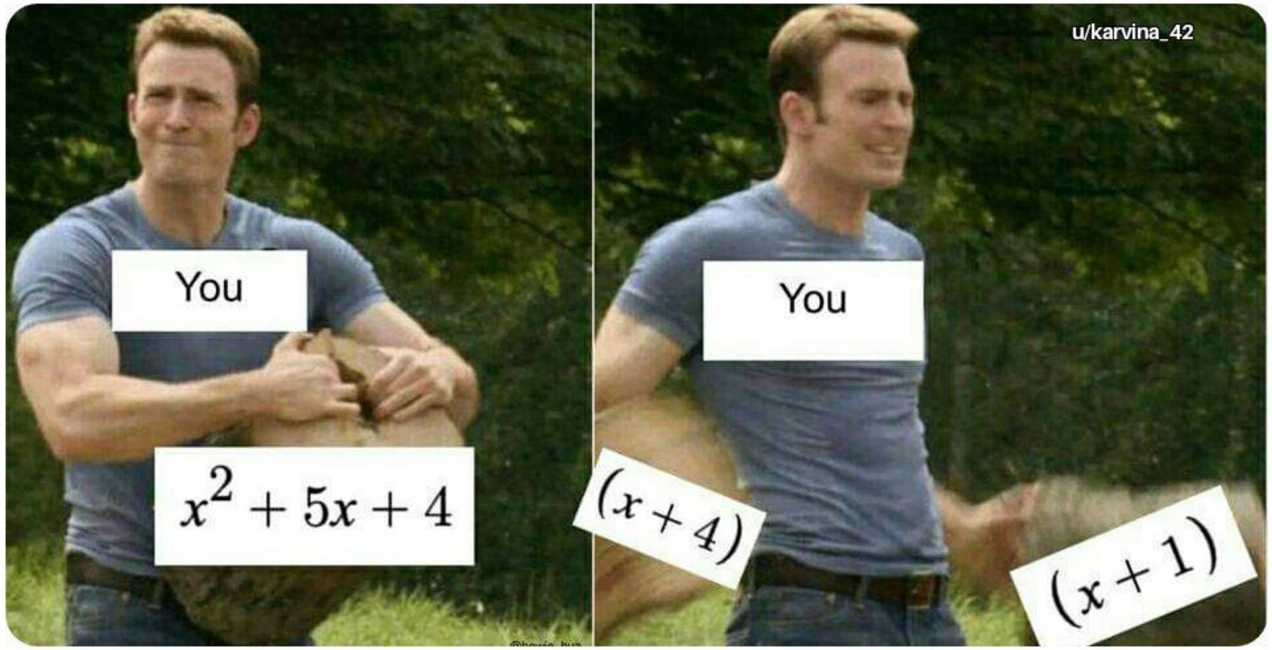
Factorisations and Diophantine Equations

MHS Maths Extension Group

What is factoring?

To **factor**, or to break an expression into factors, is to write the expression (often an integer or polynomial) as a product of different terms.



What are Diophantine Equations?

A **Diophantine equation** is an equation relating integer (or sometimes natural number or whole number) quantities.

Some Diophantine Equations

Pythagorean theorem: $a^2 + b^2 = c^2$

Linear Diophantine Equation: $ax + by = c$

Fermat's Last Theorem: $a^n + b^n = c^n$ for positive integers n>2

Pells's equation: $x^2 - dy^2 = 1$

Useful Factorisation Techniques

- Grouping
- DOPS
- Completing the Square
- Difference and Sums of Powers
- Look for “hidden” quadratics/polynomials

Grouping and SFFT

Look to group terms – rearranging the terms can help

Adding/Getting rid of terms can help you group!

Try breaking apart terms if you are stuck.

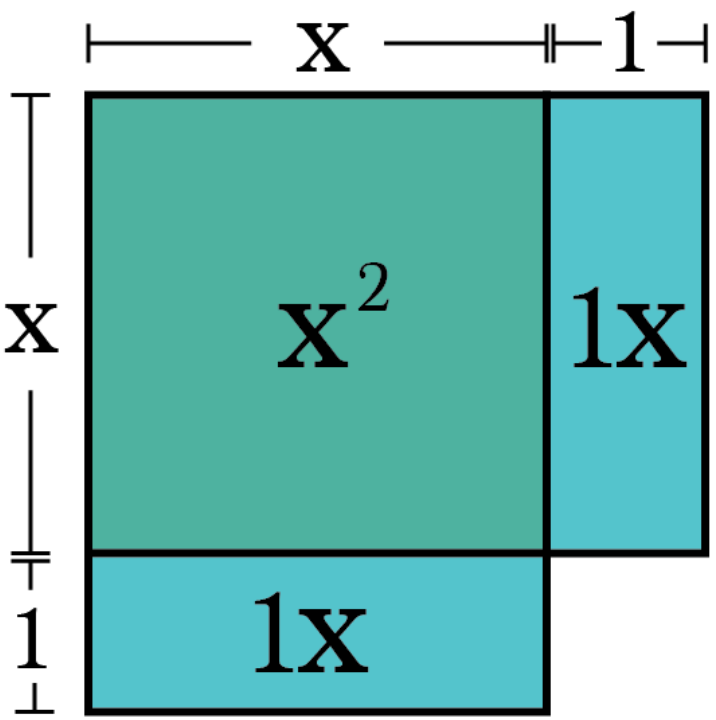
Example 4.2.4

Find $x, y \in \mathbb{Z}$ such that

$$xy = x + y + 3.$$

DOPS and Completing the Square

Use basic techniques e.g., difference of perfect squares and completing the square



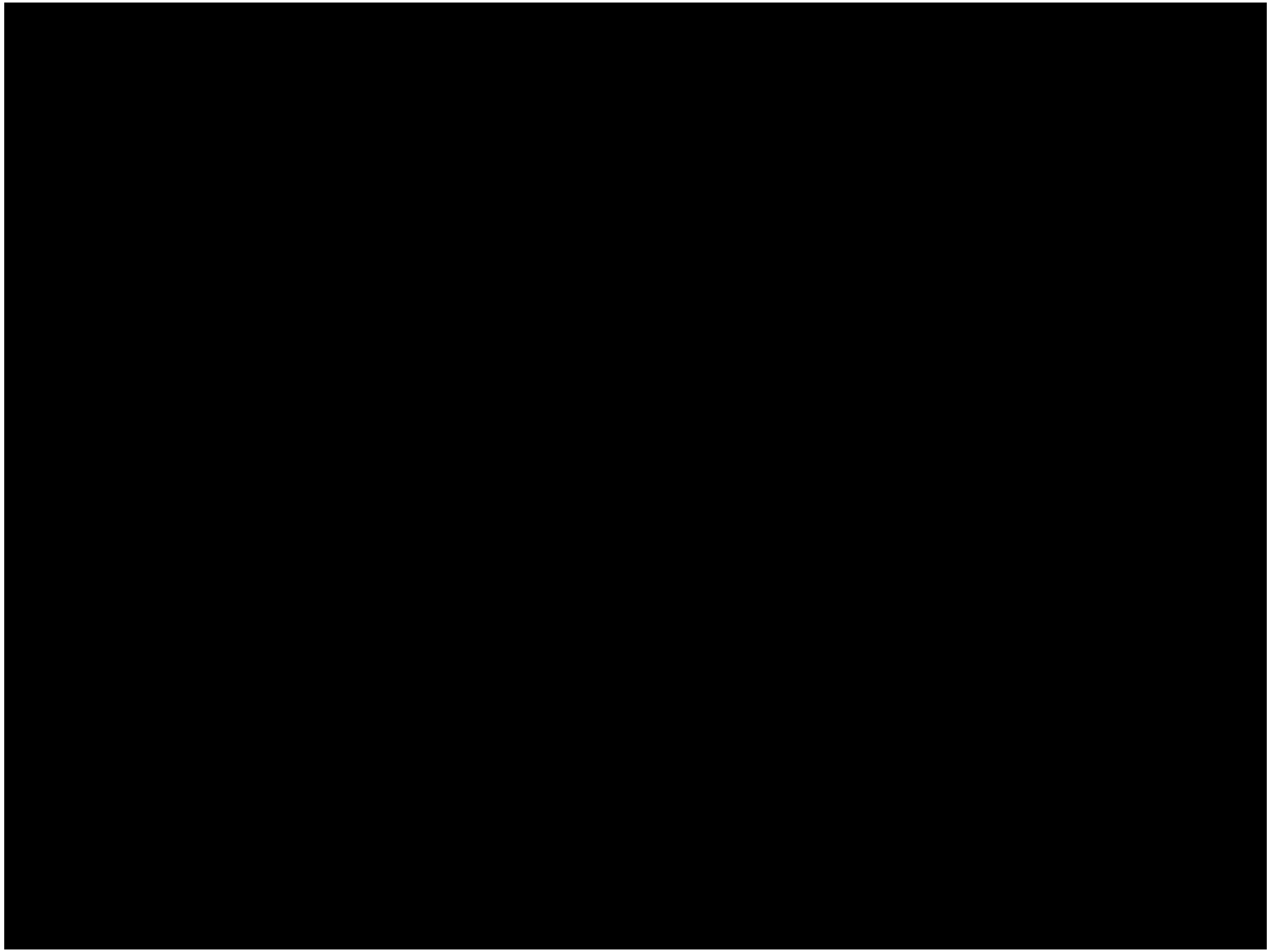
DOPS and Completing the Square

Whenever you see $x^2 + x$ or $x^2 - x$, you can multiply by 4 to complete the square.

Example 4.2.3

Find $x, y \in \mathbb{Z}$ such that
$$x^2 + y^2 = x + y + 2.$$

Don’t underestimate “basic” techniques



Differences and Sums of Powers

For **any** positive integer n:

$x^n - y^n$= $(x - y)(x^{n-1} + x^{n-2}y + \ \ldots + x\ y^{n-2} + y^{n-1})$

For an **odd** positive integer n:

$x^n + y^n$= $(x - y)(x^{n-1} + x^{n-2}y + \ \ldots + x\ y^{n-2} + y^{n-1})$

Hidden Quadratics/Polynomials

Some quadratics are disguised as seemingly complicated expressions.

$$e^{2x} - e^{x+1} = -1$$

Solving Diophantine Equations using Factoring

Solving $xy = 100$ where x and y are integers is easy right?

List the possible values of x (factors of 100) and then the corresponding values of y

e.g., $x = 25, y = 4$

Solving Diophantine Equations using Factoring

In general, try to rearrange a Diophantine equation so we have an integer on one side, and a factored expression on the other side.

Then simply it’s a matter of assigning the factors of the integer to the factors.

Try this past AMC

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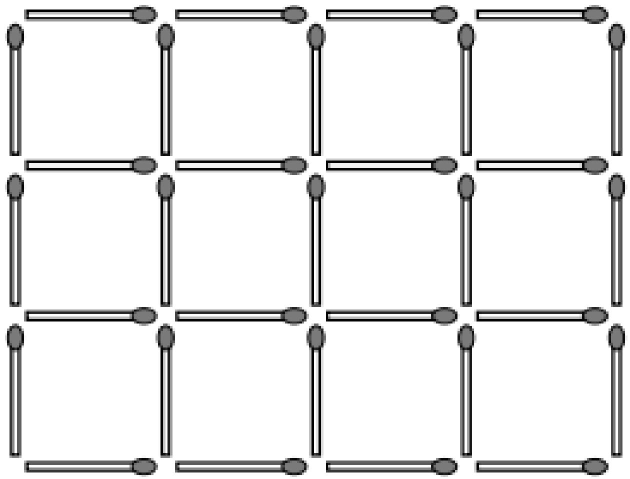
As shown in the diagram, you can create a grid of squares 3 units high and 4 units wide using 31 matches. I would like to make a grid of squares a units high and b units wide, where $a < b$ are positive integers. Determine the sum of the areas of all such rectangles that can be made, each using exactly 337 matches.



▶ *Alternative 1*

There are $a+1$ rows of horizontal matches and each row contains b matches. There are $b+1$ columns of vertical matches and each column contains a matches. So the total number of matches is $(a+1)b+(b+1)a = 2ab+a+b$.

We would like to solve the equation $2ab+a+b = 337$, where $a < b$ are positive integers. By multiplying the equation by 2 and adding 1 to both sides, we obtain

$$4ab+2a+2b+1 = 675 \qquad \Rightarrow \qquad (2a+1)(2b+1) = 675$$

The only ways to factorise 675 into two positive integers are

$$1\times 675, \quad 3\times 225, \quad 5\times 135, \quad 9\times 75, \quad 15\times 45, \quad 25\times 27$$

We must have $2a+1$ correspond to the smaller factor and $2b+1$ to the larger factor. So the solutions we obtain for (a,b) are

$$(0,337), \quad (1,112), \quad (2,67), \quad (4,37), \quad (7,22), \quad (12,13)$$

We must disregard the first solution, but one can check that the remaining ones are all valid. So the sum of the areas of all such rectangles is

$$1\times 112+2\times 67+4\times 37+7\times 22+12\times 13 = 704$$

hence (704).

Any Questions?