Melbourne High School
Maths Extension Group 2023
Term 3 Week 8 Handout Solutions

Problems: functions and functional equations

1. For f to be invertible, we need $g(f(x)) = x$ for all x in the domain of f. But $g(f(x)) = \sqrt{x} = |x|$, which is not equal to x when x is negative. Also, f is not injective because $f(a) = f(-a)$ and so the inverse does not exist.

2. Let $y = 0$, then we have $f(x) = x + f(0) = x + 2$. Hence $f(1998) = 1998 + 2 = 2000$.

3. To prove injectivity, if $f(x) = f(y)$ then clearly $f(f(x)) = f(f(y))$, and hence $x = y$ from the definition of an involution.

   To prove surjectivity, take z to be a real number in the co-domain of f. It suffices to show there exists x such that $f(x) = z$. From $f(x) = z$ we have $f(f(x)) = f(z)$ or $f(z) = x$. Thus, the x we needed to show that exists is $f(z)$.

4. Subbing in $y = 0$ yields $f(x) = x^2$. Thus $f(x) = x^2$ is the only possible solution. Trying this solution in the original equation, we have $f(x+y) + f(x-y) = 2x^2 + 2y^2$, not $2x^2 - 2y^2$. Thus the only possible candidate fails the test, hence there are no solutions.

5. Subbing in $y = 0$ we get

   $$-2f(0) = -2$$

   $$f(0) = 1$$

   Subbing in $x = 0$ we obtain

   $$f(y) + 2f(-y) = 3 - y$$

   and by replacing y with $-y$ we obtain

   $$f(-y) + 2f(y) = 3 + y$$

   Solving the system of equations, treating $f(y)$ and $f(-y)$ as variables, we get $f(y) = y+1$. It is easy to verify $f(x) = x+1$ satisfies the conditions of the problem, thus it is the only solution.