Melbourne High School
Maths Extension Group 2023
First Meeting Problems Solutions

1. Solve to get the solution $t = 9$.

2. The square has side length $2a$; hence its area is $4a^2$. The circle has area πa^2, so the probability a randomly chosen point in the square is also in the circle is $\pi a^2/4a^2 = \pi/4$.

3. Solving $x + 5 = 5x$ yields $x = 5/4$.

4. Compute the first few partial sums to spot a pattern, viz.

$$\begin{aligned} \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} &= \frac{3+1}{6} = \frac{2}{3} \\ \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} &= \frac{2}{3} + \frac{1}{12} = \frac{3}{4} \\ \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{4\cdot 5} &= \frac{3}{4} + \frac{1}{20} = \frac{4}{5} \end{aligned}$$

If this pattern continues, the desired result is 99/100. (Extension: for those who have learnt proof, try proving that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$$

by induction.)

5. The number must be divisible by both 5 and 3, and so the last digit must be a 0. To be divisible by 3, the sum of the digits must be divisible by 3; we need three 2s. Hence the smallest number that is gobbus is 2220.

6. All triples that have a product of 36 are: $(1, 1, 36)$, $(2, 1, 18)$, $(2, 2, 9)$, $(3, 3, 4)$, $(4, 9, 1)$, $(6, 6, 1)$. If knowing the sum of the three numbers is not enough information, then it must be either $(2, 2, 9)$ or $(6, 6, 1)$, as both sum to 13. Since there is a jar with the most cookies in it, the answer is $(2, 2, 9)$.

7. First rearrange to find that $a + b = 1/4$. Square both sides and we get $a^2 + b^2 + 2ab = 1/16$. Substitute in $a^2 + b^2 = 16$ and the equation becomes $2ab = -255/16$; hence $ab = -255/32$. Note that $1/a + 1/b = (a + b)/ab$, so we have

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} &= \frac{1/4}{-255/32} \\ &= \frac{-8}{255}. \end{aligned}$$

8. $2, 5, 13, 17, 29, \ldots$, i.e., all the primes in the form $4k + 1$ (except for 2).

9. Rearrange and factorise to get $(5x-y)(3x-2y) = 100$. Now we just need to enumerate the factors of 100 and solve for x and y.

It is wise to consider that since x and y are positive integers, $5x - y > 3x - 2y$, so we only need to consider half the pairs of factors of 100. Beginning with $5x - y = 100$ and

$3x-2y=1$, we see that this means $7x=199$, which has no solution in positive integers. Then we move on to $5x-y=50$ and $3x-2y=2$. Solving these two simultaneous equations yields $7x=98$, or $x=14$, which implies $y=20$.

Continuing in this fashion, we find one other solution, when $5x-y=20$ and $3x-2y=5$. In this case $x=5$ and $y=5$. So all the solutions in positive integers to our original equation are $(5,5)$ and $(14,20)$.

10. Reduce modulo 10 to get the final digit of $2^3=8$.

If not familiar with modular arithmetic, consider the last digit of the first few powers of 2: $2,4,8,16,32,64,128,256,512,\ldots$. Note how the the last digits go $2,4,8,6,2,4,8,6,2,\ldots$. That is, the last digits repeat every fourth term in the sequence. Since 2023 is 3 more than a multiple of 4, its last digit is the third number in the repeated sequence $2,4,8,6$. Hence the last digit of 2^{2023} is 8.

11. Without loss of generality, assume $a \geq b \geq c$. This will be needed for an application of the rearrangement inequality.

Square both sides of the equation $a+b+c=1$ to get

$$a^2+b^2+c^2+2(ab+bc+ac)=1.$$

Substituting this into $a^2+b^2+c^2+1$ yields

$$\begin{aligned} a^2+b^2+c^2+1 &= a^2+b^2+c^2+(a^2+b^2+c^2+2(ab+bc+ac)) \\ &= 2(a^2+b^2+c^2)+2(ab+bc+ac). \end{aligned}$$

By the rearrangement inequality, $a^2+b^2+c^2 \geq ab+bc+ac$, so we have

$$\begin{aligned} 2(a^2+b^2+c^2)+2(ab+bc+ac) &\geq 2(ab+bc+ac)+2(ab+bc+ac) \\ &= 4(ab+bc+ac). \end{aligned}$$

Hence $a^2+b^2+c^2+1 \geq 4(ab+bc+ac)$.

12. Consider a set $S_n=\{1,2,3,\ldots,n\}$. Let $a_1,a_2,a_3,\ldots,a_{2^n}$ be the alternating sums of its 2^n subsets, and let A_n equal the sum of all alternating sums of all the subsets of S_n. By these definitions we have

$$\begin{aligned} S_n &= \sum_{i=0}^{n} a_i \\ &= a_1+a_2+a_3+\cdots+a_{2^n}. \end{aligned}$$

Now consider the set $S_{n+1}=\{1,2,3,\ldots,n,n+1\}$. Note that 2^n of its subsets are the same as those of S_n; the other 2^n subsets can be formed by taking each subset of S_n and adding $n+1$ to it. Now let $b_1,b_2,b_3,\ldots,b_{2^n}$ be the alternating sums of the 2^n subsets that are not subsets of S_n. In other words, these are the alternating sums of all the subsets of S_n with $n+1$ added to them. Since $n+1$ is the biggest element in each of these sets, each alternating sum begins with $n+1$, followed by the rest of the numbers in the set.

We see that $b_i=(n+1)-a_i$ for $1 \leq i \leq 2^n$. To see why this is true consider the set $\{1,2,3,4\}$; its alternating sum is $4-3+2-1=2$. If we add 5 to the set, the alternating sum becomes $5-4+3-2+1=5-(4-3+2-1)$.

So we have

$$\begin{aligned}
b_1 &= (n+1) - a_1\\
b_2 &= (n+1) - a_2\\
b_3 &= (n+1) - a_3\\
&\vdots\\
b_{2^n} &= (n+1) - a_{2^n}.
\end{aligned}$$

Adding these together we get

$$\begin{aligned}
b_1 + b_2 + b_3 + \cdots + b_{2^n} &= (n+1)\cdot 2^n - (a_1 + a_2 + a_3 + \cdots + a_{2^n})\\
&= (n+1)\cdot 2^n - S_n
\end{aligned}$$

The sum of the alternating sums of all subsets of S_{n+1} is the sum of the alternating sums of all subsets of S_n plus all of $b_1, b_2, b_3, \cdots, b_{2^n}$. Hence

$$\begin{aligned}
A_{n+1} &= S_n + b_1 + b_2 + b_3 + \cdots + b_{2^n}\\
&= S_n + (n+1)\cdot 2^n - S_n\\
&= (n+1)\cdot 2^n.
\end{aligned}$$

Thus we have found a closed-form expression for A_{n+1}. We can just plug in $n = 9$ to get $A_{10} = 10 \cdot 2^9$.

13. This can be done with just a scientific calculator. First we calculate the 5 raised to powers of 2 modulo 397. This is doable because we get the next power by squaring the previous residue and reducing modulo 397. Since our residues are guaranteed to be less than 397, our calculators do not blow up, as they would do if we tried computing 5^{261}.

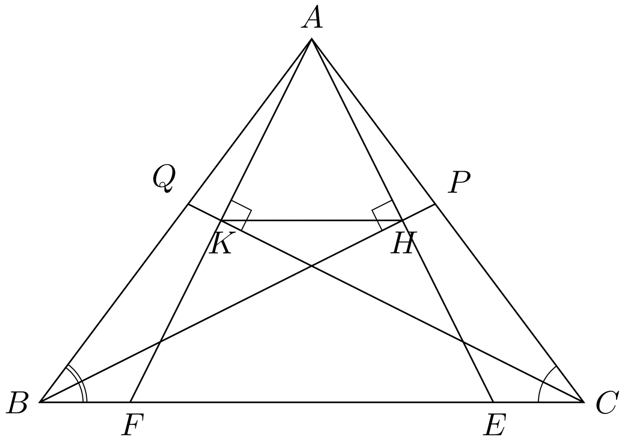
$$\begin{aligned}
5^1 &\equiv 5 \pmod{397}\\
5^2 &\equiv 25 \pmod{397}\\
5^4 &\equiv 228 \pmod{397}\\
5^8 &\equiv 374 \pmod{397}\\
5^{16} &\equiv 132 \pmod{397}\\
5^{32} &\equiv 353 \pmod{397}\\
5^{64} &\equiv 348 \pmod{397}\\
5^{128} &\equiv 19 \pmod{397}\\
5^{256} &\equiv 361 \pmod{397}
\end{aligned}$$

Since $261 = 256 + 4 + 1$, we get

$$\begin{aligned}
5^{261} &\equiv 5^{256} \cdot 5^4 \cdot 5^1\\
&\equiv 361(228)(5) \pmod{397}\\
&\equiv 248 \pmod{397}.
\end{aligned}$$

Here's an example of how to evaluate one of those powers of 5 modulo 397 on a simple calculator. Let's say we wanted to compute $5^{16} \pmod{397}$. We see that $5^8 \equiv 374 \pmod{397}$, so we evaluate $374^2/397$ on our calculator. We get something like $352.332\ldots$. Round down and take away that multiple of 397 from 374^2 to get $5^{16} \pmod{397}$. In other words, now we evaluate $374^2 - 352 \cdot 397$ to get 132; this number is $5^{16} \pmod{397}$. Think about this a few times to understand why it works.

14.



Since $\angle BHE$ is also a right angle, $\angle HBE = \angle HBA$ by definition, and HB is a common side, $\triangle AHB$ is congruent to $\triangle BHE$ by ASA. This means $AH = HE$, as they are corresponding sides of the two triangles. Similarly, $\triangle AKC$ is congruent to $\triangle KFC$, again by ASA. This implies $AK = KF$.

Since $AK = KF$ and $AH = HE$, $\triangle AFE$ is similar to $\triangle AKH$ by SAS; from this we conclude that $\angle AKH = \angle AFE$ and $\angle AHK = \angle AEF$. Thus HK is parallel to FE, and because FE lies on BC, HK is also parallel to BC, as required.

15. Firstly, since $F_0 = 0$ we can get rid of the the F_0 term. Our power series becomes

$$G(x) = F_1 x + F_2 x^2 + F_3 x^3 + \cdots.$$

Replace each coefficient, except for the first, with its recursive definition:

$$G(x) = F_1 x + (F_0 + F_1)x^2 + (F_1 + F_2)x^3 + (F_2 + F_3)x^4 + \cdots.$$

Now rearrange and group the first terms of each set of parentheses together. Do likewise with the second term of each set of parentheses.

$$G(x) = F_1 x + (F_0 x^2 + F_1 x^3 + F_2 x^4 + \cdots) + (F_1 x^2 + F_2 x^3 + F_3 x^4 + \cdots).$$

Notice that by factoring out each set of parentheses we get $G(x)$ again.

$$\begin{aligned} G(x) &= F_1 x + (F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \cdots)x^2 + (F_1 x + F_2 x^2 + F_3 x^3 + \cdots)x \\ &= x + x^2 G(x) + xG(x) \end{aligned}$$

Rearranging yields

$$G(x) = \frac{x}{1 - x - x^2}.$$