Year 11/12

1. (3 points) There are twelve lockers, numbered from 110 to 121. The keys to these twelve lockers are numbered 1 to 12. Each locker number is divisible by the number on its key.

   Determine the key number for each locker.

Total for Question 1: 3

2. (a) (1 point) Two lines can divide the plane into at most how many regions?
   (b) (1 point) Three lines can divide the plane into at most how many regions?
   (c) (1 point) If we add a fourth line, what is the maximum number of intersection points that line can make with the existing lines?
   (d) (1 point) Four lines can divide the plane into at most how many regions?
   (e) (3 points) What is the maximum number of regions that n lines can divide the plane into?

Total for Question 2: 7

3. (a) (1 point) How many ways are there to create a two-element subset from the set $\{1, 2, 3, 4\}$?
   (b) (2 points) How many ways are there to split the set $\{1, 2, 3, 4\}$ into two disjoint nonempty sets? (Disjoint means the two sets share no elements.) For example, if we had the set $\{1, 2, 3\}$ a valid splitting would be $\{1, 2\}$ and $\{3\}$.
   (c) (2 points) How many ways are there to split the set $\{1, 2, 3, 4, 5\}$ into two disjoint nonempty sets?
   (d) (2 points) Denote by $S(n, k)$ the number of ways to split a set of n elements into k disjoint nonempty sets. Suppose we know what $S(n-1, 2)$ is. What happens when we add another element to the set? What happens to each of the $S(n-1, 2)$ ways to do the partitioning?
   (e) (2 points) What is $S(n, 2)$?
   (f) (3 points (bonus)) If there are $S(n-1, k-1)$ ways to partition a set of $n-1$ elements into $k-1$ nonempty disjoint sets, what is $S(n, k)$ in terms of $S(n-1, k)$ and $S(n-1, k-1)$?

Total for Question 3: 9

4. (a) (2 points) Is the sum or difference of two rational numbers always rational? Why or why not?
   (b) (2 points) Is the product or quotient of two rational numbers always rational? Why or why not?
   (c) (2 points) When one rational number is raised to another is the result guaranteed to be rational?
   (d) (3 points) It is known that $\sqrt{2}$ is irrational, but it is not known whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. Do there exist two irrational numbers such that when one is raised to the other the result is rational? Prove your conjecture.

Total for Question 4: 9

5. (a) (1 point) If two normal six-sided dice are rolled, what is the probability that the sum of the two numbers is 2?

   (b) (1 point) If two normal six-sided dice are rolled, what is the probability that the sum of the two numbers is 7?

   (c) (2 points) Consider a biased six-sided die in which the probability of rolling n is p_n. This die is rigged so that when two of them are rolled, every possible sum of the two numbers is equally likely. What is p_2 in terms of p_1?

   (d) (1 point) What is p_3 in terms of p_1?

   (e) (2 points) What is p_4 in terms of p_1? Do you notice a pattern?

   (f) (2 points) Let $p_i = a_{i-1}p_1$. So $a_0 = 1$, $p_2 = a_1p_1$, $p_3 = a_2p_1$, $p_4 = a_3p_1$ and so on. What are $a_0a_1 + a_1a_0$, $a_0a_2 + a_1a_1 + a_2a_0$, $a_0a_3 + a_1a_2 + a_2a_1 + a_3a_0$, etc. all equal to?

   (g) (1 point) Since we have a six-sided die,

   $$p_1 + p_2 + \cdots + p_6 = 1.$$

   What is p_1 in terms of $a_0, a_1, a_2, \ldots, a_5$?

   (h) (2 points) Let $s_i = a_0 + a_1 + \cdots + a_i$. What are s_0, s_1, s_2? What is s_5?

   (i) (3 points) Is such a six-sided die possible?

Total for Question 5: 15

6. (a) (1 point) What is $1 + 2 + 3 + \cdots + 200$?

   (b) (2 points) Find a formula for $1 + 2 + \cdots + n$ and prove it.

   (c) (1 point) Define

   $$F_k(n) = 1^k + 2^k + \cdots + n^k.$$

   (So in the previous part you found a formula for $F_1(n)$.) More compactly it may be written as

   $$F_k(n) = \sum_{i=1}^{n} i^k.$$

   What is

   $$1^3 + (2^3 - 1^3) + (3^3 - 2^3) + \cdots + (100^3 - 99^3)?$$

   (d) (2 points) By considering

   $$1^3 + \sum_{i=1}^{n}((i+1)^3 - i^3)$$

   find a formula for $F_2(n)$.

   (e) (2 points) Find a formula for $F_3(n)$.

   (f) (1 point) The Binomial Theorem states that

   $$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

If

$$(i+1)^k - i^k = \sum_{j=0}^{x} \binom{k}{j} i^j$$

what is x in terms of k?

(g) (3 points) By considering the general telescoping sum

$$1^k + \sum_{i=1}^{n}((i+1)^k - i^k) = (n+1)^k$$

show that

$$F_{k-1}(n) = \frac{(n+1)^k - 1}{k} + \frac{1}{k}\sum_{i=0}^{k-2}\binom{k}{i} F_i(n).$$

Total for Question 6: 12

Total: 55