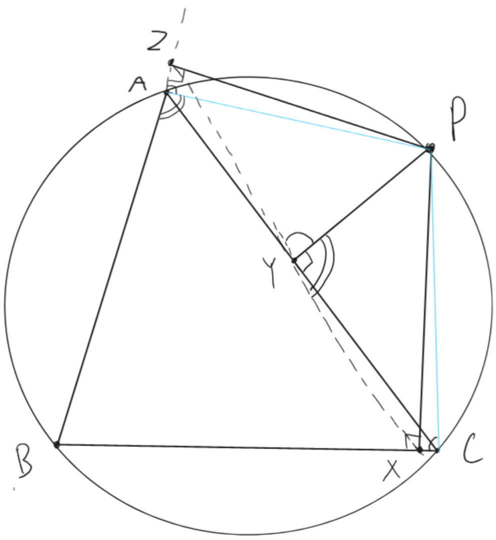
Melbourne High School
Maths Extension Group 2023
Term 3 Holiday Handout Solutions

Problems

1. We have $x = b^2+b+1$ and $x = 2(b-2)^2+(b-2)+b = 2b^2-7b+8$. Hence $(2b^2-7b+8)-(b^2+b+1) = b^2-8b+7 = (b-7)(b-1) = 0$. Clearly $b \neq 1$, so $b = 7$ and thus $x = 49+7+1 = 57$.

2. Considering mod 3, WLOG let $x_1 = 3$, and thus we are left with $x_2x_3x_4+x_3x_4x_5 = x_3x_4(x_2+x_5)$. Clearly $3 \nmid x_3x_4$ so $3|x_2+x_5$, so (x_3, x_4) can be $(1,2), (1,5), (4,2), (4,5)$ and $(2,1), (5,1), (2,4), (5,4)$ thus giving 8 ways, and x_3x_4 can be permuted in 2 ways after designating x_3 and x_4. So there are $5 \times 8 \times 2 = 80$ ways.

3. Let $\angle PYZ = \theta$. Since $\angle PZA + \angle PYA = 180^\circ$, $PZAY$ is cyclic, so $\angle PYZ = \angle PAZ = \theta$ (subtended by same arc). Thus $\angle PAB = 180^\circ - \theta$, and so $PCB = \theta$. Since $\angle PYX = \angle PXC = 90^\circ$, $PYXC$ is cyclic. Thus $PYX = 180^\circ - \theta$. Since $\angle PYZ + \angle PYX = 180^\circ$, we are done.



4. Let t be real number that is not ± 1, and $t = \frac{x-3}{x+1}$ thus $x = \frac{3+t}{1-t}$. Rewriting the given equation in terms of t we have

$$f(t) + f(\frac{t-3}{t+1}) = \frac{3+t}{1-t}.$$

Similarly, let $t = \frac{3+x}{1-x}$, so then we have $x = \frac{t-3}{t+1}$ and $\frac{x-3}{x+1} = \frac{3+t}{1-t}$. Rewriting the given equation in terms of t again, we have

$$f(\frac{3+t}{1-t}) + f(t) = \frac{t-3}{t+1}.$$

Adding the two rewritten equations we get

$$2f(t) + f(\frac{t-3}{t+1}) + f(\frac{3+t}{1-t}) = \frac{3+t}{1-t} + \frac{t-3}{t+1}$$

And since $f(\frac{t-3}{t+1}) + f(\frac{3+t}{1-t}) = t$, it implies

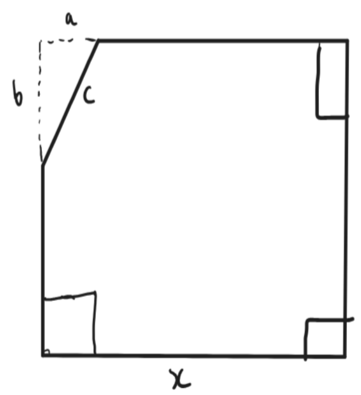
$$2f(t) + t = \frac{8t}{1-t^2}.$$

Thus the function is $f(t) = \frac{4t}{1-t^2} - \frac{t}{2}$. It is easy to check that this satisfies the given equation.

5. A convex pentagon with side integer side lengths and an odd perimeter can have two right angles as a unit equilateral triangle on top of a unit square suffices.

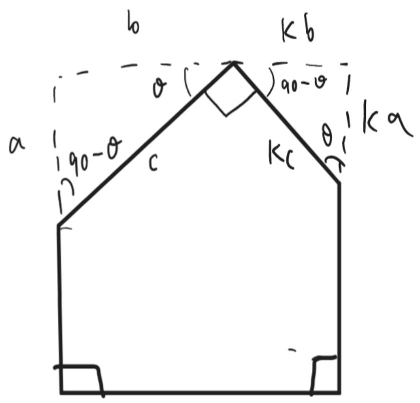
   Clearly a pentagon can't have 5 right angles. If a pentagon has 4 right angles, then the remaining angle is $540 - 4 \times 90 = 180°$ which is a contradiction. Hence we consider when a pentagon has 3 right angles. There are two cases, when exactly two of the right angles are adjacent, and when all three of the right angles are adjacent.

   Case 1: Three of the right angles are adjacent.



   Then the perimeter is $2(x+y) + (c-a-b)$. Clearly $2(x+y)$ is even, and $c-a-b$ is even since $a^2+b^2 = c^2$ implies $a+b$ has the same parity as c.

   Case 2: Exactly two of the right angles are adjacent.



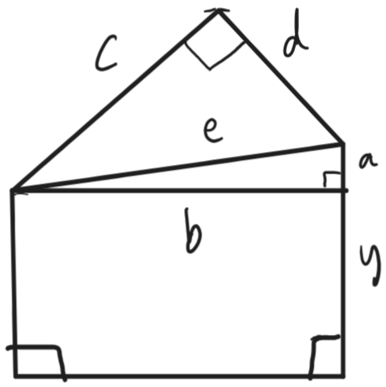
   Let x and y be the sides of the rectangle formed with the dotted lines. Then from similar triangles we deduce the perimeter is

$$2(x+y) - a - b - kb - ka + c + kc$$

$$= 2(x+y) + (c-a-b)(1+k)$$

   Which is even as $2(x+y)$ and $c-a-b$ are even.

   Alternatively, since $a^2+b^2 = e^2 = c^2+d^2$, $a+b$ has the same parity as $c+d$. Meaning $a+b+c+d$ is even. Since the perimeter is $a+b+c+d+2y$, the pentagon has an even perimeter.



Hence a pentagon with integer side lengths and odd perimeter can have at most two right angles.

6. The key observation is that $11 \times 29 = 319$. This means that exactly 319 ordered pairs, and hence 319 numbers of the form $11x + 29y$, will be formed. So all we need to prove that those numbers are incongruent to each modulo 319, as that would mean there is a bijection between those numbers and the numbers $1, 2, \dots, 319$.

   This problem is a special case of the general theorem that if a runs through a complete set of residues modulo m and a' runs through a complete set of residues modulo m', and m and m' are coprime, then the mm' numbers

$$am' + a'm$$

   form a complete set of residues modulo mm'. To prove it, suppose that

$$a_1 m' + a_1' m \equiv a_2 m' + a_2' m \pmod{mm'}. \tag{1}$$

   Clearly the congruence also holds relative to the modulus m, so the terms with m in them disappear and we are left with

$$a_1 m' \equiv a_2 m' \pmod{m}.$$

   Since m' and m are coprime we can divide both sides by m' and we have

$$a_1 \equiv a_2 \pmod{m}.$$

   Both a_1 and a_2 are running through the residues of m and therefore both are less than m; hence $a_1 = a_2$.

   The congruence in (1) also holds relative to the modulus m', so we have

$$a_1' m \equiv a_2' m \pmod{m'}.$$

   Again we can deduce that $a_1' \equiv a_2' \pmod{m'}$ from which it follows that $a_1' = a_2'$. Since $a_1 = a_2$ and $a_1' = a_2'$ two numbers of the form $a'm + am'$ that are congruent are actually the same number, and this proves the result.

   The problem is a case of the theorem where $m = 11$ and $m' = 29$; clearly they are coprime so the theorem proved above can be applied.