Homework 6

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1. PROBLEM 1 - WIEN'S DISPLACEMENT CONSTANT (NEWMAN 6.13)

Exercise 6.13 is about using Planck's radiation law to determine Wien's displacement constant for blackbody radiation, using the constants c as the speed of light, h for Planck's constant, and k_B for Boltzmann's constant. Planck's radiation law is given below:

$$I(\lambda) = \frac{2\pi hc^2\lambda^{-5}}{e^{hc/\lambda k_B T} - 1}$$

Part a)

Part a) is just solving for the peak, so we will use the following logic to determine the value of λ for which the radiation is at its maximum.

$$\frac{dI}{d\lambda} = \frac{2ch\pi}{k_B\lambda^7 T}\frac{5k_BT\lambda + e^{\frac{ch}{k_BT\lambda}}(ch - 5k_BT\lambda)}{(e^{\frac{ch}{k_BT\lambda}} - 1)^2}$$

$$0 = 5k_BT\lambda + e^{\frac{ch}{k_BT\lambda}}(ch - 5k_BT\lambda)$$

$$0 = 5 + e^{\frac{ch}{k_BT\lambda}}(\frac{ch}{k_BT\lambda} - 5)$$

Using what we have here, we're going to make an x substiution into these equations, because it's a bit messy as it is.

$$x = \frac{ch}{k_BT\lambda}$$

And here we go...

$$\begin{aligned} 0 &= 5 + e^x(x - 5) \\ &= 5 + xe^x - 5e^x \\ &= 5e^{-x} + x - 5 \end{aligned}$$

And we have our proof!

Part b)

For Part b) I used both Newton and Binary search methods to find the root of the function, they had slightly different values, but varied by an order of magnitude less than the $\epsilon = 10^{-6}$. The root of the function is $x = 4.965114$, leading to a value of $b = 2.897771 * 10^{-3}$

Part c)

Given that $\lambda = \frac{b}{T} = 502$ nm, we can solve for T, giving us the equation:

$$T = \frac{b}{\lambda}$$

Providing the answer that $T = 5.77245 * 10^3 K$.

2. PROBLEM 2 - LAGRANGE POINTS (NEWMAN 6.16)

This problem involves finding the Lagrange point for two bodies orbiting under the influence of gravity.

Part a)

Part a) wants us to show that if the earth is much more massive than both the moon and the satellite intended to be placed into the Lagrange point, the following equation would be satisfied for the distance r to L_1

$$\frac{GM}{r^2} - \frac{Gm}{(R-r)^2} = \omega^2 r$$

R is the distance from the earth to the moon, r is the listance from the center of earth to the L_1 point, G is the gravitational constant, m is the mass of the moon, M is the mass of the earth, and ω is the angular velocity of both the moon and the satellite. We're going to show that the above equation is true with the following logic:

$$F_M = \frac{GMm}{r^2}$$

$$F_m = \frac{Gm^2}{(R-r)^2}$$

$$F_c = m\omega^2 r$$

Given that the force of the moon pulling on the satellite is going to be in the direction opposite that of the Earth pulling on it, we're going to subtract F_M F_m to get F_{tot}. Now we just set it equal to the centripetal force to satisfy that the orbits are circular

$$F_M - F_m = F_{tot} = \frac{GMm}{r^2} - \frac{Gm^2}{(R-r)^2}$$

$$\frac{GMm}{r^2} - \frac{Gm^2}{(R-r)^2} = m\omega^2 r$$

$$\frac{GM}{r^2} - \frac{Gm}{(R-r)^2} = \omega^2 r$$

Part b)

We solved this problem using the secant method, with the following given parameters:

$$G = 6.674 * 10^{-11} m^3 kg^{-1} s^{-2}$$
$$M = 5.974 * 10^{24} kg$$
$$m = 7.348 * 10^{22} kg$$
$$R = 3.844 * 10^{8} m$$
$$\omega = 2.662 * 10^{-6} s^{-1}$$

We found the lagrange point to be located at $3.260 * 10^8$ meters from the earth's center.

Lagrange Points Extra Credit

Lagrange Points are where the gravitational pull from 2 large masses is equal to the centripetal force for a smaller mass to move with them in a regular and constant pattern. The ARTEMIS Satellite occupies a Lagrange Point between the earth and the moon, and the James-Webb telescope is supposed to occupy one of the Sun-Earth Lagrange points

3. PROBLEM 3 - CMB STUFF

This problem primarily relates to taking in data from the Cosmic Microwave Background (CMB). This will primarily involve examing the usefulness of interpolation, as well as using methods of differentiation and integration.

Part a)

Here we have both the power spectrum data, as well as the data we've computed using 200 sample points using Spline Interpolation and Scipy. As you can see below, it visually matches very nicely.

However, we also want to check how the interpolated fit compares in terms of fractional error, so we plot that as well against the fractional precision of $\frac{3}{l}$.

Here we can see that it manages to stay below the threshold for all plotted values of l. Thus we can say that our spline does do well enough for the purposes of accurate fitting to cosmological parameters.

Part b)

This part had me plotting the slope of the power spectrum, I plotted it on a log scale and found that it again stayed below the $\frac{3}{l}$ line.

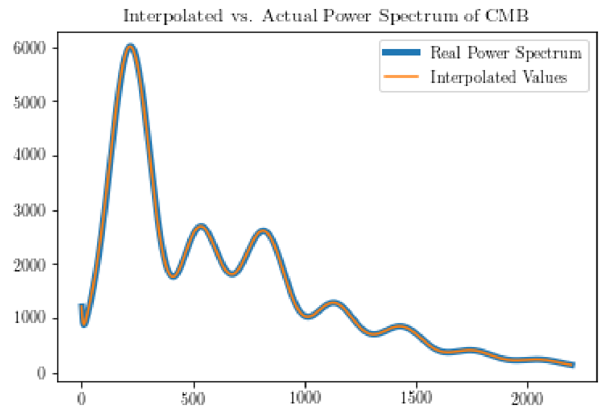

FIG. 1: Actual Data vs. Interpolated Data Power Spectrum

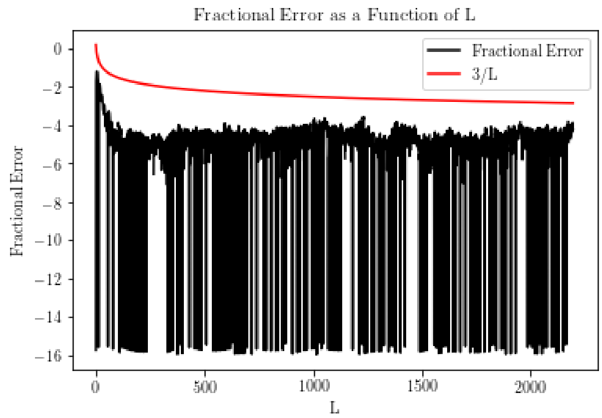

FIG. 2: Fractional Error of Interpolated Values

Part c)

Part c) has us take the following integral:

$$\int_1^\infty \frac{ldl}{2\pi} C_l^{TT}$$

I used a modified Simpson's Integral, and evaluated it to get a value of .0871

4. SURVEY RESPONSE

This homework was a tricky one, and I spent a fair amount of time on it. I figured out how to do problem 1 and 2 with Katharine Bancroft, and we breezed through those. I spent a significant amount of time trying to figure out the CMB problem, and even still I'm not quite 100 percent confident in my answers. This assignment is rather late though, so I'm just going to turn it in and hope for the best.

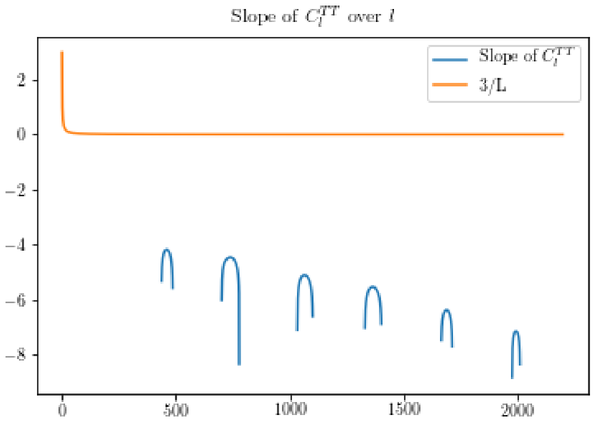

FIG. 3: Fractional Error of Interpolated Values