Melbourne High School
Maths Extension Group 2023
First Meeting Problems

1. Solve the equation $4t/3 + 2 = 2(t - 2)$.

2. A square circumscribes a circle of radius a. Find the probability that a randomly chosen point in the square is also in the circle.

3. Find a number that is the same when added to 5 as when multiplied by 5.

4. Evaluate
$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{99 \cdot 100}.$$

5. Call an integer to be *gobbus* if every digit is either 0 or 2 and it is divisible by 15. What is the smallest number that is gobbus?

6. I have three jars of cookies. The product of the number of cookies in each jar is 36. Unfortunately, even if I tell you the sum of the number of cookies in each jar, you won't be able to determine how many there are in each jar, but I do know that the jar with the most cookies is currently open. How many cookies are there in each jar?

7. If $1/(a + b) = 4$ and $a^2 + b^2 = 16$ find $1/a + 1/b$.

8. Find all primes less than 100 that are the sum of two squares.

9. Find all solutions in positive integers x and y of the equation $15x^2 + 2y^2 = 100 + 13xy$.

10. Find the last digit of 2^{2023}.

11. If $a, b, c > 0$ and $a + b + c = 1$, prove that $a^2 + b^2 + c^2 + 1 \geq 4(ab + bc + ca)$.

12. Define the *alternating sum* of a set of positive integers as follows: List the elements of the set in decreasing order. Take the first number in the list, subtract the second, add the third, subtract the fourth, and so on. For example, the alternating sum of the set $3, 5, 11, 7$ is $11 - 7 + 5 - 3 = 6$. Find the sum of the alternating sums of all the subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

13. Evaluate $5^{261} \pmod{397}$.

14. Suppose BP and CQ are bisectors of the angles B, C of triangle ABC and suppose AH, AK are the perpendiculars from A to BP, CQ. Prove that KH is parallel to BC.

15. The Fibonacci sequence is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, where $F_0 = 0$ and $F_1 = 1$. Consider the power series
$$\begin{aligned} G(x) &= \sum_{i=0}^{\infty} F_i x^i \\ &= F_0 x^0 + F_1 x^1 + F_2 x^2 + F_3 x^3 + F_4 x^4 + F_5 x^5 + F_6 x^6 + \cdots. \end{aligned}$$

Find a closed-form expression for $G(x)$.