Year 11/12

1. (3 points) There are twelve lockers, numbered from 110 to 121. The keys to these twelve lockers are numbered 1 to 12. Each locker number is divisible by the number on its key.

   Determine the key number for each locker.

   Solution:

110	111	112	113	114	115	116	117	118	119	120	121
10	3	8	1	6	5	4	9	2	7	12	11

2. (a) (1 point) Two lines can divide the plane into at most how many regions?

   Solution: 4

   (b) (1 point) Three lines can divide the plane into at most how many regions?

   Solution: 7 (If you try it on paper, you will see that the third line can only intersect the existing 2 lines in at most 2 places, resulting in 3 new regions. Therefore the number of regions is $4 + 3 = 7$.)

   (c) (1 point) If we add a fourth line, what is the maximum number of intersection points that line can make with the existing lines?

   Solution: 3

   (d) (1 point) Four lines can divide the plane into at most how many regions?

   Solution: 11

   (e) (3 points) What is the maximum number of regions that n lines can divide the plane into?

   Solution: $1 + n(n + 1)/2$

3. (a) (1 point) How many ways are there to create a two-element subset from the set $\{1, 2, 3, 4\}$?

   Solution: There are $\binom{4}{2} = 6$ ways.

   (b) (2 points) How many ways are there to split the set $\{1, 2, 3, 4\}$ into two disjoint nonempty sets? (Disjoint means the two sets share no elements.) For example, if we had the set $\{1, 2, 3\}$ a valid splitting would be $\{1, 2\}$ and $\{3\}$.

Solution: The valid ways are: $\{1\}$ and $[\{2,3,4\}$, $\{2\}$ and $\{1,3,4\}$, $\{3\}$ and $\{1,2,4\}$, $\{4\}$ and $\{1,2,3\}$, $\{1,2\}$ and $\{3,4\}$, $\{1,3\}$ and $\{2,4\}$, and $\{1,4\}$ and $\{2,3\}$. So 7 ways.

(c) (2 points) How many ways are there to split the set $\{1,2,3,4,5\}$ into two disjoint nonempty sets?

Solution: By listing all the ways again, we find that there are 15 ways.

(d) (2 points) Denote by $S(n,k)$ the number of ways to split a set of n elements into k disjoint nonempty sets. Suppose we know what $S(n-1,2)$ is. What happens when we add another element to the set? What happens to each of the $S(n-1,2)$ ways to do the partitioning?

Solution: For each way to do the partitioning, we may add the new number n to either set to obtain a new partitioning. There is also one new partition that is not obtained by adding n to existing one, and that is the splitting into $\{n\}$ and $\{1,2,\ldots,n-1\}$. Therefore we have

$$S(n,2) = 2S(n-1,2) + 1.$$

(e) (2 points) What is $S(n,2)$?

Solution: We have $S(1,2) = 0$, $S(2,2) = 1$, $S(3,2) = 3$, $S(4,2) = 7$, and $S(5,2) = 15$. From this pattern and the recursive formula found above it is easy to prove by induction that $S(n,2) = 2^{n-1} - 1$.

(f) (3 points (bonus)) If there are $S(n-1,k-1)$ ways to partition a set of $n-1$ elements into $k-1$ nonempty disjoint sets, what is $S(n,k)$ in terms of $S(n-1,k)$ and $S(n-1,k-1)$?

Solution: We need to think about what happens when we add the nth element. Clearly we may put this object into a separate set by itself and partition the other $n-1$ objects into $k-1$ nonempty disjoint subsets, because all up there would be k sets. There are $S(n-1,k-1)$ ways to do this. We may also take the $n-1$ objects and partition them into k sets and add the nth element to any one of them to get another valid partitioning. How many ways are there to do this? In each of the $S(n-1,k)$ ways to partition $n-1$ objects into k nonempty sets, there are k different subsets we can put the nth object into. Therefore in this case there are $kS(n-1,k)$ ways. In total then we have

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

4. (a) (2 points) Is the sum or difference of two rational numbers always rational? Why or why not?

   Solution: Yes, because $a/b + c/d = (ad + bc)/bd$ and both $ad + bc$ and bd are integers.

   (b) (2 points) Is the product or quotient of two rational numbers always rational? Why or why not?

   Solution: Yes, because $(a/b)(c/d) = ac/bd$ and both ac and bd are integers.

   (c) (2 points) When one rational number is raised to another is the result guaranteed to be rational?

   Solution: No; an example is $2^{1/2}$.

   (d) (3 points) It is known that $\sqrt{2}$ is irrational, but it is not known whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. Do there exist two irrational numbers such that when one is raised to the other the result is rational? Prove your conjecture.

   Solution: The answer is yes. Let $A = \sqrt{2}^{\sqrt{2}}$. We don't know if A is rational or irrational, but it doesn't matter. If A is rational, then it is an example of a rational number that is an irrational number raised to another irrational number. If A is irrational, then $A^{\sqrt{2}} = \sqrt{2}^2 = 2$ and this number is the desired example.

5. (a) (1 point) If two normal six-sided dice are rolled, what is the probability that the sum of the two numbers is 2?

   Solution: Out of the 36 possible pairs the only one that has a sum of 2 is $(1, 1)$, so the probability is $1/36$.

   (b) (1 point) If two normal six-sided dice are rolled, what is the probability that the sum of the two numbers is 7?

   Solution: There are 6 ways to get a 7: $(1, 6)$, $(2, 5$, $(3, 4)$, $(4, 3)$, $(5, 2)$, and $(6, 1)$. Therefore the probability is $1/6$.

   (c) (2 points) Consider a biased six-sided die in which the probability of rolling n is p_n. This die is rigged so that when two of them are rolled, every possible sum of the two numbers is equally likely. What is p_2 in terms of p_1?

   Solution: The probability of getting a 3 is $p_2p_1 + p_1p_2$, which by hypothesis is equal to the probability of getting a 2, or p_1^2. Therefore $2p_2 = p_1$, which implies that $p_2 = p_1/2$.

(d) (1 point) What is p_3 in terms of p_1?

> **Solution:** Following the previous procedure we find that $p_3 = 3p_1/8$.

(e) (2 points) What is p_4 in terms of p_1? Do you notice a pattern?

> **Solution:** $p_4 = 5p_1/16$. The pattern is $1/2$, $(1/2)(3/4)$, $(1/2)(3/4)(5/6)$, and so on.

(f) (2 points) Let $p_i = a_{i-1}p_1$. So $a_0 = 1$, $p_2 = a_1p_1$, $p_3 = a_2p_1$, $p_4 = a_3p_1$ and so on. What are $a_0a_1 + a_1a_0$, $a_0a_2 + a_1a_1 + a_2a_0$, $a_0a_3 + a_1a_2 + a_2a_1 + a_3a_0$, etc. all equal to?

> **Solution:** If $p_1p_2 + p_2p_1 = p_1^2$ then
>
> $$a_0p_1a_1p_1 + a_0p_1a_1p_1 = p_1^2$$
>
> which implies
>
> $$(a_0a_1 + a_1a_0)p_1^2 = p_1^2.$$
>
> Therefore $a_0a_1 + a_1a_0 = 1$. Similarly the other combinations of a's can be found to be 1 too.

(g) (1 point) Since we have a six-sided die,

$$p_1 + p_2 + \cdots + p_6 = 1.$$

What is p_1 in terms of $a_0, a_1, a_2, \ldots, a_5$?

> **Solution:** Since the die is six-sided, we have $p_1 + p_2 + \cdots + p_6 = 1$. We can express this in terms of p_1:
>
> $$a_0p_1 + a_1p_1 + a_2p_1 + \cdots + a_5p_1 = 1$$
>
> and hence
>
> $$p_1 = \frac{1}{a_0 + a_1 + \cdots + a_5}.$$

(h) (2 points) Let $s_i = a_0 + a_1 + \cdots + a_i$. What are s_0, s_1, s_2? What is s_5?

> **Solution:** $s_0 = a_0 = 1$, $s_1 = a_0 + a_1 = 3/2$, $s_2 = 1 + 1/2 + 3/8 = 15/8$. Note that $s_2 = (3/2)(5/4)$, $s_3 = (3/2)(5/4)(7/6)$, $s_4 = (3/2)(5/4)(7/6)(9/8)$ and so on. Therefore
>
> $$s_5 = \frac{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} = \frac{693}{256}.$$

(i) (3 points) Is such a six-sided die possible?

Solution: From the previous part we find that $p_1 = 256/693$. This means that the probability of rolling two of these dice and getting a sum of $2, 3, \ldots, 12$ must all be $p_1^2 = 65536/480249$. But this is impossible because there are 11 possible sums and if they are equally likely each must have a probability of $1/11$. Therefore such a die is not possible.

6. (a) (1 point) What is $1 + 2 + 3 + \cdots + 200$?

Solution: 10100

(b) (2 points) Find a formula for $1 + 2 + \cdots + n$ and prove it.

Solution: The formula is

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

It is easily proved by induction. When $n = 1$ we have $1 = 1(1+1)/2$ so the base case is done. Now assume that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

for some natural number k. Add $k + 1$ to both sides:

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

and the induction is complete.

(c) (1 point) Define

$$F_k(n) = 1^k + 2^k + \cdots + n^k.$$

(So in the previous part you found a formula for $F_1(n)$.) More compactly it may be written as

$$F_k(n) = \sum_{i=1}^{n} i^k.$$

What is

$$1^3 + (2^3 - 1^3) + (3^3 - 2^3) + \cdots + (100^3 - 99^3)?$$

Solution: This sum is an example of a telescoping sum. All the terms vanish except for the 100^3 term in the last set of parentheses, so the answer is 100^3.

(d) (2 points) By considering

$$1^3 + \sum_{i=1}^{n} ((i+1)^3 - i^3)$$

find a formula for $F_2(n)$.

Solution: We know that

$$1^3 + \sum_{i=1}^{n}((i+1)^3 - i^3) = (n+1)^3.$$

Expanding both sides using the Binomial Theorem we get

$$1^3 + \sum_{i=1}^{n}(3i^2 + 3i + 1) = n^3 + 3n^2 + 3n + 1.$$

After removing the 1 from both sides, on the LHS we can break up the sum and pull out the constant factors, like so:

$$3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 = n^3 + 3n^2 + 3n.$$

But note that $\sum_{i=1}^{n} i^2 = F_2(n)$ and $\sum_{i=1}^{n} i = F_1(n)$, so we have

$$3F_2(n) + 3F_1(n) + n = n^3 + 3n^2 + 3n.$$

Then we just need to rearrange for $F_2(n)$ after substituting in $F_1(n) = n(n+1)/2$ to find that

$$F_2(n) = \frac{2n^3 + 3n^2 + n}{6}.$$

(e) (2 points) Find a formula for $F_3(n)$.

Solution: Using the same method as above, we find that

$$F_3(n) = \frac{n^4 + 2n^3 + n^2}{4}.$$

(f) (1 point) The Binomial Theorem states that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

If

$$(i+1)^k - i^k = \sum_{j=0}^{x} \binom{k}{j} i^j$$

what is x in terms of k?

Solution: The leading term vanishes, so instead of going from 0 to k the loop only goes up to $k-1$. Therefore $x = k-1$.

(g) (3 points) By considering the general telescoping sum

$$1^k + \sum_{i=1}^{n} ((i+1)^k - i^k) = (n+1)^k$$

show that

$$F_{k-1}(n) = \frac{(n+1)^k - 1}{k} + \frac{1}{k} \sum_{i=0}^{k-2} \binom{k}{i} F_i(n).$$

Solution: Expand the $(n+1)^k$ term using the Binomial Theorem, applying the previous part:

$$(n+1)^i = 1^k + \sum_{i=1}^{k} \sum_{j=0}^{k-1} \binom{k}{j} i^j.$$

Now expand the inner sigma notation:

$$(n+1)^k = 1^k + \sum_{i=1}^{k} \left(\binom{k}{0} i^0 + \binom{k}{1} i^1 + \binom{k}{2} i^2 + \cdots + \binom{k}{k-1} i^{k-1} \right).$$

We may take out the binomial coefficients, as they are independent of the sum variable i, so we have

$$(n+1)^k = 1^k + \binom{i}{0} \sum_{i=1}^{k} i^0 + \binom{k}{1} \sum_{i=1}^{k} i^1 + \binom{k}{2} \sum_{i=1}^{k} i^2 + \cdots + \binom{k}{k-1} \sum_{i=1}^{k} i^{k-1}.$$

We're interested in $F_{k-1}(n)$, so we notice that the binomial coefficient $\binom{k}{k-1} = k$ and isolate that term. The other sums are $F_0(n)$, $F_1(n)$, $F_2(n)$ and so on. Therefore we get

$$(n+1)^k = 1^k + \binom{k}{0} F_0(n) + \binom{k}{1} F_1(n) + \binom{k}{2} F_2(n) + \cdots + \binom{k}{k-2} F_{k-2}(n) + kF_{k-1}(n)$$

or more compactly,

$$(n+1)^k = 1 + \sum_{i=0}^{k-2} \binom{k}{i} F_i(n) + kF_{k-1}(n).$$

Rearranging for $F_{k-1}(n)$ we get the desired result.

Total: 55