Combinatorial Games

Maths Extension Group MHS

Why Combinatorial Games?

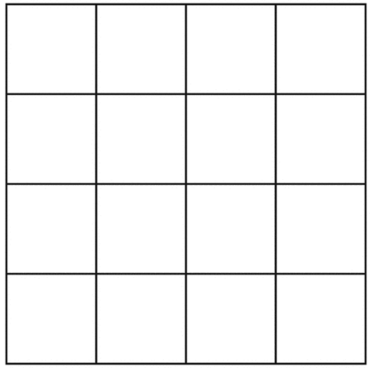
- Improved problem solving skills
- Unique subject that requires minimal theory
- Beat your friends in games!

What are Combinatorial Games?

- There are two players who take turns to move

- There is no chance involved e.g. rolling dice

- Each player’s move are known to the other player



Combinatorial Games

<u>Analyse different outcomes and states in combinatorial games</u>

-Who has a winning strategy?

-Who doesn’t have a winning strategy?

-What is the winning strategy?

Combinatorial Games

Some well known combinatorial games are: Chess, Tic Tac Toe, Nim and Checkers

Fundamental Theorem of Combinatorial Games

Theorem: In a combinatorial game, either there exists a winning strategy for one of the players or both of them can force a draw.

Now we will learn a corollary of this theorem that can be very useful.

Position Analysis

Position Analysis is a strategy to help find the winning strategy.

In a combinatorial game with no draws, every position of the game can be categorised as *winning* or *losing:*

- Winning positions: it is impossible to move to a losing position

- Losing position: it is impossible to move to a winning position


CAN YOU
SOLVE THE
ROGUE AI
RIDDLE?

Position Analysis

- Describe the winning positions
- Describe the losing positions
- Come up with a winning strategy

General Strategy and Tips

- Play the game
- Try noticing patterns in winning positions
- Try to form a winning strategy
- Prove your winning strategy

- It’s also useful to know how some games generally work (e.g. how chess pieces move)

2020 AMO q2

Amy and Bec play the following game. Initially there are three piles, each containing 2020 stones. The players take turns to make a move, with Amy going first. Each move consists of choosing one of the available, removing the unchosen pile(s) from the game, and then dividing the chosen pile into 2 or 3 non-empty piles. A player loses the game if they are unable to move.

Prove Bec has the winning strategy

Solution

So Bec has the winning strategy here.

Call a pile “bad” if it has 1 more than a multiple of 3 stones

Call a pile “good” if it has some number of stones that is not 1 more than a multiple of 3

I claim that if Bec can force Amy to have a bad pile every turn then that Bec will win the game. Why?

Solution

Because the number of stones is strictly decreasing! That means after every turn the number of stones is always less than it was before.

So if Amy is always on a pile with 1 more than multiple of 3 stones, she will eventually end up with 1 stone.

Can you divide 1 stone into 2 or 3 non-empty sets???

Solution

Suppose it’s Amy’s turn and she has all bad piles. Then I claim it is impossible for her to force Bec to have a bad pile. Why?

Have a look at examples: can you partition a number 1 more than a multiple of 3 into 2 or 3 parts, where all the parts are 1 more than a multiple of 3? (Hint: You can’t)

Solution

Proof: WLOG let the pile Amy chooses to divide to $3k+1$ stones where k is an integer.

Then it is impossible to have: $3k+1 = 3j + 1 + 3i + 1$

It is also impossible to have: $3k+1 = 3x + 1 + 3y + 1 + 3z + 1$

For some integers j,i,x,y,z

Solution

So we know it’s impossible to force Bec to have a bad pile if Amy has a bad pile. In other words, Bec will always have a winning pile if Amy has a bad pile.

I claim that if Bec has a good pile, she can force Amy to have a bad pile next turn. Why?

Solution

Suppose Bec has a pile with 2 more than a multiple of 3 stones

Then let her have $3k+2$ stones for some integer k.

It is obvious there exists integers m, n that satisfy $3k + 2 = 3m + 1 + 3n + 1$

Can anyone prove the case where Amy has a pile that has a multiple of 3 stones?

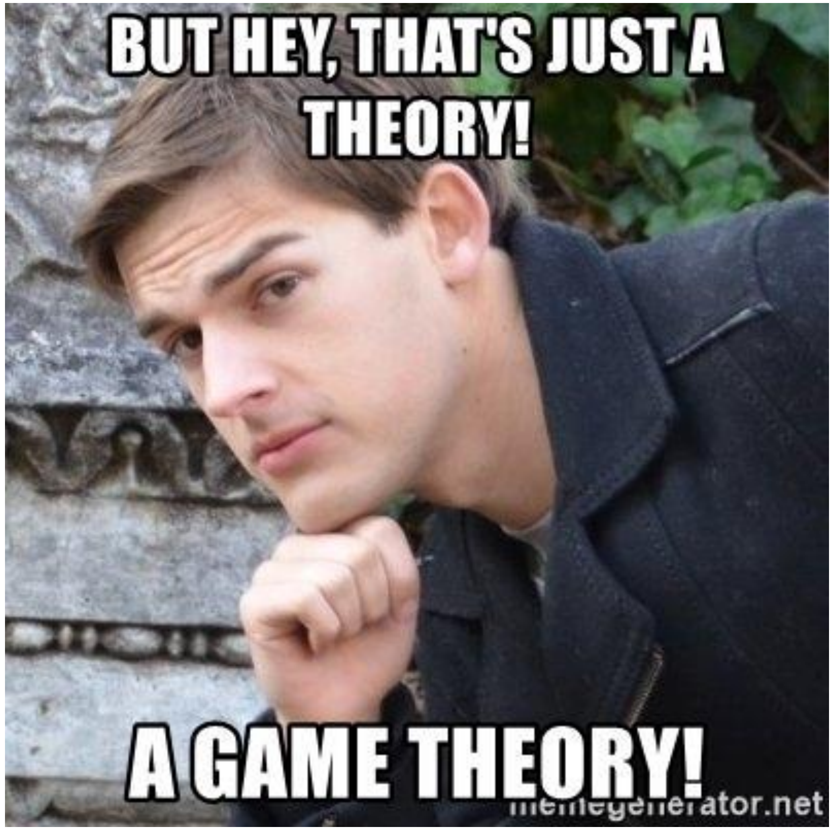
Solution

Hence Bec can always force Amy to have a bad pile and Amy can’t do anything about it.

To finish it off our specific question, why does Bec win if Amy starts with 2020 stones?

Conclusion

- Try playing the game yourself
- Make a guess on how to win
- Prove your strategy
- Apply it to your problem



Your Turn!

Try playing the out the problems on the worksheet with your friends!

Any Questions?