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Pricing Black–Scholes options with correlated credit risk

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Abstract

This paper presents an improved method of pricing vulnerable Black–Scholes options under assumptions which are appropriate in many business situations. An analytic pricing formula is derived which allows not only for correlation between the option's underlying asset and the credit risk of the counterparty, but also for the option writer to have other liabilities. Further, the proportion of nominal claims paid out in default is endogenous to the model and is based on the terminal value of the assets of the counterparty and the amount of other equally ranking claims. Numerical examples compare the results of this model with those of other pricing formulas based on alternative assumptions, and illustrate how the model can be calibrated using market data.

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1. Introduction

The rapid growth in the use of over-the-counter options by a diverse set of counterparties has motivated increased attention on the implications of counter-

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party credit risk. Unlike the listed options and futures options markets, there is no organizing exchange in over-the-counter markets requiring that options positions be marked to market on a daily basis and sufficient collateral posted. The holders of over-the-counter options are exposed to potential credit risk due to the possibility of their counterparty being unable to make the necessary payments at the exercise date. Accordingly, options which are vulnerable to counterparty credit risk have lower values than otherwise identical non-vulnerable options. This difference is important to all participants in the over-the-counter options market when pricing these products. It also needs to be considered by the banks and other financial institutions making markets in these products when accounting and allocating capital for their derivatives portfolios, particularly given the current trend towards greater regulation of capital adequacy.

This paper derives an analytic formula for the value of vulnerable Black–Scholes options when the proportion of nominal claims paid out in default depends on the amount of other liabilities as well as the assets of the counterparty. The model also allows for correlation between the credit risk of the counterparty and the asset underlying the option. The contribution of the pricing formula which is derived under these assumptions is an improved ability to price vulnerable Black–Scholes options in many business situations. It extends Johnson and Stulz (1987) by allowing the option writer to have other liabilities which rank equally with payments under the option. It also extends Hull and White (1995) and Jarrow and Turnbull (1995) by relaxing the assumption of independence between the assets of the counterparty and the asset underlying the option, and by specifying a payout ratio which is endogenous to the model. [1] Further, the pricing formula can be evaluated without numerical integration, and can easily be calibrated using the yield on outstanding risky bonds.

This paper is organized as follows. Section 2 reviews existing models of credit risk and the effect of credit risk on option prices. Section 3 outlines a model of expected credit loss and derives the expected present value of a claim of a fixed nominal amount when credit risk is present. Section 4 extends this analysis to vulnerable Black–Scholes call and put options. Section 5 discusses how the pricing formula is a general case of the standard Black–Scholes pricing formula when there is no credit risk, the Hull and White (1995) model assuming independence between the asset underlying the option and the credit risk of the counterparty, and other options pricing formulas involving two correlated assets. Section 6 provides numerical examples, and compares the effect of credit risk on option prices with the results of Johnson and Stulz (1987), and Hull and White (1995). A method of estimating model parameters from market data is discussed in Section 7, followed by a brief conclusion in Section 8.

[1] This paper considers only Black–Scholes options in contrast with Hull and White (1995) and Jarrow and Turnbull (1995) who also considered fixed income options and other derivative securities.

2. Review of credit risk models

The effect of credit risk on realized bond yields has been well studied empirically by Altman (1989), Altman (1990), Altman (1992), and Altman (1993). Two theoretical approaches have been used to model this effect when pricing bonds subject to credit risk. The first defines an event of default as occurring when the value of the assets of the firm is less than some boundary, for example the value of outstanding debt, at the maturity of the bond under consideration. Examples of this approach include Black and Scholes (1973), Merton (1974), Black and Cox (1976), Smith and Warner (1979), Lee (1981), Ho and Singer (1982), Pitts and Selby (1983), Yawitz et al. (1985) and Chance (1990). The payout ratio in default typically depends on the terminal value of the assets of the bond issuer which is endogenous to the model. This link between the payout ratio and the reason for default has intuitive appeal and is one of the strengths of this approach.

The second approach to modelling the effect of credit risk on bond prices allows greater flexibility in the timing of an event of default. Examples of this approach include Jonkhart (1979), Longstaff and Schwartz (1995), Nielsen et al. (1995) and Madan and Unal (1994). In these models, default can occur at any time prior to the maturity of the bond if an exogenous default boundary is hit. It should be noted, however, that the greater flexibility concerning the timing of default is not without its cost. In order to make these models tractable, the payout ratio is usually assumed to be fixed and not to depend on the cause or timing of default, nor on the assets of the bond issuer.

Existing literature on the effect on option prices of credit risk in the option writer involves extending one of the above approaches to the case when the nominal amount of the claim depends on the value of the asset underlying the option. Three papers stand out in the area: Johnson and Stulz (1987), Hull and White (1995) and Jarrow and Turnbull (1995).

Johnson and Stulz (1987) follow the first approach when deriving pricing formulas for vulnerable European options. They assume that the option is the sole liability of the counterparty and model default as occurring when the value of the option is greater than the value of the assets of the counterparty upon exercise. In this case the option holder receives all the assets of the counterparty. This approach implies a variable default boundary equal to the value of the option upon the exercise date. As a result, credit risk arises not only from possible deterioration in the value of the assets of the counterparty, but also from growth in the value of the option itself. Thus even if the assets of the counterparty have not deteriorated over the life of the option, a credit loss may still occur if the assets have not grown sufficiently in order to keep up with the growth in the value of the option.

This approach is appropriate if the expected payoff on the option is relatively large as compared to the total value of the counterparty's assets. Johnson and Stulz (1987) provide a base case example in which the non-vulnerable value of the option is \$3.07, as compared to a value of the total assets of the writer of only \$5.00. In this case a variable default boundary may be a good approximation even when other small liabilities exist since default, if it occurs, will likely be due to growth in the value of the option. It should also be noted that this model implies there is very little credit risk if the expected payoff on the option is small as compared to the total value of the counterparty's assets. In this case it is unlikely that the option will grow in value sufficiently for a credit loss to occur. Due to the assumption that the counterparty has no liabilities apart from the option, credit risk is minimal because there is no other mechanism by which default could occur.

These assumptions are not appropriate in most business settings. Despite a few well-publicized exceptions, neither derivatives market makers nor users generally take positions in individual options which are as large in comparison with their total assets as in the example in Johnson and Stulz (1987). Option positions are generally insignificant on an individual basis as compared to the total assets of the counterparty. Further, the option is not often the only liability of the option writer. Usually there are other claims which will compete with the option holder for a share of the assets of the defaulted option writer. As a result, the model in Johnson and Stulz (1987) may not properly measure the credit risk in most business situations.

The model in Johnson and Stulz (1987) also implies that there will be significant credit risk in a call option even when the assets of the counterparty are positively correlated with the underlying asset. This point is illustrated in their numerical examples in which the value of a vulnerable option is substantially below the non-vulnerable value for both positive and negative correlation. [2] This result is in contrast with market practice to consider credit risk as very small when the assets of the counterparty are positively correlated with the asset underlying the option.

Another implication of the Johnson and Stulz (1987) model is that credit risk could easily be eliminated through proper hedging by the option writer. With proper hedging, the total assets of the writer would never fall below the value of the option since in the absence of other claims, the net value of the other assets of the writer would never be negative. This result would not hold in the presence of other claims on the counterparty. Thus the Johnson and Stulz (1987) model may understate the credit risk on a hedged option position when the counterparty has other liabilities.

In contrast to Johnson and Stulz (1987), Hull and White (1995) assume competing claims can exist which rank equally with the option in default. They

[2] Specifically, values of the vulnerable option are \$1.96, \$1.82 and \$1.69 for correlation coefficients of $+0.5$, 0 and -0.5 respectively, as compared to \$3.07 for the non-vulnerable option in Table I of Johnson and Stulz (1987). These results are reproduced in Table 1 of this paper, with prices for similar vulnerable options according to the model developed herein.

take the second approach to modelling the effect of credit risk on American and European options. Default can occur at any time prior to maturity of the option if the value of the assets of the option writer falls below a specified default boundary. They assume that in default only a proportion of the nominal amount of a claim is paid out. They allow this proportion to be a general function of a number of variables, but do not relate it directly to the value of the assets of the counterparty. Their numerical examples treat this proportion as exogenous. They derive an analytic solution for the value of vulnerable options, but only when the process for the asset underlying the option is independent from the credit risk of the counterparty.

Jarrow and Turnbull (1995) also take the second approach when considering the effect of credit risk on fixed income and other options. They model not only the effect of credit risk of the option writer, but make the further contribution of modelling the effect on option prices of credit risk in the asset underlying the option. They use a foreign currency analogy where both the term structure of credit spreads and the term structure of non-vulnerable interest rates are exogenous, but assume that these processes are independent for simplicity. As in Hull and White (1995), they also assume for simplicity that the payout ratio is exogenous.

The exogenous payout ratio assumed in these models implies that the value of the assets of the financially distressed firm and the nominal amount of the claims on that firm are fixed at the time of default. This implication is inappropriate because of the length of time typically involved in the resolution of financial distress which has been reported by Wruck (1990) to average two and one half years. During that time the assets of the counterparty may recover and claims be paid out in full. Further, the exercise date of most over-the-counter options will be within this period. As a result, holders of options written by distressed counterparties may choose not to calculate the settlement amount under the option contract, but wait for the exercise date when calculating their claims. Accordingly, the imposition of an exogenous payout rate may overstate the effect of credit risk on option values.

As discussed above, the independence assumption is made in the Hull and White (1995) and Jarrow and Turnbull (1995) models in order to keep the pricing formulas reasonably tractable. [3] As noted by Hull and White (1995), this assumption is likely appropriate when the option writer is a large, well diversified financial institution. It is not realistic in many other situations, such as when an option position is written as a part of a hedging program. For example, natural resource producers, particularly gold and oil companies, often hedge in order to

[3] As noted by a reviewer, the independence assumption was not made in Johnson and Stulz (1987). As discussed above, this paper extends Johnson and Stulz (1987) by relaxing the assumption that the option writer has no other liabilities.

dampen the effect of resource price volatility on their earnings. Industrial companies are often involved in options-based raw material price hedging programs, for example, caps, floors and collars on heating oil. Writer's positions in options are usually part of such hedging programs. It is almost certain that the credit risk of the option writer is correlated with the asset underlying the option, given the nature of the risks which are being hedged. Modelling this correlation is important not only when pricing these options, but also when banks and other financial institutions allocate regulatory capital to such transactions.

3. A model of expected credit loss

This paper departs from Hull and White (1995) and Jarrow and Turnbull (1995) and follows the first approach to modelling credit risk as in Black and Scholes (1973), Merton (1974), Black and Cox (1976), Smith and Warner (1979), Lee (1981), Ho and Singer (1982), Pitts and Selby (1983), Yawitz et al. (1985) and Chance (1990). This approach allows for the presence of other liabilities and a proportional recovery of nominal claims in default, but has the advantage of explicitly relating the payout ratio to the value of the assets of the counterparty. It also allows correlation between the assets of the counterparty and the asset underlying the option. As discussed above, these assumptions are appropriate in many business situations and should improve the quality of pricing vulnerable options accordingly.

It is assumed that the value of the assets of the counterparty, V, is large as compared to the expected payoff under the option. V is defined to include the current market value of all assets of the counterparty as well as the marked to market value of all derivative and other contracts, including the writer's position in the option being valued and any hedging related thereto. [4] The default event itself is not directly considered. Instead, and of more relevance to the option holder, the expected credit loss at the expiry of the option is modelled. At the exercise date T, a credit loss occurs if V_T is less than some amount D^*. In contrast with Johnson and Stulz (1987), this amount is not set to the value of the option but roughly corresponds to the amount of claims D outstanding at time T. [5] D^* may be less than D due to the possibility of a counterparty continuing in operation even while V_T is less than D. In the event of a credit loss, only the proportion $(1-\alpha)V_T/D$ of the nominal claim [6] is paid out by the counterparty, where α represents the

[4] As in all models reviewed in this paper, the effect on the value of the firm of contracts with bad credits is included in this definition.

[5] It is also assumed that there are no claims senior to these liabilities. This model could easily be extended to cover the presence of senior claims as well by conditioning Eq. (9) over three ranges of terminal asset values.

[6] In other words, the amount paid in default depends on the nominal amount of the claim as well as on the net worth of the defaulting party.

deadweight costs associated with bankruptcy expressed as a percentage of the value of the assets of the counterparty. These deadweight costs include the direct cost of the bankruptcy or reorganization process, as well as the effects of distress on the business operations of the firm. If V_T is greater than D^* no credit loss occurs and the time T claim receives full payment.[7]

Note that in this model, default is not restricted to occur only at time T. It is assumed, however, that if default occurs prior to T, the deadweight costs associated with bankruptcy are applied only if the assets of the counterparty have not recovered to above the default boundary by time T. Accordingly, the amount $(1-\alpha)V_T/D$ representing the value of the assets of the counterparty available for distribution to all creditors does not depend on the precise time at which default occurs.

It is assumed that V follows geometric Brownian motion with instantaneous volatility σ_V. It is also assumed that V is a traded security; although V is not likely traded directly, the market value of the assets of the counterparty behaves as if it were a traded asset. It can easily be shown using the risk neutrality approach of Cox and Ross (1976) and Harrison and Pliska (1981) that for the purpose of pricing derivative securities dependent on V, the appropriate risk neutral process for V is:

$$\frac{dV}{V} = r\,dt + \sigma_V\,dw$$

where r denotes the riskfree rate and w follows a standard Wiener process. This process implies that $\ln V_T$ is normally distributed with mean of $(r-\sigma_V^2/2)(T-t)$ and standard deviation of $\sigma_V(T-t)^{1/2}$.

The expected actual payout B^* corresponding to a nominal claim of B on the counterparty can be written as

$$B^* = \mathrm{E}^*\left[B|V_T \geq D^*\right] + \mathrm{E}^*\left[B(1-\alpha)V_T/D|V_T < D^*\right] \tag{1}$$

where E^* denotes risk neutral expectations. This expression indicates that the expected future payout on a nominal claim of amount B is comprised of two parts which are conditional on the terminal value of the assets of the counterparty. The claim is paid out in full if the assets of the counterparty are above the default boundary. If the assets of the counterparty are below the default boundary, the payout is only a proportion of the nominal claim where the proportion depends on the value of the assets as well as other liabilities of the counterparty.

[7] As noted by a reviewer, this approach is similar to the risky debt model of Longstaff and Schwartz (1995) in which the assets of the counterparty follow geometric Brownian motion and default occurs if an exogenous default boundary is hit. Longstaff and Schwartz (1995) do not link the payout ratio to the value of the assets upon default, but assume an exogenous payout rate.

It can be shown [8] that if B is a fixed nominal amount, the undiscounted expected value of the amount actually to be recovered can be expressed as

$$B^* = B\left(\mathrm{N}_1(b_2) + \mathrm{e}^{r(T-t)}\mathrm{N}_1(d_2)(1-\alpha)V_t/D\right) \tag{2}$$

where

$$b_2 = \frac{\ln V_t/D^* + \left(r - \sigma_V^2/2\right)(T-t)}{\sigma_V\sqrt{T-t}}$$

$$d_2 = -\left[b_2 + \sigma_V\sqrt{T-t}\right]$$

and N_1 is the standard univariate normal cumulative distribution function.

When B equals one, Eq. (2) may be used to value a zero coupon bond issued by the counterparty. If r^* is the yield on a traded zero coupon bond, the following relationship must hold:

$$\mathrm{e}^{-r^*(T-t)} = \mathrm{e}^{-r(T-t)}\left(\mathrm{N}_1(b_2) + \mathrm{e}^{r(T-t)}\mathrm{N}_1(d_2)(1-\alpha)V_t/D\right).$$

The terms in large brackets represent the undiscounted effect of credit risk on a cash flow of fixed nominal amount. This effect can be estimated through credit spreads on traded bonds as follows:

$$\mathrm{e}^{-(r^*-r)(T-t)} = \mathrm{N}_1(b_2) + \mathrm{e}^{r(T-t)}\mathrm{N}_1(d_2)(1-\alpha)V_t/D \tag{3}$$

where $r - r^*$ represents the difference in yield between risky zero coupon bonds of the counterparty and a similar term riskless zero coupon bond.

4. Vulnerable Black–Scholes options

The model of credit risk in the preceding section is similar to other credit risk models based on Merton (1974) as discussed above. The main contribution of this paper is the extension of this model to claims with nominal amounts equal to the payoff under a Black–Scholes option, which is the topic of this section.

Vulnerable Black–Scholes options can be valued in a similar manner when the approach is generalized to include another variable, S, the value of the asset underlying the option. It is assumed that S and V follow the diffusion processes:

$$\frac{dS}{S} = \mu_S\,dt + \sigma_S\,dz,$$

$$\frac{dV}{V} = \mu_V\,dt + \sigma_V\,dw,$$

where z and w follow standard Wiener processes with instantaneous correlation ρ. It is assumed that both S and V are traded; although V is not likely traded directly,

[8] This derivation is available from the author upon request.

the market value of the assets of the counterparty behaves as if it were a traded asset, as discussed above.

Based on these assumptions, it can easily be shown in the usual way that for the purposes of pricing derivative securities depending on S and V, the appropriate risk neutral processes are:

$$\frac{dS}{S} = r\,dt + \sigma_S\,dz, \tag{4}$$

$$\frac{dV}{V} = r\,dt + \sigma_V\,dw \tag{5}$$

Applying Ito's lemma results in the following processes for $\ln S$ and $\ln V$:

$$d\ln S = \left(r - \sigma_S^2/2\right)dt + \sigma_S\,dz, \tag{6}$$

$$d\ln V = \left(r - \sigma_V^2/2\right)dt + \sigma_V\,dw. \tag{7}$$

From Eqs. (6) and (7), it is clear that $\ln S_T$ and $\ln V_T$ are bivariate normally distributed [9] as

$$\mathrm{n}_2\left(\ln S_t + \left(r - \frac{\sigma_S^2}{2}\right)(T-t),\right.$$
$$\left.\ln V_t + \left(r - \frac{\sigma_V^2}{2}\right)(T-t), \sigma_S\sqrt{T-t}, \sigma_V\sqrt{T-t}, \rho\right), \tag{8}$$

where n_2 is the standard bivariate normal probability density function. This joint distribution of $\ln S_T$ and $\ln V_T$ above implies that S_T and V_T are bivariate lognormally distributed, [10] i.e.,

$$S_T, V_T \sim \Lambda_2$$

where bivariate lognormal distribution Λ_2 corresponds to the risk neutral processes described in Eqs. (4) and (5) above.

The value of a vulnerable call C^* is the expectation of the value of the cash flow from a non-vulnerable call C times the value of a claim on the risky counterparty as specified in Eq. (1) above. Accordingly, the value of C^* may be written as follows:

$$C^* = \mathrm{e}^{-r(T-t)}\mathrm{E}^*\left[\max(S_T - K, 0)\left(\left[1 | V_T \geq D^*\right]\right.\right.$$
$$\left.\left. + \left[(1-\alpha)V_T/D | V_T < D^*\right]\right)\right] \tag{9}$$

[9] See Abramowitz and Stegun (1972) for properties of the bivariate normal distribution.

[10] See Aitchison and Brown (1966) for properties of the lognormal distribution.

where E^* denotes risk neutral expectations over S_T and V_T and K is the exercise price.[11] Eq. (9) can be restated as follows:

$$\begin{aligned} C^* = \mathrm{e}^{-r(T-t)}\big(&\mathrm{E}^*\left[S_T|S_T \geq K, V_T \geq D^*\right] - \mathrm{E}^*\left[K|S_T \geq K, V_T \geq D^*\right] \\ &+\mathrm{E}^*\left[S_T(1-\alpha)V_T/D|S_T \geq K, V_T < D^*\right] \\ &-\mathrm{E}^*\left[K(1-\alpha)V_T/D|S_T \geq K, V_T < D^*\right]\big). \end{aligned} \tag{10}$$

Eq. (10) illustrates that the value of a vulnerable European call comprises four terms which are conditional on whether S_T is greater than the exercise price K and whether V_T is greater or less than the default boundary D^*. It can be shown[12] that Eq. (10) can be expressed as the following pricing formula for the value of a vulnerable European call option:

$$\begin{aligned} C^* = \; & S_t\big(\mathrm{N}_2(a_1, a_2, \rho) + \mathrm{e}^{(r+\rho\sigma_S\sigma_V)(T-t)}(1-\alpha)V_t/D\,\mathrm{N}_2(c_1, c_2, -\rho)\big) \\ & -\mathrm{e}^{-r(T-t)}K\big(\mathrm{N}_2(b_1, b_2, \rho) + \mathrm{e}^{r(T-t)}(1-\alpha)V_t/D\,\mathrm{N}_2(d_1, d_2, -\rho)\big) \\ = \; & S_t\,\mathrm{N}_2(a_1, a_2, \rho) - \mathrm{e}^{-r(T-t)}K\,\mathrm{N}_2(b_1, b_2, \rho) \\ & +(1-\alpha)V_t/D\big(S_t\mathrm{e}^{(r+\rho\sigma_S\sigma_V)(T-t)}\,\mathrm{N}_2(c_1, c_2, -\rho) - K\,\mathrm{N}_2(d_1, d_2, -\rho)\big) \end{aligned} \tag{11}$$

where the arguments to N_2, the bivariate normal cumulative distribution function, are as follows:

$$a_1 = \frac{\ln(S_t/K) + \left(r + \sigma_S^2/2\right)(T-t)}{\sigma_S\sqrt{T-t}} = b_1 + \sigma_S\sqrt{T-t},$$

$$a_2 = \frac{\ln(V_t/D^*) + \left(r - \sigma_V^2/2 + \rho\sigma_S\sigma_V\right)(T-t)}{\sigma_V\sqrt{T-t}} = b_2 + \rho\sigma_S\sqrt{T-t},$$

$$b_1 = \frac{\ln(S_t/K) + \left(r - \sigma_S^2/2\right)(T-t)}{\sigma_S\sqrt{T-t}},$$

$$b_2 = \frac{\ln(V_t/D^*) + \left(r - \sigma_V^2/2\right)(T-t)}{\sigma_V\sqrt{T-t}},$$

$$c_1 = \frac{\ln(S_t/K) + \left(r + \sigma_S^2/2 + \rho\sigma_S\sigma_V\right)(T-t)}{\sigma_S\sqrt{T-t}} = b_1 + (\sigma_S + \rho\sigma_V)\sqrt{T-t},$$

[11] If there is no credit risk, D^* equals zero and Eq. (9) immediately simplifies to the standard expression for a non-vulnerable European call option.

[12] This derivation is available from the author upon request.

$$c_2 = -\frac{\ln(V_t/D^*) + (r + \sigma_V^2/2 + \rho\sigma_S\sigma_V)(T-t)}{\sigma_V\sqrt{T-t}}$$
$$= -\left[b_2 + (\sigma_V + \rho\sigma_S)\sqrt{T-t}\right],$$
$$d_1 = \frac{\ln(S_t/K) + (r - \sigma_S^2/2 + \rho\sigma_S\sigma_V)(T-t)}{\sigma_S\sqrt{T-t}} = b_1 + \rho\sigma_V\sqrt{T-t},$$
$$d_2 = -\frac{\ln(V_t/D^*) + (r + \sigma_V^2/2)(T-t)}{\sigma_V\sqrt{T-t}} = -\left[b_2 + \sigma_V\sqrt{T-t}\right].$$

Similarly, the pricing formula for a vulnerable European put option is as follows:

$$\begin{aligned} P^* = {} & K\,\mathrm{e}^{-r(T-t)}\big(\mathrm{N}_2(-b_1, b_2, \rho) \\ & + (1-\alpha)V_t/D\,\mathrm{e}^{r(T-t)}\,\mathrm{N}_2(-d_1, d_2, -\rho)\big) - S_t\big(\mathrm{N}_2(-a_1, a_2, \rho) \\ & + \mathrm{e}^{(r+\rho\sigma_S\sigma_V)(T-t)}(1-\alpha)V_t/D\,\mathrm{N}_2(-c_1, c_2, -\rho)\big) \end{aligned} \tag{12}$$

It can also be shown that a European put or call option on a stock paying a continuous dividend yield q can be valued using the above formulas when r is replaced with $r-q$ in the first set of parameters to the bivariate normal distribution (the a_1, b_1, c_1 and d_1 terms), and S_t is discounted by $\mathrm{e}^{-q(T-t)}$. With these adjustments, Eqs. (11) and (12) can be used to value various other products such as currency options and options on futures.

5. Specific cases of the pricing formula

5.1. Non-vulnerable options

It is easily seen from Eq. (9) that if there is no risk of bankruptcy, i.e., D^* equals zero, then the price of the vulnerable call equals the Black–Scholes price. This result can also be verified by evaluating the limit of Eq. (11) as D^* goes to zero.

The effect on the parameters to the bivariate normal distribution function as D^* goes to zero is as follows:

$$\lim_{D^* \to 0} a_2 = \lim_{D^* \to 0} b_2 = \infty,$$
$$\lim_{D^* \to 0} c_2 = \lim_{D^* \to 0} d_2 = -\infty.$$

Recognizing $\mathrm{N}_2(x, \infty, \rho)$ as the marginal distribution of x which is distributed as $\mathrm{N}_1(x)$, and noting that $\mathrm{N}_2(x, -\infty, \rho)$ equals zero allows Eq. (11) to be rewritten as

$$C^* = C = S_t\,\mathrm{N}_1(a_1) - \mathrm{e}^{-r(T-t)}K\,\mathrm{N}_1(b_1)$$

which is the standard Black–Scholes result.

5.2. Independence

When the credit risk of the counterparty is independent from the process for the asset underlying the option ρ equals zero and Eq. (11) simplifies as follows:

$$\begin{aligned} C^* = &\quad S_t \mathrm{N}_1(a_1)\big(\mathrm{N}_1(a_2) + \mathrm{e}^{r(T-t)}\mathrm{N}_1(c_2)(1-\alpha)V_t/D\big) \\ &\quad -\mathrm{e}^{-r(T-t)}K\,\mathrm{N}_1(b_1)\big(\mathrm{N}_1(b_2) + \mathrm{e}^{r(T-t)}\mathrm{N}_1(d_2)(1-\alpha)V_t/D\big) \\ = &\quad \big(S_t \mathrm{N}_1(a_1) - \mathrm{e}^{-r(T-t)}K\,\mathrm{N}_1(b_1)\big)\big(\mathrm{N}_1(b_2) + \mathrm{e}^{r(T-t)}\mathrm{N}_1(d_2)(1-\alpha)V_t/D\big) \end{aligned}$$

since $c_1 = a_1$, $b_1 = d_1$, $d_2 = c_2$, and $a_2 = b_2$.

Note that the expression in the right-hand set of brackets corresponds to the undiscounted expected value of a claim of fixed nominal amount in Eq. (2). This result is similar to Hull and White (1995) under the independence assumption, but expressly relates the undiscounted expected value of a claim of fixed nominal amount to the assets of the counterparty. As discussed above, this amount can be estimated by considering the difference between the yield on traded bonds and the riskfree rate.

5.3. Certain default

Although the pricing of options that are certain to be in default is not of interest directly, the formulas in this paper can be applied assuming default is certain to value options on the product of two correlated assets. For example, the pricing formula of Margrabe (1978) for the value of an option to exchange one asset for another can also be shown to be a special case of Eq. (11) in the limit as D^* approaches infinity.

The expected present value of the payoff from an option to exchange asset X for asset Y can be rewritten as

$$\mathrm{e}^{-r(T-t)}\mathrm{E}^*\big[\max(X_T - Y_T, 0) = \mathrm{e}^{-r(T-t)}E^*\big[\max(Y_T(G_T - 1), 0)\big] \quad (13)$$

where $G_T = X_T/Y_T$. Assuming both X and Y follow geometric Brownian motion allows Ito's lemma to be applied to determine the process for $\ln G$ as follows:

$$\mathrm{d}\ln G = \tfrac{1}{2}\big(\sigma_Y^2 - \sigma_X^2\big)\mathrm{d}t + \sigma_G\,\mathrm{d}z$$

where

$$\sigma_G = \sqrt{\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2}\,.$$

It can also be shown that the instantaneous correlation between G and Y can be expressed as

$$\rho_{GY} = \frac{\sigma_X}{\sigma_Y}\left(\rho_{XY} - \frac{\sigma_Y}{\sigma_X}\right).$$

Eq. (13) is similar to Eq. (10) in the limit as D^* approaches infinity. In this case the first two terms disappear in Eq. (11) which can be rewritten as

$$C^* = (1-\alpha)V_t/D\left(S_t e^{(r_S^* + \sigma_S^2/2 + \rho\sigma_S\sigma_V)(T-t)} N_2(c_1, \infty, -\rho) - K N_2(d_1, \infty, -\rho)\right)$$

where

$$c_1 = \frac{\ln(S_t/K) + \left(r_S^* + \sigma_S^2 + \rho\sigma_S\sigma_V\right)(T-t)}{\sigma_S\sqrt{T-t}},$$

$$d_1 = c_1 - \sigma_S\sqrt{T-t},$$

and r_S^* is the drift of the risk neutral process for ln S. Setting $D = K = 1$, $\alpha = 0$ and substituting S and V for G and Y respectively yields

$$e^{-r(T-t)} E^*\left[\max(X_T - Y_T, 0)\right] = X_0 N_1(v_1) - Y_0 N_1(v_2)$$

where

$$v_1 = \frac{\ln(X_t/Y_t) + \sigma_G^2(T-t)/2}{\sigma_G\sqrt{T-t}},$$

$$v_2 = v_1 - \sigma_G\sqrt{T-t},$$

which is the desired result.

In a similar manner, it can also be shown that the cross currency option pricing results of Wei (1991) are special cases of Eqs. (11) and (12).

6. Numerical examples

As discussed above, the Johnson and Stulz (1987), Hull and White (1995), and Jarrow and Turnbull (1995) models may have the effect of overstating the impact of credit risk on the value of Black–Scholes options in many business situations. This greater effect of credit risk is due to the difference in their assumptions concerning default and the amount recovered in bankruptcy as compared to those of the model presented herein. Although direct comparison with these models is not possible because of the differing assumptions, the numerical examples of both Johnson and Stulz (1987) and Hull and White (1995) are reproduced in Tables 1 and 2, with values for similar vulnerable options based on the model of this paper and the approximation to the bivariate normal integral of Drezner (1978). [13]

The calculations in Table 1 indicate that the effect of credit risk on vulnerable option prices is generally less than that reported by Johnson and Stulz (1987),

[13] Jarrow and Turnbull (1995) do not provide numerical examples for vulnerable Black–Scholes options.

Table 1
Values of vulnerable call options [a]

Case	$\alpha=0$	$\alpha=0.25$	$\alpha=0.5$	$\alpha=1.0$	BS value	JS value
Base case	3.00	2.84	2.67	2.33	3.07	1.96
$\sigma_S=0.2$	2.12	1.99	1.87	1.62	2.17	1.77
$\sigma_S=0.4$	3.90	3.68	3.47	3.04	3.98	2.04
$\sigma_V=0.2$	3.03	2.87	2.72	2.40	3.07	1.92
$\sigma_V=0.4$	2.98	2.80	2.62	2.27	3.07	2.00
$\rho=-0.5$	2.70	2.22	1.73	0.76	3.07	1.69
$\rho=0$	2.88	2.55	2.21	1.54	3.07	1.82
$T-t=0.0833$	1.44	1.36	1.27	1.10	1.46	1.33
$T-t=0.5833$	4.07	3.85	3.62	3.18	4.19	2.18
$K=30$	10.16	9.29	8.40	6.65	10.58	4.50
$K=50$	0.43	0.42	0.41	0.38	0.44	0.34
$S_t=30$	0.15	0.14	0.14	0.14	0.15	0.13
$S_t=50$	10.54	9.68	8.83	7.11	10.64	4.24
$V_t=3$	2.09	1.57	1.05	0.02	3.07	1.34
$V_t=10$	3.07	3.07	3.07	3.07	3.07	2.77

[a] Calculations of values of vulnerable call options based on following parameter values: $\sigma_S=0.3$, $\sigma_V=0.3$, $\rho=0.5$, $r=0.04833$, $T-t=0.3333$, $K=40$, $S_t=40$, $V_t=5$, $D=5$, $D^*=5$, unless otherwise noted. Black–Scholes values, and values reported by Johnson and Stulz (1987) are reported in the last two columns for comparison.

except when deadweight costs of bankruptcy are very high. This result is due to the reduced emphasis in the model developed in this paper on the ability of the assets of the option writer to keep up with growth in the value of the option. As discussed above, this reduced emphasis is appropriate when there are other liabilities of the option writer which may also cause default.

Table 1 also indicates that in contrast to Johnson and Stulz (1987), the model in this paper calculates the effect of credit risk on Black–Scholes option values as being much less for positive correlation between the assets of the counterparty and the asset underlying the option, as compared to when this correlation is negative. This result is consistent with market practice and improves the quality of pricing these options as discussed above.

Table 2 compares reductions in option prices due to credit risk as calculated by the model in this paper with the results of Hull and White (1995). In general, the effect of credit risk calculated using the model of this paper is less than that reported by Hull and White (1995) for European options, and is more similar to their calculations for American options. This difference is due to the relatively low payout ratio, assumed to be 50% in all cases by Hull and White (1995), as compared to the model of this paper which bases the payout ratio on the value of

Table 2
Percentage reduction in call values due to credit risk. [a]

		$D = D^*$			
ρ	α	90	92	94	96
−0.8	α = 0.00	0.00	0.05	0.16	0.41
	= 0.25	0.22	0.75	2.10	4.79
	= 0.50	0.42	1.46	4.03	9.16
	= 1.00	0.84	2.86	7.91	17.92
	HW – European	0.86	3.09	8.93	20.43
	– American	0.00	0.00	0.10	1.73
−0.4	α = 0.00	0.01	0.02	0.08	0.21
	= 0.25	0.10	0.36	1.04	2.49
	= 0.50	0.20	0.70	2.00	4.78
	= 1.00	0.39	1.37	3.93	9.35
	HW – European	0.43	1.61	4.74	11.94
	– American	0.01	0.09	0.59	3.03
0.0	α = 0.00	0.00	0.01	0.02	0.07
	= 0.25	0.03	0.11	0.36	0.97
	= 0.50	0.06	0.21	0.69	1.87
	= 1.00	0.11	0.42	1.36	3.67
	HW – European	0.15	0.62	2.22	6.97
	– American	0.03	0.15	0.73	3.24
0.4	α = 0.00	0.00	0.00	0.00	0.01
	= 0.25	0.00	0.01	0.05	0.19
	= 0.50	0.00	0.02	0.10	0.37
	= 1.00	0.01	0.04	0.19	0.72
	HW – European	0.01	0.10	0.61	2.84
	– American	0.01	0.06	0.40	2.07
0.8	α = 0.00	0.00	0.00	0.00	0.00
	= 0.25	0.00	0.00	0.00	0.00
	= 0.50	0.00	0.00	0.00	0.00
	= 1.00	0.00	0.00	0.00	0.00
	HW – European	0.00	0.00	0.06	0.91
	– American	0.00	0.00	0.06	0.89

[a] Percentage reduction due to default risk in prices of call options based on following parameter values: $\sigma_S = 0.15$, $\sigma_V = 0.05$, $r = 0.05$, $q = 0.05$, $T - t = 1$, $K = 1$, $S_t = 1$, $V_t = 100$. Values reported by Hull and White (1995, Table 2) are also provided for comparison.

the assets of the counterparty which may recover subsequent to an event of default.

7. Estimation of model parameters

Although σ_V and ρ can not be measured directly using market information, their values can be estimated from stock price data. Noting that V_t, the market value of the assets of the counterparty, equals the market value of all of its securities allows E_t, the value of the equity of the firm, to be written as $E_t = V_t - B_t^*$ where B_t^* is the market value of the risky debt of the firm. B_t^* depends on V_t as in Eq. (2). Applying Ito's lemma to $G_t = \ln E_t = \ln(V_t - B_t^*)$ allows σ_E, the volatility of E_t, to be written in terms of σ_V as

$$\sigma_E = \sigma_V V_t / E_t \left((1-\alpha)\,\mathrm{N}_1(d_2) + \frac{\alpha}{\sigma_V \sqrt{T-t}}\,\mathrm{n}_1(d_2) \right).$$

Rearranging this equation yields an expression for σ_V as follows:

$$\begin{aligned}\sigma_V &= \frac{E_t}{V_t}\left((1-\alpha)\,\mathrm{N}_1(d_2) + \frac{\alpha}{\sigma_V \sqrt{T-t}}\,\mathrm{n}_1(d_2) \right)^{-1} \sigma_E \\ &= \frac{E_t}{V_t}\,\mathrm{N}_1(d_2)^{-1}\sigma_E \end{aligned} \tag{14}$$

when $\alpha = 0$. Based on this relationship between σ_V and σ_E, the instantaneous

Table 3
Calculation of yield differentials due to credit risk. [a]

Case	$T-t$	Annual yield differential (bps)	σ_V
Base case	1	0	0.150
	2	2	0.151
	3	10	0.152
	4	24	0.153
	5	41	0.155
$\sigma_E = 0.4$	1	1	0.200
	2	25	0.202
	3	72	0.209
	4	129	0.219
	5	194	0.234
$\alpha = 1.00$	1	0	0.153
	2	4	0.183
	3	20	0.210
	4	43	0.236
	5	70	0.262
$\sigma_E = 0.7$, $D_0 = D_0^* = 80$	1	309	0.153
	2	570	0.183
	3	683	0.210
	4	740	0.236
	5	779	0.262

[a] Annual yield differentials between riskless bonds and bonds subject to credit risk based on following parameter values: $\sigma_E = 0.3$, $V_t = 100$, $D = 50$, $D^* = 50$, $r = 0.05$, $T - t = 1$, $\alpha = 0.25$, unless otherwise noted. Calculated values for σ_V based on σ_E are also provided.

Table 4
Example of model calibration and use of pricing formula. [a]

D	D^*	σ_V	ρ	C^*	C	% Reduction
50	48.65	0.229	0.4	14.29	14.38	0.6
			0.0	13.95	14.38	3.0
			−0.4	13.22	14.38	8.1
60	57.45	0.183	0.4	14.32	14.38	0.4
			0.0	14.05	14.38	2.3
			−0.4	13.45	14.38	6.5

[a] Calculations illustrating model calibration and use of pricing formula are based on following parameter values: $T-t=5$, $S_t=40$, $r^*=0.07$, $r=0.05$, $K=40$, $\sigma_E=0.40$, $V_t=100$, $\sigma_S=0.3$ and $\alpha=0.25$. Values for D^* and σ_V are implied from r^*-r and D.

correlation between the total assets of the firm and the asset underlying the option will equal the correlation between the equity of the firm and the underlying asset.

Table 3 provides examples of yield differentials between riskless bonds and risky bonds issued by the counterparty implied by this approach using Eq. (3) above. It also provides values for σ_V implied by values of σ_E in Eq. (14).

As discussed in Section 2 above, this pricing model may be calibrated with the yield on outstanding risky bonds issued by the counterparty. Table 4 illustrates this approach, and provides values of example vulnerable options based on the calibrated model.

8. Conclusions

A pricing formula for vulnerable Black–Scholes options has been derived which allows for correlation between the credit risk of the counterparty and the asset underlying the option. This formula allows the option writer to have other liabilities in addition to the potential claim under the option. The payout ratio in default is based on the assets as well as other liabilities of the counterparty. This formula is shown through numerical examples to improve the quality of pricing vulnerable Black–Scholes options as compared to models based on alternative assumptions. Further, the model is relatively easy to use, and can be calibrated using market data.

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