MEG first meeting solutions

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Solutions

1. Since the last digit repeats every 4 numbers and 2023 is 3 mod 4, we deduce the last digit 2^{2023} is $2^3 = 8$.

2. For a number to be divisible by 15, it has to be divisible by 5 and 3. By the divisibility test for 3, we get that the sum of the occurrences of the digit 2 has to be a multiple of 6. Since the number is also divisible by 5, the last digit is 0. Hence the smallest number that is gobbus is 2220.

3. Solve to equation to get 9.

4. Square $a + b + c = 1$ to get $a^2 + b^2 + c^2 + 2(ab + bc + ca) = 1$. Now substitute this into the inequality we get

$$a^2 + b^2 + c^2 + a^2 + b^2 + c^2 + 2(ab + bc + ca) \geq 4(ab + bc + ca)$$

$$2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca)$$

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

Which holds true by the rearrangement inequality.

5. Consider the subset T that contains 10, then there exists a subset T' which contains all the elements of T except for 10 and only those elements. However, since T' has one less element, each element of T' has one fewer element preceding it than it does in T, and hence their signs are opposite. Thus $T + T' = 10$.

There are 2^9 subsets with 10, for each of those subsets, simply add 10 to get a subset with the same elements but with 10. Hence the alternating sum of all the subsets is 10×2^9.

6. Lemma: A line parallel to the base of a triangle cuts the side proportionally; and conversely. (Proof by the definition of similar triangles and parallel lines)

We have $\triangle AHB \cong \triangle BHE(AAS)$ meaning $AK = KF$, we also have $\triangle AKC \cong FCK(AAS)$ meaning $AK = KF$. By SAS, $\triangle AKH \simeq \triangle AFE$.

By the converse of the lemma above, KH is parallel to FE and since FE have the same gradient as BC, clearly KH is parallel to BC as required.