Year 10

1. (3 points) There are twelve lockers, numbered from 110 to 121. The keys to these twelve lockers are numbered 1 to 12. Each locker number is divisible by the number on its key.

   Determine the key number for each locker.

Solution:

110	111	112	113	114	115	116	117	118	119	120	121
10	3	8	1	6	5	4	9	2	7	12	11

2. (2 points) Three consecutive numbers sum to 141. What are the numbers?

Solution: $46, 47, 48$

3. (a) (2 points) When is the sum of 2 consecutive numbers divisible by 2?

   Solution: Never, because $n + (n+1) = 2n + 1$ is odd.

   (b) (2 points) When is the sum of 3 consecutive numbers divisible by 3?

   Solution: Always, as $n + (n+1) + (n+2) = 3n + 3$.

   (c) (2 points) When is the sum of 4 consecutive numbers divisible by 4?

   Solution: Never, as $n+(n+1)+(n+2)+(n+3) = 4n+10 \equiv 2 \pmod 4$.

   (d) (2 points) When is the sum of 5 consecutive numbers divisible by 5?

   Solution: Always, as $n + (n+1) + (n+2) + (n+3) + (n+4) = 5n + 10$.

   (e) (2 points) When is the sum of n consecutive numbers divisible by n?

   Solution: The sum of n consecutive numbers starting from k is kn plus the $n-1$th triangular number, so it is

$$n\left(k + \frac{n-1}{2}\right).$$

   The resulting number is divisible by n if $(n-1)/2$ is an integer, and that is when n is odd.

4. (3 points) Prove it possible to pair up the numbers $0, 1, 2, 3, \ldots, 61$ in such a way that when we sum each pair, the product of the 31 numbers we get is a perfect fifth power.

> **Solution:** Pair 0 with 1, and k with $63 - k$ for all $2 \leq k \leq 31$. This would result in a product equal to $1 \times 63^{30} = (63^6)^5$ which is a perfect 5^{th} power.

5. (a) (1 point) Two lines can divide the plane into at most how many regions?

   > **Solution:** 4

   (b) (1 point) Three lines can divide the plane into at most how many regions?

   > **Solution:** 7 (If you try it on paper, you will see that the third line can only intersect the existing 2 lines in at most 2 places, resulting in 3 new regions. Therefore the number of regions is $4 + 3 = 7$.)

   (c) (1 point) If we add a fourth line, what is the maximum number of intersection points that line can make with the existing lines?

   > **Solution:** 3

   (d) (1 point) Four lines can divide the plane into at most how many regions?

   > **Solution:** 11

   (e) (3 points) What is the maximum number of regions that n lines can divide the plane into?

   > **Solution:** $1 + n(n + 1)/2$

6. (a) (2 points) Is the sum or difference of two rational numbers always rational? Why or why not?

   > **Solution:** Yes, because $a/b + c/d = (ad + bc)/bd$ and both $ad + bc$ and bd are integers.

   (b) (2 points) Is the product or quotient of two rational numbers always rational? Why or why not?

   > **Solution:** Yes, because $(a/b)(c/d) = ac/bd$ and both ac and bd are integers.

   (c) (2 points) When one rational number is raised to another is the result guaranteed to be rational?

Solution: No; an example is $2^{1/2}$.

(d) (3 points) It is known that $\sqrt{2}$ is irrational, but it is not known whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. Do there exist two irrational numbers such that when one is raised to the other the result is rational? Prove your conjecture.

Solution: The answer is yes. Let $A = \sqrt{2}^{\sqrt{2}}$. We don't know if A is rational or irrational, but it doesn't matter. If A is rational, then it is an example of a rational number that is an irrational number raised to another irrational number. If A is irrational, then $A^{\sqrt{2}} = \sqrt{2}^2 = 2$ and this number is the desired example.

7. (a) (2 points) Prove that $x^2 + y^2 \geq 2xy$ for real numbers x, y.

Solution: Moving $2xy$ to the left we get $(x-y)^2 \geq 0$. Which is true.

(b) (2 points) Prove that $2a^2 + b^2 + c^2 \geq 2(ab + ac)$ for real numbers a, b, c.

Solution: Using the result from a), we add the two following inequalities

$$a^2 + b^2 \geq 2ab$$

$$a^2 + c^2 \geq 2ac$$

to get

$$2a^2 + b^2 + c^2 \geq 2(ab + ac).$$

(c) (2 points) Prove that $3(a^2 + b^2 + c^2 + d^2) \geq 2(ab + ac + ad + bc + bd + cd)$ for real numbers a, b, c, d.

Solution: Same deal as part b), just with more inequalities. We add up

$$a^2 + b^2 \geq 2ab$$

$$a^2 + c^2 \geq 2ac$$

$$a^2 + d^2 \geq 2ad$$

$$b^2 + c^2 \geq 2bc$$

$$b^2 + d^2 \geq 2bd$$

$$c^2 + d^2 \geq 2cd$$

to get

$$3(a^2 + b^2 + c^2 + d^2) \geq 2(ab + ac + ad + bc + bd + cd).$$

(d) (3 points) Real numbers a, b, c, d, e are linked by the two equations:

$$e = 40 - a - b - c - d$$
$$e^2 = 400 - a^2 - b^2 - c^2 - d^2$$

Determine the largest value for e.

Solution: From the first equation we have

$$(40-e)^2 = (a+b+c+d)^2 = a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)$$

Using the result from part c), we get that

$$(40-e)^2 \leq 4(a^2+b^2+c^2+d^2)$$

Using the second equation we get

$$(40-e)^2 \leq 4(400-e^2)$$

After expanding the brackets and simplifying, it follows that

$$e(80-5e) \geq 0$$

Implying that $0 \leq e \leq 16$. Thus the largest value for e that satisfies the given equations is 16. This value of e is attainable when $a = b = c = d = 6$.

8. (a) (1 point) How many ways are there to create a two-element subset from the set $\{1, 2, 3, 4\}$?

Solution: There are $\binom{4}{2} = 6$ ways.

(b) (2 points) How many ways are there to split the set $\{1, 2, 3, 4\}$ into two disjoint nonempty sets? (Disjoint means the two sets share no elements.) For example, if we had the set $\{1, 2, 3\}$ a valid splitting would be $\{1, 2\}$ and $\{3\}$.

Solution: The valid ways are: $\{1\}$ and $[\{2, 3, 4\}$, $\{2\}$ and $\{1, 3, 4\}$, $\{3\}$ and $\{1, 2, 4\}$, $\{4\}$ and $\{1, 2, 3\}$, $\{1, 2\}$ and $\{3, 4\}$, $\{1, 3\}$ and $\{2, 4\}$, and $\{1, 4\}$ and $\{2, 3\}$. So 7 ways.

(c) (2 points) How many ways are there to split the set $\{1, 2, 3, 4, 5\}$ into two disjoint nonempty sets?

Solution: By listing all the ways again, we find that there are 15 ways.

(d) (2 points) Denote by $S(n, k)$ the number of ways to split a set of n elements into k disjoint nonempty sets. Suppose we know what $S(n-1, 2)$ is. What happens when we add another element to the set? What happens to each of the $S(n-1, 2)$ ways to do the partitioning?

Solution: For each way to do the partitioning, we may add the new number n to either set to obtain a new partitioning. There is also one new partition that is not obtained by adding n to existing one, and that is the splitting into $\{n\}$ and $\{1, 2, \ldots, n-1\}$. Therefore we have

$$S(n, 2) = 2S(n-1, 2) + 1.$$

(e) (2 points) What is $S(n, 2)$?

Solution: We have $S(1, 2) = 0$, $S(2, 2) = 1$, $S(3, 2) = 3$, $S(4, 2) = 7$, and $S(5, 2) = 15$. From this pattern and the recursive formula found above it is easy to prove by induction that $S(n, 2) = 2^{n-1} - 1$.

(f) (3 points (bonus)) If there are $S(n-1, k-1)$ ways to partition a set of $n-1$ elements into $k-1$ nonempty disjoint sets, what is $S(n, k)$ in terms of $S(n-1, k)$ and $S(n-1, k-1)$?

Solution: We need to think about what happens when we add the nth element. Clearly we may put this object into a separate set by itself and partition the other $n-1$ objects into $k-1$ nonempty disjoint subsets, because all up there would be k sets. There are $S(n-1, k-1)$ ways to do this. We may also take the $n-1$ objects and partition them into k sets and add the nth element to any one of them to get another valid partitioning. How many ways are there to do this? In each of the $S(n-1, k)$ ways to partition $n-1$ objects into k nonempty sets, there are k different subsets we can put the nth object into. Therefore in this case there are $kS(n-1, k)$ ways. In total then we have

$$S(n, k) = S(n-1, k-1) + kS(n-1, k).$$

Total: 52