Melbourne High School
Maths Extension Group 2023
Term 4 Week 2 Handout Solutions

Expected value problems

1. Each letter has a $\frac{1}{12}$ chance of being delivered to the right envelope. So since there are 12 letters, the expected number of letters that are placed to the proper envelope is $12 \times \frac{1}{12} = 1$.

2. Define $X_i = 1$ if coin flip i and $i+1$ are heads, and $X_i = 0$ otherwise. Then we seek $E(X_1 + X_2 + ... + X_9)$ Note that the chance that any pair of consecutive coin flips is $(\frac{1}{2})^2 = \frac{1}{4}$, so $E(X_i) = \frac{1}{4}$ and

$$E(X_1) + E(X_2) + ... + E(X_9) = 9 \times \frac{1}{4} = \frac{9}{4}.$$

3. Number the babies $1, 2, ..., 2006$. Define $X_i = 1$ if baby i is poked and $X_i = 0$ otherwise. Then we seek $E(X_1 + X_2 + ... X_{2006})$. Any baby has $\frac{1}{4}$ chance of being unpoked (if both its neighbours miss). Hence $E(X_i) = \frac{1}{4}$ for each i and

$$E(X_1 + X_2 + ... + X_{2006}) = E(X_1) + E(X_2) + ... E(X_{2006}) = 2006 \times \frac{1}{4} = \frac{1003}{2}.$$

4. Clearly the first digit has to be 1. Then each other digit has a $\frac{3}{9} = \frac{1}{3}$ chance of being 1. So the expected value is $2^9 + \frac{1}{3}(2^8 + 2^7 + ... + 1) = 2^9 + \frac{1}{3}(2^9 - 1) = \frac{2047}{3}$.

5. We expect there to be $\frac{1}{n} \times n = 1$ fixed point on average. But also, a permutation with k fixed points occurs with chance of $\frac{p_n(k)}{n!}$ and such a permutation has k fixed points, so the expected number of fixed points is

$$\sum_{k=0}^{n} \frac{k \cdot p_n(k)}{n!}.$$

Which is also equal to 1. Thus

$$\sum_{k=0}^{n} \frac{k \cdot p_n(k)}{n!} = 1.$$

Multiplying both sides by $n!$ gives our desired result.