Melbourne High School
Maths Extension Group 2023
Term 1 Holiday Problems

1. Suppose you have n distinct objects, each with an equal probability of being selected. After an object is selected, it is replaced. Each selection of an object is considered a "turn." What is the expected value of turns that you need to complete before you have selected at least one of each object?

2. Your good friend Harold is about to start reading Virginia Woolf's *Between the Acts*. The book has 199 pages, which you instantly recognise as being prime.

   Since Harold is much smarter than you, he doesn't read books in the correct page order. He begins by choosing a random number between 1 and 199—call this number x. To determine the nth page that he will read, he multiplies n by x, and if the number is greater than 199, he subtracts off a multiple of 199 so that he has a page number that is actually in the book. He then reads that page before calculating the next page he will read.

   Show that not only will Harold read each page of the book, but he will also never read a page more than once.

3. Find all natural numbers n such that $2^{2^n} + 5$ is prime.

4. Let N be a regular nonagon, having O as the centre of its circumcircle, and let PQ and QR be adjacent edges of N. The midpoint of PQ is A and the midpoint of the radius perpendicular to QR is B. Determine the angle between AO and AB.

5. Let $0 < a < 1$. Solve
$$x^{a^x} = a^{x^a}$$
   for positive numbers x.

6. Find the sum of all positive integers n for which
$$n^2 - 19n + 99$$
   is a perfect square.

7. Prove that the fraction
$$\frac{21n+4}{14n+3}$$
   cannot be reduced further for all natural number values of n.

8. Show that we can colour the elements of the set $S = \{1, 2, \dots, 2007\}$ with 4 colours such that any subset of S with 10 elements, whose elements form an arithmetic sequence, is not all one colour.

9. Two points K and L are chosen inside triangle ABC and a point D is chosen on the side AB. Suppose that B, K, L, C are concyclic, $\angle AKD = \angle BCK$ and $\angle ALD = \angle BCL$. Prove that $AK = AL$.

10. (This is a long problem.)

    (a) Evaluate $(a+b)^2 \pmod 2$, $(a+b)^3 \pmod 3$, and $(a+b)^5 \pmod 5$.

(b) The Binomial Theorem states that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Prove that

$$(a+b)^p \equiv a^p + b^p \pmod{p}$$

for any prime p

(c) Suppose you have 6 different applied mathematics books that you would like place in 3 bins. How many ways are there to distribute the books such that the first bin has 1 book, the second 2 books, and the third 3 books?

(d) Now suppose you have n different applied mathematics books and m bins. Someone supplies you a list of numbers whose sum is n: $k_1, k_2, k_3, \dots, k_m$. How many ways are there to distribute the books such that the ith bin receives k_i books? That is, the first bin should receive k_1 books, the second k_2 books, and so on. Note that $k_1 + k_2 + k_3 + \cdots + k_m = n$.

(e) If you have completed the last part correctly, you have just discovered *multinomial coefficients*, which, as you might be able tell by the name, generalise the venerable binomial coefficients.

Let n be a positive integer and let $k_1, k_2, k_3, \cdots k_m$ be m positive integers that sum to n. This multinomial coefficient is denoted

$$\binom{n}{k_1, k_2, \dots, k_m}.$$

If your answer to the previous part involved binomial coefficients, make use of a telescoping product to show that

$$\binom{n}{k_1, k_2, \dots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}.$$

(f) Expand $(a+b+c)^3$, taking note of the coefficients and the powers of a, b, and c in each term.

(g) By first doing some more examples if necessary, come up with a conjecture about the expansion of

$$(x_1 + x_2 + x_3 + \cdots + x_m)^n$$

using multinomial coefficients.

(h) Prove your conjecture by induction (you will need the binomial theorem).

(i) If you have made it this far in the problem, well done.

Prove that

$$(x_1 + x_2 + x_3 + \cdots + x_m)^p \equiv x_1^p + x_2^p + x_3^p + \cdots x_m^p \pmod{p}$$

for any prime p. (You can induction, but you could also look at the definition of a multinomial coefficient directly.)

(j) Hence, show that

$$a^p \equiv a \pmod{p}$$

for any prime p and any positive integer a.

11. Find $(102^{73} + 55)^{37}$ modulo 111.

12. We define the Fibonacci numbers by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, where $F_0 = 0$ and $F_1 = 1$. Earlier in the term we discovered that the generating function

$$G(x) = F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \cdots$$

has the closed form expression

$$\frac{x}{1 - x - x^2}.$$

In this problem, we will use this result to derive a closed-form expression for the nth Fibonacci number.

(a) Begin by showing that the closed-form expression for $G(x)$ can be written

$$\frac{x}{(1-\alpha x)(1-\beta x)}$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

(b) This fraction can be split into the sum of two fractions:

$$\frac{A}{1-\alpha x} + \frac{B}{1-\beta x}.$$

Show that $A = 1/\sqrt{5}$ and $B = -1/\sqrt{5}$.

(c) Show that

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

(This value makes sense for $|x| < 1$.)

(d) Hence show that

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

This is known as *Binet's Formula.*