Meg term 3 holiday problems

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September 2023

1 Introduction

1. (AIMO 2017/1) The number x is 111 when written in base b, but it is 212 when written in base $b-2$. What is x in base 10?

2. (AIME II 2021/3) Find the number of permutations x_1, x_2, x_3, x_4, x_5 of numbers $1, 2, 3, 4, 5$ such that the sum of five products

$$x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2$$

is divisible by 3.

3. (Simson Line) Let ABC be a triangle and P be any point on the circumcircle of ABC. Let X, Y, Z be the feet of the perpendiculars from P onto lines BC, CA and AB. Prove that points X, Y, Z are collinear.

4. Find all functions $f : \mathbb{R}\backslash\{-1, 1\} \to \mathbb{R}$ for which

$$f(\frac{x-3}{x+1}) + f(\frac{3+x}{1-x}) = x$$

for all $x, y \in \mathbb{R}\backslash\{-1, 1\}$.

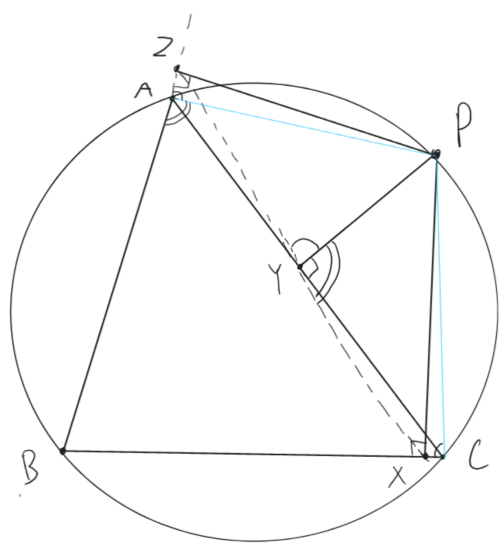
5. (AMO 2022/1) Prove that a convex pentagon with integer side lengths and an odd perimeter can have two right angles, but cannot have more than two right angles.

2 Solutions

1. We have $x = b^2 + b + 1$ and $x = 2(b-2)^2 + (b-2) + b = 2b^2 - 7b + 8$. Hence $(2b^2 - 7b + 8) - (b^2 + b + 1) = b^2 - 8b + 7 = (b-7)(b-1)$. Clearly $b \neq 1$, so $b = 7$ and thus $x = 49 + 7 + 1 = 57$.

2. Considering mod 3, WLOG let $x_1 = 3$, and thus we are left with $x_2x_3x_4 + x_3x_4x_5 = x_3x_4(x_2 + x_5)$. Clearly $3 \nmid x_3x_4$ so $3|x_2 + x_5$, so (x_3, x_4) can be $(1,2), (1,5), (4,2), (4,5)$ and $(2,1), (5,1), (2,4), (5,4)$ thus giving 8 ways, and x_3x_4 can be permuted in 2 ways after designating x_3 and x_4. So there are $5 \times 8 \times 2 = 80$ ways.

3. Let $\angle PYZ = \theta$. Since $\angle PZA + \angle PYA = 180°$, $PZAY$ is cyclic, so $\angle PYZ = \angle PAZ = \theta$ (subtended by same arc). Thus $\angle PAB = 180° - \theta$, and so $PCB = \theta$. Since $\angle PYX = \angle PXC = 90°$, $PYXC$ is cyclic. Thus $PYX = 180° - \theta$. Since $\angle PYZ + \angle PYX = 180°$, we are done.



4. Let t be real number that is not ± 1, and $t = \frac{x-3}{x+1}$ thus $x = \frac{3+t}{1-t}$. Rewriting the given equation in terms of t we have

$$f(t) + f(\frac{t-3}{t+1}) = \frac{3+t}{1-t}.$$

Similarly, let $t = \frac{3+x}{1-x}$, so then we have $x = \frac{t-3}{t+1}$ and $\frac{x-3}{x+1} = \frac{3+t}{1-t}$. Rewriting the given equation in terms of t again, we have

$$f(\frac{3+t}{1-t}) + f(t) = \frac{t-3}{t+1}.$$

Adding the two rewritten equations we get

$$2f(t) + f(\frac{t-3}{t+1}) + f(\frac{3+t}{1-t}) = \frac{3+t}{1-t} + \frac{t-3}{t+1}$$

And since $f(\frac{t-3}{t+1}) + f(\frac{3+t}{1-t}) = t$, it implies

$$2f(t) + t = \frac{8t}{1-t^2}.$$

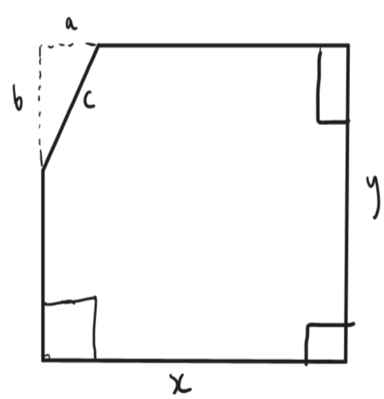
Thus the function is $f(t) = \frac{4t}{1-t^2} - \frac{t}{2}$. It is easy to check that this satisfies the given equation.

5. A convex pentagon with side integer side lengths and an odd perimeter can have two right angles as a unit equilateral triangle on top of a unit square suffices.

Clearly a pentagon can't have 5 right angles. If a pentagon has 4 right angles, then the remaining angle is $540 - 4 \times 90 = 180°$ which is a contradiction. Hence we consider when a pentagon has 3 right angles.

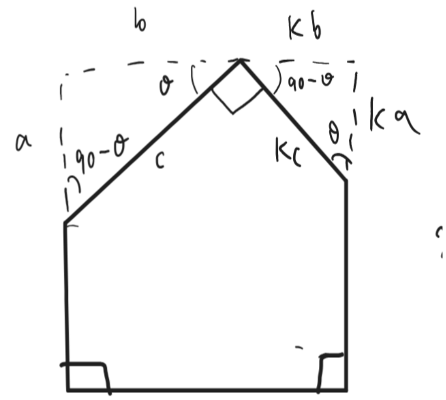
There are two cases, when exactly two of the right angles are adjacent, and when all three of the right angles are adjacent.

Case 1: Three of the right angles are adjacent.



Then the perimeter is $2(x+y)+(c-a-b)$. Clearly $2(x+y)$ is even, and $c-a-b$ is even since $a^2+b^2=c^2$ implies $a+b$ has the same parity as c.

Case 2: Exactly two of the right angles are adjacent.



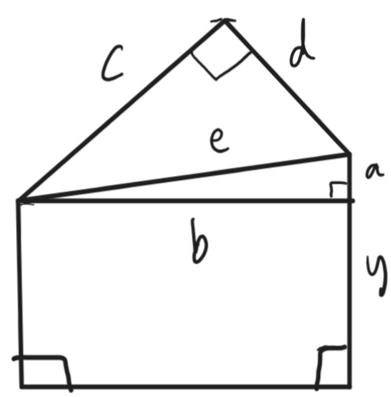
Let x and y be the sides of the rectangle formed with the dotted lines. Then from similar triangles we deduce the perimeter is

$$2(x+y)-a-b-kb-ka+c+kc$$

$$=2(x+y)+(c-a-b)(1+k)$$

Which is even as $2(x+y)$ and $c-a-b$ are even.

Alternatively, since $a^2+b^2=e^2=c^2+d^2$, $a+b$ has the same parity as $c+d$. Meaning $a+b+c+d$ is even. Since the perimeter is $a+b+c+d+2y$, the pentagon has an even perimeter.



Hence a pentagon with integer side lengths and odd perimeter can have at most two right angles.