exp(r\Delta t) - d}{u - d}.$$

Pricing with a BT? use risk-neutral pricing…

Consider a European call, u and d given,

- follow the tree forward $\rightarrow$ *intrinsic values* of the derivative,
- follow the tree backward $\rightarrow$ *risk-neutral prices*

Let's see how it works in practice…

Example: European call in a BT

Consider a 2-step model with

$$u = 1.02, \quad d = 0.99, \quad S_0 = 100, \quad K = 99 \quad \text{and} \quad T = 1.$$

Assume that $r = 0$ so that $\Phi_r = 1$.

- Write the evolution of the price:

Example: European call in a BT

Consider a 2-step model with

$$u = 1.02, \quad d = 0.99, \quad S_0 = 100, \quad K = 99 \quad \text{and} \quad T = 1.$$

Assume that $r = 0$ so that $\Phi_r = 1$.

- Write the evolution of the price:

$$\begin{tikzcd}
 & & 104.04 \\
 & 102 \arrow[ur] \arrow[dr] & \\
100 \arrow[ur] \arrow[dr] & & 100.98 \\
 & 99 \arrow[ur] \arrow[dr] & \\
 & & 98.01
\end{tikzcd}$$

Example: European call in a BT

- Write the value of the derivative at the last layer (maturity):

Example: European call in a BT

- Write the value of the derivative at the last layer (maturity):

$$104.04 \longrightarrow 5.04$$

$$100.98 \longrightarrow 1.98$$

$$98.01 \longrightarrow 0.00$$

Example: European call in a BT

- risk neutral probability: $p^\star = \frac{1-d}{u-d} = \frac{1}{3}$,
- follow the tree backward using risk-neutral pricing:

Example: European call in a BT

- risk neutral probability: $p^\star = \frac{1-d}{u-d} = \frac{1}{3}$,
- follow the tree backward using risk-neutral pricing:

$$\begin{tikzcd}
5.04 \arrow[dr] & & \\
& 3.00 \arrow[dr] & \\
1.98 \arrow[ur] \arrow[dr] & & 1.44 \\
& 0.66 \arrow[ur] & \\
0.00 \arrow[ur] & &
\end{tikzcd}$$

- for example, $5.04p^\star + 1.98(1 - p^\star) = 3.0$.

LA >>> Coding a simple BT

Head to your notebook and reproduce the pricing steps with…

- $u = 1.02, d = 0.99, S_0 = 100, K = 99, T = 1, N = 2$

```
Forward
100
[  99.   102.]
[  98.01  100.98  104.04]
Backward
[ 0.     1.98  5.04]
[ 0.66  3.   ]
1.44
```

Set u, d so that BT matches LogNormal model

It can be shown that setting

$$u \;=\; \exp(\sigma\sqrt{\Delta t}), \quad \text{and} \quad d \;=\; \frac{1}{u},$$

the BT converges to the same LogNormal model than the one observed empirically, *in the risk-neutral world*.

Recap...

Are these concepts clear?

- *recombining BT*,
- *calibration of a BT*,
- *pricing with a BT*.

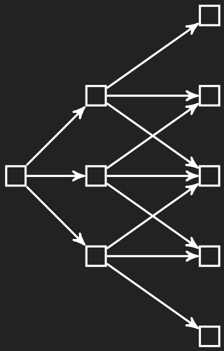
Part II – Advanced models

Part II – Advanced models

Key concepts:

- *recombining trinomial tree (TT)*
- *pricing in a TT*
- *the Black-Scholes PDE*
- *performance computing*

Trinomial tree = binomial tree on steroids



- go up (u), down ($d = 1/u$) or mid ($m = 1$)
- can be more accurate
- more flexible

Pricing in a TT = similar than in BT

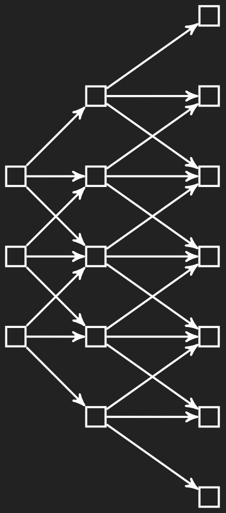
Same process (*forward* then *backward* pass) with

- risk-neutral probabilities $p_u^\star, p_m^\star, p_d^\star$ (*cf. content*),
- to have the TT converge to the LogNormal model, can set

$$u, d \;=\; \exp(\pm\lambda\sigma\sqrt{2\Delta t})$$

- λ can be tuned (eg.: to have nodes achieving specific values)

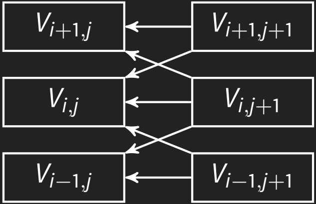
Replicating the TT for multiple initial prices



This allows to *precompute* values avoiding having to recompute a tree if the value of the underlying changes!

From tree pricing to grid pricing

If we repeat the replication, the core part of the tree will look like a grid, and the backward step will look like:



With the $V_{i,j}$ being given by a formula of the form

$$V_{i,j} \;=\; \alpha V_{i+1,j+1} + \beta V_{i,j+1} + \gamma V_{i-1,j+1}.$$

Grid pricing or a finite-difference solver…

The equation can be written as

$$V_{:j} = \mathbf{T}\, V_{:j+1}$$

where $V_{:j}$ refers to the jth column of the grid and $\mathbf{T}$ is a tridiagonal matrix. You may recognise the particular form of a *finite-difference solver* used to solve *Partial differential equations* (PDE).

The Black-Scholes PDE for pricing…

In fact, it can be shown that pricing with the LogNormal model amounts to solving the following PDE:

$$\frac{\partial V(S,t)}{\partial t} + rS\frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} - rV(S,t) = 0.$$

with specific boundary conditions (see notes).

This is beyond the scope of this course… but it useful to understand that pricing can amount to solving a PDE.

The PDE perspective: a path to more methods

- solving the PDE yields the *full evolution of the price* of the option.
- numerical methods for PDE is a well studied field with a wealth of algorithms and techniques that can be used.

Performance computing = crucial in industry

- on electronic exchanges, things go *very fast*, everything is *automated*.
- if conditions change, you cannot afford to recompute everything.
- need for pre-computations (caching) and interpolation of results,
- crucial to be competitive!

Recap...

Are these concepts clear?

- *recombining TT*,
- *calibration & pricing with a TT*,
- *the Black-Scholes PDE*,
- *the need for performance computing*.

Market Making Competition

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- *write your own numerical model (binomial / trinomial tree)*

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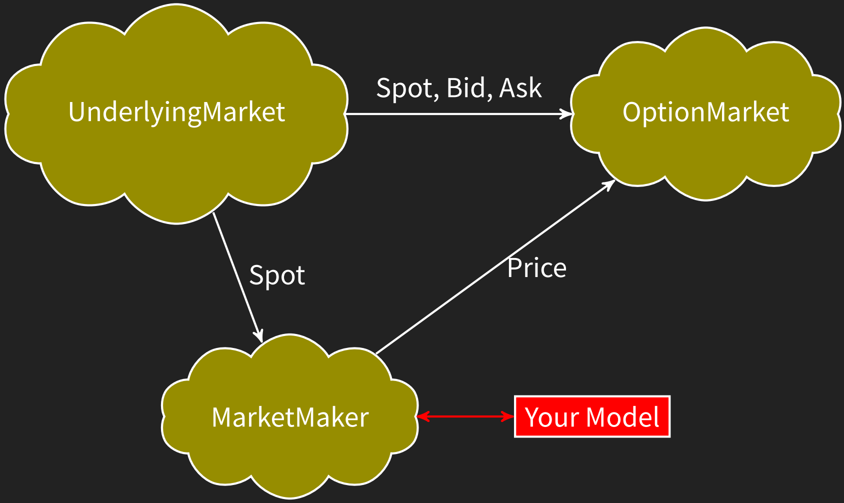
- *tune it for perfomance*

Market Making Competition

- *write your own numerical model (binomial / trinomial tree)*
- *get it to converge to the correct price*
- *tune it for perfomance*
- *beat the competition!*

Competition The Game



Competition How to go about developing it?

Unzip the `students.zip` file into a directory of your choice

Competition How to go about developing it?

Unzip the `students.zip` file into a directory of your choice

IPython Notebook

- start `jupyter-notebook`
- read through `TestYourModel` notebook
- use this notebook as a starting point

Python Script

- Your starting point is the `models/MyPricer.py` file
- Edit / run / improve

Competition Demonstration