Solving Equations MEG

Tom Yan

August 2023

1 Introduction

1. Find all x such that $-4 < \frac{1}{x} < 3$.

2. Solve the following system of equations

$$xy = 12\sqrt{6}$$
$$yz = 54\sqrt{2}$$
$$zx = 48\sqrt{3}.$$

3. Find all real numbers x for which

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}.$$

4. (USSR 1990) Mr Fat is going to pick three non-zero real numbers and Mr Taf is going to arrange the three numbers as coefficients of a quadratic equation

$$\_\_\_\_x^2 + \_\_\_\_x + \_\_\_\_ = 0$$

Mr Fat wins the game if and only if the resulting equation has two distinct rational solutions. Who has the winning strategy?

5. (ARML 1997) Find a triple of rational numbers (a, b, c) such that

$$\sqrt[3]{\sqrt[3]{2} - 1} = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}.$$

1. If x is positive we have $1 < 3x$ so $x > \frac{1}{3}$, if x is negative $\frac{1}{3} < 3$ will always be true. Meanwhile, we have $1 < -4x$ so $x < -\frac{1}{4}$. Therefore, all x such that $x > \frac{1}{3}$ or $x < -\frac{1}{4}$ works.

2. Solving for x and substituting it into the third equation we get $z\frac{12\sqrt{6}}{y} = 45\sqrt{3}$, we multiply this by the second equation to get $z^2 = 4 \times 54$ so $z = \pm 6\sqrt{6}$. From there, its pretty easy to sub this value back into the equations to get two sets of solutions for (x, y, z), $(\pm 4\sqrt{2}, \pm 3\sqrt{3}, \pm 6\sqrt{6})$.

3. We let $a = 2^x$ and $b = 3^x$. The equation becomes $\frac{a^3+b^3}{a^2b+b^2a} = \frac{7}{6}$. Which after some simplifying we get $6a^2 - 13ab + 6b^2 = 0$, which is $(2a - 3b)(3a - 2b) = 0$. Therefore $2^{x+1} = 3^{x+1}$ or $2^{x-1} = 3^{x-1}$, which implies $x = -1$ and $x = 1$.

4. Mr Fat has the winning strategy, because by choosing a set of distinct rational nonzero numbers a, b, c, such that $a + b + c = 0$ will make him win. Let a', b', c' be a random permutation of a, b, c and let $f(x) = a'x^2 + b'x + c'$. Then $f(1) = a' + b' + c' = a + b + c = 0$, and so 1 is a solution. Since the product of two numbers is $\frac{c'}{a'}$ by Vieta's, the other solution is clearly $\frac{c'}{a'}$, which is different from 1. Thus Mr Fat can guarantee two distinct solutions.

5. Official Solution: Let $x = \sqrt[3]{\sqrt[3]{2} - 1}$ and $y = \sqrt[3]{2}$. So $y^3 = 2$ and $x = \sqrt[3]{y-1}$. Note that

$$1 = y^3 - 1 = (y-1)(y^2 + y + 1)$$

and

$$y^2 + y + 1 = \frac{3y^2 + 3y + 3}{3} = \frac{(y+1)^2}{3}$$

which implies that

$$x^3 = y - 1 = \frac{1}{y^2 + y + 1} = \frac{3}{(y+1)^3}$$

or

$$x = \frac{\sqrt[3]{3}}{y+1}.$$

On the other hand $3 = y^3 + 1 = (y+1)(y^2 - 1 + 1)$ from which it follows that

$$\frac{1}{y+1} = \frac{y^2 - y + 1}{3}.$$

Thus we have

$$x = \sqrt[3]{\frac{1}{9}}(\sqrt[3]{4} - \sqrt[3]{2} + 1).$$

Consequently $(a, b, c) = (\frac{4}{9}, -\frac{2}{9}, \frac{1}{9})$ is a desired triple.