ECON501 Problem Set 1

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Problem 2

(a) The seller problem is

$$\max \sum_i p_i(t(\theta_i) - c(q(\theta_i)))$$

subject to IC:

$$\forall i \neq j \quad \theta_i v(q(\theta_i)) - t(\theta_i) \geq \theta_i v(q(\theta_j)) - t(\theta_j)$$

and IR:

$$\forall i \quad \theta_i v(q(\theta_i)) - t(\theta_i) \geq 0$$

Now, manipulating IC, we get that for any $i > j$,

$$\theta_i v(q(\theta_i)) - t(\theta_i) \geq \theta_i v(q(\theta_j)) - t(\theta_j)$$

$$\theta_i(v(q(\theta_i)) - v(q(\theta_j))) \geq t(\theta_i) - t(\theta_j)$$

Similarly from IC,

$$\theta_j v(q(\theta_i)) - t(\theta_j) \geq \theta_j v(q(\theta_i)) - t(\theta_i)$$

$$t(\theta_i) - t(\theta_j) \geq \theta_j(v(q(\theta_i)) - v(q(\theta_i)))$$

Putting these two together, we get:

$$\theta_i(v(q(\theta_i)) - v(q(\theta_j))) \geq t(\theta_i) - t(\theta_j) \geq \theta_j(v(q(\theta_i)) - v(q(\theta_i)))$$

$$(\theta_i - \theta_j)(v(q(\theta_i)) - v(q(\theta_j))) \geq 0$$

By our supposition, $i > j$, so $\theta_i > \theta_j$, and hence the first term in the product is positive. This implies the second term must also be positive, and hence

$$v(q(\theta_i)) - v(q(\theta_j)) \geq 0$$

$$v(q(\theta_i)) \geq v(q(\theta_j))$$

Since v is monotonically increasing by assumption, this implies $q(\theta_i) \geq q(\theta_j)$. Hence q is monotonic if IC holds.

(b) From IC, for $i > 1$, $\theta_i > \theta_1$, and hence we have

$$\theta_i v(q(\theta_i)) - t(\theta_i) \geq \theta_i v(q(\theta_1)) - t(\theta_1) > \theta_1 v(q(\theta_1)) - t(\theta_1)$$

But the expression on the right is > 0 by IR, hence IR for all $i > 1$ are redundant.

Additionally, we note that from our previous formulation of IC in part a, we have for $i \neq j$,

$$\theta_i v(q(\theta_i)) - t(\theta_i) \geq \theta_i v(q(\theta_j)) - t(\theta_j)$$

$$\theta_i (v(q(\theta_i)) - v(q(\theta_j))) \geq t(\theta_i) - t(\theta_j)$$

We claim that IC for consecutive i, j is sufficient. We show this for the case where $i > j$ (the case where $i < j$ is similar, using the opposite directional IC constraint). Then we have from the consecutive IC ($i - j = \pm 1$):

$$\theta_i (v(q_i) - v(q_{i-1})) \geq t_i - t_{i-1}$$

$$\theta_{i-1} (v(q_{i-1}) - v(q_{i-2})) \geq t_{i-1} - t_{i-2}$$

$$\vdots$$

$$\theta_{j+1} (v(q_{j+1}) - v(q_j)) \geq t_{j+1} - t_j$$

Summing, we get

$$\sum_{k=j+1}^{i} \theta_k (v(q_k) - v(q_{k-1})) \geq t_i - t_j$$

Since the RHS telescopes. But since $i \geq k$, we have $\theta_i \geq \theta_k$, and hence

$$\sum_{k=j+1}^{i} \theta_i (v(q_k) - v(q_{k-1})) \geq \sum_{k=j+1}^{i} \theta_k (v(q_k) - v(q_{k-1})) \geq t_i - t_j$$

But the LHS telescopes, and we get

$$\theta_i (v(q_i) - v(q_j)) \geq t_i - t_j$$

and hence we have shown IC for $i > j$ from consecutive IC. So the only non-redundant constraints are consecutive IC and IR for 1.

(c) IR1 must bind (otherwise we can increase all transfers by ϵ). Now, consider the constraint IC for $k, k-1$:

$$\theta_k v(q_k) - t_k \geq \theta_k v(q_{k-1}) - t_{k-1}$$

Suppose this did not bind. Then consider increasing the transfers $t_k, t_{k+1}, t_{k+2} ... t_n$ by ϵ. Clearly, this doesn't break any of the ICs above k or below $k-1$. Clearly, IC for $k, k-1$ still holds as long as ϵ is small. Additionally, IC for $k-1, k$ is

$$\theta_{k-1}v(q_{k-1}) - t_{k-1} \geq \theta_{k-1}v(q_k) - t_k$$

so increasing t_k without changing t_{k-1} maintains this constraint. Hence, we have IC $k, k-1$ must bind. Lastly, since q is monotonic in θ, we have $v(q_{k-1}) - v(q_k) \leq 0$, and hence since IC $k, k-1$ binds,

$$\theta_{k-1}(v(q_{k-1}) - v(q_k)) \geq \theta_k(v(q_{k-1}) - v(q_k)) = t_{k-1} - t_k$$

$$\theta_{k-1}v(q_{k-1}) - t_{k-1} \geq \theta_{k-1}v(q_k) - t_k$$

so IC $k-1, k$ also holds (but does not necessarily bind).

Problem 3

From class, we know that

$$q(\theta) = \arg\max v(q)\psi(\theta) - c(q) = \arg\max v(q)\psi(\theta) - q$$

We know that in order for this FOC to be valid, we need $\psi(\theta) > 0$, and hence under the regularity assumption, there exists a unique θ^* such that

$$\psi(\theta^*) = 0 \iff \theta^* - \frac{1 - F(\theta^*)}{f(\theta^*)} = 0$$

Since regularity implies ψ is increasing in θ, we have for $\theta \leq \theta^*$, $q(\theta) = 0$, and for $\theta > \theta^*$, we have the FOC

$$v'(q)\psi(\theta) = 1$$

$$v'(q(\theta)) = 1/\psi(\theta)$$

Also, we know that

$$t(\theta) = \theta v(q(\theta)) - \int_0^\theta v(q(x))\ dx$$

$$t'(\theta) = v(q(\theta)) + \theta v'(q(\theta))q'(\theta) - v(q(\theta)) = v'(q(\theta))q'(\theta) = \frac{\theta}{\psi(\theta)}q'(\theta)$$

Hence

$$t(\theta) = t(0) + \int_0^\theta \frac{x}{\psi(x)}q'(x)\ dx$$

Since $t(0) = 0v(q(0)) = 0$,

$$t(\theta) = \int_0^\theta \frac{x}{\psi(x)}q'(x)\ dx$$

Integrating the RHS by parts,

$$t(\theta) = \frac{x}{\psi(x)} q(x) \Big|_0^\theta - \int_0^\theta \frac{\psi(x) - x\psi'(x)}{\psi(x)^2} q(x) \ dx$$

$$t(\theta) = \frac{\theta}{\psi(\theta)} q(\theta) - \int_0^\theta \frac{\psi(x) - x\psi'(x)}{\psi(x)^2} q(x) \ dx$$

Since $q(0) = 0$. Dividing by $q(\theta)$, we get

$$\frac{t(\theta)}{q(\theta)} = \frac{\theta}{\psi(\theta)} - \frac{1}{q(\theta)} \int_0^\theta \frac{\psi(x) - x\psi'(x)}{\psi(x)^2} q(x) \ dx$$

Taking the derivative wrt θ, we get

$$\frac{\partial}{\partial\theta} \frac{t(\theta)}{q(\theta)} = \frac{\psi(\theta) - \theta\psi'(\theta)}{(\psi(\theta))^2} + \frac{q'(\theta)}{q(\theta)^2} \int_0^\theta \frac{\psi(x) - x\psi'(x)}{\psi(x)^2} q(x) \ dx - \frac{\psi(\theta) - \theta\psi'(\theta)}{\psi(\theta)^2}$$

$$= \frac{q'(\theta)}{q(\theta)^2} \int_0^\theta \frac{\psi(x) - x\psi'(x)}{\psi(x)^2} q(x) \ dx$$

Now, we know $q' > 0$, $q^2 > 0$, $\psi^2 > 0$. Then $\psi(\theta) \leq \theta\psi'(\theta)$ is a sufficient condition for this expression to be negative, since this makes the integrand negative at all values (equivalently, we can require $\theta/\psi(\theta)$ is decreasing).

Problem 4

(a)

(b)

Problem 5

(a) Suppose agent i realizes type θ_i. By truthful reporting, the expected payout is given by the probability of winning the good times the expected payout given the good was won:

$$\theta_i \left(\theta_i - 2 * (\theta_i/2)\right) = \theta_i(0) = 0$$

However, by reporting some $\theta_i - \epsilon$, the expected payout is then

$$(\theta_i - \epsilon) \left(\theta_i - 2 * ((\theta_i - \epsilon)/2)\right) = (\theta_i - \epsilon)(\epsilon) > 0$$

Hence truthful reporting cannot be an equilibrium, since both players gain strictly higher expected payoff by underreporting.

(b)

Problem 6

(a)

(b)

Problem 7

Problem 8