Expected Value MEG

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1 Introduction

1. I have 12 addressed letters to mail, and 12 corresponding pre-addressed envelopes. For some wacky reason, I decide to put the letters into envelopes at random, one letter per envelope. What is the expected number of letters that get placed into their proper envelopes?

2. I flip a coin 10 times. What is the expected number of pairs of consecutive tosses that comes up heads? (For example, the sequence THHTHHHTHH has 4 pairs of consecutive HH's).

3. (HMMT 2006) At a nursery, 2006 babies sit in a circle. Suddenly, each baby randomly pokes either the baby to its left or to its right. What is the expected value of the number of unpoked babies?

4. A 10 digit binary number with four 1's is chosen at random. What is its expected value?

5. (IMO 1987/1) Let $p_n(k)$ be the number of permutations of the set $\{1, 2 \ldots, n\}$, $n \geq 1$, which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k \cdot p_n(k) = n!.$$

(A *fixed point* of a permutation in this case is an element i such that i is in the i^{th} position of the permutation. For example, the permutation $(3, 2, 5, 4, 1)$ of $\{1, 2, 3, 4, 5\}$ has two fixed points: the 2 in the 2^{nd} spot and the 4 in the 4^{th} spot)

2 Solutions

1. Each letter has a $\frac{1}{12}$ chance of being delivered to the right envelope. So since there are 12 letters, the expected number of letters that are placed to the proper envelope is $12 \times \frac{1}{12} = 1$.

2. Define $X_i = 1$ if coin flip i and $i+1$ are heads, and $X_i = 0$ otherwise. Then we seek $E(X_1 + X_2 + \ldots + X_9)$ Note that the chance that any pair of consecutive coin flips is $(\frac{1}{2})^2 = \frac{1}{4}$, so $E(X_i) = \frac{1}{4}$ and

$$E(X_1) + E(X_2) + \ldots + E(X_9) = 9 \times \frac{1}{4} = \frac{9}{4}.$$

3. Number the babies $1, 2, \ldots, 2006$. Define $X_i = 1$ if baby i is poked and $X_i = 0$ otherwise. Then we seek $E(X_1 + X_2 + \ldots X_{2006})$. Any baby has $\frac{1}{4}$ chance of being unpoked (if both its neighbours miss). Hence $E(X_i) = \frac{1}{4}$ for each i and

$$E(X_1 + X_2 + \ldots + X_{2006}) = E(X_1) + E(X_2) + \ldots E(X_{2006}) = 2006 \times \frac{1}{4} = \frac{1003}{2}.$$

4. Clearly the first digit has to be 1. Then each other digit has a $\frac{3}{9} = \frac{1}{3}$ chance of being 1. So the expected value is $2^9 + \frac{1}{3}(2^8 + 2^7 + \ldots + 1) = 2^9 + \frac{1}{3}(2^9 - 1) = \frac{2047}{3}$.

5. We expect there to be $\frac{1}{n} \times n = 1$ fixed point on average. But also, a permutation with k fixed points occurs with chance of $\frac{p_n(k)}{n!}$ and such a permutation has k fixed points, so the expected number of fixed points is

$$\sum_{k=0}^{n} \frac{k \cdot p_n(k)}{n!}.$$

Which is also equal to 1. Thus

$$\sum_{k=0}^{n} \frac{k \cdot p_n(k)}{n!} = 1.$$

Multiplying both sides by $n!$ gives our desired result.