Year 10

1. (3 points) There are twelve lockers, numbered from 110 to 121. The keys to these twelve lockers are numbered 1 to 12. Each locker number is divisible by the number on its key.

   Determine the key number for each locker. Total for Question 1: 3

2. (2 points) Three consecutive numbers sum to 141. What are the numbers?

   Total for Question 2: 2

3. (a) (2 points) When is the sum of 2 consecutive numbers divisible by 2?
   (b) (2 points) When is the sum of 3 consecutive numbers divisible by 3?
   (c) (2 points) When is the sum of 4 consecutive numbers divisible by 4?
   (d) (2 points) When is the sum of 5 consecutive numbers divisible by 5?
   (e) (2 points) When is the sum of n consecutive numbers divisible by n?

   Total for Question 3: 10

4. (3 points) Prove it is possible to pair up the numbers $0, 1, 2, 3, \ldots, 61$ in such a way that when we sum each pair, the product of the 31 numbers we get is a perfect fifth power. Total for Question 4: 3

5. (a) (1 point) Two lines can divide the plane into at most how many regions?
   (b) (1 point) Three lines can divide the plane into at most how many regions?
   (c) (1 point) If we add a fourth line, what is the maximum number of intersection points that line can make with the existing lines?
   (d) (1 point) Four lines can divide the plane into at most how many regions?
   (e) (3 points) What is the maximum number of regions that n lines can divide the plane into?

   Total for Question 5: 7

6. (a) (2 points) Is the sum or difference of two rational numbers always rational? Why or why not?
   (b) (2 points) Is the product or quotient of two rational numbers always rational? Why or why not?
   (c) (2 points) When one rational number is raised to another is the result guaranteed to be rational?
   (d) (3 points) It is known that $\sqrt{2}$ is irrational, but it is not known whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. Do there exist two irrational numbers such that when one is raised to the other the result is rational? Prove your conjecture.

   Total for Question 6: 9

7. (a) (2 points) Prove that $x^2 + y^2 \geq 2xy$ for real numbers x, y.
   (b) (2 points) Prove that $2a^2 + b^2 + c^2 \geq 2(ab + ac)$ for real numbers a, b, c.

(c) (2 points) Prove that $3(a^2 + b^2 + c^2 + d^2) \geq 2(ab + ac + ad + bc + bd + cd)$ for real numbers a, b, c, d.

(d) (3 points) Real numbers a, b, c, d, e are linked by the two equations:

$$e = 40 - a - b - c - d$$

$$e^2 = 400 - a^2 - b^2 - c^2 - d^2$$

Determine the largest value for e.

Total for Question 7: 9

8. (a) (1 point) How many ways are there to create a two-element subset from the set $\{1, 2, 3, 4\}$?

(b) (2 points) How many ways are there to split the set $\{1, 2, 3, 4\}$ into two disjoint nonempty sets? (Disjoint means the two sets share no elements.) For example, if we had the set $\{1, 2, 3\}$ a valid splitting would be $\{1, 2\}$ and $\{3\}$.

(c) (2 points) How many ways are there to split the set $\{1, 2, 3, 4, 5\}$ into two disjoint nonempty sets?

(d) (2 points) Denote by $S(n, k)$ the number of ways to split a set of n elements into k disjoint nonempty sets. Suppose we know what $S(n-1, 2)$ is. What happens when we add another element to the set? What happens to each of the $S(n-1, 2)$ ways to do the partitioning?

(e) (2 points) What is $S(n, 2)$?

(f) (3 points (bonus)) If there are $S(n-1, k-1)$ ways to partition a set of $n-1$ elements into $k-1$ nonempty disjoint sets, what is $S(n, k)$ in terms of $S(n-1, k)$ and $S(n-1, k-1)$?

Total for Question 8: 9

Total: 52