Maths Games Team selection test

Tom Yan

June 2023

TOTAL MARKS: $2+2+2+3+3+4+5=21$

1: A/N AMC Senior 2020/14

2: N AMC Senior 2019/11

3. G AIMO 2006/1

4: G AMC 2020/24

5. A AMC Upper Primary 2018/29

6. A/N Crux Mathematicorum

7. C/G AMO 2005/2

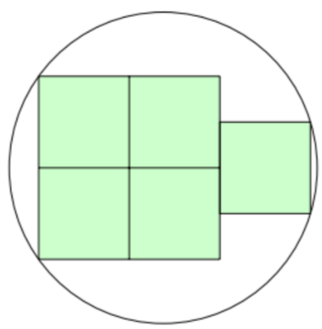
1 Introduction

1. Given that x and y are both integers and $2^{x+1} + 2^x = 3^{y+2} - 3^y$, what is the value of $x + y$? (2 marks)

2. The 5-digit number $P679Q$ is divisible by 72. What is the digit P equal to? (2 marks)

3. Consider a cube of edge $9cm$. In the centre of three different and not opposite faces a square hole is made which goes through to the opposite face. Each side of each hole has width $3cm$. What is the surface area, in cm^2, of the remaining solid? (2 marks)

4. Five squares of unit area are circumscribed by a circle as shown. What is the radius of the circle? (3 marks)



5. Jan and Jill are both on a circular track.
Jill runs at a steady pace, completing each circuit 72 seconds.
Jan walks at a steady pace in the opposite direction and meets Jill every 56 seconds. How long does it take Jan to walk each circuit? (3 marks)

6. Show that n^4-20n^2+4 is composite for all integers n, where $n > 4$. (4 marks)

7. Consider a polyhedron whose faces are convex polygons. Show that it has at least two faces with the same number of edges. (5 marks)

2 Solutions:

1. Solution: Solving,

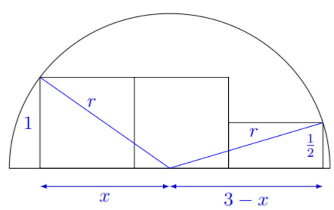
$$2^{x+1} + 2^x = 3^{y+2} - 3^y$$
$$2^x(2+1) = 3^y(3^2 - 1)$$
$$2^x \times 3 = 8 \times 3^y$$

Clearly $x = 3$ and $y = 1$ is a possible solution. Due to uniqueness of prime factorisation, this is the only solution where x and y are integers. Consequently $x + y = 3 + 1 = 4$.

2. Solution: let $N = P679Q$, the N is divisible by 9 and divisible by 8. Since any number of thousands is divisible by 8, and 800 is divisible by 8, the 5-digit number $P6800$ is a known multiple of 8 near N. From this $N = P6792$ and $Q = 2$.
For N to be divisible by 9, its digit sum $P + 6 + 7 + 9 + 2 = P + 24$ must be divisible by 9. Then $P + 24 = 27$ and $P = 3$.

3. Solution: Each of the six external faces of the solid is a square $9cm \times 9cm$ with a $3cm \times 3cm$ square removed. Its area is $81 - 9 = 72cm^2$.
The tunnel from the centre of each face towards the centre of the solid has four walls, each $3cm \times 3cm$, so the area of each tunnel is $4 \times 3 \times 3 = 36cm^2$. Hence the total area is $6 \times (72 + 36)cm^2 = 648cm^2$.

4. Solution: In the upper circle, label the lengths as shown and then equate the two radii using Pythagoras' theorem.



We have $r^2 = x^2 + 1$ and $r^2 = (3 - x)^2 + (1/2)^2$

$$= x^2 - 6x + \frac{37}{4}$$

So

$$x^2 = 1 = x^2 - 6x + \frac{37}{4}$$
$$6x = \frac{33}{4} \to x = \frac{11}{8}$$

Then $r = \sqrt{x^2 + 1} = \sqrt{\frac{185}{64}} = \frac{\sqrt{185}}{8}$

5. Solution: In 56 seconds, Jill runs $\frac{56}{72} = \frac{7}{9}$ of the track, and so in the same 56 seconds, Jan has walked along the other $\frac{2}{9}$ of the track. In half this time which is 28 seconds, Jan will have walked $\frac{1}{9}$ of the track. Then Jan will take $9 \times 28 = 252$ seconds to walk the complete track.

6. $n^4 - 20n^2 + 4 = n^4 - 4n^2 + 4 - 16n^2 = (n^2 - 2)^2 - (4n)^2$, Thus we have difference of perfect squares, which factors as

$$(n^2 - 2)^2 - (4n)^2 = (n^2 - 4n - 2)(n^4 + 4n - 2)$$

To show that this expression is always composite, it suffices to show that both factors are always at least 2. For $n \geq 5$, we have $n^2 - 4n - 2 = (n-2)^2 - 6 \geq 3^2 - 6 = 3$, and $n^2 + 4n - 2 = (n+2)^2 - 6 \geq 7^2 - 6 = 43$, so both factors are greater than 2. Therefore, $n^4 - 20n^2 + 4$ is composite for all integers $n > 4$.

7. Suppose for the sake of contradiction that there exists a polyhedron whose faces are convex but no two of its faces have the same number of edges. Let its face with with the largest number of edges be called X and let X have n edges. Since there cannot be another face with n edges, all other faces must have either $n-1, n-2, \ldots, 5, 4$ or 3 edges. This makes a total of $n-3$ possibilities. However since X has n edges there must be at least n other faces in the polyhedron, one for each such edge. But we have already shown that there are only $n-3$ possibilities for the number of edges of these n other faces. This means that two of these other faces must have the same number of edges - a contradiction.
Thus the original assumption is wrong and hence there are two faces with the same number of edges.