Lecture 21
Options Pricing

- Readings
  - BM, chapter 20
  - Reader, Lecture 21

Outline

- Last lecture:
  - Examples of options
  - Derivatives and risk (mis)management
  - Replication and Put-call parity
- This lecture
  - Binomial option valuation
  - Black Scholes formula

Put-Call parity and early exercise

- Put-call parity:

$$\mathbf{C = S + P - K / (1+r)^T}$$

- Put-call parity gives us an important result about exercising American call options.

$$\begin{aligned} C &= S + P - K/(1+r)^T \\ &\geq S - K/(1+r)^T \\ &> S - K. \end{aligned}$$

- In words, the value of a European (and hence American) call is strictly larger than the payoff of exercising it today.

Early Exercise

- In the absence of dividends, you should thus <u>never</u> exercise an American call prior to expiration.
  - What should you do instead of exercising if you're worried that your currently "**in-the-money**" (S > K) option will expire "**out-of-the-money**" (S < K)?
  - What is the difference between the value of a European and an American call option?
- **Note:** If the stock pays dividends, you <u>might</u> want to exercise the option just before a dividend payment.
- **Note:** This only applies to an American call option.
  - You might want to exercise an American <u>put</u> option before expiration, so you receive the strike price earlier.

Key Questions About Derivatives

- How should a derivative be valued relative to its underlying asset?
- Can the payoffs of a derivative asset be replicated by trading only in the underlying asset (and possibly cash)?
  - If we can find such a replicating strategy, the current value of the option must equal the initial cost of the replicating portfolio.
  - This also allows us to create a non-existent derivative by following its replicating strategy.
- This is the central idea behind all of modern option pricing theory.

Notation

- We shall be using the following notation a lot:

  S = Value of underlying asset (stock)

  C = Value of call option

  P = Value of put option

  K = Exercise price of option

  r = One period riskless interest rate

  R = 1 + r

Factors Affecting Option Value

- The main factors affecting an option's value are:

Factor	**Call**	**Put**
S, stock price	+	-
K, exercise price	-	+
σ, Volatility	+	+
T, expiration date (Am.)	+	+
T, expiration date (Eu.)	+	?
r	+	-
Dividends	-	+

- What is not on this list?

Binomial Option Valuation

- Consider a European call option on a stock, price S, exercise price K, and 1 year to expiration.
- Suppose over the next year the stock price will either move up to uS, or down to dS. For example:

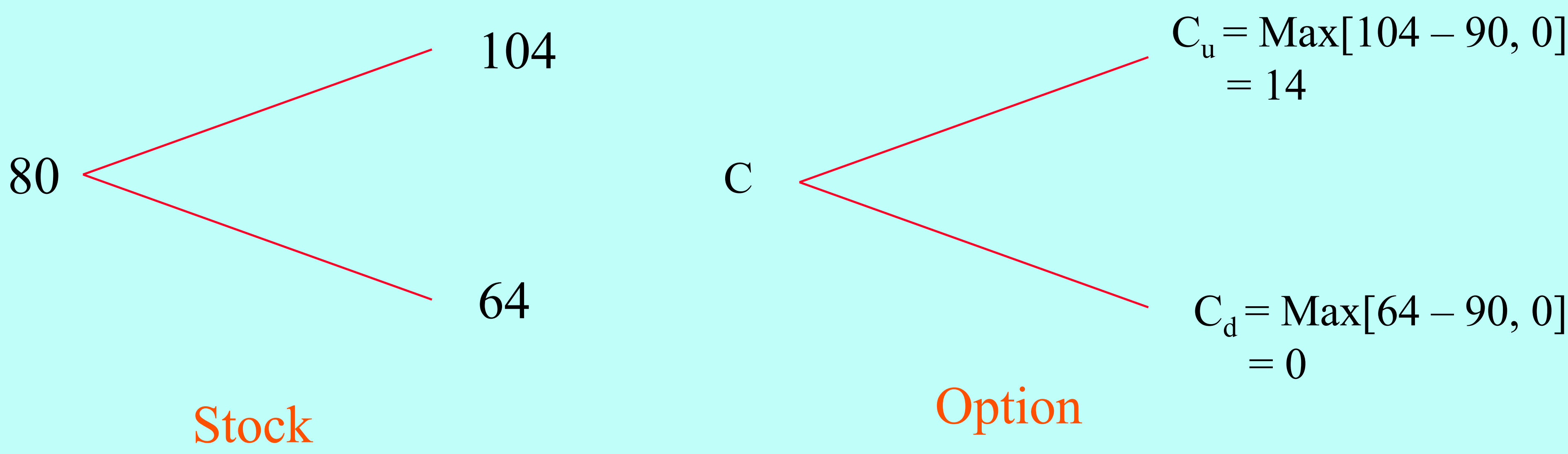
u x S = 104
(u = 1.3)

S = 80

d x S = 64
(d = 0.8)

Example

- Write C for the price of a call option today, and C_u and C_d for the price in one year in the two possible states, e.g. (suppose K = 90):



Example

- Consider forming a portfolio today by
  - Buying 0.35 shares
  - Borrowing \$20.3637 (assume interest rate = 10%)
- Cost of portfolio today = 0.35 x 80 – 20.3637
  = \$7.6363
- What's it worth next year?

0.35 x 104 – (20.3637 x 1.1) = **\$14.00**

7.6363

0.35 x 64 – (20.3637 x 1.1) = **\$0.00**

- Do these payoffs look familiar?
- How much would you pay for the call option?

Replicating Portfolios

- This is an example of a **replicating portfolio.**
- Its payoff is the same as that of the call option, regardless of whether the stock goes up or down
- The current value of the option must therefore be the same as the value of the portfolio, \$7.6363
  - What if the option were trading for \$5 instead?
  - Note that this result does not depend on the probability of an up vs. a down movement in the stock price.
- The call option is thus equivalent to a portfolio of the underlying stock plus borrowing.
- How do we construct the replicating portfolio in general?

Forming a Replicating Portfolio in General

- Form a portfolio today by
  - Buying Δ shares
  - Lending \$B
- Cost today = $\Delta S + B$ (= option price)
- Its value in one year depends on the stock price:

$$\Delta uS + B(1+r)$$

$$\Delta S + B$$

$$\Delta dS + B(1+r)$$

Replicating Portfolio

- We can make the two possible portfolio values equal to the option payoffs by solving:

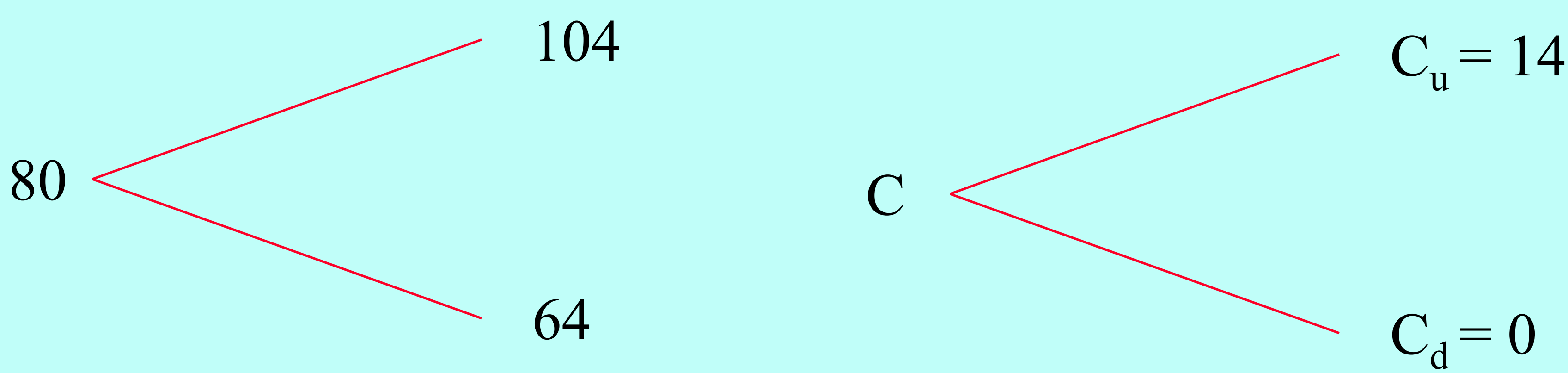
$$\Delta u S + B(1+r) = C_u,$$
$$\Delta d S + B(1+r) = C_d.$$

- Solving these equations, we obtain

$$\Delta = \frac{C_u - C_d}{(u-d)S}, \qquad B = \frac{uC_d - dC_u}{(u-d)(1+r)}.$$

Example

- $S = 80$, $u = 1.3$, $d = 0.8$
- $K = 90$, $r = 10\%$.



Example

- From the formulae,

$$\Delta = \frac{14 - 0}{(1.3 - 0.8)80} = 0.35,$$

$$B = \frac{1.3(0) - 0.8(14)}{(1.3 - 0.8)1.1} = -20.3637.$$

- Hence

$$\begin{aligned} C &= \Delta S + B \\ &= 0.35 \times 80 - 20.3637 \\ &= \$7.6363 \end{aligned}$$

Delta and hedging

- "Delta" (Δ) is the standard terminology used in options markets for the number of units of the underlying asset in the replicating portfolio
  - For a call option, Δ is between 0 and 1
  - For a put option, Δ is between 0 and –1 (see HW 11)

option value = (asset price x "delta") + lending

- For small changes, Δ measures the change in the option's value per $1 change in the value of the underlying asset.
- A position in the option can be **hedged** using a short position in Δ of the underlying asset.

"Risk-neutral" probabilities

- Define R = (1+r), and let p = (R - d) / (u - d).
- A little algebra shows that we can write:

$$C = \frac{pC_u + (1-p)C_d}{1+r}$$

- I.e. to value the call (or <u>any</u> derivative)
  - Calculate its "expected" value next period pretending p is the probability of prices going up.
  - Discount the expected value back at the riskless rate to obtain the price today.
  - **Note**: We don't need the <u>true</u> probability of an up movement, just the **"pseudo-probability",** p.
    - p would be the true probability if everyone were <u>risk-neutral</u>.

Example

- Using the previous example,

$$p = \frac{R-d}{u-d} = \frac{1.1-0.8}{1.3-0.8} = 0.6$$

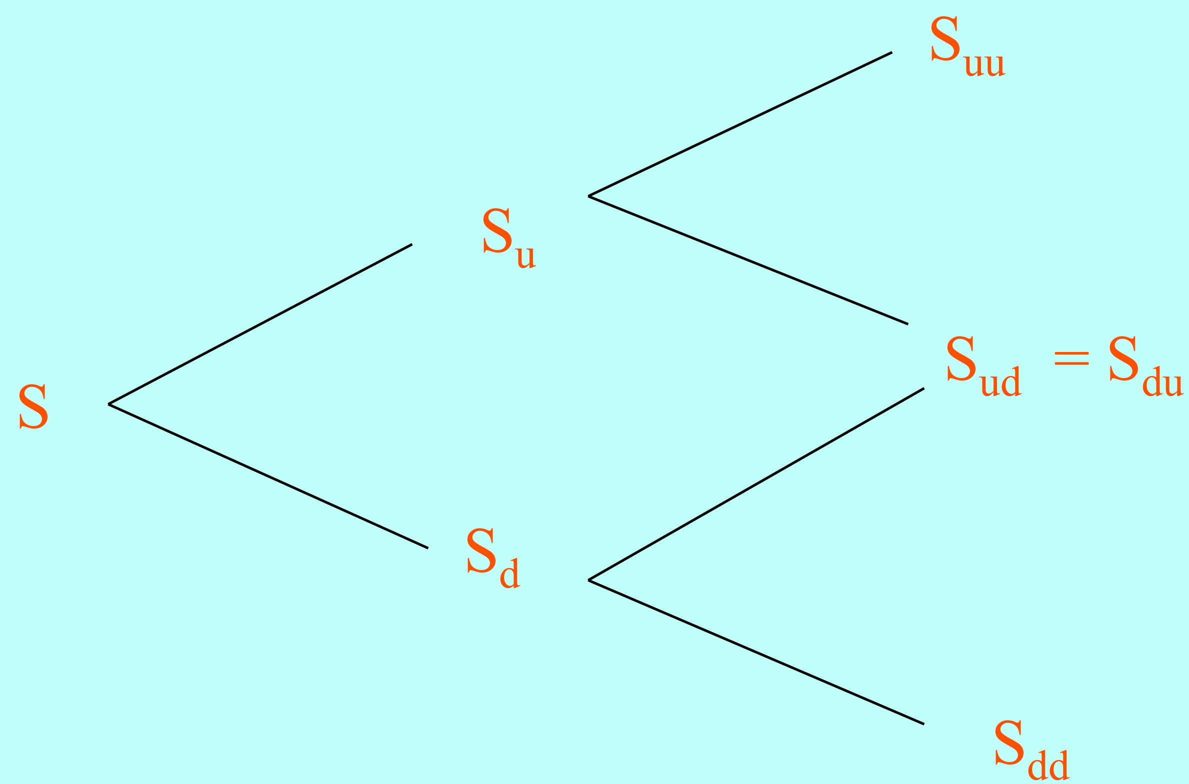
- Hence the call price equals

$$C = \frac{pC_u + (1-p)C_d}{1+r}$$
$$= \frac{(0.6\times 14)+(0.4\times 0)}{1.1} = 7.6363.$$

Shortcomings of Binomial Model

- The binomial model provides many insights:
  - Risk neutral pricing
  - Replicating portfolio containing only stock + borrowing
  - Allows valuation/hedging using underlying stock
- But it allows only two possible stock returns.
- To get around this:
  - Split year into a number of smaller subintervals
  - Allow one up/down movement per subperiod
  - n subperiods give us n+1 values at end of year.

Example, two subperiods



- Start at the end, and work backwards through tree.
- See reader pp. 164 – 167 for details.

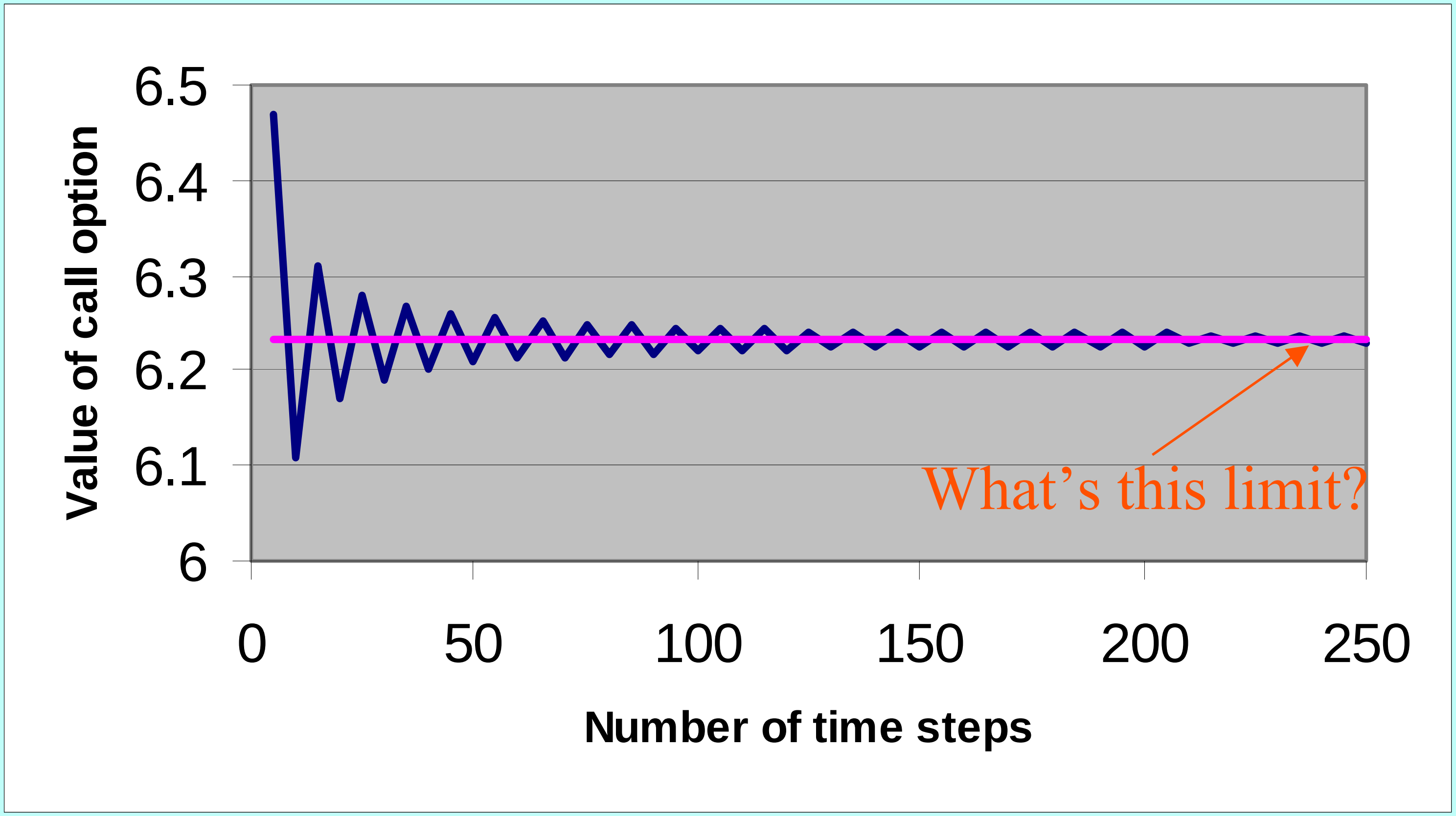
How big should up/down movements be?

- For a given expiration date, we keep overall volatility right as we split into n subperiods [each of length t (= T/n)], by picking

$$u = e^{\sigma\sqrt{t}},\ \ d = 1/u = e^{-\sigma\sqrt{t}}.$$

- As n gets larger, the distribution of the asset price at maturity approaches a **lognormal** distribution, with expected return r, and annualized volatility σ.
- What happens to option prices as we increase the number of time steps?

Binomial prices vs. # steps (S=K=60, T=0.5, σ=30%, r=8%)



Black-Scholes formula

- In the limit, the price of a European call option converges to the **Black-Scholes formula**,

$$C = S\,N(x) - Ke^{-rt}\,N\left(x - \sigma\sqrt{t}\right),$$

$$\text{where} \quad x \equiv \frac{\left[\log(S/K) + \left(r + \sigma^2/2\right)t\right]}{\sigma\sqrt{t}}$$

- r here is a continuously-compounded interest rate.

Interpretation of Black-Scholes

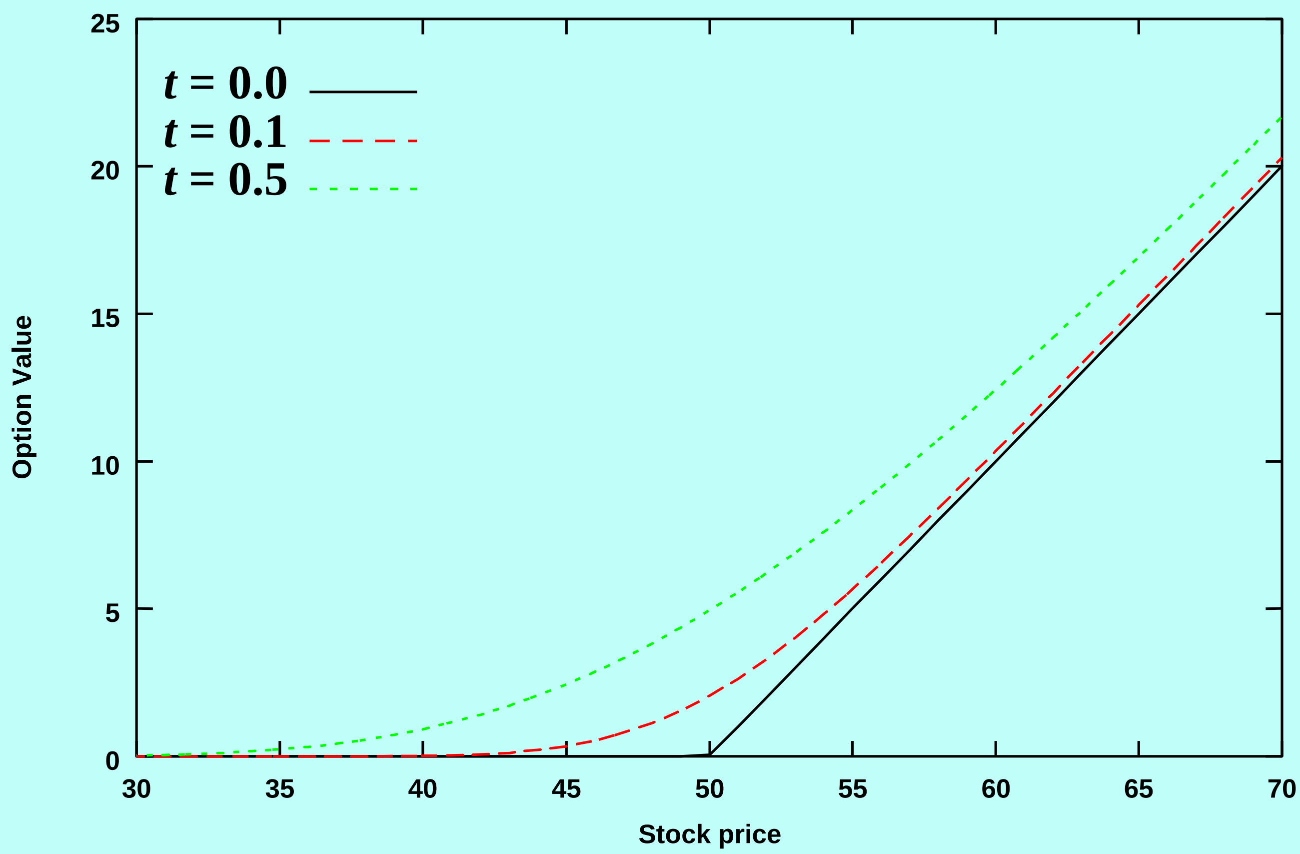
- This is just a special case of our old formula

$$C = \Delta S + B$$

- The formula tells us the values of Δ and B in the replicating portfolio.
- Note that, for a European call,
  - Δ is always between 0 and 1.
  - B is negative, and between 0 and -PV(K) (i.e. borrow).

Black-Scholes Call Prices for different Maturities

$K = 50$
$r = 0.06$
$\sigma = 0.3$



Black-Scholes put formula

- Combining the Black-Scholes call result with put-call parity, we obtain the Black-Scholes put value,

$$P = Ke^{-rt}\left[1 - N\left(x - \sigma\sqrt{t}\right)\right] - S\left[1 - N(x)\right],$$

$$\text{where} \qquad x \equiv \frac{\left[\log(S/K) + \left(r + \sigma^2/2\right)t\right]}{\sigma\sqrt{t}}$$

- Note that, for a European put,
  - Δ is always between 0 and -1 (i.e. short).
  - B is positive, and between 0 and PV(K) (i.e. lend).