Melbourne High School
Maths Extension Group 2023
Some of Euler's Gems: Solutions

1. (a) Suppose that n is composite and write $n = mp$ for some prime p and integer m. From the formula for the sum of a geometric series, the fraction

$$\frac{2^{pm}-1}{2^m-1}$$

is the sum of the first p terms of a geometric series with first term 1 and common ratio 2^m. That is,

$$\frac{2^{pm}-1}{2^m-1} = 1 + 2^m + 2^{2m} + \cdots + 2^{(p-1)m}.$$

Hence $2^n - 1$ is divisible by $2^m - 1$ and therefore can only be prime if $m = 1$ and $n = p$.

(b)

n	$\sigma(n)$
1	1
2	3
3	4
4	7
5	6
6	12
7	8
8	15
9	13
10	18
11	12
12	28
13	14
14	24
15	24
16	31
17	18
18	39
19	20
20	42

(c) Since m and n are coprime, the divisors of mn are exactly

$$a_1b_1, a_1b_2, \ldots, a_1b_{d(n)}, a_2b_1, \ldots, a_2b_{d(n)}, \ldots, a_{d(m)}b_1, \ldots, a_{d(m)}b_{d(n)}.$$

Hence

$$\begin{aligned}\sigma(mn) &= a_1(b_1 + b_2 + \cdots + b_{d(n)}) + a_2(b_1 + b_2 + \cdots + b_{d(n)}) + \cdots + a_{d(m)}(b_1 + b_2 + \cdots + b_{d(n)}) \\ &= (a_1 + a_2 + \cdots + a_{d(m)})(b_1 + b_2 + \cdots + b_{d(n)}) \\ &= \sigma(m)\sigma(n)\end{aligned}$$

(d) The divisors of p_k are exactly

$$1, p, p^2, \dots, p^k.$$

Therefore

$$\sigma(p^k) = 1 + p + p^2 + \cdots + p^k = \frac{p^{k+1} - 1}{p - 1}.$$

(e) We know that $\sigma(N) = \sigma(2^{p-1}(2^p - 1)) = \sigma(2^{p-1})\sigma(2^p - 1)$, so we can do each separately. From the preceding exercise we have

$$\sigma(2^{p-1}) = \frac{2^p - 1}{2 - 1} = 2^p - 1.$$

Since $2^p - 1$ is prime, its only divisors are $2^p - 1$ and 1, so

$$\sigma(2^p - 1) = 2^p - 1 + 1 = 2^p.$$

Therefore

$$\sigma(N) = \sigma(2^{p-1}(2^p - 1)) = (2^p - 1)2^p = 2(2^{p-1}(2^p - 1)) = 2N.$$

Therefore $2^{p-1}(2^p - 1)$ is perfect.

(f) If N is even, it divides some power of 2. Let 2^k be the highest power of 2 that divides N; this means $k \geq 1$. Hence $N/2^k$ must be odd; otherwise 2^k would not be the highest power of 2 that divides N. Therefore $N = 2^k m$ for $k \geq 1$ and odd m.

(g) We have

$$\sigma(N) = \sigma(2^k)\sigma(m) = (2^{k+1} - 1)\sigma(m).$$

But $\sigma(N) = 2N$ because N is a perfect number, so we have

$$2N = (2^{k+1} - 1)\sigma(m).$$

Clearly $2N = 2^{k+1}m$. Substituting this in yields

$$2^{k+1}m = (2^{k+1} - 1)\sigma(m)$$

as required.

(h) Substituting in $\sigma(m) = 2^{k+1}c$, we get

$$2^{k+1}m = (2^{k+1} - 1)2^{k+1}c.$$

Cancelling 2^{k+1} from both sides we have

$$m = (2^{k+1} - 1)c.$$

(i) If $c > 1$, then m is at least divisible by 1, m, and c. (We note that $m \neq c$ because $k \geq 1$.) Then

$$\sigma(m) \geq 1 + m + c = 1 + (2^{k+1} - 1)c + c = 1 + 2^{k+1}c.$$

(j) But $\sigma(m) = 2^{k+1}c$, so the preceding inequality implies

$$2^{k+1}c \geq 1 + 2^{k+1}c$$

which implies $0 \geq 1$. Clearly this is a contradiction, so $c = 1$.

(k) If $c = 1$ then $\sigma(m) = 2^{k+1} = m + 1$. The number m is at least divisible by itself and 1, so the sum of its divisors is at least $m + 1$, but if $\sigma(m)$ actually equals $m + 1$ then no other number can divide it except for 1 and m; hence m must be prime.

(l) From $m = 2^{k+1} - 1$ we have

$$N = 2^k(2^{k+1} - 1).$$

We know that $2^{k+1} - 1$ is prime, so $k + 1$ must also be prime. If $k + 1$ is prime, then k is one less than a prime, say p. So if we put $k = p - 1$, then we get

$$N = 2^{p-1}(2^p - 1).$$

This completes the proof.

2. (a) Starting from the 1/3 term, we note that $1/3+1/4 > 1/4+1/4 = 1/2$, since $1/3 > 1/4$. Similarly, $1/5 + 1/6 + 1/7 + 1/8 > 1/8 + 1/8 + 1/8 + 1/8 = 1/2$. This can be repeated to infinity to show that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots.$$

Clearly the RHS goes to infinity, and therefore the LHS must diverge too.

(b) In the first graph, where the rectangles are underneath the graph, the first rectangle is excluded because $\ln(x)$ is the area under the graph of $1/x$ from 1 to x. So we get $H_n - 1 < \ln(n)$. From the second graph, $H_n > \ln(n)$. Putting these two inequalities together yields

$$\ln(n) < H_n < \ln(n) + 1.$$

(c) We have

$$H_{kn} \approx \ln(kn) + \gamma_1$$

and

$$H_n \approx \ln(n) + \gamma_2.$$

As n approaches infinity, γ_1 and γ_2 will approach the same value, so we have

$$H_{kn} - H_n = \ln(kn) - \ln(n) = \ln(k).$$

i. From the preceding formula, $\ln 2 = H_{2n} - H_n$ as n goes to infinity. At first, it's not obvious how to subtract these two values because the first n terms will disappear. The trick is the same as that of recognising that the even numbers and all the numbers are both still countable infinities even though the even numbers seems to be an infinity that is half as small, in a sense. Countable infinity means that there is a one-to-one correspondence with the positive integers, which the even numbers have. We associate 1 with 2, 2 with 4, 3 with 6 and so on. The same principle applies here. Instead of subtracting in order, we change the order of the series slightly (we're allowed to this because the series is convergent) and subtract a term of H_n from every second term of H_{2n}; in this way an infinite number of terms can be subtracted without losing an infinite number of terms at the beginning. Hopefully this is clear:

$$\begin{aligned}\ln 2 &= 1 + \left(\frac{1}{2} - 1\right) + \frac{1}{3} + \left(\frac{1}{4} - \frac{1}{2}\right) + \frac{1}{5} + \left(\frac{1}{6} - \frac{1}{3}\right) + \cdots \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\end{aligned}$$

ii. Apply the same method as for $\ln 2$.

iii. Apply the same method as for $\ln 2$.

The formula is simply a compact form of the algorithm we use to subtract the two infinite series. From the preceding examples we note that the algorithm consists of doing the subtraction every nth term (if we're calculating $\ln n$). This subtraction is equivalent to multiplying the fraction by $1-n$. The combination of the floor and ceiling functions evaluate to either 0 or 1 depending on whether k is a multiple of n and therefore determine which fractions to multiply by $1-n$.

(d) Let

$$\ln(1+x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Putting in $x=0$ we get $a_0 = 0$. Differentiating both sides yields

$$\frac{1}{(1+x)} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots .$$

Again putting in $x=0$, we find that $a_1 = 1$. We continue this process to find the other coefficients. Differentiating again we have

$$\frac{-1}{(1+x)^2} = 2a_2 + 6a_3 x + \cdots .$$

Putting in $x=0$ results in $-1 = 2a_2$; hence $a_2 = -1/2$. Eventually we find that $a_3 = 1/3$, $a_4 = -1/4$, and so on, leaving us with

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

(e) Put $x = -1/t$ and substitute it in to get

$$\ln\left(\frac{t-1}{t}\right) = -\frac{1}{t} - \frac{1}{2t^2} - \frac{1}{3t^3} - \cdots .$$

Flip the fraction in the logarithm on the LHS to get rid of all the minus signs:

$$\ln\left(\frac{t}{t-1}\right) = \frac{1}{t} + \frac{1}{2t^2} + \frac{1}{3t^3} + \cdots .$$

Since the series only makes sense for $|x| < 1$, the restriction on t is that $|-1/t| < 1$. From this $t < -1$ or $t > 1$.

(f)

$$\sum_{k=2}^{n} \ln\left(\frac{k}{k-1}\right) = \sum_{k=2}^{n} \left(\frac{1}{k} + \frac{1}{2k^2} + \frac{1}{3k^3} + \cdots\right)$$

By adding the logarithms on the LHS, we get a telescoping product (or sum if we break the logarithms into the difference of two logarithms) that leaves us with $\ln(n) - \ln(1) = \ln(n)$. By expanding the sum on the RHS, we get

$$\ln(n) = \left(\frac{1}{2} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 8} + \cdots +\right) + \left(\frac{1}{3} + \frac{1}{2\cdot 9} + \frac{1}{3\cdot 27} + \cdots\right) + \cdots + \left(\frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots\right).$$

If we take the first term of each bracket, we get the sum

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

which is just $H_n - 1$. Similarly, if we take the second term of each bracket, we get

$$\frac{1}{2\cdot 4} + \frac{1}{2\cdot 9} + \frac{1}{2\cdot 16} + \cdots + \frac{1}{2n^2} = \frac{1}{2}(H_n^{(2)} - 1).$$

The third term of each bracket gives us

$$\frac{1}{3}(H_n^{(3)} - 1).$$

Continuing this rearrangement of the terms, we find that

$$\ln(n) = H_n - 1 + \frac{1}{2}(H_n^{(2)} - 1) + \frac{1}{3}(H_n^{(3)} - 1) + \frac{1}{4}(H_n^{(4)} - 1) + \cdots.$$

(g) The limit of the sum should a number around 0.577216.

3. (a) Each partition of n contributes a unit to the coefficient of x^n. Each infinite sum represents the number of times a particular number appears in the partition—the first bracket represents the number of 1s, the second the number of 2s, the third the number of 3s, and so on. A partition of n is really just selecting a term from each bracket such that the indices add to n when the terms are multiplied. Multiplying out the brackets results in an enumeration of all the possible ways of choosing different terms from each bracket, and therefore each coefficient will be all the ways to partition that number.

(b) The first 6 values of $p(n)$ seem to yield the prime numbers. However, this would-be glorious theorem is destroyed when we find that $p(7) = 15$.

(c) Using the formula

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

the result follows immediately.

(d) $p_d(7) = 5$.

(e) $p_o(7) = 5$.

(f) From each bracket now we can only choose x^i or 1; this is equivalent to either including i in the partition or not. Whereas before we could choose x^{2i}, x^{3i}, and so on to include more than one of i in the partition, only being able to choose x^i means only one of i ends up in the final partition, and therefore the coefficients of the expanded product are the number of partitions that use only unique numbers.

This generating function is equivalent to finding how many subsets of a set $\{1, 2, 3, \dots, n\}$ sum to a particular number, since a partition with distinct elements is the same as choosing elements of that set that sum to that particular number.

(g) This generating function can be written as

$$(1 + x + x^2 + x^3 + \cdots)(1 + x^3 + x^6 + x^9 + \cdots)(1 + x^5 + x^{10} + x^{15} + \cdots)\cdots.$$

This is the same as the original generating function except that the brackets determining which even numbers ended up in the partition are missing. Therefore by expanding the brackets only partitions that use odd numbers will be counted in the coefficient of each term.

(h) This problem shows the power of generating functions. If we can manipulate our generating functions so that they are equal, we have completed the proof.

Euler's proof involves transforming each factor in $F_d(x)$ into a fraction using the difference of two squares, as follows.

$$\begin{aligned} F_d(x) &= \frac{(1+x)(1-x)}{1-x} \cdot \frac{(1+x^2)(1-x^2)}{1-x^2} \cdot \frac{(1+x^3)(1-x^3)}{1-x^3} \cdots \\ &= \frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)\cdots} \\ &= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} \\ &= F_o(x) \end{aligned}$$