ECON501 Problem Set 6

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Spring 2021

Problem 2

8.E.1 We write out the normal form of this game. The pure strategies are AA, AN, NA, NN, where A means attack and N means not attack, and the first letter denotes the action if the general sees strong type.

	AA	AN	NA	NN
AA	$\frac{M}{4} - \frac{s+w}{2}, \frac{M}{4} - \frac{s+w}{2}$	$\frac{M}{2} - \frac{s+w}{4}, \frac{M}{4} - \frac{s}{2}$	$\frac{3M}{4} - \frac{s+w}{4}, -\frac{w}{2}$	$M, 0$
AN	$\frac{M}{4} - \frac{s}{2}, \frac{M}{2} - \frac{s+w}{4}$	$\frac{M}{4} - \frac{s}{4}, \frac{M}{4} - \frac{s}{4}$	$\frac{M}{2} - \frac{s}{4}, \frac{M}{4} - \frac{w}{4}$	$\frac{M}{2}, 0$
NA	$-\frac{w}{2}, \frac{3M}{4} - \frac{s+w}{4}$	$\frac{M}{4} - \frac{w}{4}, \frac{M}{2} - \frac{s}{4}$	$\frac{M}{4} - \frac{w}{4}, \frac{M}{4} - \frac{w}{4}$	$\frac{M}{2}, 0$
NN	$0, M$	$0, \frac{M}{2}$	$0, \frac{M}{2}$	$0, 0$

We note NA is strictly dominated by AN since $w > s$, so we can restrict our attention to

	AA	AN	NN
AA	$\frac{M}{4} - \frac{s+w}{2}, \frac{M}{4} - \frac{s+w}{2}$	$\frac{M}{2} - \frac{s+w}{4}, \frac{M}{4} - \frac{s}{2}$	$M, 0$
AN	$\frac{M}{4} - \frac{s}{2}, \frac{M}{2} - \frac{s+w}{4}$	$\frac{M}{4} - \frac{s}{4}, \frac{M}{4} - \frac{s}{4}$	$\frac{M}{2}, 0$
NN	$0, M$	$0, \frac{M}{2}$	$0, 0$

If $M < s$, then NN, NN is the only Nash equilibrium.

Note that the best reply to NN is always AA. If $M/2 < s$ then the best reply to AA is NN. So if $M/2 < s$, then (AA, NN) and (NN, AA) are Nash equilibria.

If $w > M > s$, then the best reply to AN is AN. Hence, if this holds (note this is not mutually exclusive with $M/2 < s$), then (AN, AN) is a Nash equilibrium.

Note that the best reply to AA is never AA, so the only potential other Nash equilibria are (AA, AN) and (AN, AA). In order for AA to be a best response to AN, we require $M > w$. In order for AN to be the best response to AA, we need $M/2 > s$. Hence, for $M > w$ and $M/2 > s$, (AA, AN) and (AN, AA) are Nash equilibria.

8.E.3

9.C.2

9.C.3

9.C.4

Problem 3

The strategies for the sender are LL, LR, RL, RR, denoting the actions after observing t_1 and t_2. The receiver strategies are uu, ud, du, dd, the actions after observing L and R. In the subsequent parts, we denote μ_L the receiver belief in t_1 after observing L and μ_R the receiver belief in t_1 after observing R.

(a) We start with pooling equilibria. Note that in no case after observing t_2 will the sender want to play L, so the pooling equilibria must have the sender send R always. In this case, the belief of the receiver on t_1 versus t_2 is just $(0.5, 0.5)$, and hence u is optimal. Hence, (RR, uu) is a PBE, supported by belief $\mu_L \geq 0.5$ and $\mu_R = 0.5$. Additionally, (RR, du) is also a PBE, supported by $\mu_L \leq 0.5$ and $\mu_R = 0.5$. (since the beliefs off-path can be arbitrary).

For separating equilibria, we note that the sender always plays R after realization t_2. Then after realizing t_1, the separating pure equilibria dictates the sender plays L. Hence, the only separating PBE strategy for the sender is LR. The only consistent beliefs are then $\mu_R = 0$, $\mu_L = 1$, and the resulting strategy for the receiver is ud. So the only separating PBE is (LR, ud) with $\mu_R = 0$, $\mu_L = 1$.

(b) We start with pooling equilibria. If player 1 plays LL, then $\mu_L = 1/2$, so player 2 plays u after observing L. Off-path, player 2 cannot want to play d after observing R, else t_1 type would want to deviate to R. Hence, a PBE here is (LL, uu), where $\mu_L = 1/2$ and $\mu_R \leq 2/3$.

If player 1 plays RR, then $\mu_R = 1/2$, so player 2 plays u after observing R. But this means the t_1 type would rather deviate to L, so player 1 cannot play RR in a PBE. Hence the only pooling PBE is the (LL, uu) one we described earlier.

Now, for separating PBE. If player 1 plays LR, player 2's beliefs are $\mu_L = 1$, $\mu_R = 0$. So player 2 plays du. Note that player 1 does not want to deviate for either type, so this is a PBE: (LR, du) with $\mu_L = 1$, $\mu_R = 0$. Now, if player 1 plays RL, player 2's beliefs are $\mu_L = 0$, $\mu_R = 1$, so player 2's best response is ud. Once again, we observe that neither type of player 1 would like to deviate. Hence this is also a PBE: (RL, ud) with $\mu_L = 0$, $\mu_R = 1$.

Problem 4

First, we find the Nash equilibria. These are (L, R), $(L, pL + (1-p)R)$ and (M, L), where $0 \leq p \leq 0.7$.

We find a PBE (no restrictions on beliefs off path) first for the first two equilibria. Since player 1 playing M, R, is off path, we can allow player 2's belief over the information set to be what we want. If it is $\alpha M + (1-\alpha)R$ for $\alpha < 1/2$, then (L, R) is supported. If it is $(1/2)M + (1/2))R$, then player 2 can take play $pL + (1-p)R$, $p \in [0, 0.7]$ at his information set, since that is optimal given his belief and player 1 playing L is still optimal.

Finally, for (M, L), we note the belief at player 2's information set is M with certainty. Then, L is supported for player 2 under this belief. Hence, this is also a PBE.

Problem 5

We first look for Nash equilibria where player 1 plays a pure strategy. Suppose player 1 plays R for sure. Then player 3 will always play L for sure. Then player 2 can mix between L and R. Let player 2 play $pL+(1-p)R$. In order for player 1 to not want to deviate, $(1-p) \leq 1/4$ or $p \geq 3/4$. Hence, $(R, pL+(1-p)R, L)$ is a Nash equilibrium, for $p \geq 3/4$. Now suppose player 1 plays L for sure. Player 2 will always play R, since it always gives a strictly higher payout. Player 3 will then also play R since it gives the highest payout. Hence, (L, R, R) is also a Nash equilibrium.

Finally, we look for Nash equilibria where player 1 mixes. Suppose player 1 plays $(1-a)L+aR$. In order to want to mix, we know that player 1 always gets payoff 1 when playing R. So player 1 must have expected payoff at least 1 playing L. Let player 2 play R with probability b, and player 3 play R with probability c. The mixture indifference condition for player 1 implies

$$4b(1-c)+bc=1$$

$$4b-3bc=1$$

$$b(4-3c)=1$$

If $b=1$, then we must have $c=1$. If $c=1$, then $b=1$. So if players 2, 3 play (R,R), then player 1 can mix. In order for it to be optimal for 2 to play R, the expected value from playing R must exceed the expected value from playing L, or

$$1 \geq (1-a)+4a = 1+3a$$

This requires $a=0$, but we already counted (L,R,R). Hence, if player 1 mixes, we must have players 2 and 3 also both mix.

Now, we find the Nash equilibrium where all 3 players mix. We already have the indifference condition for player 1 is :

$$b(4-3c)=1$$

The indifference condition for mixing for player 2 is:

$$1 = 4ac + a(1-c) = 3ac + a = a(3c+1)$$

The indifference condition for player 3 is that players 1 and 2 play (L,R) and (R,L) with equal probability, or

$$a(1-b)=b(1-a)$$

$$a-ab=b-ab$$

$$a=b$$

So we get

$$a(4-3c)=1$$

$$a(3c+1)=1$$

This implies

$$4 - 3c = 3c + 1$$

$$c = 1/2$$

$$a = 2/5$$

$$b = 2/5$$

Hence, the other Nash equilibrium is

$$\left(\frac{3}{5}L + \frac{2}{5}R, \frac{3}{5}L + \frac{2}{5}R, \frac{1}{2}L + \frac{1}{2}R\right)$$

Now, for PBE, we need to construct beliefs for players 2 and 3 that support the equilibrium. Consider $(R, pL + (1-p)R, L)$, for $p \geq 3/4$. To support this, we have player 2's belief be that player 1 played R for sure, and player 3's belief is that players 1 and 2 played (R, L). Alternatively, for $p = 0$, the equilibrium (R, R, L) is also supported for player 2 believing player 1 played R and player 3 believing players 1 and 2 played some mixture of L, L and R, L (this is fine since player 3's belief is off-path). Similarly, for (L, R, R), we have player 2's belief is that player 1 played L and player 3's belief is that players 1 and 2 played (L, R).

Now, for the mixed-strategy Nash equilibrium, we can make this a PBE by the following beliefs: player 2 must believe with probability 3/5 that player 1 played L, and player 3 has the following belief:

$$\frac{3}{7}(L, L) + \frac{2}{7}(L, R) + \frac{2}{7}(R, L)$$

Problem 6

For pure-strategy Nash equilibria, we note that we have (R, C, L) and (R, S, R). Note that player 1 guarantees maximum payoff by playing 1: the only way player 1 could mix is if players 2 and 3 always play C, L. Clearly, if player 3 plays L, player 2 is best off playing C. Further, in order for player 3 to want to play L, we need player 1 playing R with higher probability than L. Hence, $(\alpha L + (1-\alpha)R, C, L)$ is Nash for $\alpha \leq 1/2$. If player 1 doesn't mix, the only way for player 2 to want to mix is if player 3 also mixes, but player 3 only wants to mix when player 2 plays S. Hence, for players 1 and 2 playing R, S, player 3 can play $\beta L + (1-\beta)R$ with $\beta \leq 1/2$, and this will be a Nash eq. Hence $(R, S, \beta L + (1-\beta)R)$ with $\beta \leq 1/2$ are the other Nash equilibria.

Now, for PBE, we need beliefs of players 2 and 3 to support the strategies. Let μ_2 be the belief of player 2 in the left side, and μ_3 be the belief of player 3 in the left side. For (R, C, L), this is supported by $\mu_2 = 0, \mu_3 = 0$. We can also support (R, S, R) with $\mu_2 = 0$, $\mu_3 \geq 1/2$ (in order for player 3 to play R). Further, $(\alpha L + (1-\alpha)R, C, L)$ for $\alpha \leq 1/2$ is supported for $\mu_2 = \alpha$, $\mu_3 = \alpha$. Lastly, $(R, S, \beta L + (1-\beta)R)$ with $\beta \leq 1/2$ is supported by beliefs $\mu_2 = 0, \mu_3 = 1/2$.