ECON501 Problem Set 4

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Problem 2

(a) Let the probability of low type be p, and let q be the probability the high type chooses e_2^*. Then Bayes rule implies the wages are

$$w(e_2^*) = \theta_H$$

$$w(e_1^*) = \frac{p\theta_L + (1-p)(1-q)\theta_H}{p + (1-p)(1-q)}$$

To satisfy incentives for staying on-path, we set $w(e) = \theta_L$ for $e \neq e_1^*, e_2^*$. Then the on-path incentives require that

$$u(w(e_1^*)) - c(e_1^*, \theta_L) \geq u(w(e_2^*)) - c(e_2^*, \theta_L)$$

in order for the low type to choose e_1^* and

$$u(w(e_2^*)) - c(e_2^*, \theta_H) = u(w(e_1^*)) - c(e_1^*, \theta_H)$$

in order for high type to randomize. Rearranging the first

$$c(e_2^*, \theta_L) - c(e_1^*, \theta_L) \geq u(w(e_2^*)) - u(w(e_1^*))$$

and the second gives

$$u(w(e_2^*)) - u(w(e_1^*)) = c(e_2^*, \theta_H) - c(e_1^*, \theta_H)$$

But note that because of strict single crossing,

$$c(e_2^*, \theta_L) - c(e_1^*, \theta_L) \geq c(e_2^*, \theta_H) - c(e_1^*, \theta_H)$$

And hence the inequality automatically follows from the equality. Hence the inequality constraint is redundant. We now just have to check that there are no deviations off-path. Since cost is increasing, and any off-path e yields θ_L wage, the best possible deviation is to 0, so we require

$$u(w(e_1^*)) - c(e_1^*, \theta_L) \geq u(\theta_L) - c(0, \theta_L)$$

$$u(w(e_2^*)) - c(e_2^*, \theta_H) \geq u(\theta_L) - c(0, \theta_H)$$

Note by the existing constraint,

$$u(w(e_2^*)) - u(w(e_1^*)) = c(e_2^*, \theta_H) - c(e_1^*, \theta_H)$$

$$u(w(e_2^*)) - c(e_2^*, \theta_H) = u(w(e_1^*)) - c(e_1^*, \theta_H) \geq u(w(e_1^*)) - c(e_1^*, \theta_L)$$

Hence the off-path deviation IC is redundant for the high type. So the only nonredundant constraints are:

$$u(w(e_2^*)) - u(w(e_1^*)) = c(e_2^*, \theta_H) - c(e_1^*, \theta_H)$$

$$u(w(e_1^*)) - c(e_1^*, \theta_L) \geq u(\theta_L) - c(0, \theta_L)$$

and these characterize e_1^*, e_2^* and q.

(b) As before, let the probability of low type be p, and let q be the probability the low type chooses e_2^*. Then Bayes rule implies the wages are

$$w(e_2^*) = \frac{pq\theta_L + (1-p)\theta_H}{pq + (1-p)}$$

$$w(e_1^*) = \theta_L$$

As before, we set $w(e) = \theta_L$ for $e \neq e_1^*, e_2^*$. The on-path IC constraints are

$$u(w(e_1^*)) - c(e_1^*, \theta_L) = u(w(e_2^*)) - c(e_2^*, \theta_L)$$

$$u(w(e_2^*)) - c(e_2^*, \theta_H) \geq u(w(e_1^*)) - c(e_1^*, \theta_H)$$

Rewriting, we have

$$c(e_2^*, \theta_L) - c(e_1^*, \theta_L) = u(w(e_2^*)) - u(w(e_1^*))$$

$$u(w(e_2^*)) - u(w(e_1^*)) \geq c(e_2^*, \theta_H) - c(e_1^*, \theta_H)$$

But strict single crossing implies that $c(e_2^*, \theta_L) - c(e_1^*, \theta_L) \geq c(e_2^*, \theta_H) - c(e_1^*, \theta_H)$, and hence the equality implies the inequality. Again, we check IC for off-path deviations

$$u(w(e_1^*)) - c(e_1^*, \theta_L) \geq u(\theta_L) - c(0, \theta_L)$$

$$u(w(e_2^*)) - c(e_2^*, \theta_H) \geq u(\theta_L) - c(0, \theta_H)$$

Note that

$$u(w(e_1^*)) - c(e_1^*, \theta_L) = u(w(e_2^*)) - c(e_2^*, \theta_L) \leq u(w(e_2^*)) - c(e_2^*, \theta_H)$$

Hence the first inequality implies the second. Therefore, the nonredundant constraints are

$$u(w(e_1^*)) - c(e_1^*, \theta_L) = u(w(e_2^*)) - c(e_2^*, \theta_L)$$

$$u(w(e_1^*)) - c(e_1^*, \theta_L) \geq u(\theta_L) - c(0, \theta_L)$$

(c) For part (a): Take w as specified in part (a). We know by strict single crossing that

$$u(w(e_1^*)) - c(e_1^*, \theta_L) > u(w(e_2^*)) - c(e_2^*, \theta_L)$$

Since this equality is strict and c is continuouss, we can pick some ε such that

$$u(w(e_1^*)) - c(e_1^*, \theta_L) > u(w(e_2^*)) - c(e_2^* - \varepsilon, \theta_L)$$

Consider what happens if the high type chooses $e_2^* - \varepsilon$. Clearly, it is unreasonable to believe that this level of effort was chosen by the low type, since the choice of ε ensures that the low type prefers choosing e_1^*. So the reasonable belief is that this type corresponds to e_2^*, and paying θ_H means that this would be a profitable deviation for the high type. (The only reason the high type doesn't decrease their e is because of the empty threat of $w(e) = \theta_L$, which does not make sense). Hence this fails the intuitive criterion.

Consider part (b). Take w as in part (b). By strict single crossing, for any $e^* > e_2^*$

$$c(e^*, \theta_L) - c(e_2^*, \theta_L) > c(e^*, \theta_H) - c(e_2^*, \theta_H)$$

Rewriting,

$$c(e^*, \theta_L) - c(e_2^*, \theta_L) > c(e^*, \theta_H) - c(e_2^*, \theta_H)$$

Note that $\theta_H > w(e_2^*)$. So $u(\theta_H) - u(w(e_2^*)) > 0$.

$$f(e^*) = c(e^*, \theta_L) - c(e_2^*, \theta_L)$$

$$g(e^*) = c(e^*, \theta_H) - c(e_2^*, \theta_H)$$

Then we can rewrite the strict single crossing condition as for all $e^* > e_2^*$,

$$f(e) > g(e)$$

Note that at $f(e_2^*) = g(e_2^*) = 0 < u(\theta_H) - u(w(e_2^*))$. Since both f and g are increasing and unbounded because c is increasing and unbounded, and $f(e) > g(e)$ at all $e > e_2^*$, we can select some e' such that $f(e') > u(\theta_H) - u(w(e_2^*)) > g(e')$. That is,

$$c(e', \theta_L) - c(e_2^*, \theta_L) > u(\theta_H) - u(w(e_2^*))$$

$$u(\theta_H) - u(w(e_2^*)) > c(e', \theta_H) - c(e_2^*, \theta_H)$$

or

$$u(w(e_2^*)) - c(e_2^*, \theta_L) > u(\theta_H) - c(e', \theta_L)$$

$$u(\theta_H) - c(e', \theta_H) > u(w(e_2^*)) - c(e_2^*, \theta_H)$$

Now, consider choosing e'. Clearly, as we just showed, this is not incentivized for the low type, since even a payment θ_H would not justify the switch to e'. Hence, it is unreasonable to believe the low type chose this e'. However, if you pay θ_H, then this is a profitable deviation for the high type. That is, this equilibrium relies on the empty threat to pay θ_L for observing this e' which is unreasonable. Hence both equilibria fail the intuitive criterion.

Problem 3

Let $\underline{w}$ be the wage paid to workers who don't take the test. Then workers with type above $\underline{w}$ will take the test, and be paid their type, and workers blow $\underline{w}$ will not take the test. Then Bayesian updating requires that

$$\underline{w} = E[\theta|\theta \leq \underline{w}]$$

However, this equality can only be satisfied if $\underline{w} = \underline{\theta}$, as otherwise the right hand side is less than the left hand side. Therefore, all workers with type above $\underline{\theta}$ take the test, and are paid as their type, and workers with $\underline{\theta}$ can take the test or not or mix, and either way will be paid $\underline{\theta}$.

Problem 4

(a) Assuming increasing cost of effort, clearly inducing $e = 0$ can be satisfied by offering $w = 0$ regardless of output, as this satisfies IC, IR, and LL without costing the principal anything. Even if we observe effort, this contract is still optimal, and hence this achieves the first-best.

(b) Let w_0 denote the low output payment, and w_1 denote the high output payment. Then we have the following problem for the principal:

$$\max p_1(\pi_1 - w_1) + (1 - p_1)(\pi_0 - w_0)$$

subject to

$$p_1 w_1 + (1 - p_1)w_0 - c \geq p_0 w_1 + (1 - p_0)w_0$$

$$p_1 w_1 + (1 - p_1)w_0 - c \geq 0$$

$$w_1, w_0 \geq 0$$

Note the first and third constraints imply the second. Rewriting the IC constraint,

$$p_1 w_1 - p_0 w_1 + (1 - p_1)w_0 - (1 - p_0)w_0 \geq c$$

$$(p_1 - p_0)(w_1 - w_0) \geq c$$

Since $p_1 > p_0$, $c > 0$, we need $w_1 > w_0$. Hence, the only other constraint is $w_1 > w_0 \geq 0$. We must have $w_0 = 0$, else we can decrease both w_1 and w_0 equally without affecting IC. Then we just require

$$w_1 \geq \frac{c}{p_1 - p_0}$$

This is tight at optimum, otherwise we can decrease w_1 and increase the principal's payoff. Hence, the contract is

$$w_0 = 0$$

$$w_1 = \frac{c}{p_1 - p_0}$$

Note that $p_1 - p_0 < 1$, so $w_1 > c$. Hence first-best is not achieved, since if effort was observable the principal can just offer c upon observing effort and 0 otherwise.

Problem 5

(a) When effort is observable, the principal problem is

$$\max_{e_i} 10f(\pi_H|e_i) - w(e_i)$$

subject to

$$v(w(e_i)) \geq g(e_i)$$

We can rewrite the constraint as

$$w(e_i) \geq g(e_i)^2$$

Since the constraint must bind at optimum, we get the reduced problem

$$\max_{e_i} 10f(\pi_H|e_i) - g(e_i)^2$$

We can just plug in numbers and check now which e_i optimizes this. The optimal choice is e_1 for a constant wage of 25/9.

(b) As in the hint, we let v_H and v_L be the induced utilities from $\sqrt{w(\pi_H)}$ and $\sqrt{w(\pi_L)}$ respectively. In order for e_2 to be implementable, we need

$$\frac{1}{2}v_H + \frac{1}{2}v_L - g(e_2) \geq 0$$

$$\frac{1}{2}v_H + \frac{1}{2}v_L - g(e_2) \geq \frac{2}{3}v_H + \frac{1}{3}v_L - \frac{5}{3}$$

$$\frac{1}{2}v_H + \frac{1}{2}v_L - g(e_2) \geq \frac{1}{3}v_H + \frac{2}{3}v_L - \frac{4}{3}$$

Rewriting, we get

$$\frac{1}{2}(v_H + v_L) \geq g(e_2)$$

$$\frac{1}{6}(v_H - v_L) \leq \frac{5}{3} - g(e_2)$$

$$\frac{1}{6}(v_H - v_L) \geq g(e_2) - \frac{4}{3}$$

Combining the last two,

$$\frac{1}{2}(v_H + v_L) \geq g(e_2)$$

$$g(e_2) - \frac{4}{3} \leq \frac{1}{6}(v_H - v_L) \leq \frac{5}{3} - g(e_2)$$

Note for $g(e_2) = 8/5$ the second inequality chain is impossible, since $4/15 \not\leq 1/15$. The second expression gives

$$2g(e_2) \leq 3$$

$$g(e_2) \leq \frac{3}{2}$$

So this is required for e_2 to be implementable

(c) When effort is not observable, we can only implement e_1 or e_3. To implement e_3, we can just offer constant wage 16/9, and the expected principal payoff is just 14/9. To implement e_1, the problem is given by

$$\max \frac{2}{3}(10 - v_H^2) - \frac{1}{3}v_L^2$$

subject to

$$\frac{2}{3}v_H + \frac{1}{3}v_L - \frac{5}{3} \geq 0$$

$$\frac{2}{3}v_H + \frac{1}{3}v_L - \frac{5}{3} \geq \frac{1}{3}v_H + \frac{2}{3}v_L - \frac{4}{3}$$

Rewriting, dropping constant terms from the maximization objective, we get

$$\min 2v_H^2 + v_L^2$$

$$2v_H + v_L \geq 5$$

$$v_H - v_L \geq 1$$

Now, note the IR constraint must bind, else we can lower v_H and v_L equally, and this will improve the objective. Also, the IC constraint must also bind, else we can decrease v_H by ϵ and increase v_L by 2ϵ and we get

$$2(v_H - \epsilon)^2 + (v_L + 2\epsilon)^2 = 2(v_H^2 - 2v_H\epsilon + \epsilon^2) + (v_L^2 + 4v_L\epsilon + 4\epsilon^2)$$

$$= (2v_H^2 + v_L^2) + 4v_L\epsilon + 4\epsilon^2 - 4v_H\epsilon + 2\epsilon^2$$

$$= (2v_H^2 + v_L^2) - 2\epsilon(2(v_H - v_L) - 3\epsilon)$$

Since by the contradiction assumption $v_H - v_L > 1$, as long as $\epsilon < 2/3$,

$$(2v_H^2 + v_L^2) - 2\epsilon(2(v_H - v_L) - 3\epsilon) < (2v_H^2 + v_L^2) - 2\epsilon(2 - 3\epsilon) < 2v_H^2 + v_L^2$$

Hence the IC constraint must bind. Then we uniquely have

$$v_H = 2, v_L = 1$$

so the original principal's value is

$$\frac{2}{3}(10-4)-\frac{1}{3}(1)=\frac{11}{3}$$

Note that $11/3 > 14/9$, hence the optimal contract induces effort e_1 and offers wage:

$$w_H = 4, w_L = 1$$

(d) If effort is observable, the reduced problem is

$$\max_{e_i} 10f(\pi_H|e_i) - g(e_i)^2$$

We can check for e_3 this is $14/9$, for e_2 this is $61/25$, and for e_1 this is

$$10x - 8$$

As $x \to 1$, this approaches 2. In any case, the optimal contract induces effort e_2 and pays $64/25$. If effort is not observable, inducing e_3 reqires payment $16/9$ for payoff $14/9$. For inducing e_2, we have

$$\max \frac{1}{2}(10-v_H^2)-\frac{1}{2}v_L^2$$

subject to

$$\frac{1}{2}v_H+\frac{1}{2}v_L-\frac{8}{5}\geq 0$$

$$\frac{1}{2}v_H+\frac{1}{2}v_L-\frac{8}{5}\geq xv_H+(1-x)v_L-\sqrt{8}$$

$$\frac{1}{2}v_H+\frac{1}{2}v_L-\frac{8}{5}\geq \frac{1}{3}v_H+\frac{2}{3}v_L-\frac{4}{3}$$

rewriting the constraints,

$$\frac{1}{2}v_H+\frac{1}{2}v_L-\frac{8}{5}\geq 0$$

$$(x-\frac{1}{2})(v_H-v_L)\leq \sqrt{8}-\frac{8}{5}$$

$$\frac{1}{6}(v_H-v_L)\geq \frac{8}{5}-\frac{4}{3}$$

IR must bind, else we can uniformly decrease v_H and v_L. By a similar argument in the previous part, the third constraint (second IC) must bind, else we can decrease v_H and increase v_L by some ϵ and get a better objective. Hence we get

$$v_H+v_L=\frac{16}{5}$$

$$v_H-v_L=\frac{8}{5}$$

$$v_H=\frac{12}{5}, v_L=\frac{4}{5}$$

So the maximized objective is then

$$(5 - 72/25) - (8/25) = (5 - 80/25) = 5 - 16/5 = 9/5$$

To induce e_1, we have

$$\max x(10 - v_H^2) - (1 - x)v_L^2$$

subject to

$$xv_H + (1 - x)v_L - \sqrt{8} \geq 0$$

$$xv_H + (1 - x)v_L - \sqrt{8} \geq \frac{1}{2}v_H + \frac{1}{2}v_L - \frac{8}{5}$$

$$xv_H + (1 - x)v_L - \sqrt{8} \geq \frac{1}{3}v_H + \frac{2}{3}v_L - \frac{4}{3}$$

Rewriting in the usual way,

$$xv_H + (1 - x)v_L \geq \sqrt{8}$$

$$(x - 1/2)(v_H - v_L) \geq \sqrt{8} - \frac{8}{5}$$

$$(x - 1/3)(v_H - v_L) \geq \sqrt{8} - \frac{4}{3}$$

or

$$xv_H + (1 - x)v_L \geq \sqrt{8}$$

$$v_H - v_L \geq \frac{1}{x - 1/2}(\sqrt{8} - \frac{8}{5})$$

$$v_H - v_L \geq \frac{1}{x - 1/3}(\sqrt{8} - \frac{4}{3})$$

as $x \to 1$, the IC constraints become

$$v_H - v_L \geq 2\sqrt{8} - \frac{16}{5}$$

$$v_H - v_L \geq \frac{3}{2}\sqrt{8} - 2$$

So the second constraint is redundant. IR must bind, so we get

$$v_H = \sqrt{8}, v_L = 16/5 - \sqrt{8}$$

And the payoff to the principal is

$$x(10 - v_H^2) - (1 - x)v_L^2 = 2$$

Note that this is larger than 9/5 and 14/9. So the optimal contract induces effort e_1, with wages

$$w_H = 8, w_L = \left(16/5 - \sqrt{8}\right)^2$$

Note that when effort is unobservable, the optimal contract induces higher effort than when unobservable.

Problem 6

The principal problem is

$$\max_{e,w} f(\pi_H|e)(\pi_H - w_{e,H}) + (1 - f(\pi_H|e))(\pi_L - w_{e,L})$$

subject to

$$f(\pi_H|e)v(w_{e,H}) + (1 - f(\pi_H|e))v(w_{e,L}) - c(e) \geq 0$$

$$e \in \arg\max_e f(\pi_H|e)v(w_{e,H}) + (1 - f(\pi_H|e))v(w_{e,L}) - c(e)$$

Rewriting IC,

$$e \in \arg\max_e f(\pi_H|e)(v(w_{e,H}) - v(w_{e,L})) + v(w_{e,L}) - c(e)$$

For the first order approach to be valid, we need the objective in the IC constraint to be concave. Sufficient conditions would be requiring c to be convex, $f(\pi_H|e)$ to be concave in e, and $v(w_{e,H}) - v(w_{e,L})$ to be positive. Under these conditions, the FOC is

$$\left(\frac{\partial}{\partial e} f(\pi_H|e)\right)(v(w_{e,H}) - v(w_{e,L})) - c'(e) = 0$$

Note that at optimum, IR binds, so we get

$$f(\pi_H|e)v(w_{e,H}) + (1 - f(\pi_H|e))v(w_{e,L}) = c(e)$$

Solving for $v(w_{e,L})$ and $v(w_{e,H})$, we get

$$v(w_{e,H}) - v(w_{e,L}) = \frac{c'(e)}{\frac{\partial}{\partial e} f(\pi_H|e)}$$

$$f(\pi_H|e)(v(w_{e,H}) - v(w_{e,L})) + v(w_{e,L}) = c(e)$$

$$f(\pi_H|e)\frac{c'(e)}{\frac{\partial}{\partial e} f(\pi_H|e)} + v(w_{e,L}) = c(e)$$

$$v(w_{e,L}) = c(e) - f(\pi_H|e)\frac{c'(e)}{\frac{\partial}{\partial e} f(\pi_H|e)}$$

$$v(w_{e,H}) = c(e) - f(\pi_H|e)\frac{c'(e)}{\frac{\partial}{\partial e} f(\pi_H|e)} + \frac{c'(e)}{\frac{\partial}{\partial e} f(\pi_H|e)} = c(e) + \frac{(1 - f(\pi_H|e))c'(e)}{\frac{\partial}{\partial e} f(\pi_H|e)}$$

Then the unconstrained principal problem is

$$\max_e f(\pi_H|e)\left(\pi_H - v^{-1}\left(c(e) + \frac{(1 - f(\pi_H|e))c'(e)}{\frac{\partial}{\partial e} f(\pi_H|e)}\right)\right) + (1 - f(\pi_H|e))\left(\pi_L - v^{-1}\left(c(e) - f(\pi_H|e)\frac{c'(e)}{\frac{\partial}{\partial e} f(\pi_H|e)}\right)\right)$$

and so the choice of e maximizes this, and the contract is

$$w_H = v^{-1}\left(c(e) + \frac{(1 - f(\pi_H|e))c'(e)}{\frac{\partial}{\partial e} f(\pi_H|e)}\right)$$

$$w_L = v^{-1}\left(c(e) - f(\pi_H|e)\frac{c'(e)}{\frac{\partial}{\partial e} f(\pi_H|e)}\right)$$