Melbourne High School
Maths Extension Group 2023
Term 1 Week 8 Handout solutions

1. The octagon is made of 8 congruent triangles each with two side lengths of 1 unit and angle 45° between them. Therefore the area of the octagon is $8 \times (1/2)(\sqrt{2}/2) = 2\sqrt{2}$. Similarly, the square is made up of four triangles, each with two side lengths of 1 unit. Therefore its area is $4 \times (1/2) = 2$. The ratio between the area of the square and the octagon is $1/\sqrt{2}$.

2. The circumference of the circle is 9π, so the angle subtended at the centre by arc BC is 120°. This is double the angle subtended by the same arc at the circumference, so $\angle BAC = 60^\circ$ and $\angle ABC = 30^\circ$. Use the sine ratio to find that $AC = 9/2$, and then use the formula for the area of a triangle to find that the area is $(81\sqrt{3})/8$.

3. The idea is to express π as a continued fraction. Begin by considering the fractional part of π, which is approximately 0.14159. This is approximately equal to 1/7, so $\pi \approx 3 + (1/7) = 22/7$. This does not give the required number of decimal places yet so we repeat the process; $0.14159... \approx 7 + 0.625133...$ and $0.625133^{-1} \approx 16$. So

$$\pi \approx 3 + \frac{1}{7 + \frac{1}{16}} \approx \frac{355}{113}.$$

   This fraction does approximate π to six decimal places.

4. (a) We see that

$$1 + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{4} + \cdots < 1 + \frac{1}{1} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{4} + \cdots.$$

   The RHS can be expressed as a telescoping that gets closer and closer to 2. Therefore the LHS, the sum we're interested in, converges to a finite value less than 2.

   (b) The solution is given in the question.

   (c) Plugging in $x = 0$ to the RHS gives us

$$1 = a(-\pi^2)(-4\pi^2)(-9\pi^2)(-16\pi^2)\cdots.$$

   Rearranging for a, we get

$$a = \frac{1}{(-\pi^2)(-4\pi^2)(-9\pi^2)(-16\pi^2)\cdots}.$$

   Putting this back into the original equation and distributing each factor in the denominator to the corresponding bracket we get

$$\begin{aligned} \frac{\sin x}{x} &= \frac{1}{-\pi^2}(x^2 - \pi^2)\frac{1}{-4\pi^2}(x^2 - 4\pi^2)\frac{1}{-9\pi^2}(x^2 - 9\pi^2)\cdots \\ &= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots. \end{aligned}$$

   (d) The coefficient of x^2 is

$$-\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right).$$

   (e) The coefficient of the x^2 term in the original expansion was $-1/6$, so we have

$$\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots = \frac{1}{6}.$$

   Multiplying both sides by π^2 yields the desired result.

5. (a) Note that

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots$$

and so

$$\zeta(s) - \frac{1}{2^s}\zeta(s) = \left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \cdots.$$

Now apply the same process again:

$$\frac{1}{3^s}\left(1 - \frac{1}{2^s}\right) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \cdots$$

and therefore

$$\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \cdots.$$

If we repeat the process over all primes p, we are left with only 1 on the RHS, and therefore

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{7^s}\right)\cdots\zeta(s) = 1.$$

(b) Rearranging for $\zeta(s)$, we have

$$\zeta(s) = \frac{1}{\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{7^s}\right)\cdots}.$$

Plugging in $s = 2$ gives us the desired result.

(c) Two numbers are coprime if they share no common prime factors. The probability that two randomly chosen are not both divisible by a prime p is $1 - 1/p^2$. Multiplying the product over all primes p gives us the a probability of

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{7^s}\right)\cdots.$$

This is just the reciprocal of $\zeta(s)$, so the required probability is $6/\pi^2$.

(d) This is the same as before except that $s = 4$, so the probability is the reciprocal of $\zeta(4) = \pi^4/90$. Therefore the probability is $90/\pi^4$.