# Year 11/12

1. (a) (1 point) Two lines can divide the plane into at most how many regions?

    > **Solution:** 4

   (b) (1 point) Three lines can divide the plane into at most how many regions?

    > **Solution:** 7 (If you try it on paper, you will see that the third line can only intersect the existing 2 lines in at most 2 places, resulting in 3 new regions. Therefore the number of regions is $4 + 3 = 7$.)

   (c) (1 point) If we add a fourth line, what is the maximum number of intersection points that line can make with the existing lines?

    > **Solution:** 3

   (d) (1 point) Four lines can divide the plane into at most how many regions?

    > **Solution:** 11

   (e) (3 points) What is the maximum number of regions that $n$ lines can divide the plane into?

    > **Solution:** $1 + n(n+1)/2$

2. (a) (1 point) How many ways are there to create a two-element subset from the set $\{1, 2, 3, 4\}$?

    > **Solution:** There are $\binom{4}{2} = 6$ ways.

   (b) (2 points) How many ways are there to split the set $\{1, 2, 3, 4\}$ into two disjoint nonempty sets? (Disjoint means the two sets share no elements.) For example, if we had the set $\{1, 2, 3\}$ a valid splitting would be $\{1, 2\}$ and $\{3\}$.

    > **Solution:** The valid ways are: $\{1\}$ and $[\{2, 3, 4\}, \{2\}$ and $\{1, 3, 4\}, \{3\}$ and $\{1, 2, 4\}, \{4\}$ and $\{1, 2, 3\}, \{1, 2\}$ and $\{3, 4\}, \{1, 3\}$ and $\{2, 4\}$, and $\{1, 4\}$ and $\{2, 3\}$. So 7 ways.

   (c) (2 points) How many ways are there to split the set $\{1, 2, 3, 4, 5\}$ into two disjoint nonempty sets?

    > **Solution:** By listing all the ways again, we find that there are 15 ways.

   (d) (2 points) Denote by $S(n, k)$ the number of ways to split a set of $n$ elements into $k$ disjoint nonempty sets. Suppose we know what $S(n - 1, 2)$ is. What

happens when we add another element to the set? What happens to each of the $S(n-1, 2)$ ways to do the partitioning?

> **Solution:** For each way to do the partitioning, we may add the new number $n$ to either set to obtain a new partitioning. There is also one new partition that is not obtained by adding $n$ to existing one, and that is the splitting into $\{n\}$ and $\{1, 2, \ldots, n-1\}$. Therefore we have
>
> $$S(n, 2) = 2S(n-1, 2) + 1.$$

(e) (2 points) What is $S(n, 2)$?

> **Solution:** We have $S(1, 2) = 0$, $S(2, 2) = 1$, $S(3, 2) = 3$, $S(4, 2) = 7$, and $S(5, 2) = 15$. From this pattern and the recursive formula found above it is easy to prove by induction that $S(n, 2) = 2^{n-1} - 1$.

(f) (3 points (bonus)) If there are $S(n-1, k-1)$ ways to partition a set of $n-1$ elements into $k-1$ nonempty disjoint sets, what is $S(n, k)$ in terms of $S(n-1, k)$ and $S(n-1, k-1)$?

> **Solution:** We need to think about what happens when we add the $n$th element. Clearly we may put this object into a separate set by itself and partition the other $n-1$ objects into $k-1$ nonempty disjoint subsets, because all up there would be $k$ sets. There are $S(n-1, k-1)$ ways to do this. We may also take the $n-1$ objects and partition them into $k$ sets and add the $n$th element to any one of them to get another valid partitioning. How many ways are there to do this? In each of the $S(n-1, k)$ ways to partition $n-1$ objects into $k$ nonempty sets, there are $k$ different subsets we can put the $n$th object into. Therefore in this case there are $kS(n-1, k)$ ways. In total then we have
>
> $$S(n, k) = S(n-1, k-1) + kS(n-1, k).$$

3. (a) (2 points) Is the sum or difference of two rational numbers always rational? Why or why not?

> **Solution:** Yes, because $a/b + c/d = (ad + bc)/bd$ and both $ad + bc$ and $bd$ are integers.

(b) (2 points) Is the product or quotient of two rational numbers always rational? Why or why not?

> **Solution:** Yes, because $(a/b)(c/d) = ac/bd$ and both $ac$ and $bd$ are integers.

(c) (2 points) When one rational number is raised to another is the result guaranteed to be rational?

> **Solution:** No; an example is $2^{1/2}$.

(d) (3 points) It is known that $\sqrt{2}$ is irrational, but it is not known whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. Do there exist two irrational numbers such that when one is raised to the other the result is rational? Prove your conjecture.

> **Solution:** The answer is yes. Let $A = \sqrt{2}^{\sqrt{2}}$. We don't know if $A$ is rational or irrational, but it doesn't matter. If $A$ is rational, then it is an example of a rational number that is an irrational number raised to another irrational number. If $A$ is irrational, then $A^{\sqrt{2}} = \sqrt{2}^2 = 2$ and this number is the desired example.

4. (a) (1 point) If two normal six-sided die are rolled, what is the probability that the sum of the two numbers is 2?

> **Solution:** Out of the 36 possible pairs the only one that has a sum of 2 is $(1,1)$, so the probability is $1/36$.

(b) (1 point) If two normal six-sided die are rolled, what is the probability that the sum of the two numbers is 7?

> **Solution:** There are 6 ways to get a 7: $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$, and $(6,1)$. Therefore the probability is $1/6$.

(c) (2 points) Consider a biased six-sided die in which the probability of rolling $n$ is $p_n$. This die is rigged so that when two of them are rolled, every possible sum of the two numbers is equally likely. What is $p_2$ in terms of $p_1$?

> **Solution:** The probability of getting a 3 is $p_2 p_1 + p_1 p_2$, which by hypothesis is equal to the probability of getting a 2, or $p_1^2$. Therefore $2p_2 = p_1$, which implies that $p_2 = p_1/2$.

(d) (1 point) What is $p_3$ in terms of $p_1$?

> **Solution:** Following the previous procedure we find that $p_3 = 3p_1/8$.

(e) (2 points) What is $p_4$ in terms of $p_1$? Do you notice a pattern?

> **Solution:** $p_4 = 5p_1/16$. The pattern is $1/2$, $(1/2)(3/4)$, $(1/2)(3/4)(5/6)$, and so on.

(f) (2 points) Let $p_i = a_{i-1}p_1$. So $a_0 = 1$, $p_2 = a_1 p_1$, $p_3 = a_2 p_1$, $p_4 = a_3 p_1$ and so on. What are $a_0 a_1 + a_1 a_0$, $a_0 a_2 + a_1 a_1 + a_2 a_0$, $a_0 a_3 + a_1 a_2 + a_2 a_1 + a_3 a_0$, etc. all equal to?

**Solution:** If $p_1 p_2 + p_2 p_1 = p_1^2$ then

$$a_0 p_1 a_1 p_1 + a_0 p_1 a_1 p_1 = p_1^2$$

which implies

$$(a_0 a_1 + a_1 a_0) p_1^2 = p_1^2.$$

Therefore $a_0 a_1 + a_1 a_0 = 1$. Similarly the other combinations of $a$'s can be found to be 1 too.

(g) (1 point) Since we have a six-sided die,

$$p_1 + p_2 + \cdots + p_6 = 1.$$

What is $p_1$ in terms of $a_0, a_1, a_2, \ldots, a_5$?

**Solution:** Since the die is six-sided, we have $p_1 + p_2 + \cdots + p_6 = 1$. We can express this in terms of $p_1$:

$$a_0 p_1 + a_1 p_1 + a_2 p_1 + \cdots + a_5 p_1 = 1$$

and hence

$$p_1 = \frac{1}{a_0 + a_1 + \cdots + a_5}.$$

(h) (2 points) Let $s_i = a_0 + a_1 + \cdots + a_i$. What are $s_0, s_1, s_2$? What is $s_5$?

**Solution:** $s_0 = a_0 = 1$, $s_1 = a_0 + a_1 = 3/2$, $s_2 = 1 + 1/2 + 3/8 = 15/8$. Note that $s_2 = (3/2)(5/4)$, $s_3 = (3/2)(5/4)(7/6)$, $s_4 = (3/2)(5/4)(7/6)(9/8)$ and so on. Therefore

$$s_5 = \frac{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} = \frac{693}{256}.$$

(i) (3 points) Is such a six-sided die possible?

**Solution:** From the previous part we find that $p_1 = 256/693$. This means that the probability of rolling two of these dice and getting a sum of $2, 3, \ldots, 12$ must all be $p_1^2 = 65536/480249$. But this is impossible because there are 11 possible sums and if they are equally likely each must have a probability of $1/11$. Therefore such a die is not possible.

5. (a) (1 point) What is $1 + 2 + 3 + \cdots + 200$?

> **Solution:** 10100

(b) (2 points) Find a formula for $1 + 2 + \cdots + n$ and prove it.

> **Solution:** The formula is
> $$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$
> It is easily proved by induction. When $n = 1$ we have $1 = 1(1+1)/2$ so the base case is done. Now assume that
> $$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$
> for some natural number $k$. Add $k+1$ to both sides:
> $$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$
> and the induction is complete.

(c) (1 point) Define
$$F_k(n) = 1^k + 2^k + \cdots + n^k.$$
(So in the previous part you found a formula for $F_1(n)$.) More compactly it may be written as
$$F_k(n) = \sum_{i=1}^{n} i^k.$$
What is
$$1^3 + (2^3 - 1^3) + (3^3 - 2^3) + \cdots + (100^3 - 99^3)?$$

> **Solution:** This sum is an example of a telescoping sum. All the terms vanish except for the $100^3$ term in the last set of parentheses, so the answer is $100^3$.

(d) (2 points) By considering
$$1^3 + \sum_{i=1}^{n} ((i+1)^3 - i^3)$$
find a formula for $F_2(n)$.

> **Solution:** We know that
> $$1^3 + \sum_{i=1}^{n} ((i+1)^3 - i^3) = (n+1)^3.$$

Expanding both sides using the Binomial Theorem we get

$$1^3 + \sum_{i=1}^{n}(3i^2 + 3i + 1) = n^3 + 3n^2 + 3n + 1.$$

After removing the 1 from both sides, on the LHS we can break up the sum and pull out the constant factors, like so:

$$3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 = n^3 + 3n^2 + 3n.$$

But note that $\sum_{i=1}^{n} i^2 = F_2(n)$ and $\sum_{i=1}^{n} i = F_1(n)$, so we have

$$3F_2(n) + 3F_1(n) + n = n^3 + 3n^2 + 3n.$$

Then we just need to rearrange for $F_2(n)$ after substituting in $F_1(n) = n(n+1)/2$ to find that

$$F_2(n) = \frac{2n^3 + 3n^2 + n}{6}.$$

(e) (2 points) Find a formula for $F_3(n)$.

**Solution:** Using the same method as above, we find that

$$F_3(n) = \frac{n^4 + 2n^3 + n^2}{4}.$$

(f) (1 point) The Binomial Theorem states that

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

If

$$(i + 1)^k - i^k = \sum_{j=0}^{x} \binom{k}{j} i^j$$

what is $x$ in terms of $k$?

**Solution:** The leading term vanishes, so instead of going from 0 to $k$ the loop only goes up to $k - 1$. Therefore $x = k - 1$.

(g) (3 points) By considering the general telescoping sum

$$1^k + \sum_{i=1}^{n}((i + 1)^k - i^k) = (n + 1)^k$$

show that

$$F_{k-1}(n) = \frac{(n+1)^k - 1}{k} + \frac{1}{k}\sum_{i=0}^{k-2}\binom{k}{i}F_i(n).$$

**Solution:** Expand the $(n+1)^k$ term using the Binomial Theorem, applying the previous part:

$$(n+1)^i = 1^k + \sum_{i=1}^{k}\sum_{j=0}^{k-1}\binom{k}{j}i^j.$$

Now expand the inner sigma notation:

$$(n+1)^k = 1^k + \sum_{i=1}^{k}\left(\binom{k}{0}i^0 + \binom{k}{1}i^1 + \binom{k}{2}i^2 + \cdots + \binom{k}{k-1}i^{k-1}\right).$$

We may take out the binomial coefficients, as they are independent of the sum variable $i$, so we have

$$(n+1)^k = 1^k + \binom{i}{0}\sum_{i=1}^{k}i^0 + \binom{k}{1}\sum_{i=1}^{k}i^1 + \binom{k}{2}\sum_{i=1}^{k}i^2 + \cdots + \binom{k}{k-1}\sum_{i=1}^{k}i^{k-1}.$$

We're interested in $F_{k-1}(n)$, so we notice that the binomial coefficient $\binom{k}{k-1} = k$ and isolate that term. The other sums are $F_0(n)$, $F_1(n)$, $F_2(n)$ and so on. Therefore we get

$$(n+1)^k = 1^k + \binom{k}{0}F_0(n) + \binom{k}{1}F_1(n) + \binom{k}{2}F_2(n) + \cdots + \binom{k}{k-2}F_{k-2}(n) + kF_{k-1}(n)$$

or more compactly,

$$(n+1)^k = 1 + \sum_{i=0}^{k-2}\binom{k}{i}F_i(n) + kF_{k-1}(n).$$

Rearranging for $F_{k-1}(n)$ we get the desired result.

Total: 52