Chapter 03

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1 3.2.4

How many elements does S_3 have?

$$
\begin{aligned}
|S_0| &= 0 \\
|S_1| &= 3 \\
|S_2| &= 3^3 + 3 \cdot 3 + 3 = 39 \\
|S_3| &= 39^3 + 39 \cdot 3 + 3 = 59439
\end{aligned}
$$

2 3.2.5

Show that the sets S_i are cumulative*—that is, for each i we have $S_i \subseteq S_{i+1}$.*

Proof. By induction on i. Our inductive hypothesis is:

$$H(i) : S_i \subseteq S_{i+1}$$

Case $(i = 0)$. By definition, $S_0 = \emptyset$, which is a subset of any set.
Case $(i > 0)$. We want to show:

$$\forall t.\ t \in S_i \implies t \in S_{i+1}$$

By definition of S_i, t must belong to one of three sets:

- $\{\ \texttt{true}, \texttt{false}, \texttt{zero}\ \}$
  These are in S_{i+1} by definition.
- $\{\ \texttt{succ t}_1, \texttt{pred t}_1, \texttt{iszero t}_1 \mid \texttt{t}_1 \in S_{i-1}\ \}$
  By the inductive hypothesis, we know that $\texttt{t}_1 \in S_i$.
  Then $\texttt{succ t}_1, \texttt{pred t}_1, \texttt{iszero t}_1 \in S_{i+1}$, by construction.
- $\{\ \texttt{if t}_1 \texttt{ then t}_2 \texttt{ else t}_3 \mid \texttt{t}_1,\ \texttt{t}_2,\ \texttt{t}_3 \in S_{i-1}\ \}$
  Similarly, by the inductive hypothesis, $\texttt{t}_1, \texttt{t}_2, \texttt{t}_3 \in S_i$,
  and $\texttt{if t}_1 \texttt{ then t}_2 \texttt{ else t}_3 \in S_{i+1}$ by construction.

□

3 3.5.5

Spell out the induction principle used in the preceding proof, in the style of Theorem 3.3.4.

If, for each derivation d, given $P(c)$ for all immediate subderivations c of d, we can show $P(d)$, then $P(d)$ holds for all d.

4 3.5.10

Rephrase Definition 3.5.9 as a set of inference rules.

$$\frac{}{t \longrightarrow^* t}\ \textsc{M-Zero}$$

$$\frac{t \longrightarrow^* t' \qquad t \longrightarrow t''}{t \longrightarrow^* t''}\ \textsc{M-Step}$$

5 3.5.13

Suppose we add a new rule:

$$\frac{}{\texttt{if true then } \texttt{t}_2 \texttt{ else } \texttt{t}_3 \longrightarrow \texttt{t}_3}\ \textsc{E-Funny1}$$

to the ones in figure 3–1. Which of the above theorems (3.5.4, 3.5.7, 3.5.8, 3.5.11, 3.5.12) remain valid?

- 3.5.4 (Determinancy of One-Step Evaluation) no longer holds. As a counterexample:
  - `if true then true else false` $\longrightarrow$ `true` by (E-IfTrue)
  - `if true then true else false` $\longrightarrow$ `false` by (E-Funny1)
- 3.5.7 (Every value is in normal form.) still holds.
- 3.5.8 (If `t` is in normal form, then `t` is a value.) still holds.
- 3.5.11 (Uniqueness of Normal Forms) no longer holds. The same counterexample as 3.5.4 can be used.
- 3.5.12 (Termination of Evaluation) still holds.

Suppose instead that we add this rule:

$$\frac{\mathtt{t_2} \longrightarrow \mathtt{t_2}'}{\mathtt{if\ t_1\ then\ t_2\ else\ t_3} \longrightarrow \mathtt{if\ t_1\ then\ t_2' \ else\ t_3}} \text{ E-FUNNY2}$$

Now which of the above theorems remain valid? Do any of the proofs need to change?

- 3.5.4 no longer holds. As a counterexample:

  ```
  if (if true then true else true)
  then (if true then true else true)
  else true
  ```

  We can proceed by E-IF or E-FUNNY2.

- 3.5.7 still holds.
- 3.5.8 still holds.
- 3.5.11 still holds, but the proof needs to change.
- 3.5.12 still holds.

6 3.5.14

Show that Theorem 3.5.4 is also valid for the evaluation relation on arithmetic expressions: if $\mathtt{t} \longrightarrow \mathtt{t}'$ *and* $\mathtt{t} \longrightarrow \mathtt{t}''$, *then* $\mathtt{t}' = \mathtt{t}''$.

Proof. By structural induction on $\mathtt{t} \longrightarrow \mathtt{t}'$. Our inductive hypothesis is:

$$H(\mathtt{t} \longrightarrow \mathtt{t}') : \mathtt{t} \longrightarrow \mathtt{t}'' \implies \mathtt{t}' = \mathtt{t}''$$

Case (E-SUCC). Here $\mathtt{t}$ is in the form $\mathtt{succ\ t_1}$. No other evaluation rules apply. Suppose $\mathtt{succ\ t_1} \longrightarrow \mathtt{succ\ t_1}'$ and $\mathtt{succ\ t_1} \longrightarrow \mathtt{succ\ t_1}''$. By the inductive hypothesis, if $\mathtt{t_1} \longrightarrow \mathtt{t_1}''$, then $\mathtt{t_1}' = \mathtt{t_1}''$, and therefore $\mathtt{t}' = \mathtt{t}''$.

Case (E-PREDZERO). Here $\mathtt{t}$ is in the form $\mathtt{pred}\ 0$. No other evaluation rules apply, so $\mathtt{t}'$ and $\mathtt{t}''$ can only be 0.

Case (E-PREDSUCC). This case is analogous to E-PREDZERO.

Case (E-PRED). This case is analogous to E-SUCC.

Case (E-ISZEROZERO). This case is analogous to E-PREDZERO.

Case (E-ISZEROSUCC). This case is analogous to E-PREDZERO.

Case (E-ISZERO). This case is analogous to E-SUCC.

Case (Other). The proof of all previous Boolean cases remains the same.

□

7 3.5.17

Show that the small-step and big-step semantics for this language coincide, i.e. $\mathtt{t} \longrightarrow^* \mathtt{v}$ *iff* $\mathtt{t} \Downarrow \mathtt{v}$.

We prove one direction at a time.

Forward. By induction on the number of steps in the derivation of $\mathtt{t} \longrightarrow^* \mathtt{v}$. Our inductive hypothesis is:

$$H(n) : \mathtt{t} \longrightarrow^n \mathtt{v} \implies \mathtt{t} \Downarrow \mathtt{v}$$

Case (n = 0). Here, we have $\mathtt{t} = \mathtt{v}$, and $\mathtt{v} \Downarrow \mathtt{v}$ by B-Value.

Case (n > 0). We proceed by case analysis on the last step, $\mathtt{t}' \longrightarrow \mathtt{v}$, where:

$$\mathtt{t} \longrightarrow^{n-1} \mathtt{t}' \longrightarrow \mathtt{v}$$

Case (E-IfTrue). Here, we have $\mathtt{t}_1 \longrightarrow^* \mathtt{true}$ and $\mathtt{t}_2 \longrightarrow^* \mathtt{v}_2$. By the inductive hypothesis, $\mathtt{t}_1 \Downarrow \mathtt{true}$ and $\mathtt{t}_2 \Downarrow \mathtt{v}_2$, and the conclusion follows from B-IfTrue.

Case (E-IfFalse). Analogous to E-IfTrue.

Case (E-If). Outcome is not a value.

Case (E-Succ). Analogous to E-IfTrue, by way of B-Succ.

Case (E-PredZero). Analogous to E-IfTrue, by way of B-PredZero.

Case (E-PredSucc). Analogous to E-IfTrue, by way of B-PredSucc.

Case (E-Pred). Outcome is not a value.

Case.

□

Backward.

$$H(\mathtt{t} \Downarrow \mathtt{v}) : \mathtt{t} \longrightarrow^* \mathtt{v}$$

Case (B-Value). Follows from M-Zero.

Case (B-IfTrue). By the inductive hypothesis, $\mathtt{t}_1 \longrightarrow^* \mathtt{true}$ and $\mathtt{t}_2 \longrightarrow^* \mathtt{v}_2$. Because only E-If applies, we have:

$$\begin{aligned}
&\mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3 \\
&\longrightarrow^* \mathtt{if}\ \mathtt{true}\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3 \\
&\longrightarrow \mathtt{t}_2 \\
&\longrightarrow^* \mathtt{v}_2
\end{aligned}$$

And we conclude this case by transitivity of $\longrightarrow^*$.

Case (B-IfFalse). Analogous to B-IfTrue.

Case (B-Succ). Analogous to B-IfTrue, but by way of E-Succ.

Case (B-PredZero). Analogous to B-IfTrue, but by way of E-Pred and then E-PredZero.

Case (B-PredSucc). Analogous to B-IfTrue, but by way of E-Pred and then E-PredSucc.

Case (B-IsZeroZero). Analogous to B-IfTrue, but by way of E-IsZero and then E-IsZeroZero.

Case (B-IsZeroSucc). Analogous to B-IfTrue, but by way of E-IsZero and then E-IsZeroSucc.

□

8 3.5.18

Suppose we want to change the evaluation strategy of our language so that the `then` *and* `else` *branches of an* `if` *expression are evaluated (in that order)* before *the guard is evaluated. Show how the evaluation rules need to change to achieve this effect.*

We replace E-If with three separate rules:

$$\frac{\mathtt{t}_2 \longrightarrow \mathtt{t}_2{}'}{\mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3 \longrightarrow \mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2{}'\ \mathtt{else}\ \mathtt{t}_3}\ \text{E-Then}$$

$$\frac{\mathtt{t}_3 \longrightarrow \mathtt{t}_3{}'}{\mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{v}_2\ \mathtt{else}\ \mathtt{t}_3 \longrightarrow \mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{v}_2\ \mathtt{else}\ \mathtt{t}_3{}'}\ \text{E-Else}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1{}'}{\mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{v}_2\ \mathtt{else}\ \mathtt{v}_3 \longrightarrow \mathtt{if}\ \mathtt{t}_1{}'\ \mathtt{then}\ \mathtt{v}_2\ \mathtt{else}\ \mathtt{v}_3}\ \text{E-If}$$