# Year 9

1. In an AFL game, a goal is worth 6 points and a behind is worth 1 point.

   (a) (1 point) How many points does a team have if they score 4 goals and 8 behinds?

   (b) (2 points) How many different ways are there to score some number of goals and some number of behinds such that the score is the product of the number of goals and the number of behinds?

   (c) (2 points) By considering a graph, show that you have found all the ways that this could occur.

   Total for Question 1: 5

2. (a) (2 points) What is the area of the largest square that can fit inside a unit circle?

   (b) (2 points) What is the area of the largest circle that can fit inside a unit square?

   (c) (3 points) A triangle is placed inside a unit square so that of its vertices either lie on the square's edges or inside the square. What areas are possible for this triangle?

   (d) (2 points) What is the side length of the largest square that can fit inside an equilateral triangle that has side length 1 unit?

   Total for Question 2: 9

3. (3 points) What is the volume of a regular tetrahedron with side length 1 unit?

4. (a) (1 point) How many positive integer solutions are there to the equation $2x + 3y = 25$?

   (b) (2 points) How many integer solutions are there to the equation $2x + 3y = 25$?

   (c) (2 points) How many integer solutions are there to the equation $51x + 24y = 17$?

   (d) (1 point) Consider the expression $9x + 15y$. Put in as many integer values of $x$ and $y$ as you can. Which number are all the resulting numbers multiples of?

   (e) (2 points) When does the equation $ax + by = c$ always have integer solutions?

   Total for Question 4: 8

5. (a) (2 points) When is the sum of 2 consecutive numbers divisible by 2?

   (b) (2 points) When is the sum of 3 consecutive numbers divisible by 3?

   (c) (2 points) When is the sum of 4 consecutive numbers divisible by 4?

   (d) (2 points) When is the sum of 5 consecutive numbers divisible by 5?

   (e) (2 points) When is the sum of $n$ consecutive numbers divisible by $n$?

   Total for Question 5: 10

6. (a) (1 point) Two lines can divide the plane into at most how many regions?

   (b) (1 point) Three lines can divide the plane into at most how many regions?

   (c) (1 point) If we add a fourth line, what is the maximum number of intersection points that line can make with the existing lines?

(d) (1 point) Four lines can divide the plane into at most how many regions?

(e) (3 points) What is the maximum number of regions that $n$ lines can divide the plane into?

<div align="right">Total for Question 6: 7</div>

7. (3 points) Prove it possible to pair up the numbers $0, 1, 2, 3, \ldots, 61$ in such a way that when we sum each pair, the product of the 31 numbers we get is a perfect fifth power.

<div align="right">Total for Question 7: 3</div>

Total: 45

# Year 10

1. (a) (2 points) When is the sum of 2 consecutive numbers divisible by 2?

   (b) (2 points) When is the sum of 3 consecutive numbers divisible by 3?

   (c) (2 points) When is the sum of 4 consecutive numbers divisible by 4?

   (d) (2 points) When is the sum of 5 consecutive numbers divisible by 5?

   (e) (2 points) When is the sum of $n$ consecutive numbers divisible by $n$?

2. (3 points) Prove it possible to pair up the numbers $0, 1, 2, 3, \ldots, 61$ in such a way that when we sum each pair, the product of the 31 numbers we get is a perfect fifth power.

3. (a) (1 point) Two lines can divide the plane into at most how many regions?

   (b) (1 point) Three lines can divide the plane into at most how many regions?

   (c) (1 point) If we add a fourth line, what is the maximum number of intersection points that line can make with the existing lines?

   (d) (1 point) Four lines can divide the plane into at most how many regions?

   (e) (3 points) What is the maximum number of regions that $n$ lines can divide the plane into?

4. (a) (2 points) Is the sum or difference of two rational numbers always rational? Why or why not?

   (b) (2 points) Is the product or quotient of two rational numbers always rational? Why or why not?

   (c) (2 points) When one rational number is raised to another is the result guaranteed to be rational?

   (d) (3 points) It is known that $\sqrt{2}$ is irrational, but it is not known whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. Do there exist two irrational numbers such that when one is raised to the other the result is rational? Prove your conjecture.

5. (a) (2 points) Prove that $x^2 + y^2 \geq 2xy$ for real numbers $x, y$.

   (b) (2 points) Prove that $2a^2 + b^2 + c^2 \geq 2(ab + ac)$ for real numbers $a, b, c$.

   (c) (2 points) Prove that $3(a^2 + b^2 + c^2 + d^2) \geq 2(ab + ac + ad + bc + bd + cd)$ for real numbers $a, b, c, d$.

   (d) (3 points) Real numbers $a, b, c, d, e$ are linked by the two equations:

   $$e = 40 - a - b - c - d$$

   $$e^2 = 400 - a^2 - b^2 - c^2 - d^2$$

   Determine the largest value for $e$.

6. (a) (1 point) How many ways are there to create a two-element subset from the set $\{1, 2, 3, 4\}$?

(b) (2 points) How many ways are there to split the set $\{1, 2, 3, 4\}$ into two disjoint nonempty sets? (Disjoint means the two sets share no elements.) For example, if we had the set $\{1, 2, 3\}$ a valid splitting would be $\{1, 2\}$ and $\{3\}$.

(c) (2 points) How many ways are there to split the set $\{1, 2, 3, 4, 5\}$ into two disjoint nonempty sets?

(d) (2 points) Denote by $S(n, k)$ the number of ways to split a set of $n$ elements into $k$ disjoint nonempty sets. Suppose we know what $S(n - 1, 2)$ is. What happens when we add another element to the set? What happens to each of the $S(n - 1, 2)$ ways to do the partitioning?

(e) (2 points) What is $S(n, 2)$?

(f) (3 points (bonus)) If there are $S(n - 1, k - 1)$ ways to partition a set of $n - 1$ elements into $k-1$ nonempty disjoint sets, what is $S(n, k)$ in terms of $S(n-1, k)$ and $S(n - 1, k - 1)$?

Total: 47

# Year 11/12

1. (a) (1 point) Two lines can divide the plane into at most how many regions?

   (b) (1 point) Three lines can divide the plane into at most how many regions?

   (c) (1 point) If we add a fourth line, what is the maximum number of intersection points that line can make with the existing lines?

   (d) (1 point) Four lines can divide the plane into at most how many regions?

   (e) (3 points) What is the maximum number of regions that $n$ lines can divide the plane into?

2. (a) (1 point) How many ways are there to create a two-element subset from the set $\{1, 2, 3, 4\}$?

   (b) (2 points) How many ways are there to split the set $\{1, 2, 3, 4\}$ into two disjoint nonempty sets? (Disjoint means the two sets share no elements.) For example, if we had the set $\{1, 2, 3\}$ a valid splitting would be $\{1, 2\}$ and $\{3\}$.

   (c) (2 points) How many ways are there to split the set $\{1, 2, 3, 4, 5\}$ into two disjoint nonempty sets?

   (d) (2 points) Denote by $S(n, k)$ the number of ways to split a set of $n$ elements into $k$ disjoint nonempty sets. Suppose we know what $S(n - 1, 2)$ is. What happens when we add another element to the set? What happens to each of the $S(n - 1, 2)$ ways to do the partitioning?

   (e) (2 points) What is $S(n, 2)$?

   (f) (3 points (bonus)) If there are $S(n - 1, k - 1)$ ways to partition a set of $n - 1$ elements into $k-1$ nonempty disjoint sets, what is $S(n, k)$ in terms of $S(n-1, k)$ and $S(n - 1, k - 1)$?

3. (a) (2 points) Is the sum or difference of two rational numbers always rational? Why or why not?

   (b) (2 points) Is the product or quotient of two rational numbers always rational? Why or why not?

   (c) (2 points) When one rational number is raised to another is the result guaranteed to be rational?

   (d) (3 points) It is known that $\sqrt{2}$ is irrational, but it is not known whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. Do there exist two irrational numbers such that when one is raised to the other the result is rational? Prove your conjecture.

4. (a) (1 point) If two normal six-sided die are rolled, what is the probability that the sum of the two numbers is 2?

   (b) (1 point) If two normal six-sided die are rolled, what is the probability that the sum of the two numbers is 7?

   (c) (2 points) Consider a biased six-sided die in which the probability of rolling $n$ is $p_n$. This die is rigged so that when two of them are rolled, every possible sum of the two numbers is equally likely. What is $p_2$ in terms of $p_1$?

(d) (1 point) What is $p_3$ in terms of $p_1$?

(e) (2 points) What is $p_4$ in terms of $p_1$? Do you notice a pattern?

(f) (2 points) Let $p_i = a_{i-1}p_1$. So $a_0 = 1$, $p_2 = a_1 p_1$, $p_3 = a_2 p_1$, $p_4 = a_3 p_1$ and so on. What are $a_0 a_1 + a_1 a_0$, $a_0 a_2 + a_1 a_1 + a_2 a_0$, $a_0 a_3 + a_1 a_2 + a_2 a_1 + a_3 a_0$, etc. all equal to?

(g) (1 point) Since we have a six-sided die,

$$p_1 + p_2 + \cdots + p_6 = 1.$$

What is $p_1$ in terms of $a_0, a_1, a_2, \ldots, a_5$?

(h) (2 points) Let $s_i = a_0 + a_1 + \cdots + a_i$. What are $s_0, s_1, s_2$? What is $s_5$?

(i) (3 points) Is such a six-sided die possible?

5. (a) (1 point) What is $1 + 2 + 3 + \cdots + 200$?

(b) (2 points) Find a formula for $1 + 2 + \cdots + n$ and prove it.

(c) (1 point) Define
$$F_k(n) = 1^k + 2^k + \cdots + n^k.$$

(So in the previous part you found a formula for $F_1(n)$.) More compactly it may be written as
$$F_k(n) = \sum_{i=1}^{n} i^k.$$

What is
$$1^3 + (2^3 - 1^3) + (3^3 - 2^3) + \cdots + (100^3 - 99^3)?$$

(d) (2 points) By considering
$$1^3 + \sum_{i=1}^{n}((i+1)^3 - i^3)$$

find a formula for $F_2(n)$.

(e) (2 points) Find a formula for $F_3(n)$.

(f) (1 point) The Binomial Theorem states that
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

If
$$(i+1)^k - i^k = \sum_{j=0}^{x} \binom{k}{j} i^j$$

what is $x$ in terms of $k$?

(g) (3 points) By considering the general telescoping sum

$$1^k + \sum_{i=1}^{n}((i+1)^k - i^k) = (n+1)^k$$

show that

$$F_{k-1}(n) = \frac{(n+1)^k - 1}{k} + \frac{1}{k}\sum_{i=0}^{k-2}\binom{k}{i}F_i(n).$$

Total: 52