ECON501 Problem Set 1

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Problem 2

For each player i, let H_i denote the set of information sets for this player, and let $A(h_i)$ denote the actions available to player i at information set $h_i \in H_i$. Let σ_i be a behavioral strategy for player i, and to slightly abuse notation, let $\sigma_i(h_i) \in \Delta(A(h_i))$, and $\sigma = \times_{h_i \in H_i} \sigma_i(h_i)$.

Let $a(h_i)$ denote some action available at information set $h_i \in H_i$. Consider the mixed strategy σ_i' defined such that

$$\sigma'(\times_{h_i \in H_i} a(h_i)) = \prod_{h_i \in H_i} \sigma_i(h_i)(a(h_i))$$

We show that σ_i' induces the same distribution over outcomes as behavioral strategies σ. Fix an arbitrary outcome z. By the tree properties, there exists one unique path to z from the game tree root. (Due to imperfect recall, it is possible that some of the nodes on this path belong to the same information set). Consider the sequence h^* of information sets encountered along this path; label them $h_{i_1}^1, h_{i_2}^2, ...h_{i_n}^n$ where i_n is the individual acting at the nth information set. Note that some of these can be the same information set due to imperfect recall. Further, due to the fact this path is unique and well-defined, there exist a unique sequence of actions a_1, a_2, ... a_n that result in the given outcome, $a_k \in A(h_{i_k}^k)$. Then the probability of seeing outcome z under behavioral strategy σ is

$$\prod_{h_{i_k}^k \in h^*} \sigma_{i_k}(h_{i_k}^k)(a_k)$$

The probability of outcome z under the mixed strategy σ' is given by the probability each individual plays a mixed strategy that has the right actions at the relevant information sets on the path to z:

$$\prod_i \left(\sum_{a_i} (\sigma'(a_i))\, 1\{\forall k \in \{1, 2, ...n\}, (i_k \neq i) | (a_k = a_i(h_i^k))\} \right)$$

Note that the indicator is 1 if for all k, either $i_k \neq i$ (i is not the individual acting at the kth node) or $a_k = a_i(h_i^k)$, agent i takes action a_k at the information set h_i^k.

Plugging in the definition of $\sigma'(a_i)$, we have

$$\prod_i \left(\sum_{a_i} \left(\prod_{h_i \in H_i} \sigma_i(h_i)(a(h_i)) \right) 1\{\forall k \in \{1, 2, ...n\}, (i_k \neq i) | (a_k = a_i(h_{i_k}^k))\} \right)$$

Since this sums over a_i, and the indicator is 0 iff $\exists k$ such that $i_k = i$ and $a_k \neq (h_i^k)$, the sum computes the probability that a given mixed strategy realization takes the appropriate actions at each information set h_i^k, and hence this sum of products factors as

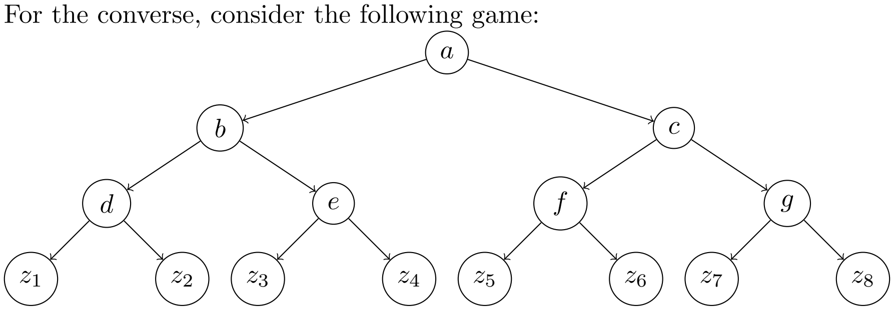
$$\prod_i \left(\prod_{h_{i_k}^k \in h^*,\ i_k = i} \sigma_i(h_i)(a_k) \right)$$

Simplifying the product indices, we get

$$\prod_{h_{i_k}^k \in h^*} \sigma_{i_k}(h_i)(a_k)$$

which is the same probability as before. Since z was chosen arbitrarily, σ' is outcome equivalent to σ, and by construction every behavior strategy has an outcome-equivalent mixed strategy.

For the converse, consider the following game:



(I am bad at tikzpicture so I will specify the details of the game manually). Player 1 has information sets $\{a\}$, and $\{d, e, f, g\}$, and player 2 acts at information set $\{b, c\}$. All the moves are either L or R, for both players, for left or right. The mixed strategies for player 1 are in $\Delta(\{LL, LR, RL, RR\})$. Consider strategy $\{1/2, 0, 0, 1/2\}$ for player 1, and suppose player 2 just plays strategy L. The mixed strategy outcome distribution is $(1/2)z_1 + (1/2)z_6$. However, any behavioral strategy that assigns nonzero probability to both z_1 and z_6 must mix at the second information set $\{d, e, f, g\}$, and hence must also assign nonzero probability to z_2 and z_5, which results in a distinct outcome distribution from the mixed strategy. Hence, no behavior strategy is outcome equivalent to this mixed strategy, and we are done.

Problem 3

Fix player i. Let H_i denote the set of information sets, and let $A(h_i)$ denote the set of actions of player i at information set $h_i \in H_i$. Then we have the set of behavioral strategies is given by

$$\times_{h_i \in H_i} \Delta \left(A(h_i) \right)$$

So the dimension of this is

$$\sum_{h_i \in H_i} \left(|A(h_i)| - 1 \right)$$

The set of mixed strategies is given by

$$\Delta \left(\times_{h_i \in H_i} A(h_i) \right)$$

This has dimension

$$\left(\prod_{h_i \in H_i} |A(h_i)|\right) - 1$$

Problem 4

No, consider the game where the actions of player 1 are A, B and the actions of player 2 are A, B. Suppose that the outcomes are 0 for everyone unless both players play A, in which case the outcomes are 1 for everyone. No matter which player is put as moving 'first' in the extensive form game, there will always be two distinct terminal nodes with outcome 0. Hence, this is a counterexample, and not any arbitrary normal form game can be written as an extensive form game with **unique** terminal nodes.