Mixed School Problem Solving Questions

1. Define the "reverse" of a number to be the same digits written backwards. For example, the reverse of 135 is 531, and the reverse of 4630 is 0364.

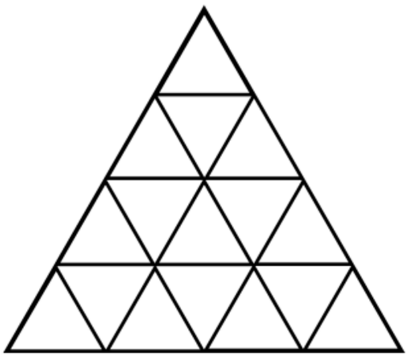
   Prove that the sum of any four-digit number and its reverse is divisible by 11.

   Hint: Represent the 4-digit number as $ABCD$.

2. (a) Suppose you have an analogue clock with an hour hand and a minute hand. If it is currently midnight, at what time will the two hands form a right angle? Give your answer to the nearest second, in the form HH:MM:SS.

   (b) How many times will the hour and minute hand form a right-angle over the course of 24 hours?

3. How many triangles are there in the image below?



4. (a) Find the smallest integer x such that $480x$ is a perfect square.

   (b) What is the general strategy to find the smallest integer x such that $N \times x$ is a perfect square for positive integer N?

5. (a) Prove that there is no smallest positive number.

   (b) Explain why there is no biggest number.

6. (a) Must a quadratic pass through the x axis? Why or why not?

   (b) Must a cubic pass through the x axis? Why or why not?

7. The scales at Jasper's house only work if two people weigh themselves at the same time. He has four people in his house who weigh themselves in pairs obtaining 89kg, 105kg, 112kg, 113kg, 120kg and 136kg. What is the difference between the weights of the heaviest and lightest people in the house?

Year 9

1. (2 points) Three consecutive numbers sum to 141. What are the numbers?

Total for Question 1: 2

2. (3 points) There are twelve lockers, numbered from 110 to 121. The keys to these twelve lockers are numbered 1 to 12. Each locker number is divisible by the number on its key.

   Determine the key number for each locker.

Total for Question 2: 3

3. In an AFL game, a goal is worth 6 points and a behind is worth 1 point.
   (a) (1 point) How many points does a team have if they score 4 goals and 8 behinds?
   (b) (2 points) How many different ways are there to score some number of goals and some number of behinds such that the score is the product of the number of goals and the number of behinds?
   (c) (2 points) By considering a graph, show that you have found all the ways that this could occur.

Total for Question 3: 5

4. (a) (2 points) What is the area of the largest square that can fit inside a unit circle?
   (b) (2 points) What is the area of the largest circle that can fit inside a unit square?
   (c) (3 points) A triangle is placed inside a unit square so that of its vertices either lie on the square's edges or inside the square. What areas are possible for this triangle?
   (d) (2 points) What is the side length of the largest square that can fit inside an equilateral triangle that has side length 1 unit?

Total for Question 4: 9

5. (3 points) What is the volume of a regular tetrahedron with side length 1 unit?

Total for Question 5: 3

6. (a) (1 point) How many positive integer solutions are there to the equation $2x + 3y = 25$?
   (b) (2 points) How many integer solutions are there to the equation $2x + 3y = 25$?
   (c) (2 points) How many integer solutions are there to the equation $51x+24y = 17$?
   (d) (1 point) Consider the expression $9x + 15y$. Put in as many integer values of x and y as you can. Which number are all the resulting numbers multiples of?
   (e) (2 points) When does the equation $ax + by = c$ always have integer solutions?

Total for Question 6: 8

7. (a) (2 points) When is the sum of 2 consecutive numbers divisible by 2?
   (b) (2 points) When is the sum of 3 consecutive numbers divisible by 3?

(c) (2 points) When is the sum of 4 consecutive numbers divisible by 4?

(d) (2 points) When is the sum of 5 consecutive numbers divisible by 5?

(e) (2 points) When is the sum of n consecutive numbers divisible by n?

Total for Question 7: 10

8. (a) (1 point) Two lines can divide the plane into at most how many regions?

   (b) (1 point) Three lines can divide the plane into at most how many regions?

   (c) (1 point) If we add a fourth line, what is the maximum number of intersection points that line can make with the existing lines?

   (d) (1 point) Four lines can divide the plane into at most how many regions?

   (e) (3 points) What is the maximum number of regions that n lines can divide the plane into?

Total for Question 8: 7

9. (3 points) Prove it is possible to pair up the numbers $0, 1, 2, 3, \ldots, 61$ in such a way that when we sum each pair, the product of the 31 numbers we get is a perfect fifth power.

Total for Question 9: 3

Total: 50

Year 10

1. (3 points) There are twelve lockers, numbered from 110 to 121. The keys to these twelve lockers are numbered 1 to 12. Each locker number is divisible by the number on its key.

   Determine the key number for each locker. Total for Question 1: 3

2. (2 points) Three consecutive numbers sum to 141. What are the numbers?

   Total for Question 2: 2

3. (a) (2 points) When is the sum of 2 consecutive numbers divisible by 2?
   (b) (2 points) When is the sum of 3 consecutive numbers divisible by 3?
   (c) (2 points) When is the sum of 4 consecutive numbers divisible by 4?
   (d) (2 points) When is the sum of 5 consecutive numbers divisible by 5?
   (e) (2 points) When is the sum of n consecutive numbers divisible by n?

   Total for Question 3: 10

4. (3 points) Prove it is possible to pair up the numbers $0, 1, 2, 3, \ldots, 61$ in such a way that when we sum each pair, the product of the 31 numbers we get is a perfect fifth power. Total for Question 4: 3

5. (a) (1 point) Two lines can divide the plane into at most how many regions?
   (b) (1 point) Three lines can divide the plane into at most how many regions?
   (c) (1 point) If we add a fourth line, what is the maximum number of intersection points that line can make with the existing lines?
   (d) (1 point) Four lines can divide the plane into at most how many regions?
   (e) (3 points) What is the maximum number of regions that n lines can divide the plane into?

   Total for Question 5: 7

6. (a) (2 points) Is the sum or difference of two rational numbers always rational? Why or why not?
   (b) (2 points) Is the product or quotient of two rational numbers always rational? Why or why not?
   (c) (2 points) When one rational number is raised to another is the result guaranteed to be rational?
   (d) (3 points) It is known that $\sqrt{2}$ is irrational, but it is not known whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. Do there exist two irrational numbers such that when one is raised to the other the result is rational? Prove your conjecture.

   Total for Question 6: 9

7. (a) (2 points) Prove that $x^2 + y^2 \geq 2xy$ for real numbers x, y.
   (b) (2 points) Prove that $2a^2 + b^2 + c^2 \geq 2(ab + ac)$ for real numbers a, b, c.

(c) (2 points) Prove that $3(a^2 + b^2 + c^2 + d^2) \geq 2(ab + ac + ad + bc + bd + cd)$ for real numbers a, b, c, d.

(d) (3 points) Real numbers a, b, c, d, e are linked by the two equations:

$$e = 40 - a - b - c - d$$

$$e^2 = 400 - a^2 - b^2 - c^2 - d^2$$

Determine the largest value for e.

Total for Question 7: 9

8. (a) (1 point) How many ways are there to create a two-element subset from the set $\{1, 2, 3, 4\}$?

(b) (2 points) How many ways are there to split the set $\{1, 2, 3, 4\}$ into two disjoint nonempty sets? (Disjoint means the two sets share no elements.) For example, if we had the set $\{1, 2, 3\}$ a valid splitting would be $\{1, 2\}$ and $\{3\}$.

(c) (2 points) How many ways are there to split the set $\{1, 2, 3, 4, 5\}$ into two disjoint nonempty sets?

(d) (2 points) Denote by $S(n, k)$ the number of ways to split a set of n elements into k disjoint nonempty sets. Suppose we know what $S(n-1, 2)$ is. What happens when we add another element to the set? What happens to each of the $S(n-1, 2)$ ways to do the partitioning?

(e) (2 points) What is $S(n, 2)$?

(f) (3 points (bonus)) If there are $S(n-1, k-1)$ ways to partition a set of $n-1$ elements into $k-1$ nonempty disjoint sets, what is $S(n, k)$ in terms of $S(n-1, k)$ and $S(n-1, k-1)$?

Total for Question 8: 9

Total: 52

Year 11/12

1. (3 points) There are twelve lockers, numbered from 110 to 121. The keys to these twelve lockers are numbered 1 to 12. Each locker number is divisible by the number on its key.

   Determine the key number for each locker.

Total for Question 1: 3

2. (a) (1 point) Two lines can divide the plane into at most how many regions?
   (b) (1 point) Three lines can divide the plane into at most how many regions?
   (c) (1 point) If we add a fourth line, what is the maximum number of intersection points that line can make with the existing lines?
   (d) (1 point) Four lines can divide the plane into at most how many regions?
   (e) (3 points) What is the maximum number of regions that n lines can divide the plane into?

Total for Question 2: 7

3. (a) (1 point) How many ways are there to create a two-element subset from the set $\{1, 2, 3, 4\}$?
   (b) (2 points) How many ways are there to split the set $\{1, 2, 3, 4\}$ into two disjoint nonempty sets? (Disjoint means the two sets share no elements.) For example, if we had the set $\{1, 2, 3\}$ a valid splitting would be $\{1, 2\}$ and $\{3\}$.
   (c) (2 points) How many ways are there to split the set $\{1, 2, 3, 4, 5\}$ into two disjoint nonempty sets?
   (d) (2 points) Denote by $S(n, k)$ the number of ways to split a set of n elements into k disjoint nonempty sets. Suppose we know what $S(n - 1, 2)$ is. What happens when we add another element to the set? What happens to each of the $S(n - 1, 2)$ ways to do the partitioning?
   (e) (2 points) What is $S(n, 2)$?
   (f) (3 points (bonus)) If there are $S(n - 1, k - 1)$ ways to partition a set of $n - 1$ elements into $k-1$ nonempty disjoint sets, what is $S(n, k)$ in terms of $S(n-1, k)$ and $S(n - 1, k - 1)$?

Total for Question 3: 9

4. (a) (2 points) Is the sum or difference of two rational numbers always rational? Why or why not?
   (b) (2 points) Is the product or quotient of two rational numbers always rational? Why or why not?
   (c) (2 points) When one rational number is raised to another is the result guaranteed to be rational?
   (d) (3 points) It is known that $\sqrt{2}$ is irrational, but it is not known whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. Do there exist two irrational numbers such that when one is raised to the other the result is rational? Prove your conjecture.

Total for Question 4: 9

5. (a) (1 point) If two normal six-sided dice are rolled, what is the probability that the sum of the two numbers is 2?
   (b) (1 point) If two normal six-sided dice are rolled, what is the probability that the sum of the two numbers is 7?
   (c) (2 points) Consider a biased six-sided die in which the probability of rolling n is p_n. This die is rigged so that when two of them are rolled, every possible sum of the two numbers is equally likely. What is p_2 in terms of p_1?
   (d) (1 point) What is p_3 in terms of p_1?
   (e) (2 points) What is p_4 in terms of p_1? Do you notice a pattern?
   (f) (2 points) Let $p_i = a_{i-1}p_1$. So $a_0 = 1$, $p_2 = a_1p_1$, $p_3 = a_2p_1$, $p_4 = a_3p_1$ and so on. What are $a_0a_1 + a_1a_0$, $a_0a_2 + a_1a_1 + a_2a_0$, $a_0a_3 + a_1a_2 + a_2a_1 + a_3a_0$, etc. all equal to?
   (g) (1 point) Since we have a six-sided die,

$$p_1 + p_2 + \cdots + p_6 = 1.$$

   What is p_1 in terms of $a_0, a_1, a_2, \ldots, a_5$?
   (h) (2 points) Let $s_i = a_0 + a_1 + \cdots + a_i$. What are s_0, s_1, s_2? What is s_5?
   (i) (3 points) Is such a six-sided die possible?

Total for Question 5: 15

6. (a) (1 point) What is $1 + 2 + 3 + \cdots + 200$?
   (b) (2 points) Find a formula for $1 + 2 + \cdots + n$ and prove it.
   (c) (1 point) Define

$$F_k(n) = 1^k + 2^k + \cdots + n^k.$$

   (So in the previous part you found a formula for $F_1(n)$.) More compactly it may be written as

$$F_k(n) = \sum_{i=1}^{n} i^k.$$

   What is

$$1^3 + (2^3 - 1^3) + (3^3 - 2^3) + \cdots + (100^3 - 99^3)?$$

   (d) (2 points) By considering

$$1^3 + \sum_{i=1}^{n}((i+1)^3 - i^3)$$

   find a formula for $F_2(n)$.
   (e) (2 points) Find a formula for $F_3(n)$.
   (f) (1 point) The Binomial Theorem states that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

If

$$(i+1)^k - i^k = \sum_{j=0}^{x} \binom{k}{j} i^j$$

what is x in terms of k?

(g) (3 points) By considering the general telescoping sum

$$1^k + \sum_{i=1}^{n}((i+1)^k - i^k) = (n+1)^k$$

show that

$$F_{k-1}(n) = \frac{(n+1)^k - 1}{k} + \frac{1}{k}\sum_{i=0}^{k-2}\binom{k}{i}F_i(n).$$

Total for Question 6: 12

Total: 55