Combinatorics Introduction

Maths Extension Group MHS

Why Combinatorics?

- Very cool topic
- Common Topic; see 2022 AMC
- Combinatorics is useful for VCE

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$$

$$\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{n+r}{r}=\binom{n+r+1}{r}$$

$$\binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+\cdots+\binom{n}{r}=\binom{n+1}{r+1}$$

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2=\binom{2n}{n}$$

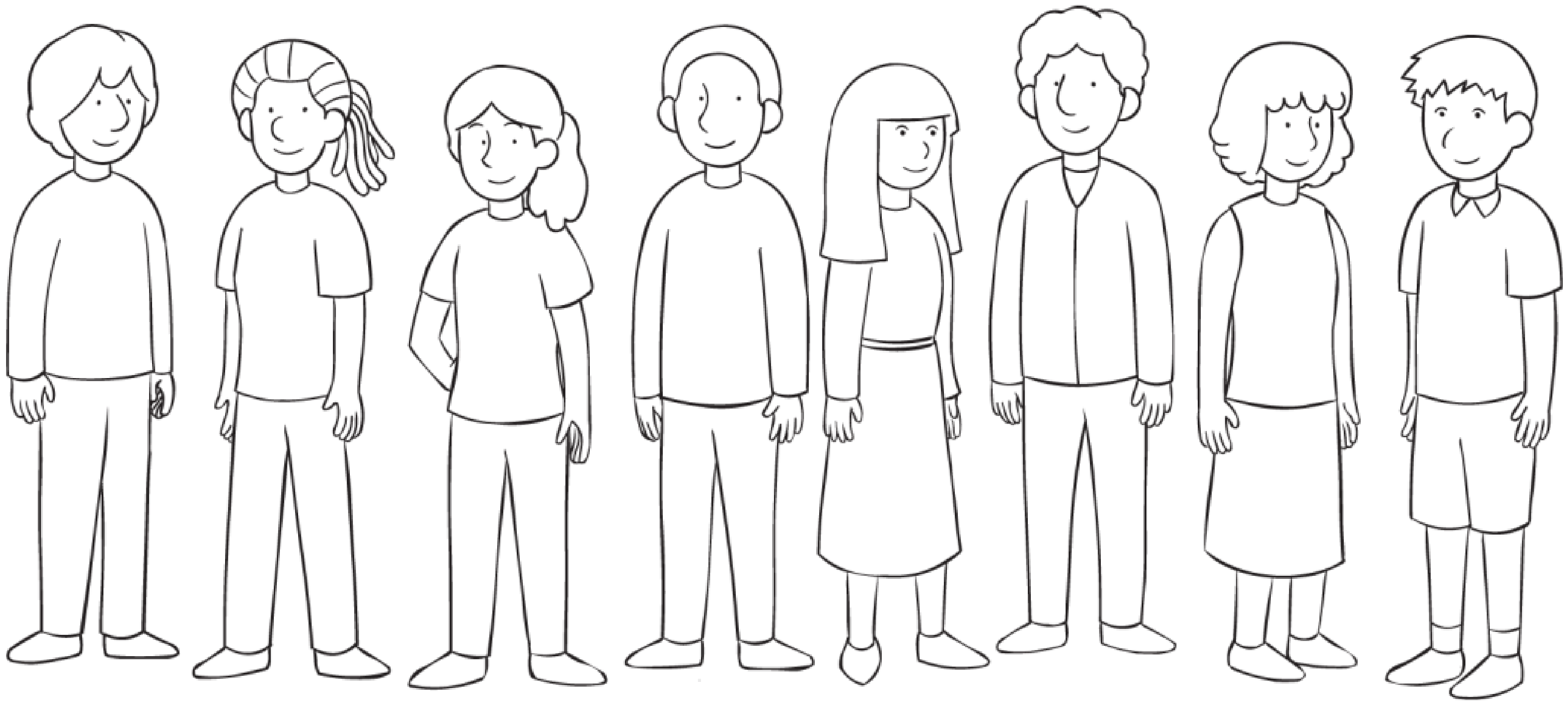
$$\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}=\binom{m+n}{r}$$

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n}{r+k}=\binom{m+n}{m+r}$$

$$\sum_{k=n-s}^{m-r}\binom{m-k}{r}\binom{n+k}{s}=\binom{m+n+1}{r+s+1}$$

Multiplication

In how many ways can 8 people stand in a line?



Multiplication

There are 8 options for the first person in the line.

There are 7 options for the second person in the line.

There are 6 options for the third person in line.

…..

There is 1 option for the last person in line

Hence there are 8 x 7 x 6 x … x 1= 8! ways

Multiplication

In general there are $n!$ ways that n people can stand in a line.

Note: $n! = n \times (n-1) \times (n-2) \times \ldots \times 1$

E.g. $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Division

In how many ways can you choose a four-flavour combination from 10 ice-cream flavours?

Division

Let’s imagine if we cared about the order of the ice-creams.

There would be 10 options for the first ice-cream, 9 for the second ice-cream etc. So there would be 10 x 9 x 8 x 7 choices.

But we do not care about the order: we have overcounted the ice-creams by a factor of 4!

Hence there are $\frac{10!}{6!4!} = 210$ choices

Division

We conveniently have a formula. If we have n distinct objects and we want to make a group of r objects there are “n choose r” ways to do so.

$$ {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} $$

Extra Stuff that is useful

- Pigeon Hole Principle (PHP)
- Principle of Inclusion Exclusion (PIE)
- Complementary counting
- Recursion

2022 AMC Junior Q30

In how many ways can 100 be written as the sum of three different positive integers? Note that we do not consider the sums formed by reordering terms to be different, so that 34 + 5 + 61 and 61 + 34 + 5 are treated as the same. (AMC Junior Q30)

Method 1

Suppose we have

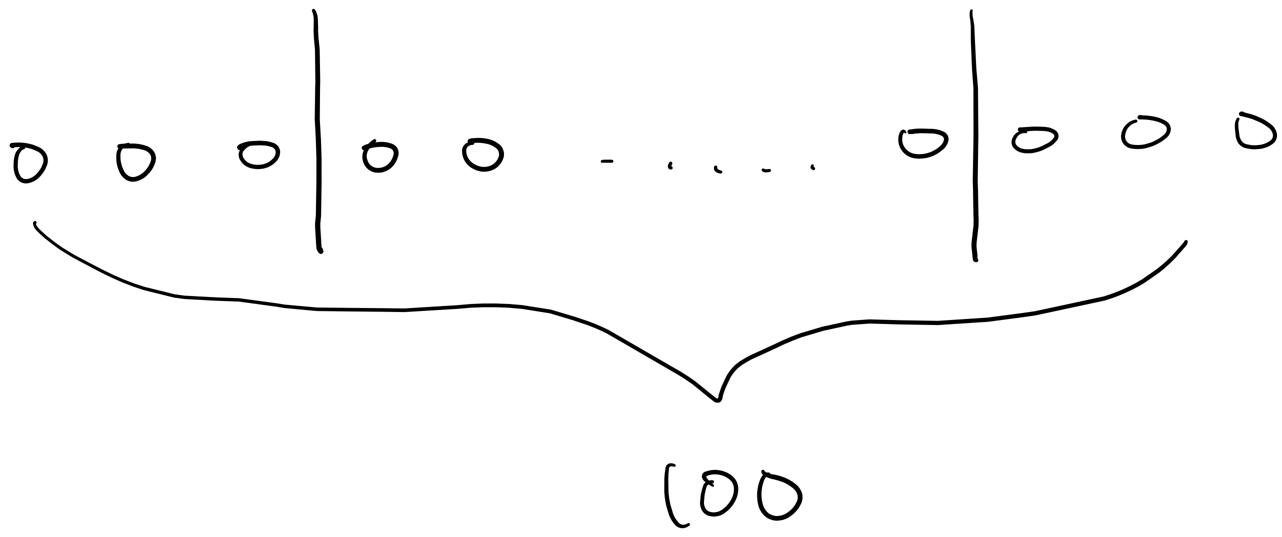
$$a+b+c=100$$

Where a, b ,c ≥ 1

Then we have $\binom{99}{2} = 4851$ possible solutions. Why?

Method 1

Imagine we have 100 balls, and we have 2 sticks. We can split it into 3 parts using the sticks, and there are 99 places to choose where to put the sticks



Method 1

But wait …. a, b, c are different.

Without loss of generality, lets look at a=b.

There are 49 cases where a=b; (a, b, c) = (1,1,98), (2,2,96), … , (49,49,2)

Hence there are 49 x 3 = 147 cases where a, b, c are not different (since a=b, b=c, a=c are symmetrical)

Method 1

So there are 4851 - 147= 4704 solutions?

But the order does not matter!

There are 6 permutations of a,b,c (why?)

So there are $\frac{4704}{6} = 784$ possible solutions

Method 2

Consider $a + b + c = 100$, where a, b, c are different and order does not matter. WLOG $a < b < c$

If $a = 1$: b can range from 2-49, since b cannot be 1, but cannot be 50 otherwise $c > b$. So there are $49 - 2 + 1 = 48$ choices

Similarly $a = 2$: b ranges from 3-48 leading to 46 choices

Similarly $a = 3$: b ranges from 4-48 leading to 45 choices

Similarly $a = 4$: b ranges from 5-47 leading to 43 choices

…

Continuing to $a = 32$: b can only be 33, leading to 1 choice

Method 2

So our final sum is 48 + 46 + 45 + 43 + … + 6 + 4 + 3 +1

We compute this by pairing 48 with 1, 46 with 3, 45 with 4, etc

So our final sum is 16 x 49 = 784

What is the pattern of our final sum? What is actually going on?

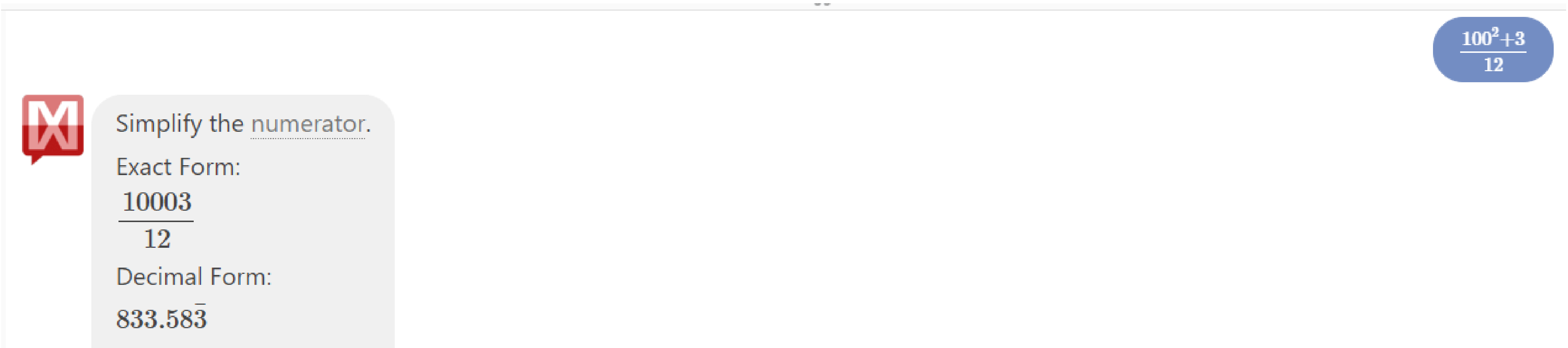
Method 2

The reason that when a is odd, there is a decrease in 1 option, however when a is even there is a decrease in two options is because if a is even, we have b=c which we must discard, conversely if a is odd we cannot have b=c in the first place.

Trying proving why we cannot have b=c if a is odd. (hint: a,b,c are integers)

Method 3

Theorem: The number of ways to partition a positive integer n into exactly 3 parts is $[\frac{n^2+3}{12}]$. Where [x] denotes the greatest integer less than or equal to x.



So are there 833 solutions?

Method 3

No. Since parts must be different. There are 49 cases where there are two parts that are the same; (1,1,98), (2,2,96), … , (49,49,2)

Hence there are 833 – 49 = 784 solutions

Conclusion?

- There are a lot of ways of approaching a combinatorics problem
- Make sure everything is counted, and **exactly** once!

Any Questions?