Investment Basics: XXXVIII. Options pricing using a binomial lattice

1. INTRODUCTION

The use of binomial trees in the numerical valuation of options was first proposed by Cox, Ross and Rubinstein (1979). Although not as instantaneously recognised as the Black-Scholes (1973) options pricing model, binomial lattices are more easily generalisable and they are often able to handle a variety of conditions for which the former model cannot be applied.

The binomial model uses a discrete time framework to trace the evolution of the key variable upon which the claim of interest is contingent. For instance, the value of the right to buy or sell an ounce of platinum within three months is obviously contingent upon the possible evolution of the underlying platinum price over the period. The relevant binomial lattice therefore describes the evolution of the platinum price over time.

Figure 1 traces this process assuming that the price of platinum "evolves" over discrete intervals of one month duration. At the end of each interval, the price is presumed to increase by 2,93% or decrease by 2,85% from its value at the beginning of the interval. Consequently, from a current price of $369,50, the price either increases to $369,50x1,0293=$380,32 or decreases to $369,50x0,9715=$358,99 over the first month. If it increases to $380,32 then the following month will see the price either increase further to $380,32x1,0293=$391,46 or decrease to $380,32x0,9715=$369,50. The choice of the percentage increase and decrease is based upon the underlying volatility of annualised platinum price relatives[1]. As may be clear from the figure, the two price changing percentages in the example are also related in a way that produces a recombining lattice[2].

Say we wish to value the right to buy platinum for $375 per ounce in three months time. At the date of expiry, the binomial lattice indicates this call option will be worth $27,93 if the underlying platinum price is $402,93, $5,32 if the price is $380,32 and worth nothing if the price is $358,99 or $33885.

Valuing the option at each of the three nodes one interval prior to expiry involves discounting each of the expiry date values and multiplying them by their probability of occurrence. In order to do this, Cox, Ross and Rubinstein (1979) developed a risk-neutral probability of the price increasing as a function of the risk-free rate of return, the percentage increase in price and the percentage decrease in price[3]. Assuming a monthly risk-free rate of return of 6,17% per annum, or 0,50% per one month interval, the relevant probability for the example is 0,579. The option value at A in Figure 2, given as $18,33, is therefore computed as ($27,93x0,579+$5,32x0,421)/1,005.

Once the option values at nodes A, B and C have been determined, the discounting process is repeated to find the option values at nodes D and E. Finally, the current value of the call option is computed as ($11,85x0,579+$1,77x0,421)/1,005=$7,57.

In an identical fashion the value of the right to sell platinum for $375 in three months time is currently valued at $7,50. The binomial lattice node values for this put option are shown in italics in Figure 2.

The conditions underlying the above options also allow the put option to be valued directly from the call option value using the well established put-call parity relationship (Stoll, 1969; Merton, 1973). This relationship was developed for European options, which cannot be exercised early, and for options on non-dividend paying stocks even when early exercise is possible. It states that value of a put option is equal to the value of a call option (having the same exercise price and maturity date) plus the present value of the exercise price minus the current value of the underlying asset. As before, the value of the put is $7,50, computed as $7,57+$375,00/1,005-$369,50.

The impact of dividing the three-month period until the option maturity into an increasing number of intervals is shown in Table 1. For a non-income paying asset such as platinum, the binomial valuation of both calls and puts clearly converges to the Black-Scholes valuation as the length of the discrete time intervals tends to zero.

The numerical illustrations presented in this paper outline the basics of the binomial lattice approach to options valuation. Given its convergence to the Black-

*Graduate School of Business, University of Cape Town, Private Bag, Rondebosch 7701. Email: mpage@gsb2.uct.ac.za

[1]The percentage increase in value, *u*, is computed as $u = \left(e^{\sigma\sqrt{t}} - 1\right) \times 100\%$ where *t* is the interval length as a fraction of a year and δ is the annualised standard deviation (volatility) of the natural logarithm of price relatives. The percentage decrease in value, *d*, is computed as d = (1/(1+0,01u)-1)x100%.

[2]A recombining lattice is one where different paths lead to identical node values. In other words, from any point, an increase followed by a decrease produces the identical outcome to the one achieved when a decrease is followed by an increase.

[3]The risk-neutral probability of an increase is computed as $p = (r_f - d)/(u - d)$.

Scholes model for standard options where the volatility and risk-free rate are assumed constant, the approach is not usually applied in these situations. However, as alluded to in the opening paragraph, the attraction of the binomial lattice lies not in its ability to replicate the Black-Scholes model, but in its ability to handle more complex options. Amongst these are:

(1) Options on uncertain income producing assets that can be exercised early. In this situation, the binomial lattice allows one to adjust the price evolution of the underlying asset for interim cash payments.

(2) Long options on bonds that cannot be valued correctly using Black-Scholes because the evolution of a bond's price does not conform to the distributional assumptions of the Black-Scholes model. The problem of price convergence for a maturing bond is addressed by using the binomial lattice to first develop the evolution of the underlying term structure curve. The evolving bond price is computed using the projected term structure at each node and the option is then valued off the derived bond price lattice.

(3) Real assets options. These are options that are implicit in many capital investment decisions but which are generally ignored in financial evaluations using the net present value decision rule. The binomial lattice approach enables one to consider the value of options to abandon, expand and to defer start-up. Numerical techniques using the binomial lattice are being increasingly applied in finite reserve analysis such as the oil industry (Paddock, Seigel and Smith, 1988).

(4) A range of complex options that includes path dependent barrier options, lookback options, options on options and American exchange options (Rubinstein, 1991).

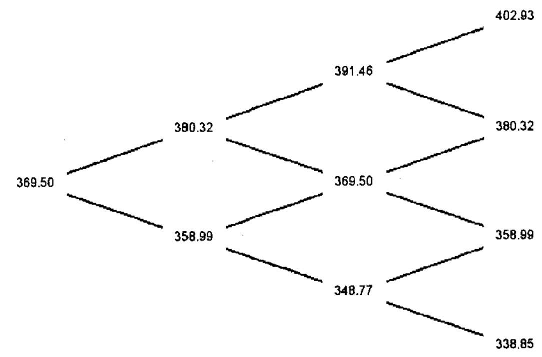

Figure 1: Platinum price evolution over three months from a current price of $369,50 and assuming an annualised standard deviation of 10%

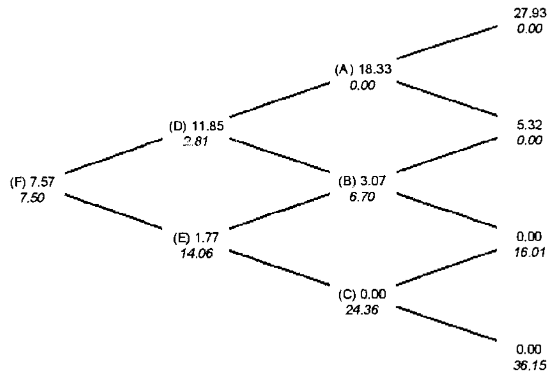

Figure 2: Binomial valuation of; (a) a call option giving the holder the right to buy platinum for $375 in three months; and, (b) a put option giving the holder the right to sell platinum on the same terms

Table 1: Binomial valuation of three month put and call options on platinum having exercise prices of $375.

Number of intervals (until maturity)	Call option value	Put option value
1	8,45	8,38
3	7,57	7,50
6	7,62	7,55
12	7,53	7,46
24	7,44	7,37
60	7,40	7,33
180	7,39	7,32
Black-Scholes	7,40	7,33

REFERENCES

Black F and Scholes M. 1973. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637-659.

Cox JC, Ross SA and Rubinstein M. 1979. Options pricing: a simplified approach, *Journal of Financial Economics*, 7:229-263.

Merton RC. 1973. The relationship between put and call prices: comment, *Journal of Finance*, 28:183-184.

Paddock, JS, Seigel D and Smith J. 1988. Option valuation of claims on physical assets: the case of offshore petroleum leases. *Quarterly Journal of Economics*, 103:479-508.

Stoll HR. 1969. The relationship between put and call prices, *Journal of Finance*, 24:801-824.

Rubinstein M. 1991. Exotic options, Finance Working Paper No. 220. Walter A. Hass School of Business Working Paper Series, University of California at Berkeley.