Holiday Problem Solutions-ish

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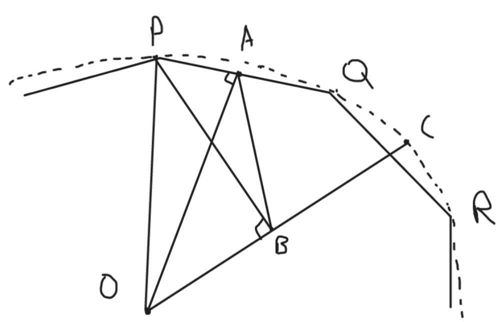
April 2023

Solutions

3. Since $2^{2^n} + 5 > 3$ and is also 0 mod 3, it is never prime.

4. Since $\angle POQ = \angle QOR = \frac{360^\circ}{9} = 40^\circ$, and C is halfway in between on the circumcircle, so $\angle POC = 60^\circ$ and hence $\triangle OPC$ is equilateral.

Quadrilateral PABO is cyclic since $\angle PAO = \angle PBO = 90^\circ$. Then $\angle OPB$ and $\angle OAB$ are subtended by the same arc so $\angle OAB = \angle OPB = \frac{60^\circ}{2} = 30^\circ$.



5. Taking $\log_a$ on both sides gives us $a^x \log_a x = x^a$. Since $f(x) = a^x$ and $g(x) = \log_a x$ are decreasing and $h(x) = x^a$ is increasing for $0 < a < 1$, they have one unique solution.

It follows that $x = a$ is the only solution.

8. Since arithmetic sequences are of the form $a, a+d, \ldots, a+9d$, we can find the number of 10 element arithmetic sequences by counting the pairs (a, d) such that $a + 9d \leq 2007$. This is equivalent to

$$d \leq \frac{2007 - a}{9}$$

If $1 \leq a \leq 9$, then $d \leq 222$, if $10 \leq a \leq 18$, then $d \leq 221$. Similarly every time a increases by 9, d has one less possible value since we are the expression $2007 - a$ is being divided by 9.

Continuing until $1990 \leq a \leq 1998$, then $d \leq 1$, we have $9(222 + 221 + \ldots + 1) = 9(\frac{222(1+222)}{2}) = 222777$ total 10 element arithmetic sequences.

Now the probability of a subset of 10 random elements in a random colouring being one colour is $1 \times (\frac{1}{4})^9 = \frac{1}{262144}$. Meaning the expected number of monochromatic arithmetic sequences of length 10 is $\frac{222777}{262144}$. Since the expected number of monochromatic arithmetic sequences of length 10 is less than 1, there must be at least 1 sequence that is not all one colour.

9. Extend the segment AK to the circumcircle of BKLC and call the intersection K', define L' similarily.

Since $\angle AL'B = \angle ACB = \angle ALD$ and $\angle AK'B = \angle BCK = \angle AKD$, we have $\triangle ALD \sim \triangle AL'B(AA)$ and $\triangle ADK \sim ABK'(AA)$.

Now considering the ratio $AK : AL$, we have

$$(AK : AD) : (AL : AD)$$

$$= (KK' : DB) : (LL' : DB)$$

$$= KK' : LL'$$

Since $KLL'L$ is a trapzium (KL is parallel to $K'L'$ by similar triangles), and is cyclic, it is isosceles and hence $KK' = LL'$. So $AK : Al = KK' : LL' = 1$ and thus $AK = AL$.