# Functions and Functional Equations

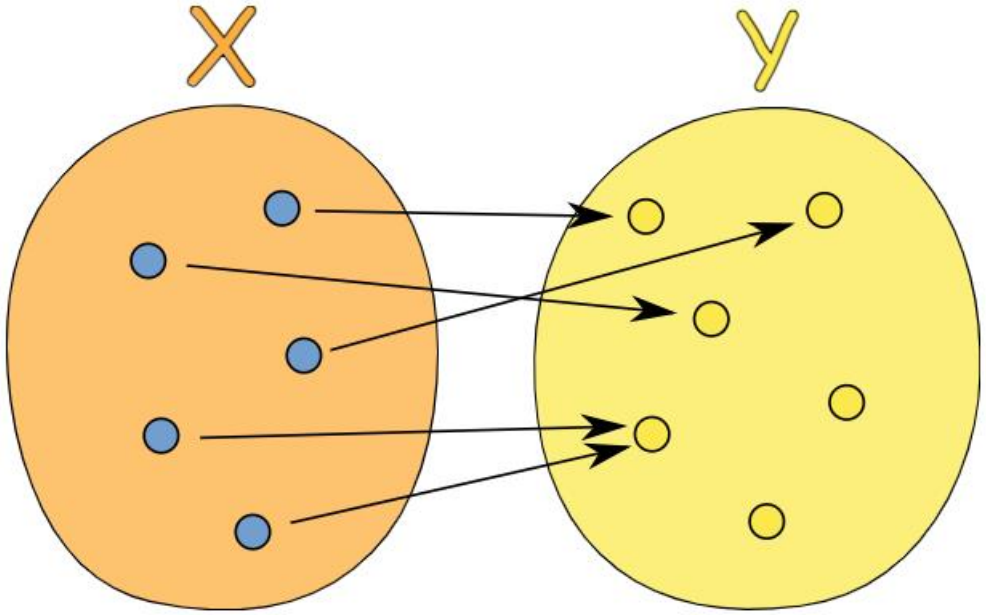
MHS Maths Extension Group

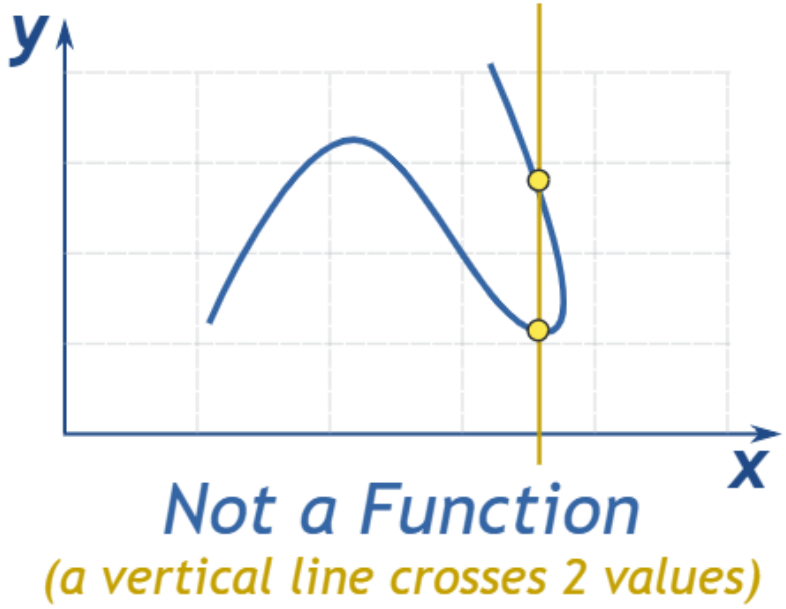
# Why Functions and Functional Equations?

- You may have encountered functions in school maths
- Functions are an important of algebra
- Functional equations strengthen your understanding of functions

# Functions Review

A function is a “machine”. It takes one thing in, and outputs something.



# Example of some functions…

- Linear functions such as $100x + 500$
- Polynomial functions such as $2x^4 + 4x^3 + 160x + 12$
- Exponential functions such as $5e^x$
- Absolute value functions like $|x| + 2$

# Domain

The domain is what can go into a function.

EVERY function has a domain, sometimes it’s just assumed “e.g., all real numbers work”.

# Codomain vs Range

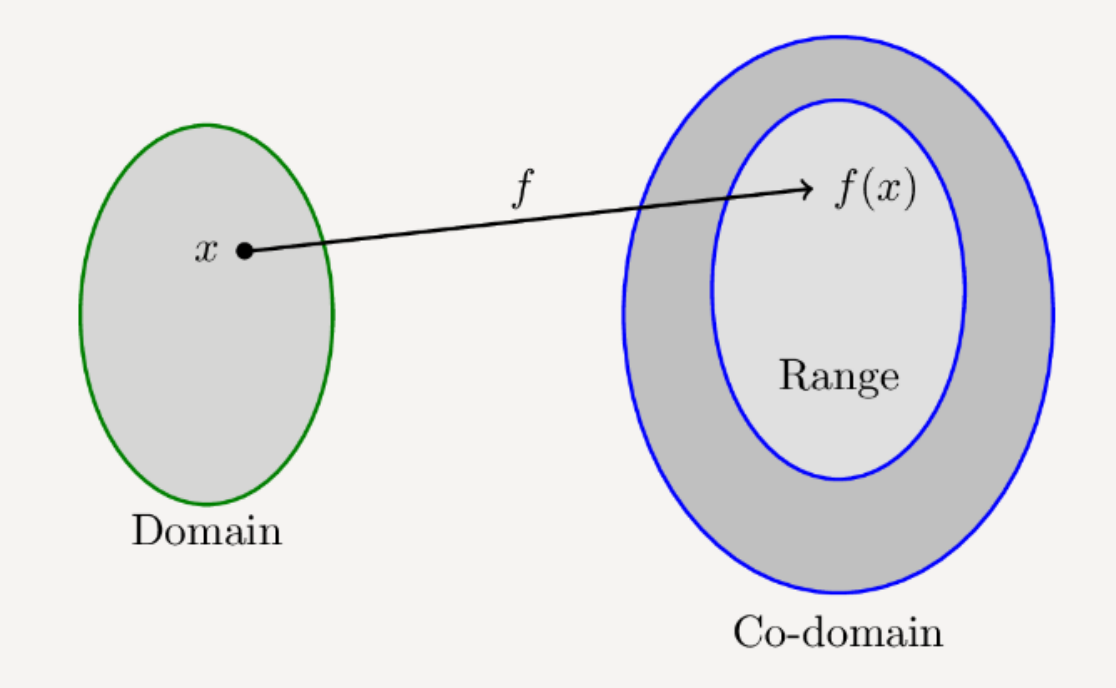
The range is a subset of the codomain:

- The codomain is what values could come out of the function.

- The range is what values come out of the function.

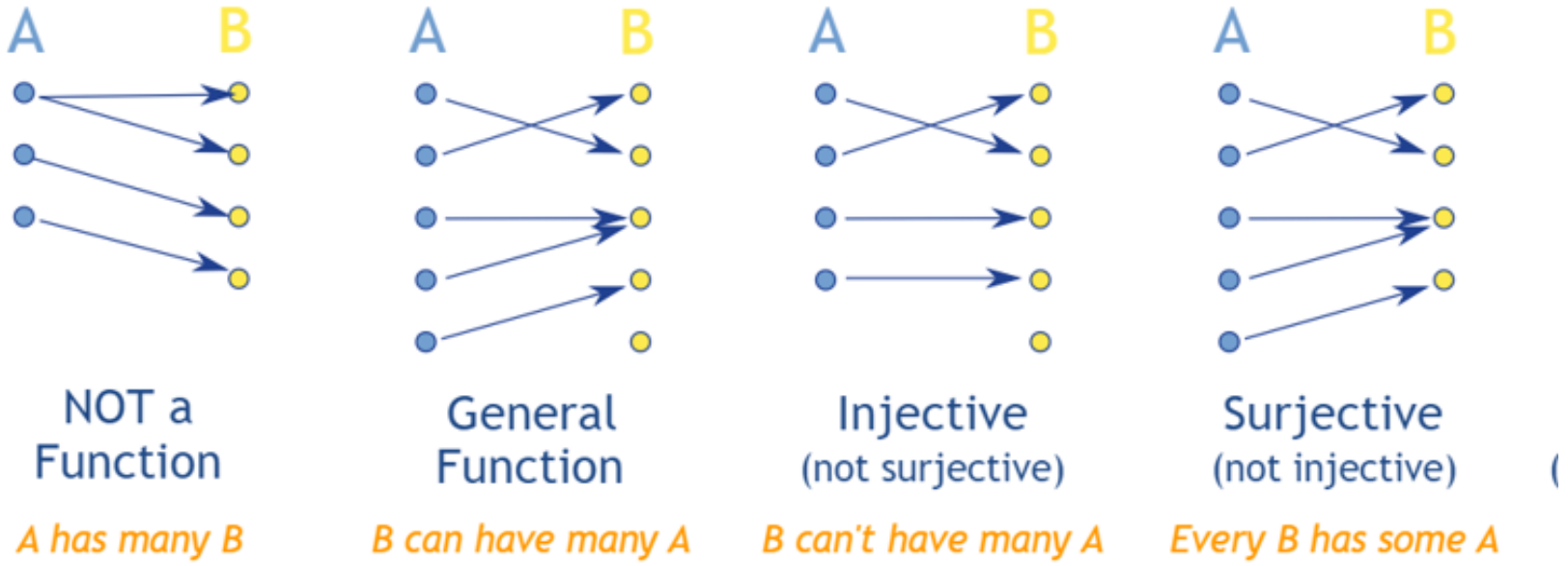
For example, we can define f(x)=2x with a domain and codomain of integers.

But we could say that the range is even integers.



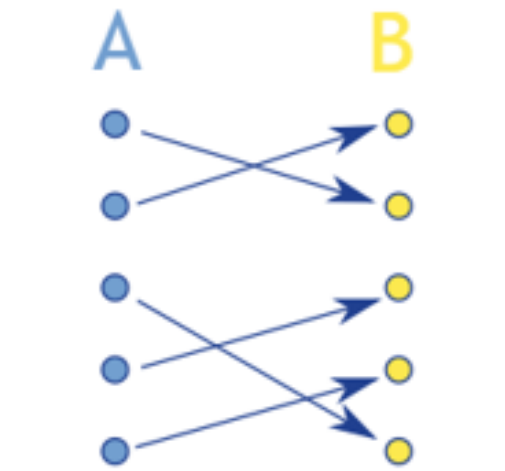
# Surjectivity, Injectivity

- A function is injective if $f(a) = f(b)$ implies $a = b$
- A function is surjective if it attains every value in its co-domain



# Bijectivity

A function is bijective, if it is BOTH injective and surjective.




Bijective
(injective, surjective)

*A to B, perfectly*

# Inverses

Functions f and g are inverse functions of each other if and only if

$g\big(f(x)\big) = x$ for all values of x in the domain of f, and
$f\big(g(x)\big) = x$ for all values of x in the domain of g

**Important:** A function has an inverse if and only if it is bijective!

# Functional equations – What are they?

Functional equations are just like the equations you have encountered, where you are solving for x, y, z, … etc.

Except the unknowns to be solved for are functions!

For example, we could solve for f given that y$f(x) = xf(y)$.

# How to solve functional equations

There are many strategies, but the basics involve substituting values and noticing interesting things about the function.

Basically, guessing and hoping you're right…

**Important:** After finding a solution, make sure to plug it into the original equation to see if it works.

# Solving Functional Equations

Don’t forget the skills from solving equations with x, y etc as variables.

Use similar techniques, such as isolating $f(x)$ to one side, or solving a system of equations.

# Functional equations: Substitutions

Usually, you want to try to sub in values to cancel terms.

Plugging in $x = 0, 1, -1$ can usually tell you some information.

All the problems in today's handout can be solved with this technique. Don’t underestimate it!

# Functional Equation example:

Let's solve $\mathrm{y}f(x) = xf(y)$ from before.

If we divide by $x\mathrm{y}$ on both sides, we have $\frac{f(x)}{x} = \frac{f(y)}{y}$

Let $g(x) = \frac{f(x)}{x}$ then $g(x) = g(y)$ and then clearly $g(x) = c$ where c is a constant.

Thus, $\mathrm{f(x)} = \mathrm{cx}$.

However, we are not done. Does this satisfy the original equation?

# Functional Equation

Yes, it does. Simply sub in $f(x) = cx$ to get $cxy$ on the LHS and the RHS.

Thus, $f(x) = cx$ indeed satisfies the conditions, and is thus the only solution.

# Any Questions?