Melbourne High School
Maths Extension Group 2023
Term 3 Week 6 Handout Solutions

Problems in solving equations

1. If x is positive we have $1 < 3x$ so $x > \frac{1}{3}$, if x is negative $\frac{1}{3} < 3$ will always be true. Meanwhile, we have $1 < -4x$ so $x < -\frac{1}{4}$. Therefore, all x such that $x > \frac{1}{3}$ or $x < -\frac{1}{4}$ works.

2. Solving for x and substituting it into the third equation we get $z\frac{12\sqrt{6}}{y} = 45\sqrt{3}$, we multiply this by the second equation to get $z^2 = 4 \times 54$ so $z = \pm 6\sqrt{6}$. From there, its pretty easy to sub this value back into the equations to get two sets of solutions for (x, y, z), $(\pm 4\sqrt{2}, \pm 3\sqrt{3}, \pm 6\sqrt{6})$.

3. We let $a = 2^x$ and $b = 3^x$. The equation becomes $\frac{a^3+b^3}{a^2b+b^2a} = \frac{7}{6}$. Which after some simplifying we get $6a^2 - 13ab + 6b^2 = 0$, which is $(2a-3b)(3a-2b) = 0$. Therefore $2^{x+1} = 3^{x+1}$ or $2^{x-1} = 3^{x-1}$, which implies $x = -1$ and $x = 1$.

4. Mr Fat has the winning strategy, because by choosing a set of distinct rational nonzero numbers a, b, c, such that $a + b + c = 0$ will make him win. Let a', b', c' be a random permutation of a, b, c and let $f(x) = a'x^2 + b'x + c'$. Then $f(1) = a' + b' + c' = a + b + c = 0$, and so 1 is a solution. Since the product of two numbers is $\frac{c'}{a'}$ by Vieta's, the other solution is clearly $\frac{c'}{a'}$, which is different from 1. Thus Mr Fat can guarantee two distinct solutions.

5. Official Solution: Let $x = \sqrt[3]{\sqrt[3]{2} - 1}$ and $y = \sqrt[3]{2}$. So $y^3 = 2$ and $x = \sqrt[3]{y-1}$. Note that
$$1 = y^3 - 1 = (y-1)(y^2 + y + 1)$$
and
$$y^2 + y + 1 = \frac{3y^2 + 3y + 3}{3} = \frac{(y+1)^2}{3}$$
which implies that
$$x^3 = y - 1 = \frac{1}{y^2 + y + 1} = \frac{3}{(y+1)^3}$$
or
$$x = \frac{\sqrt[3]{3}}{y+1}.$$
On the other hand $3 = y^3 + 1 = (y+1)(y^2 - 1 + 1)$ from which it follows that
$$\frac{1}{y+1} = \frac{y^2 - y + 1}{3}.$$
Thus we have
$$x = \sqrt[3]{\frac{1}{9}}(\sqrt[3]{4} - \sqrt[3]{2} + 1).$$
Consequently $(a, b, c) = (\frac{4}{9}, -\frac{2}{9}, \frac{1}{9})$ is a desired triple.

Gauss's Lemma

1. In this case $p = 19, P = (19-1)/2 = 9, a = 5$. Reducing the numbers
$$5, 10, \ldots, 45$$
into the range $(-19/2, 19/2)$ gives
$$5, -9, -4, 1, 6, -8, -3, 2, 7.$$
There are 4 negative numbers, so
$$\left(\frac{5}{19}\right) = (-1)^4 = 1.$$
We verify this by observing that $9^2 \equiv 5 \pmod{19}$.

2. Some tedious work convinces us that $(2|p) = 1$ for
$$p = 7, 17, 23, 31, 41, 47, 71, 73, 79, 89, 97$$
and $(2|p) = -1$ for
$$p = 3, 5, 11, 13, 19, 29, 37, 43, 53, 59, 61, 67, 83$$
for primes p under 100.

3. (a) Since
$$2u < \frac{p}{2}$$
and
$$2(u+1) > \frac{p}{2}$$
we have
$$u < \frac{p}{4}$$
and
$$u > \frac{p}{4} - 1.$$
Hence
$$\frac{p}{4} - 1 < u < \frac{p}{4}.$$
(b) If $p = 4k+1$, then
$$k + \frac{1}{4} - 1 < u < k + \frac{1}{4}$$
which implies
$$k - \frac{3}{4} < u < k + \frac{1}{4}.$$
There is only one integer that satisfies this inequality, and that is $u = k$. Similarly if $p = 4k+3$ then
$$k + \frac{3}{4} - 1 < u < k + \frac{3}{4}$$
which implies
$$k - \frac{1}{4} < u < k + \frac{3}{4}.$$
Again there is only one integer between those two bounds, and that is $u = k$.

(c) No; since $v = P - u$ the parity of u needs to be known to determine the parity of v. Therefore the quadratic character of 2 modulo p does not depend on whether $p \equiv 1$ or $p \equiv -1$ modulo 4.

(d) Considering whether $p \equiv 1, 3, 5, 7$ modulo 8 is the idea. First let $p = 8k+1$; substituting it into the inequality found in the first part we get

$$2k - \frac{3}{4} < u < 2k + \frac{1}{4}.$$

From this we deduce that $u = 2k$ and

$$v = \frac{p-1}{2} - u = 4k - 2k = 2k$$

which is even. Similarly, if $p = 8k + 7$ we have

$$2k + \frac{7}{4} - 1 < u < 2k + \frac{7}{4}.$$

This means $u = 2k + 1$; putting this in for v yields

$$v = 4k + 3 - (2k + 1) = 2k + 2$$

which is again even. Therefore $(2|p) = 1$ for $p \equiv 1, 7 \pmod 8$.

The same procedure works for the other case. Let $p = 8k+3$. The inequality for u becomes

$$2k - \frac{1}{4} < u < 2k + \frac{3}{4}.$$

So $u = 2k$; substituting this and $p = 8k + 3$ into $v = P - u$ gives

$$v = 4k + 1 - 2k = 2k + 1,$$

an odd number. The same works for $p = 8k + 5$; now the inequality is

$$2k + \frac{1}{4} < u < 2k + \frac{5}{4}$$

from which it follows that $u - 2k + 1$. Thus

$$v = 4k + 2 - (2k + 1) = 2k + 1.$$

This completes the proof that

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1, 7 \pmod 8; \\ -1 & \text{if } p \equiv 3, 5 \pmod 8. \end{cases}$$