

# ECON501 Problem Set 1

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## Problem 2

(a) The seller problem is

$$\max \sum_i p_i(t(\theta_i) - c(q(\theta_i)))$$

subject to IC:

$$\forall i \neq j \quad \theta_i v(q(\theta_i)) - t(\theta_i) \geq \theta_i v(q(\theta_j)) - t(\theta_j)$$

and IR:

$$\forall i \quad \theta_i v(q(\theta_i)) - t(\theta_i) \geq 0$$

Now, manipulating IC, we get that for any  $i > j$ ,

$$\theta_i v(q(\theta_i)) - t(\theta_i) \geq \theta_i v(q(\theta_j)) - t(\theta_j)$$

$$\theta_i (v(q(\theta_i)) - v(q(\theta_j))) \geq t(\theta_i) - t(\theta_j)$$

Similarly from IC,

$$\theta_j v(q(\theta_i)) - t(\theta_j) \geq \theta_j v(q(\theta_i)) - t(\theta_i)$$

$$t(\theta_i) - t(\theta_j) \geq \theta_j (v(q(\theta_i)) - v(q(\theta_j)))$$

Putting these two together, we get:

$$\theta_i (v(q(\theta_i)) - v(q(\theta_j))) \geq t(\theta_i) - t(\theta_j) \geq \theta_j (v(q(\theta_i)) - v(q(\theta_j)))$$

$$(\theta_i - \theta_j)(v(q(\theta_i)) - v(q(\theta_j))) \geq 0$$

By our supposition,  $i > j$ , so  $\theta_i > \theta_j$ , and hence the first term in the product is positive. This implies the second term must also be positive, and hence

$$v(q(\theta_i)) - v(q(\theta_j)) \geq 0$$

$$v(q(\theta_i)) \geq v(q(\theta_j))$$

Since  $v$  is monotonically increasing by assumption, this implies  $q(\theta_i) \geq q(\theta_j)$ . Hence  $q$  is monotonic if IC holds.

(b) From IC, for  $i > 1$ ,  $\theta_i > \theta_1$ , and hence we have

$$\theta_i v(q(\theta_i)) - t(\theta_i) \geq \theta_i v(q(\theta_1)) - t(\theta_1) > \theta_1 v(q(\theta_1)) - t(\theta_1)$$

But the expression on the right is  $> 0$  by IR, hence IR for all  $i > 1$  are redundant.

Additionally, we note that from our previous formulation of IC in part *a*, we have for  $i \neq j$ ,

$$\theta_i v(q(\theta_i)) - t(\theta_i) \geq \theta_i v(q(\theta_j)) - t(\theta_j)$$

$$\theta_i (v(q(\theta_i)) - v(q(\theta_j))) \geq t(\theta_i) - t(\theta_j)$$

We claim that IC for consecutive  $i, j$  is sufficient. We show this for the case where  $i > j$  (the case where  $i < j$  is similar, using the opposite directional IC constraint). Then we have from the consecutive IC ( $i - j = \pm 1$ ):

$$\theta_i (v(q_i) - v(q_{i-1})) \geq t_i - t_{i-1}$$

$$\theta_{i-1} (v(q_{i-1}) - v(q_{i-2})) \geq t_{i-1} - t_{i-2}$$

$$\vdots$$

$$\theta_{j+1} (v(q_{j+1}) - v(q_j)) \geq t_{j+1} - t_j$$

Summing, we get

$$\sum_{k=j+1}^i \theta_k (v(q_k) - v(q_{k-1})) \geq t_i - t_j$$

Since the RHS telescopes. But since  $i \geq k$ , we have  $\theta_i \geq \theta_k$ , and hence

$$\sum_{k=j+1}^i \theta_i (v(q_k) - v(q_{k-1})) \geq \sum_{k=j+1}^i \theta_k (v(q_k) - v(q_{k-1})) \geq t_i - t_j$$

But the LHS telescopes, and we get

$$\theta_i (v(q_i) - v(q_j)) \geq t_i - t_j$$

and hence we have shown IC for  $i > j$  from consecutive IC. So the only non-redundant constraints are consecutive IC and IR for 1.

(c) IR1 must bind (otherwise we can increase all transfers by  $\epsilon$ ). Now, consider the constraint IC for  $k, k-1$ :

$$\theta_k v(q_k) - t_k \geq \theta_k v(q_{k-1}) - t_{k-1}$$

Suppose this did not bind. Then consider increasing the transfers  $t_k, t_{k+1}, t_{k+2} \dots t_n$  by  $\epsilon$ . Clearly, this doesn't break any of the ICs above  $k$  or below  $k-1$ . Clearly, IC for  $k, k-1$  still holds as long as  $\epsilon$  is

small. Additionally, IC for  $k - 1, k$  is

$$\theta_{k-1}v(q_{k-1}) - t_{k-1} \geq \theta_{k-1}v(q_k) - t_k$$

so increasing  $t_k$  without changing  $t_{k-1}$  maintains this constraint. Hence, we have IC  $k, k - 1$  must bind. Lastly, since  $q$  is monotonic in  $\theta$ , we have  $v(q_{k-1}) - v(q_k) \leq 0$ , and hence since IC  $k, k - 1$  binds,

$$\theta_{k-1}(v(q_{k-1}) - v(q_k)) \geq \theta_k(v(q_{k-1}) - v(q_k)) = t_{k-1} - t_k$$

$$\theta_{k-1}v(q_{k-1}) - t_{k-1} \geq \theta_{k-1}v(q_k) - t_k$$

so IC  $k - 1, k$  also holds (but does not necessarily bind).

### Problem 3

From class, we know that

$$q(\theta) = \arg \max v(q)\psi(\theta) - c(q) = \arg \max v(q)\psi(\theta) - q$$

We know that in order for this FOC to be valid, we need  $\psi(\theta) > 0$ , and hence under the regularity assumption, there exists a unique  $\theta^*$  such that

$$\psi(\theta^*) = 0 \iff \theta^* - \frac{1 - F(\theta^*)}{f(\theta^*)} = 0$$

Since regularity implies  $\psi$  is increasing in  $\theta$ , we have if  $\theta \leq \theta^*$ ,  $q(\theta) = 0$ , and otherwise for  $\theta > \theta^*$ , we have the FOC

$$v'(q)\psi(\theta) = 1$$

$$v'(q(\theta)) = 1/\psi(\theta)$$

Also, we know that

$$t(\theta) = \theta v(q(\theta)) - \int_0^\theta v(q(x)) \, dx$$

$$t'(\theta) = v(q(\theta)) + \theta v'(q(\theta))q'(\theta) - v(q(\theta)) = v'(q(\theta))q'(\theta) = \frac{\theta}{\psi(\theta)}q'(\theta)$$

Hence

$$t(\theta) = t(0) + \int_0^\theta \frac{x}{\psi(x)}q'(x) \, dx$$

Since  $t(0) = 0v(q(0)) = 0$ ,

$$t(\theta) = \int_0^\theta \frac{x}{\psi(x)}q'(x) \, dx$$

Integrating the RHS by parts,

$$t(\theta) = \frac{x}{\psi(x)} q(x) \Big|_0^\theta - \int_0^\theta \frac{\psi(x) - x\psi'(x)}{\psi(x)^2} q(x) dx$$

$$t(\theta) = \frac{\theta}{\psi(\theta)} q(\theta) - \int_0^\theta \frac{\psi(x) - x\psi'(x)}{\psi(x)^2} q(x) dx$$

Since  $q(0) = 0$ . Dividing by  $q(\theta)$ , we get

$$\frac{t(\theta)}{q(\theta)} = \frac{\theta}{\psi(\theta)} - \frac{1}{q(\theta)} \int_0^\theta \frac{\psi(x) - x\psi'(x)}{\psi(x)^2} q(x) dx$$

Taking the derivative wrt  $\theta$ , we get

$$\begin{aligned} \frac{\partial}{\partial \theta} \frac{t(\theta)}{q(\theta)} &= \frac{\psi(\theta) - \theta\psi'(\theta)}{(\psi(\theta))^2} + \frac{q'(\theta)}{q(\theta)^2} \int_0^\theta \frac{\psi(x) - x\psi'(x)}{\psi(x)^2} q(x) dx - \frac{\psi(\theta) - \theta\psi'(\theta)}{\psi(\theta)^2} \\ &= \frac{q'(\theta)}{q(\theta)^2} \int_0^\theta \frac{\psi(x) - x\psi'(x)}{\psi(x)^2} q(x) dx \end{aligned}$$

Now, we know  $q' > 0$ ,  $q^2 > 0$ ,  $\psi^2 > 0$ . Then  $\psi(\theta) \leq \theta\psi'(\theta)$  is a sufficient condition for this expression to be negative, since this makes the integrand negative at all values (equivalently, we can require  $\theta/\psi(\theta)$  is decreasing).

## Problem 4

(a) The regulator maximizes:

$$\int_0^q p(x) dx - p(q)q + \alpha\Pi(q) - s$$

subject to

$$p(q)q - C(q, \theta) + s \geq 0$$

Note that given a  $q$ , we want to pick  $s$  as small as possible to make the condition bind. Hence

$$s = C(q, \theta) - p(q)q$$

so  $\Pi(q) = 0$ . So the unconstrained maximization is given by

$$\max_q \int_0^q p(x) dx - p(q)q - C(q, \theta) + p(q)q = \max_q \int_0^q p(x) dx - C(q, \theta)$$

The interior FOC of the relaxed problem is

$$p(q) - \theta = 0$$

$$1 - 2q = \theta$$

$$q = \frac{1 - \theta}{2}$$

(b) The monopoly participation constraint is:

$$p(q(\theta))q(\theta) - K - \theta q(\theta) + s(\theta) \geq 0$$

The IC constraints are then

$$p(q(\theta))q(\theta) - K - \theta q(\theta) + s(\theta) \geq p(q(\theta'))q(\theta') - K - \theta q(\theta') + s(\theta')$$

Note for  $\theta < \bar{\theta}$  we get:

$$p(q(\bar{\theta}))q(\bar{\theta}) - K - \bar{\theta}q(\bar{\theta}) \leq p(q(\bar{\theta}))q(\bar{\theta}) - K - \theta q(\bar{\theta}) \leq p(q(\theta))q(\theta) - K - \theta q(\theta)$$

Hence IR is redundant except for  $\bar{\theta}$ . Note that this IR must bind, else we can uniformly lower the subsidies by the same amount. So  $U(\bar{\theta}) = 0$ . IC can be rewritten

$$U(\theta) = \max_{\theta'} p(q(\theta'))q(\theta') - K - \theta q(\theta') + s(\theta')$$

By the envelope theorem,  $U'(\theta) = -q(\theta)$ . So

$$U(\theta) = \int_{\theta}^{\bar{\theta}} q(x) dx$$

Then we can determine subsidies:

$$U(\theta) = \int_{\theta}^{\bar{\theta}} q(x) dx = p(q(\theta))q(\theta) - K - \theta q(\theta) + s(\theta)$$

$$s(\theta) = \int_{\theta}^{\bar{\theta}} q(x) dx - p(q(\theta))q(\theta) + K + \theta q(\theta) = \int_{\theta}^{\bar{\theta}} q(x) dx - p(q(\theta))q(\theta) + C(q, \theta)$$

Then the maximization problem becomes:

$$\begin{aligned} & \max_{\underline{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} (V(q(\theta)) + \alpha \Pi(q(\theta)) - s(\theta)) f(\theta) d\theta \\ &= \max_{\underline{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} \left( \int_0^{q(\theta)} p(x) dx - p(q(\theta))q(\theta) + \alpha \left( \int_{\theta}^{\bar{\theta}} q(x) dx \right) - \left( \int_{\theta}^{\bar{\theta}} q(x) dx - p(q(\theta))q(\theta) + K + \theta q(\theta) \right) \right) f(\theta) d\theta \\ &= \max_q \int_{\underline{\theta}}^{\bar{\theta}} \left( \int_0^{q(\theta)} p(x) dx - (1 - \alpha) \left( \int_{\theta}^{\bar{\theta}} q(x) dx \right) - K - \theta q(\theta) \right) f(\theta) d\theta \\ &= \max_q \int_{\underline{\theta}}^{\bar{\theta}} \left( \int_0^{q(\theta)} p(x) dx - K - \theta q(\theta) \right) f(\theta) d\theta - (1 - \alpha) \int_{\underline{\theta}}^{\bar{\theta}} \left( \int_{\theta}^{\bar{\theta}} q(x) dx \right) f(\theta) d\theta \end{aligned}$$

$$\begin{aligned}
&= \max_q \int_{\underline{\theta}}^{\bar{\theta}} \left( \int_0^{q(\theta)} p(x) dx - K - \theta q(\theta) \right) f(\theta) d\theta - (1-\alpha) \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\bar{\theta}} q(x) f(\theta) dx d\theta \\
&= \max_q \int_{\underline{\theta}}^{\bar{\theta}} \left( \int_0^{q(\theta)} p(x) dx - K - \theta q(\theta) \right) f(\theta) d\theta - (1-\alpha) \int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^x q(x) f(\theta) d\theta dx \\
&= \max_q \int_{\underline{\theta}}^{\bar{\theta}} \left( \int_0^{q(\theta)} p(x) dx - K - \theta q(\theta) \right) f(\theta) d\theta - (1-\alpha) \int_{\underline{\theta}}^{\bar{\theta}} q(x) F(x) dx \\
&= \max_q \int_{\underline{\theta}}^{\bar{\theta}} \left( \int_0^{q(\theta)} p(x) dx - K - \theta q(\theta) \right) f(\theta) d\theta - (1-\alpha) \int_{\underline{\theta}}^{\bar{\theta}} q(x) \frac{F(x)}{f(x)} f(x) dx \\
&= \max_q \int_{\underline{\theta}}^{\bar{\theta}} \left( \int_0^{q(\theta)} p(x) dx - K - \theta q(\theta) - (1-\alpha) \frac{F(\theta)}{f(\theta)} q(\theta) \right) f(\theta) d\theta \\
&= \max_q \int_{\underline{\theta}}^{\bar{\theta}} \left( \int_0^{q(\theta)} p(x) dx - K - \psi(\theta) q(\theta) \right) f(\theta) d\theta
\end{aligned}$$

where

$$\psi(\theta) = \theta + (1-\alpha) \frac{F(\theta)}{f(\theta)}$$

Since  $\theta + F(\theta)/f(\theta)$  is increasing, we have

$$1 + \frac{\partial}{\partial \theta} \frac{F(\theta)}{f(\theta)} > 0$$

$$\frac{\partial}{\partial \theta} \frac{F(\theta)}{f(\theta)} > -1$$

$$(1-\alpha) \frac{\partial}{\partial \theta} \frac{F(\theta)}{f(\theta)} > -(1-\alpha)$$

$$1 + (1-\alpha) \frac{\partial}{\partial \theta} \frac{F(\theta)}{f(\theta)} > \alpha > 0$$

Hence  $\theta + (1-\alpha)F(\theta)/f(\theta)$  is also increasing. Now, we just have to maximize:

$$\int_0^{q(\theta)} p(x) dx - K - \psi(\theta) q(\theta)$$

at each  $\theta$ . The FOC gives

$$p(q(\theta)) - \psi(\theta) = 0$$

$$1 - 2q(\theta) - \psi(\theta) = 0$$

$$q(\theta) = \frac{1 - \psi(\theta)}{2}$$

This interior solution is only valid as long as the integrand is positive, or

$$\int_0^{q(\theta)} p(x) dx - K - \psi(\theta) q(\theta) \geq 0$$

$$K \leq \int_0^{q(\theta)} p(x) dx - \psi(\theta)q(\theta) = q(\theta) - q(\theta)^2 - \psi(\theta)q(\theta) = \frac{(1 - \psi(\theta))^2}{4}$$

But by assumption,  $K \leq \frac{(1-\theta)^2}{4} = \frac{(1-\psi(\theta))^2}{4}$ . Since  $\psi$  is increasing, there exists some  $\theta^*$  such that  $K = (1 - \psi(\theta^*))^2/4$ . For  $\theta \geq \theta^*$ , the regulator sets  $q = 0$ , and for  $\theta < \theta^*$ , the regulator sets  $q = (1 - \psi(\theta))/2$ . This is almost the case of first best, but has  $\psi(\theta)$  instead of  $\theta$ ; that is, the regulator is forced to decrease quantity in order to properly incentivize truthful reporting of  $\theta$  by the monopoly.

## Problem 5

- (a) Suppose agent  $i$  realizes type  $\theta_i$ . By truthful reporting, the expected payout is given by the probability of winning the good times the expected payout given the good was won:

$$\theta_i (\theta_i - 2 * (\theta_i/2)) = \theta_i(0) = 0$$

However, by reporting some  $\theta_i - \epsilon$ , the expected payout is then

$$(\theta_i - \epsilon) (\theta_i - 2 * ((\theta_i - \epsilon)/2)) = (\theta_i - \epsilon)(\epsilon) > 0$$

Hence truthful reporting cannot be an equilibrium, since both players gain strictly higher expected payoff by underreporting.

- (b) Fix the player 2 report as  $b_2$ . Player 1 only wants the good iff  $\theta_1 \geq 2b_2$ , or  $\theta_1/2 \geq b_2$ . Hence, it is optimal to bid  $\theta_1/2$ . Symmetrically, it is optimal for player 2 to bid  $\theta_2/2$ . Hence, each player bids half of his/her own true value. Note it is ex-post efficient, since the player with the highest type gets the good. The truthful-reporting direct mechanism then has the allocation rule:

$$\bar{q}_i(\theta) = \int_0^\theta 1 dt = \theta$$

with transfers

$$\begin{aligned} \bar{t}_i(\theta) &= \theta \bar{q}_i(\theta) - \int_0^\theta \bar{q}_i(t) dt \\ &= \theta^2 - \theta^2/2 = \theta^2/2 \end{aligned}$$

## Problem 6

- (a) In the direct mechanism, each player reports his/her type. We need to specify the allocation rule that induces maximal revenue. We sell to the agent with the highest virtual type. The virtual type of player 1 is given by:

$$\psi_1(\theta_1) = \theta_1 - \frac{1 - (\theta_1 - 1)}{1} = 2\theta_1 - 2$$

The virtual type of 2 is

$$\psi_2(\theta_2) = \theta_2 - \frac{1 - (1/2)(\theta_2 - 1)}{1/2} = 2\theta_2 - 3$$

Note that  $\psi_1(\theta_1) \geq 0$  always, and  $\psi_2(\theta) > 0$  for  $\theta_2 > 1.5$ . Hence we want to give to player 2 if  $\psi_2 > \psi_1$ , else give to player 1. This condition is also rewriteable as:

$$2\theta_2 - 3 > 2\theta_1 - 2$$

$$\theta_2 > \theta_1 + \frac{1}{2}$$

The direct mechanism is then

$$\bar{q}_1(\theta) = \frac{\theta - 1/2}{2} = \max\left(\frac{2\theta - 1}{4}, 0\right)$$

$$\bar{q}_2(\theta) = \theta - 1/2 - 1 = \min\left(\theta - \frac{3}{2}, 1\right)$$

$$\bar{t}_1(\theta) = \theta \bar{q}_1(\theta) - \int_1^\theta \bar{q}_1(t) dt$$

$$= \theta \frac{2\theta - 1}{4} - \int_1^\theta \frac{2t - 1}{4} dt$$

$$= \frac{2\theta^2 - \theta}{4} - \frac{\theta^2 - \theta}{4}$$

$$= \frac{\theta^2}{4}$$

$$\bar{t}_2(\theta) = \theta \bar{q}_2(\theta) - \int_1^\theta \bar{q}_2(t) dt$$

$$= \theta(\theta - 3/2) - \int_{3/2}^\theta t - 3/2 dt$$

$$= \theta^2 - 3\theta/2 - (\theta^2/2 - 3\theta/2 - (9/8 - 9/4))$$

$$= \max(\min(\theta^2/2 - 9/8, 2), 0)$$

- (b) We present a BNE such that the auction implements the same allocation rule. Consider the strategy for player 1, with  $b_2$  fixed. Player 1 wants the good iff  $\theta_1 > b_2 - \frac{1}{2}$ , or  $\theta_1 + (1/2) \geq b_2$ . Hence, it is optimal to bid  $\theta_1 + 1/2$ . Now, for player 2, fixing  $b_1$ , player 2 wants the good iff  $\theta_2 \geq b_1$ , and hence it is optimal to bid  $\theta_2$ . So player 1 bids  $\theta_1 + 1/2$ , and player 2 bids  $\theta_2$ , and player 2 gets the good iff  $\theta_2 > \theta_1 + (1/2)$ , exactly the revenue maximizing result we derived in the previous part.

## Problem 7

Let  $G(x)$  denote the CDF of the the highest bid of  $N - 1$  players. That is,

$$G(x) = (F(x))^{N-1}$$



We denote the pdf associated with  $G$  as  $g$ . Then we know by the revenue equivalence theorem, the interim expected payoff is

$$\int_{\underline{v}}^{v_i} G(x) dx = G(v_i)v_i + (1 - G(v_i))(-b(v_i))$$

Solving for  $b(v_i)$ , we get

$$(1 - G(v_i))b(v_i) = G(v_i)v_i - \int_{\underline{v}}^{v_i} G(x) dx$$

$$b(v_i) = \frac{G(v_i)}{1 - G(v_i)}v_i - \frac{1}{1 - G(v_i)} \int_{\underline{v}}^{v_i} G(x) dx$$

Integrating by parts, we get

$$b(v_i) = \frac{G(v_i)}{1 - G(v_i)}v_i - \frac{1}{1 - G(v_i)} \int_{\underline{v}}^{v_i} G(x) dx$$

$$= \frac{G(v_i)}{1 - G(v_i)}v_i - \frac{1}{1 - G(v_i)} \left( v_i G(v_i) - \int_{\underline{v}}^{v_i} xg(x) dx \right)$$

$$= \frac{1}{1 - G(v_i)} \int_{\underline{v}}^{v_i} xg(x) dx$$

Hence players bidding according to this function gives a symmetric equilibrium.

## Problem 8

Once again, let  $G(x)$  denote the CDF of the highest bid of  $N - 1$  players as in the previous problem, and let the pdf associated with  $G$  be  $g$ . By revenue equivalence, the interim expected payoff of player with value  $v_i$  is

$$\int_0^{v_i} G(x) = G(v_i) (v_i - E(\alpha s(v_i) + (1 - \alpha)s(\max v_{-i}) | v_i = \max v))$$

$$\int_0^{v_i} G(x) = G(v_i) (v_i - \alpha s(v_i) - (1 - \alpha)E(s(\max v_{-i}) | v_i = \max v))$$

$$\int_0^{v_i} G(x) = (v_i - \alpha s(v_i))G(v_i) - (1 - \alpha) \int_0^{v_i} s(x)g(x) dx$$

Differentiating both sides wrt  $v_i$ , we get

$$G(v_i) = (v_i - \alpha s(v_i))g(v_i) + (1 - \alpha s'(v_i))G(v_i) - (1 - \alpha)s(v_i)g(v_i)$$

$$G(v_i) = v_i g(v_i) - \alpha s(v_i)g(v_i) + G(v_i) - \alpha G(v_i)s'(v_i) - (1 - \alpha)s(v_i)g(v_i)$$

$$0 = v_i g(v_i) - \alpha G(v_i)s'(v_i) - s(v_i)g(v_i)$$

$$\alpha G(v_i)s'(v_i) + s(v_i)g(v_i) = v_i g(v_i)$$

$$s'(v_i) + s(v_i)\frac{g(v_i)}{\alpha G(v_i)} = v_i \frac{g(v_i)}{\alpha G(v_i)}$$

Let

$$\varphi(x) = e^{\int_0^x \frac{g(t)}{\alpha G(t)} dt}$$

Then

$$\varphi'(x) = \frac{g(x)}{\alpha G(x)} \varphi(x)$$

Then

$$s'(v_i) \varphi(v_i) + s(v_i) \frac{g(v_i)}{\alpha G(v_i)} \varphi(v_i) = v_i \frac{g(v_i)}{\alpha G(v_i)} \varphi(v_i)$$

$$s'(v_i) \varphi(v_i) + s(v_i) \varphi'(v_i) = v_i \varphi'(v_i)$$

$$\frac{\partial}{\partial v_i} s(v_i) \varphi(v_i) = v_i \varphi'(v_i)$$

Integrating, we get

$$s(v_i) \varphi(v_i) = 0(\varphi'(0)) + \int_0^{v_i} x \varphi'(x) dx$$

$$s(v_i) \varphi(v_i) = \int_0^{v_i} x \varphi'(x) dx$$

$$s(v_i) \varphi(v_i) = x \varphi(x) \Big|_0^{v_i} - \int_0^{v_i} \varphi(x) dx$$

$$s(v_i) \varphi(v_i) = v_i \varphi(v_i) - \int_0^{v_i} \varphi(x) dx$$

$$s(v_i) = v_i - \frac{1}{\varphi(v_i)} \int_0^{v_i} \varphi(x) dx$$

where we integrated by parts. Now, we notice that we can rewrite  $\varphi$  :

$$\varphi(x) = e^{\int_0^x \frac{g(t)}{\alpha G(t)} dt} = e^{\frac{1}{\alpha}(\log G(x) - \log G(0))} = e^{\log(G(x))/\alpha} = G(x)^{1/\alpha} = F(x)^{(N-1)/\alpha}$$

So we can plug in

$$s(v_i) = v_i - \frac{1}{F(v_i)^{(N-1)/\alpha}} \int_0^{v_i} F(x)^{(N-1)/\alpha} dx$$