Melbourne High School
Maths Extension Group 2023
Term 1 Week 7 Handout

Introductory note

As this is the first handout of the year in this style, a few notes are in order. The first section consists of problems generally related to the main topic of the meeting—in this handout's case, combinatorics. The second section will contain a range of topics I find interesting that are not directly related to competition problems. The problems in this section require some working knowledge of proof. It is up to you what you work on.

Section A: Problems in combinatorics

1. Find the number of ways I can arrange the letters of: MATHSEXTENSIONGROUP

2. 10 points in the plane are given, with no 3 collinear. 4 distinct segments joining pairs of these points are chosen at random, all such segments being equally likely. Find the probability that some 3 of the segments form a triangle whose vertices are among the 10 given points.

3. Students sit at their desks in three rows of eight. Felix, the class pet, must be passed to each student exactly once, starting with Alex in one corner and finishing with Bryn in the opposite corner. Each student can pass only to the immediate neighbour left, right, in front or behind. How many different paths can Felix take from Alex to Bryn? (AMC Intermediate Q30)

4. Ali, Beth, Chen, Dom and Ella finish a race in alphabetical order: Ali in first place, then Beth, Chen and Ella. They decide to run another race and their placings all change. Two of the runners receive a placing higher than the week before, and the other three runners receive a placing lower than the week before. Given this information, in how many orders could the five runners have finished this second race? (AMC Senior Q30)

5. Each cell of an $m \times n$ board is filled with some nonnegative integer. Two numbers in the filling are said to be adjacent if their cells share a common side. (Note that two numbers in cells that share only a corner are not adjacent). The filling is called a garden if it satisfies the following two conditions:

(i) The difference between any two adjacent numbers is either 0 or 1.
(ii) If a number is less than or equal to all of its adjacent numbers, then it is equal to 0 .

Determine the number of distinct gardens in terms of m and n.

Section B: The higher arithmetic

Number theory, or the higher arithmetic, is the branch of mathematics that studies the natural numbers. It is a wondrous and beautiful subject, full of amazing ideas that will make your study of it worthwhile.[1]

Number theory investigates many different types of natural numbers and the relationships between them. Some are listed below.

- Prime numbers. They are the basic building blocks of all the natural numbers, and thus come up frequently in any arithmetic investigation. Many of number theory's famous solved and unsolved problems have to do with primes, including the Twin Prime Conjecture and the Prime Number Theorem.

- Shapely numbers. These are numbers like the familiar triangular and square numbers. Shapely numbers extend up to pentagonal numbers, hexagonal numbers, etc.

- Powers of numbers. The most famous theorem involving these numbers is *Fermat's Last Theorem*, which asserts that a number raised to a power greater than 2 is not the sum of two like powers.

- Perfect numbers. These are the numbers that are the sum of their proper divisors (all of their divisors except for itself). For example, 6 is a perfect number; its proper divisors, which are 1, 2, and 3, have a sum of 6.

Although number theory is primarily the study of natural numbers, we often have to resort to other number systems, including the integers, rationals, reals, and sometimes even the complex numbers. For the most part, however, we will stick to the natural numbers, the integers, and sometimes the rationals.

For all the interesting numbers of number theory, there are just as many interesting questions. Here are some about the primes; some have been solved while others remain unsolved.

- Which primes are the sum of two squares?

- The sequence 3, 5, 7 is a *prime triplet*. Are there infinitely many prime triplets?

- Are there infinitely many primes of the form $2^n - 1$?

- Are there infinitely many primes of the form $4n + 1$? $4n + 3$?

- Are there infinitely many primes of the form $n^2 + 1$?

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[1] Your mileage may vary.

- For some given natural number x, how many primes are there less than or equal to x?

- Is every even number greater than or equal to 4 the sum of two primes?

Number theory asks many questions of equations and polynomials too. Generally polynomials in number theory are considered to have integer or rational coefficients. For example, the equation $x^2 - Ny^2 = 1$ for a natural number N that is not a perfect square is known as *Pell's Equation*. Amazingly, such an equation always has infinitely many solutions. What's more, those solutions are related to continued fractions!

Even the simplest equations, linear equations, present difficulties because we often consider them in several variables. Consider the equation $ax - by = n$ for natural numbers a, b, and n. When does this equation have solutions in integers or natural numbers x and y?

Then there are equations of a more elaborate kind, such as the equation $y^2 = x^3 + 17$. It's not at all obvious how to solve such an equation in the integers or rationals. In fact, it is still and unsolved problem in mathematics as to how to determine whether such equations (y^2 equals a cubic in x) have finitely many rational solutions, infinitely many rational solutions, or no rational solutions at all.

We shall consider all the above types of numbers and equations in our journey through number theory.

Pythagorean triples

Let's begin our exploration into number theory by starting with something we already know: Pythagorean triples.

A Pythagorean triple is a triple of positive integers (a, b, c) where $a^2 + b^2 = c^2$. Geometrically, c is the hypotenuse and a and b are the two shorter sides of a right-angled triangle. A natural question is whether there are infinitely many such triples. The answer is yes because if (a, b, c) is a Pythagorean triple, then so is (na, nb, nc) for any natural number n. The simple direct proof follows.

$$(na)^2 + (nb)^2 = n^2(a^2 + b^2)$$
$$= n^2 c^2$$
$$= (nc)^2$$

Triples that are just multiples of other triples aren't very interesting, so we look mainly at those triples in which a, b, and c share no common factors. These Pythagorean triples are called *primitive*. For example, $(3, 4, 5)$ is a primitive triple, while $(6, 8, 10)$ is not.

From our definition of primitive triples, we immediately note that one of a and b has to be even and the other odd. If both a and b were even, then $c^2 = a^2 + b^2$ is even. This means c would also be even, contradicting the assumption that a, b, and c share no common factors. What happens if both a and b are odd?

Since either a or b can be even and the other odd, from now on we can assume, without loss of generality, that a is odd and b is even. Note that this implies c must be odd too, because $a^2 + b^2$, being the sum of an even and an odd number, is odd.

A way to generate primitive triples would be quite a useful thing. How might we go about deriving a formula for a, b, and c? First we note that $a^2 = c^2 - b^2$, so $a^2 = (c+b)(c-b)$. The right-hand side, $(c+b)(c-b)$, is a perfect square. Since c and b share no common factors, neither do $c + b$ and $c - b$. Since they share no common factors, both $c + b$ and $c - b$ have to be squares themselves.

Since c is odd and b is even (by assumption), both $c - b$ and $c + b$ are odd. Thus we can let

$$c - b = t^2 \tag{1}$$

and

$$c + b = s^2 \tag{2}$$

for odd integers t and s. Since $c + b > c - b$, we have $s > t$. We can make one further deduction about t and s; they must share no common factors, because $c - b$ and $c + b$ share no common factors.

If you done the above interlude correctly, you should have $b = (s^2 - t^2)/2$ and $c = (s^2 + t^2)/2$. Now we just need a formula for a. From $a^2 = b^2 - c^2$, we have

$$a = \sqrt{(b+c)(b-c)}$$
$$= \sqrt{s^2 t^2}.$$

This implies $a = st$.

At the end of this long voyage, we have discovered a formula for generating primitive Pythagorean triples:

$$(a, b, c) = \left(st, \frac{s^2 - t^2}{2}, \frac{s^2 + t^2}{2} \right)$$

where $s > t \geq 1$ and s and t are odd integers that share no common factors. This formula, with slight modifications, is attributed to Euclid.

Problems

1. Show that the formula derived above is always a solution to $a^2 + b^2 = c^2$. If you can, prove that the formula always generates a primitive triple by showing that st, $(s^2 - t^2)/2$, and $(s^2 + t^2)/2$ share no common factors.

2. What do you notice about the even number in primitive Pythagorean triples? Prove that your conjecture is correct.

3. Show that one of a and b must be a multiple of 3.

4. Given a Pythagorean triple (not necessarily primitive) (a, b, c), consider its *product* to be abc. Prove that one of a, b, and c must be divisible by 5, and hence determine the highest common factor of all products of all Pythagorean triples.

5. Consider a circle of radius 1 centred at the origin with equation $x^2 + y^2 = 1$, and a line with a rational gradient m that passes through the point $(-1, 0)$. Find the coordinates of the other point of intersection between the line and the circle in terms of m. Hence deduce a second formula to generate Pythagorean triples (this one won't just generate primitive triples).

6. Given that the equation $X^4 - Y^4 = Z^2$ has no nonzero solution in integers X, Y, and Z, show that if three square numbers are in an arithmetic progression, their difference cannot also be a square.

7. Hence, or otherwise, show that a right-angled triangle with *rational* side lengths cannot have a square area.