Year 9

1. In an AFL game, a goal is worth 6 points and a behind is worth 1 point.

   (a) (1 point) How many points does a team have if they score 4 goals and 8 behinds?

   > **Solution:** 32

   (b) (2 points) How many different ways are there to score some number of goals and some number of behinds such that the score is the product of the number of goals and the number of behinds?

   > **Solution:** The problem boils down to finding solutions to the equation $6x + y = xy$. There are 5 solutions for nonnegative integers x and y but accept if trivial solution $(0, 0)$ is not given.

   (c) (2 points) By considering a graph, show that you have found all the ways that this could occur.

   > **Solution:** Rearrange the equation to get
   > $$y = \frac{6x}{x - 1}.$$
   > Via s simple algebraic manipulation we get
   > $$y = 6 + \frac{6}{x - 1}.$$
   > The 4 solutions obtained through enumeration are $(2, 12)$, $(3, 9)$, $(4, 8)$, $(7, 7)$. For $x > 7$ the denominator of the fraction becomes greater than 6 and therfore y cannot be an integer. Hence there are no solutions for $x > 7$ and thus we have found all the solutions.

   Total for Question 1: 5

2. (a) (2 points) What is the area of the largest square that can fit inside a unit circle?

   > **Solution:** The diagonal of the square is the diameter of the circle, which is 2. Therefore the side length of the square is $\sqrt{2}$ units, meaning the area is 2 square units.

   (b) (2 points) What is the area of the largest circle that can fit inside a unit square?

   > **Solution:** The radius of the circle is half the side length of the square, so the radius is $1/2$. The area of the circle is therefore $\pi(1/2)^2 = \pi/4$.

(c) (3 points) A triangle is placed inside a unit square so that of its vertices either lie on the square's edges or inside the square. What areas are possible for this triangle?

> **Solution:** It is possible for the triangle to have an area of $1/2$, as that occurs when all three vertices are three vertices of the square. Therefore it can plainly have any area between 0 and $1/2$. Can it have an area that is greater than $1/2$? The answer is no.

(d) (2 points) What is the side length of the largest square that can fit inside an equilateral triangle that has side length 1 unit?

> **Solution:** Let x be the sidelength of the square. By similar triangles we can deduce that
> $$\frac{2x}{1-x} = \sqrt{3}.$$
> Solving this equation gives $x = 2\sqrt{3} - 3$.

Total for Question 2: 9

3. (3 points) What is the volume of a regular tetrahedron with side length 1 unit?

> **Solution:** The volume of a pyramid is $1/3$ multiplied by the area of its base times its height. Using Pythagoras' Theorem we can deduce that the height of a unit tetrahedron is $\sqrt{2/3}$. The area of the base of the tetrahedron is the area of an equilateral triangle with sidelength 1, which is $(1/2)(\sqrt{3}/2) = \sqrt{3}/4$. Therefore the area of the unit regular tetrahedron is $(1/3) \times (\sqrt{3}/4) \times \sqrt{2/3} = \sqrt{2}/12$.

4. (a) (1 point) How many positive integer solutions are there to the equation $2x + 3y = 25$?

> **Solution:** Enumerating the cases and counting gives 4 solutions.

(b) (2 points) How many integer solutions are there to the equation $2x + 3y = 25$?

> **Solution:** There are infinitely many integer solutions.

(c) (2 points) How many integer solutions are there to the equation $51x + 24y = 17$?

> **Solution:** There are no integer solutions because every number of the form $51x + 24y$ is a multiple of the greatest common divisor of 51 and 24, which is 3.

(d) (1 point) Consider the expression $9x + 15y$. Put in as many integer values of x and y as you can. Which number are all the resulting numbers multiples of?

> **Solution:** The resulting numbers are multiples of 3.

(e) (2 points) When does the equation $ax + by = c$ always have integer solutions?

> **Solution:** The equation has integer solutions only when c is a multiple of the greatest common divisor of a and b.

Total for Question 4: 8

5. (a) (2 points) When is the sum of 2 consecutive numbers divisible by 2?

> **Solution:** Never, because $n + (n + 1) = 2n + 1$ is odd.

(b) (2 points) When is the sum of 3 consecutive numbers divisible by 3?

> **Solution:** Always, as $n + (n + 1) + (n + 2) = 3n + 3$.

(c) (2 points) When is the sum of 4 consecutive numbers divisible by 4?

> **Solution:** Never, as $n + (n + 1) + (n + 2) + (n + 3) = 4n + 10 \equiv 2 \pmod{4}$.

(d) (2 points) When is the sum of 5 consecutive numbers divisible by 5?

> **Solution:** Always, as $n + (n + 1) + (n + 2) + (n + 3) + (n + 4) = 5n + 10$.

(e) (2 points) When is the sum of n consecutive numbers divisible by n?

> **Solution:** The sum of n consecutive numbers starting from k is kn plus the $n - 1$th triangular number, so it is
> $$n \left(k + \frac{n - 1}{2} \right).$$
> The resulting number is divisible by n if $(n - 1)/2$ is an integer, and that is when n is odd.

Total for Question 5: 10

6. (a) (1 point) Two lines can divide the plane into at most how many regions?

> **Solution:** 4

(b) (1 point) Three lines can divide the plane into at most how many regions?

> **Solution:** 7 (If you try it on paper, you will see that the third line can only intersect the existing 2 lines in at most 2 places, resulting in 3 new regions. Therefore the number of regions is $4 + 3 = 7$.)

(c) (1 point) If we add a fourth line, what is the maximum number of intersection points that line can make with the existing lines?

> **Solution:** 3

(d) (1 point) Four lines can divide the plane into at most how many regions?

> **Solution:** 11

(e) (3 points) What is the maximum number of regions that n lines can divide the plane into?

> **Solution:** $1 + n(n+1)/2$

Total for Question 6: 7

7. (3 points) Prove it possible to pair up the numbers $0, 1, 2, 3, \ldots, 61$ in such a way that when we sum each pair, the product of the 31 numbers we get is a perfect fifth power.

> **Solution:** Pair 0 with 1, and k with $63 - k$ for all $2 \leq k \leq 31$. This would result in a product equal to $1 \times 63^{30} = (63^6)^5$ which is a perfect 5th power.

Total for Question 7: 3

Total: 45