Melbourne High School
Maths Extension Group
Term 1 Week 8 Handout

Problems

1) Find the ratio between the areas of a square and an octagon that are both inscribed in the same circle.
2) A circle of radius $9/2$ circumscribes a triangle ABC such that one of the triangle's sides passes through the centre of the circle; let this side be AB. If the arc length of BC is 3π, find the area of the triangle.
3) Given a rational number x/y, where x and y are positive integers, we define its *size* to be $x+y$. Find the rational number with smallest size that equals π to six decimal places.
4) (This problem requires calculus.) Now you will become Euler and prove the famous identity
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots. \tag{1}$$

   a) By instead considering the sum
   $$1 + \frac{1}{1}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{4} + \cdots$$
   show that the infinite sum on the RHS of (1) converges to a finite value less than 2.

   b) Assume that $\sin x$ can be written as an infinite polynomial:
   $$\sin x = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots.$$
   Substitute in $x=0$ and we get $a_0 = 0$. Differentiating both sides yields
   $$\cos x = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots.$$
   Put in $x=0$ again and we have $a_1 = 1$. Repeat this process to find that $a_3 = -1/3!$, $a_4 = 0$, $a_5 = 1/5!$, $a_6 = 0$, $a_6 = -1/7!$ and so on. Hence show that
   $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \tag{2}$$

   c) Divide both sides of (2) by x to get
   $$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots. \tag{3}$$
   The key thing to realise in (3) is that the coefficient of the x^2 term is $-1/3! = -1/6$.

   Note that the roots of $\sin x/x$ are the same as $\sin x$ except that $x=0$ is excluded. Hence the roots of the LHS of (3) are $x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. We know with normal polynomials that we can express them as a product of their roots with a scaling factor. Let's do the same with $\sin x/x$; we get the infinite product

$$\frac{\sin x}{x} = a(x^2 - \pi^2)(x^2 - 4\pi^2)(x^2 - 9\pi^2)(x^2 - 16\pi^2)\cdots.$$

Now we just need to know what the scaling factor a is.

As with normal polynomials, we can find the scaling factor by plugging in another point that is not one of the roots. Let's choose $x = 0$. Obviously the LHS is undefined, but we can use a interesting trick that is generally not allowed when we are dealing with finite polynomials. Graphing $\sin x/x$, perhaps using Desmos, we see that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

(This might be the most important limit in calculus.) Use this limit to find a and hence show that

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots.$$

d) If we expand the first two terms, we get

$$\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right) = 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2}\right)x^2 + \frac{x^4}{4\pi^4}.$$

Keep expanding the product until you notice a pattern.

e) Comparing coefficients with (3), complete Euler's proof of the Basel Problem.

5) For this problem, we definite a particular function $\zeta(s)$ to be

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots.$$

a) Show that

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{7^s}\right)\cdots\zeta(s) = 1.$$

b) Hence show that

$$\frac{1}{\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\cdots} = \frac{\pi^2}{6}.$$

c) Finally, prove that the probability of two randomly chosen integers being coprime to each other is $6/\pi^2$.

d) What is the probability that four randomly chosen integers are all coprime? (Here we mean that the GCD of the four numbers is 1; we don't mean that every pair of those four integers is coprime.)