ECON501 Problem Set 3

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Problem 2

The provision of the public good is efficient iff $\theta_1 + \theta_2 > 0$. That is,

$$k(\theta_1, \theta_2) = 1[\theta_1 + \theta_2 > 0]$$

where we use 1[] to denote a $\{0, 1\}$ indicator variable that is 1 iff the bracketed expression is true. Hence,

$$\bar{k}_i(\theta_i) = Pr_{-i}(\theta_i + \theta_{-i} > 0) = Pr_{-i}(\theta_{-i} > -\theta_i) = \frac{1 - (-\theta_i)}{2} = \frac{1 + \theta_i}{2}$$

By IC and the envelope theorem, the interim expected utility is

$$U_i(\theta_i) = U(-1) + \int_{-1}^{\theta_i} \frac{1 + t}{2} \, dt$$

$$U_i(\theta_i) = U(-1) + \frac{(\theta_i + 1)^2}{4}$$

By IR, we need $U(-1) \geq 0$. Hence

$$U_i(\theta_i) = U(-1) + \frac{(\theta_i + 1)^2}{4} \geq \frac{(\theta_i + 1)^2}{4}$$

$$E[U_i(\theta_i)] \geq E\left[\left(\frac{\theta_i + 1}{2}\right)^2\right] = \int_{-1}^{1} \frac{1}{2}\left(\frac{t + 1}{2}\right)^2 \, dt$$

$$= \frac{1}{3}$$

So

$$E[U_1(\theta_1) + U_2(\theta_2)] \geq \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

However,

$$E[U_1(\theta_1) + U_2(\theta_2)] = E[\theta_1 k(\theta_1, \theta_2) + t_1(\theta_1, \theta_2) + \theta_2 k(\theta_1, \theta_2) + t_2(\theta_1, \theta_2)]$$

By budget balance, $t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) = 0$, so

$$E[U_1(\theta_1) + U_2(\theta_2)] = E[(\theta_1 + \theta_2)k(\theta_1, \theta_2)] = E[\max(\theta_1 + \theta_2, 0)]$$

$$= \int_{-1}^{1} \left(\int_{-\theta_1}^{1} (\theta_1 + \theta_2)\frac{1}{2}\ d\theta_2 \right) \frac{1}{2} d\theta_1$$

$$= \int_{-1}^{1} \left(\frac{(\theta_1+1)^2}{4} \right) \frac{1}{2} d\theta_1$$

$$= \int_{-1}^{1} \frac{(\theta_1+1)^2}{8} d\theta_1 = 1/3 < 2/3$$

Hence it is impossible to satisfy budget balance, IC, IR, and efficiency.

Problem 3

Throughout the problem, we denote $k_i = \partial k / \partial \theta_i$.

If: Suppose k is nondecreasing and

$$t_i(\theta_i, \theta_{-i}) = t_i(\underline{\theta}_i, \theta_{-i}) - \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v(k(s, \theta_{-i}), s)}{\partial k} \frac{\partial k(s, \theta_{-i})}{\partial s} ds$$

We have to show this is truthfully implementable in dominant strategies. Fix θ_{-i}. It suffices to show $\theta_i \in \arg\max_t u(f(t, \theta_{-i}), \theta_i)$. With some algebra, we have

$$u(f(t, \theta_{-i}), \theta_i) = v(k(t, \theta_{-i}), \theta_i) + t_i(t, \theta_{-i})$$

$$= v(k(t, \theta_{-i}), \theta_i) + t_i(\underline{\theta}_i, \theta_{-i}) - \int_{\underline{\theta}_i}^{t} \frac{\partial v(k(s, \theta_{-i}), s)}{\partial k} \frac{\partial k(s, \theta_{-i})}{\partial s} ds$$

Since v is twice continuously differentiable, and k is nondecreasing, the second order conditions are satisfied so we have the FOC:

$$\frac{\partial}{\partial t} u(f(t, \theta_{-i}), \theta_i) = 0$$

$$\frac{\partial}{\partial t} v(k(t, \theta_{-i}), \theta_i) - \frac{\partial}{\partial t} \int_{\underline{\theta}_i}^{t} \frac{\partial v(k(s, \theta_{-i}), s)}{\partial k} \frac{\partial k(s, \theta_{-i})}{\partial s} ds = 0$$

$$\left(\frac{\partial v(k(t, \theta_{-i}), \theta_i)}{\partial k} - \frac{\partial v(k(t, \theta_{-i}), t)}{\partial k} \right) \frac{\partial k(t, \theta_{-i})}{\partial t} = 0$$

Then we note that for $t = \theta_i$, this is

$$\left(\frac{\partial v(k(\theta_i, \theta_{-i}), \theta_i)}{\partial k} - \frac{\partial v(k(\theta_i, \theta_{-i}), \theta_i)}{\partial k} \right) k_i(\theta_i, \theta_{-i}) = (0) k_i(\theta_i, \theta_{-i}) = 0$$

Hence the FOC is satisfied, so $\theta_i \in \arg\max_t u(f(t, \theta_{-i}), \theta_i)$. Therefore f is truthfully implementable in dominant strategies.

Only if: Suppose f is truthfully implementable in dominant strategies. Then $\theta_i \in \arg\max_t u(f(t, \theta_{-i}), \theta_i)$. Suppose, for sake of contradiction, that k is not nondecreasing in θ_i. That is, for some $\theta_i < \theta_i'$, fixed θ_{-i}, $k(\theta_i, \theta_{-i}) > k(\theta_i', \theta_{-i})$. Since v is supermodular by the assumption $\partial^2 v / \partial k \partial \theta > 0$, we have

$$v(k(\theta_i, \theta_{-i}), \theta_i') + v(k(\theta_i', \theta_{-i}), \theta_i) > v(k(\theta_i', \theta_{-i}), \theta_i') + v(k(\theta_i, \theta_{-i}), \theta_i)$$

$$v(k(\theta_i, \theta_{-i}), \theta_i') - v(k(\theta_i', \theta_{-i}), \theta_i') > v(k(\theta_i, \theta_{-i}), \theta_i) - v(k(\theta_i', \theta_{-i}), \theta_i)$$

$$v(k(\theta_i, \theta_{-i}), \theta_i') + t(\theta_i, \theta_{-i}) - v(k(\theta_i', \theta_{-i}), \theta_i') - t(\theta_i', \theta_{-i}) > v(k(\theta_i, \theta_{-i}), \theta_i) + t(\theta_i, \theta_{-i}) - v(k(\theta_i', \theta_{-i}), \theta_i) - t(\theta_i', \theta_{-i})$$

$$u(f(\theta_i, \theta_{-i}), \theta_i') - u(f(\theta_i', \theta_{-i}), \theta_i') > u(f(\theta_i, \theta_{-i}), \theta_i) - u(f(\theta_i', \theta_{-i}), \theta_i)$$

But since f is truthfully implementable, $\theta_i \in \arg\max_t u(f(t, \theta_{-i}), \theta_i)$, which implies that

$$u(f(\theta_i', \theta_{-i}), \theta_i') \geq u(f(\theta_i, \theta_{-i}), \theta_i')$$

$$u(f(\theta_i, \theta_{-i}), \theta_i) \geq u(f(\theta_i', \theta_{-i}), \theta_i)$$

or equivalently

$$0 \geq u(f(\theta_i, \theta_{-i}), \theta_i') - u(f(\theta_i', \theta_{-i}), \theta_i')$$

$$u(f(\theta_i, \theta_{-i}), \theta_i) - u(f(\theta_i', \theta_{-i}), \theta_i) \geq 0$$

which imply $u(f(\theta_i, \theta_{-i}), \theta_i) - u(f(\theta_i', \theta_{-i}), \theta_i) \geq (f(\theta_i, \theta_{-i}), \theta_i') - u(f(\theta_i', \theta_{-i}), \theta_i')$. But this directly contradicts the expression we derived earlier, that

$$u(f(\theta_i, \theta_{-i}), \theta_i') - u(f(\theta_i', \theta_{-i}), \theta_i') > u(f(\theta_i, \theta_{-i}), \theta_i) - u(f(\theta_i', \theta_{-i}), \theta_i)$$

Hence, k must be nondecreasing in θ_i.

Now, in order for $\theta_i \in \arg\max_t u(f(t, \theta_{-i}), \theta_i)$, we have the FOC:

$$\frac{\partial}{\partial t} u(f(t, \theta_{-i}), \theta_i)\Big|_{\theta_i} = 0$$

$$\left(\frac{\partial}{\partial t} v(k(t, \theta_{-i}), \theta_i) + \frac{\partial}{\partial t} t_i(t, \theta_{-i})\right)\Big|_{\theta_i} = 0$$

$$\frac{\partial v(k(\theta_i, \theta_{-i}), \theta_i)}{\partial k} k_i(\theta_i, -\theta_i) + \frac{\partial}{\partial t} t_i(t, \theta_{-i})\Big|_{\theta_i} = 0$$

$$\frac{\partial}{\partial t} t_i(t, \theta_{-i})\Big|_{\theta_i} = -\frac{\partial v(k(\theta_i, \theta_{-i}), \theta_i)}{\partial k} k_i(\theta_i, -\theta_i)$$

Then by the fundamental theorem of calculus, we have

$$t_i(\theta_i, \theta_{-i}) - t_i(\underline{\theta}_i, \theta_{-i}) = \int_{\underline{\theta}_i}^{\theta_i} -\frac{\partial v(k(t, \theta_{-i}), t)}{\partial k} k_i(t, -\theta_i)\ dt$$

$$t_i(\theta_i, \theta_{-i}) = t_i(\underline{\theta}_i, \theta_{-i}) - \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v(k(t, \theta_{-i}), t)}{\partial k} k_i(t, -\theta_i)\ dt$$

as desired.

Problem 4

We restrict our attention to VCG mechanisms as in the TA suggestion. Let the total number of individuals be N.

(a) If: Suppose $V^*(\theta) = \sum_i V_i(\theta_{-i})$. Consider the VCG mechanism where

$$h_i(\theta_{-i}) = -(N-1)V_i(\theta_{-i})$$

Then

$$t_i(\theta) = -\sum_{j\neq i} v_j(k^*(\theta), \theta_j) + (N-1)V_i(\theta_{-i})$$

so

$$\begin{aligned}
\sum_i t_i(\theta) &= \sum_i \left(-\sum_{j\neq i} v_j(k^*(\theta), \theta_j) \right) + \sum_i (N-1)V_i(\theta_{-i}) \\
&= \sum_i \left(v_i(k^*(\theta), \theta_i) - \sum_j v_j(k^*(\theta), \theta_j) \right) + (N-1)V^*(\theta) \\
&= \sum_i \left(v_i(k^*(\theta), \theta_i) - V^*(\theta) \right) + (N-1)V^*(\theta) \\
&= V^*(\theta) - NV^*(\theta) + (N-1)V^*(\theta) \\
&= 0
\end{aligned}$$

Hence this is budget balanced. k^* is efficient, this VCG mechanism is ex-post efficient. So the VCG mechanism allocation rule is a social choice function that is ex-post efficient, and truthfully implementable (since any VCG mechanism allocation rule is truthfully implementable).

Only if: Since we can restrict our attention to VCG mechanisms, suppose the VCG mechanism given by some $\{h_i\}$ is ex-post efficient, and budget balanced. Because of budget balance,

$$0 = \sum_i t_i(\theta) = \sum_i \left(-\sum_{j\neq i} v_j(k^*(\theta), \theta_j) + h_i(\theta_{-i}) \right)$$

$$0 = \sum_i \left(-\sum_{j\neq i} v_j(k^*(\theta), \theta_j) \right) + \sum_i h_i(\theta_{-i})$$

$$0 = \sum_i \left(v_i(k^*(\theta), \theta_i) - \sum_j v_j(k^*(\theta), \theta_j) \right) + \sum_i h_i(\theta_{-i})$$

$$0 = \sum_i \left(v_i(k^*(\theta), \theta_i) - V^*(\theta) \right) + \sum_i h_i(\theta_{-i})$$

$$0 = V^*(\theta) - NV^*(\theta) + \sum_i h_i(\theta_{-i})$$

$$(N-1)V^*(\theta) = \sum_i h_i(\theta_{-i})$$

$$V^*(\theta) = \sum_i \frac{1}{N-1} h_i(\theta_{-i})$$

Hence, we can take $V_i(\theta_{-i}) = \frac{1}{N-1} h_i(\theta_{-i})$.

(b) We will show this for general N, not just $N = 3$. k^* satisfies, by assumption,

$$k^*(\theta) \in \arg\max_k \sum_{i=1}^N \theta_i k - \frac{1}{2}k^2$$

$$k^*(\theta) \in \arg\max_k \left(\sum_{i=1}^N \theta_i\right) k - \frac{N}{2}k^2$$

By the FOC,

$$\left(\sum_{i=1}^N \theta_i\right) - Nk^*(\theta) = 0$$

$$k^*(\theta) = \frac{1}{N}\left(\sum_{i=1}^N \theta_i\right)$$

We must show that $V^*(\theta) = \sum_i V_i(\theta_{-i})$ for some V_i.

$$V^*(\theta) = \sum_i \left(\theta_i k^*(\theta) - \frac{1}{2}(k^*(\theta))^2\right)$$

$$V^*(\theta) = \left(\sum_i \theta_i\right) k^*(\theta) - \frac{N}{2}(k^*(\theta))^2$$

$$V^*(\theta) = \left(\sum_i \theta_i\right) \frac{1}{N}\left(\sum_i \theta_i\right) - \frac{N}{2}\left(\frac{1}{N}\left(\sum_i \theta_i\right)\right)^2$$

$$V^*(\theta) = \frac{1}{N}\left(\sum_i \theta_i\right)^2 - \frac{1}{2N}\left(\sum_i \theta_i\right)^2$$

$$V^*(\theta) = \frac{1}{2N}\left(\sum_i \theta_i\right)^2$$

$$V^*(\theta) = \frac{1}{2N}\left(\sum_i \theta_i^2 + \sum_j \sum_{k \neq j} \theta_k \theta_j\right)$$

Now, we note that

$$\sum_i \sum_{j \neq i} \theta_j^2 = \sum_i \left(-\theta_i^2 + \sum_j \theta_j^2\right) = (N-1)\sum_i \theta_i^2$$

$$\sum_i \theta_i^2 = \sum_i \left(\frac{1}{N-1} \sum_{j \neq i} \theta_j^2 \right)$$

and

$$\sum_i \sum_{j \neq i} \sum_{k \neq j,i} \theta_j \theta_k = \sum_i \sum_{j \neq i} \left(-\theta_j \theta_i + \sum_{k \neq j} \theta_j \theta_k \right)$$

$$= \sum_i \left(-\sum_{j \neq i} \theta_j \theta_i + \sum_{j \neq i} \sum_{k \neq j} \theta_j \theta_k \right)$$

$$= \sum_i \left(-\sum_{j \neq i} \theta_j \theta_i - \sum_{k \neq i} \theta_i \theta_k + \sum_j \sum_{k \neq j} \theta_j \theta_k \right)$$

$$= \sum_i \left(-2 \sum_{j \neq i} \theta_j \theta_i \right) + N \sum_j \sum_{k \neq j} \theta_j \theta_k$$

$$= (N-2) \sum_j \sum_{k \neq j} \theta_j \theta_k$$

$$\sum_j \sum_{k \neq j} = \sum_i \left(\frac{1}{N-2} \sum_{j \neq i} \sum_{k \neq j,i} \theta_j \theta_k \right)$$

Putting these together with our previous derivation, we get

$$V^*(\theta) = \frac{1}{2N} \left(\sum_i \theta_i^2 + \sum_j \sum_{k \neq j} \theta_k \theta_j \right)$$

$$= \frac{1}{2N} \left(\sum_i \left(\frac{1}{N-1} \sum_{j \neq i} \theta_j^2 \right) + \sum_i \left(\frac{1}{N-2} \sum_{j \neq i} \sum_{k \neq j,i} \theta_j \theta_k \right) \right)$$

$$= \sum_i \left(\frac{1}{2N(N-1)} \sum_{j \neq i} \theta_j^2 \right) + \sum_i \left(\frac{1}{2N(N-2)} \sum_{j \neq i} \sum_{k \neq j,i} \theta_j \theta_k \right)$$

$$= \sum_i \left(\frac{1}{2N(N-1)} \sum_{j \neq i} \theta_j^2 + \frac{1}{2N(N-2)} \sum_{j \neq i} \sum_{k \neq j,i} \theta_j \theta_k \right)$$

Then if we let

$$V_i(\theta_{-i}) = \frac{1}{2N(N-1)} \sum_{j \neq i} \theta_j^2 + \frac{1}{2N(N-2)} \sum_{j \neq i} \sum_{k \neq j,i} \theta_j \theta_k$$

we get $V^*(\theta) = \sum_i V_i(\theta_{-i})$ and hence an ex-post efficient outcome is truthfully implementable in dominant strategies.

(c) From part a, we know that $V^*(\theta) = \sum_i V_i(\theta_{-i})$ for some V_i. Then

$$\frac{\partial^I}{\partial\theta_1\ \partial\theta_2 ... \partial\theta_I} V^*(\theta) = \sum_i \frac{\partial^I}{\partial\theta_1\ \partial\theta_2 ... \partial\theta_I} V_i(\theta_{-i}) = \sum_i 0 = 0$$

Since $\partial V_i / \partial\theta_i = 0$, since there is no θ_i dependence in V_i. Hence this is a necessary condition for an ex post efficient social choice function to exist.

(d) Let $I = 2$. By efficiency,

$$V^*(\theta_1, \theta_2) = \max_k (v_1(k, \theta_1) + v_2(k, \theta_2))$$

Applying the envelope theorem,

$$\frac{\partial V^*}{\partial\theta_1} = \frac{\partial v_1(k^*(\theta), \theta_1)}{\partial\theta_1}$$

$$\frac{\partial^2 V^*}{\partial\theta_1 \partial\theta_2} = \frac{\partial^2 v_1(k^*(\theta), \theta_1)}{\partial k \partial\theta_1} \frac{\partial k^*}{\partial\theta_2}$$

Now, since $k^*(\theta)$ maximizes $v_1(k, \theta_1) + v_2(k, \theta_2)$, we get the FOC:

$$\frac{\partial v_1(k^*(\theta), \theta_1)}{\partial k} + \frac{\partial v_2(k^*(\theta), \theta_2)}{\partial k} = 0$$

Applying the implicit function theorem, we get

$$\frac{\partial k^*}{\partial\theta_2} = \frac{\frac{\partial^2 v_2(k^*(\theta), \theta_2)}{\partial k \partial\theta_2}}{\frac{\partial^2 v_1(k^*(\theta), \theta_1)}{\partial^2 k} + \frac{\partial^2 v_2(k^*(\theta), \theta_2)}{\partial^2 k}}$$

Plugging in, we get

$$\frac{\partial^2 V^*}{\partial\theta_1 \partial\theta_2} = \frac{\frac{\partial^2 v_1(k^*(\theta), \theta_1)}{\partial k \partial\theta_1} \frac{\partial^2 v_2(k^*(\theta), \theta_2)}{\partial k \partial\theta_2}}{\frac{\partial^2 v_1(k^*(\theta), \theta_1)}{\partial^2 k} + \frac{\partial^2 v_2(k^*(\theta), \theta_2)}{\partial^2 k}} \neq 0$$

since $\frac{\partial^2 v_1(k^*(\theta), \theta_1)}{\partial k \partial\theta_1} \neq 0$, $\frac{\partial^2 v_2(k^*(\theta), \theta_2)}{\partial k \partial\theta_2} \neq 0$, and $\frac{\partial^2 v_1(k^*(\theta), \theta_1)}{\partial^2 k} + \frac{\partial^2 v_2(k^*(\theta), \theta_2)}{\partial^2 k} < 0$. Since $\frac{\partial^2 V^*}{\partial\theta_1 \partial\theta_2} \neq 0$, by the contrapositive of part c, we cannot have any ex-post efficient outcome that is truthfully implementable in dominant strategies.

Problem 5

We show that if $|X| = 2$, majority vote suffices, with ties broken randomly. That is, we choose the preferred option of the majority of the individuals. This is well-defined since we assume no indifference and there are only 2 alternatives, so one of the two alternatives must have a majority wanting it. Clearly, majority vote is not dictatorial. We just have to check efficiency and dominant IC.

For efficiency, suppose the selection x is not efficient. Since $|X| = 2$, the other alternative x' must be such that $x' \succeq_{\theta_i} x$ for all i, and $x' \succ_{\theta_i} x$ for some i. Since we assume no indifference, this implies that x cannot have been the majority vote, since for every individual, $x' \succeq_{\theta_i} x$. Since no inefficient alternative can possibly be the majority vote, the majority vote must be efficient.

Now, we check IC. Fix the social choice as x. Suppose individual i submitted preference $x \succeq x'$. Then clearly i cannot do better submitting $x' \succeq x$, since i already has achieved their best possible outcome (of the two choices). Now, suppose i's preferences are $x' \succeq x$. Then by submitting a deviation ($x \succeq x'$), majority vote on the deviation must also be x (if a majority supported x before i changed votes, then a majority still must support x if i joins them). Hence no matter what preference any individual has, dominant IC is satisfied.

So majority rule is efficient, non-dictatorial, and dominant IC.