# ECON501 Problem Set 1

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## Problem 2

(a) The seller problem is

$$\max \sum_i p_i(t(\theta_i) - c(q(\theta_i)))$$

subject to IC:

$$\forall i \neq j \quad \theta_i v(q(\theta_i)) - t(\theta_i) \geq \theta_i v(q(\theta_j)) - t(\theta_j)$$

and IR:

$$\forall i \quad \theta_i v(q(\theta_i)) - t(\theta_i) \geq 0$$

Now, manipulating IC, we get that for any $i > j$,

$$\theta_i v(q(\theta_i)) - t(\theta_i) \geq \theta_i v(q(\theta_j)) - t(\theta_j)$$

$$\theta_i(v(q(\theta_i)) - v(q(\theta_j))) \geq t(\theta_i) - t(\theta_j)$$

Similarly from IC,

$$\theta_j v(q(\theta_i)) - t(\theta_j) \geq \theta_j v(q(\theta_i)) - t(\theta_i)$$

$$t(\theta_i) - t(\theta_j) \geq \theta_j(v(q(\theta_i)) - v(q(\theta_i)))$$

Putting these two together, we get:

$$\theta_i(v(q(\theta_i)) - v(q(\theta_j))) \geq t(\theta_i) - t(\theta_j) \geq \theta_j(v(q(\theta_i)) - v(q(\theta_i)))$$

$$(\theta_i - \theta_j)(v(q(\theta_i)) - v(q(\theta_j))) \geq 0$$

By our supposition, $i > j$, so $\theta_i > \theta_j$, and hence the first term in the product is positive. This implies the second term must also be positive, and hence

$$v(q(\theta_i)) - v(q(\theta_j)) \geq 0$$

$$v(q(\theta_i)) \geq v(q(\theta_j))$$

Since $v$ is monotonically increasing by assumption, this implies $q(\theta_i) \geq q(\theta_j)$. Hence $q$ is monotonic if IC holds.

(b) From IC, for $i > 1$, $\theta_i > \theta_1$, and hence we have

$$\theta_i v(q(\theta_i)) - t(\theta_i) \geq \theta_i v(q(\theta_1)) - t(\theta_1) > \theta_1 v(q(\theta_1)) - t(\theta_1)$$

But the expression on the right is $> 0$ by IR, hence IR for all $i > 1$ are redundant.

Additionally, we note that from our previous formulation of IC in part $a$, we have for $i \neq j$,

$$\theta_i v(q(\theta_i)) - t(\theta_i) \geq \theta_i v(q(\theta_j)) - t(\theta_j)$$

$$\theta_i (v(q(\theta_i)) - v(q(\theta_j))) \geq t(\theta_i) - t(\theta_j)$$

We claim that IC for consecutive $i, j$ is sufficient. We show this for the case where $i > j$ (the case where $i < j$ is similar, using the opposite directional IC constraint). Then we have from the consecutive IC ($i - j = \pm 1$):

$$\theta_i (v(q_i) - v(q_{i-1})) \geq t_i - t_{i-1}$$

$$\theta_{i-1} (v(q_{i-1}) - v(q_{i-2})) \geq t_{i-1} - t_{i-2}$$

$$\vdots$$

$$\theta_{j+1} (v(q_{j+1}) - v(q_j)) \geq t_{j+1} - t_j$$

Summing, we get

$$\sum_{k=j+1}^{i} \theta_k (v(q_k) - v(q_{k-1})) \geq t_i - t_j$$

Since the RHS telescopes. But since $i \geq k$, we have $\theta_i \geq \theta_k$, and hence

$$\sum_{k=j+1}^{i} \theta_i (v(q_k) - v(q_{k-1})) \geq \sum_{k=j+1}^{i} \theta_k (v(q_k) - v(q_{k-1})) \geq t_i - t_j$$

But the LHS telescopes, and we get

$$\theta_i (v(q_i) - v(q_j)) \geq t_i - t_j$$

and hence we have shown IC for $i > j$ from consecutive IC. So the only non-redundant constraints are consecutive IC and IR for 1.

(c) IR1 must bind (otherwise we can increase all transfers by $\epsilon$). Now, consider the constraint IC for $k, k-1$:

$$\theta_k v(q_k) - t_k \geq \theta_k v(q_{k-1}) - t_{k-1}$$

Suppose this did not bind. Then consider increasing the transfers $t_k, t_{k+1}, t_{k+2} ... t_n$ by $\epsilon$. Clearly, this doesn't break any of the ICs above $k$ or below $k-1$. Clearly, IC for $k, k-1$ still holds as long as $\epsilon$ is small. Additionally, IC for $k-1, k$ is

$$\theta_{k-1}v(q_{k-1}) - t_{k-1} \geq \theta_{k-1}v(q_k) - t_k$$

so increasing $t_k$ without changing $t_{k-1}$ maintains this constraint. Hence, we have IC $k, k-1$ must bind. Lastly, since $q$ is monotonic in $\theta$, we have $v(q_{k-1}) - v(q_k) \leq 0$, and hence since IC $k, k-1$ binds,

$$\theta_{k-1}(v(q_{k-1}) - v(q_k)) \geq \theta_k(v(q_{k-1}) - v(q_k)) = t_{k-1} - t_k$$

$$\theta_{k-1}v(q_{k-1}) - t_{k-1} \geq \theta_{k-1}v(q_k) - t_k$$

so IC $k-1, k$ also holds (but does not necessarily bind).

# Problem 3

From class, we know that

$$q(\theta) = \arg\max v(q)\psi(\theta) - c(q) = \arg\max v(q)\psi(\theta) - q$$

We know that in order for this FOC to be valid, we need $\psi(\theta) > 0$, and hence under the regularity assumption, there exists a unique $\theta^*$ such that

$$\psi(\theta^*) = 0 \iff \theta^* - \frac{1 - F(\theta^*)}{f(\theta^*)} = 0$$

Since regularity implies $\psi$ is increasing in $\theta$, we have if $\theta \leq \theta^*$, $q(\theta) = 0$, and othehrwise for $\theta > \theta^*$, we have the FOC

$$v'(q)\psi(\theta) = 1$$

$$v'(q(\theta)) = 1/\psi(\theta)$$

Also, we know that

$$t(\theta) = \theta v(q(\theta)) - \int_0^\theta v(q(x))\ dx$$

$$t'(\theta) = v(q(\theta)) + \theta v'(q(\theta))q'(\theta) - v(q(\theta)) = v'(q(\theta))q'(\theta) = \frac{\theta}{\psi(\theta)}q'(\theta)$$

Hence

$$t(\theta) = t(0) + \int_0^\theta \frac{x}{\psi(x)}q'(x)\ dx$$

Since $t(0) = 0v(q(0)) = 0$,

$$t(\theta) = \int_0^\theta \frac{x}{\psi(x)}q'(x)\ dx$$

Integrating the RHS by parts,

$$t(\theta) = \frac{x}{\psi(x)} q(x)\Big|_0^\theta - \int_0^\theta \frac{\psi(x) - x\psi'(x)}{\psi(x)^2} q(x)\ dx$$

$$t(\theta) = \frac{\theta}{\psi(\theta)} q(\theta) - \int_0^\theta \frac{\psi(x) - x\psi'(x)}{\psi(x)^2} q(x)\ dx$$

Since $q(0) = 0$. Dividing by $q(\theta)$, we get

$$\frac{t(\theta)}{q(\theta)} = \frac{\theta}{\psi(\theta)} - \frac{1}{q(\theta)} \int_0^\theta \frac{\psi(x) - x\psi'(x)}{\psi(x)^2} q(x)\ dx$$

Taking the derivative wrt $\theta$, we get

$$\frac{\partial}{\partial\theta} \frac{t(\theta)}{q(\theta)} = \frac{\psi(\theta) - \theta\psi'(\theta)}{(\psi(\theta))^2} + \frac{q'(\theta)}{q(\theta)^2} \int_0^\theta \frac{\psi(x) - x\psi'(x)}{\psi(x)^2} q(x)\ dx - \frac{\psi(\theta) - \theta\psi'(\theta)}{\psi(\theta)^2}$$

$$= \frac{q'(\theta)}{q(\theta)^2} \int_0^\theta \frac{\psi(x) - x\psi'(x)}{\psi(x)^2} q(x)\ dx$$

Now, we know $q' > 0$, $q^2 > 0$, $\psi^2 > 0$. Then $\psi(\theta) \leq \theta\psi'(\theta)$ is a sufficient condition for this expression to be negative, since this makes the integrand negative at all values (equivalently, we can require $\theta/\psi(\theta)$ is decreasing).

## Problem 4

(a) The regulator maximizes:

$$\int_0^q p(x)\ dx - p(q)q + \alpha\Pi(q) - s$$

subject to

$$p(q)q - C(q, \theta) + s \geq 0$$

Note that given a $q$, we want to pick $s$ as small as possible to make the condition bind. Hence

$$s = C(q, \theta) - p(q)q$$

so $\Pi(q) = 0$. So the unconstrained maximization is given by

$$\max_q \int_0^q p(x)\ dx - p(q)q - C(q, \theta) + p(q)q = \max_q \int_0^q p(x)\ dx - C(q, \theta)$$

The interior FOC of the relaxed problem is

$$p(q) - \theta = 0$$

$$1 - 2q = \theta$$

$$q = \frac{1-\theta}{2}$$

(b) The monopoly participation constraint is:

$$p(q(\theta))q(\theta) - K - \theta q(\theta) + s(\theta) \geq 0$$

The IC constraints are then

$$p(q(\theta))q(\theta) - K - \theta q(\theta) + s(\theta) \geq p(q(\theta'))q(\theta') - K - \theta q(\theta') + s(\theta')$$

Note for $\theta < \overline{\theta}$ we get:

$$p(q(\overline{\theta}))q(\overline{\theta}) - K - \overline{\theta}q(\overline{\theta}) \leq p(q(\overline{\theta}))q(\overline{\theta}) - K - \theta q(\overline{\theta}) \leq p(q(\theta))q(\theta) - K - \theta q(\theta)$$

Hence IR is redundant except for $\overline{\theta}$. Note that this IR must bind, else we can uniformly lower the subsidies by the same amount. So $U(\overline{\theta}) = 0$. IC can be rewritten

$$U(\theta) = \max_{\theta'} p(q(\theta'))q(\theta') - K - \theta q(\theta') + s(\theta')$$

By the envelope theorem, $U'(\theta) = -q(\theta)$. So

$$U(\theta) = \int_{\theta}^{\overline{\theta}} q(x)\ dx$$

Then we can determine subsidies:

$$U(\theta) = \int_{\theta}^{\overline{\theta}} q(x)\ dx = p(q(\theta))q(\theta) - K - \theta q(\theta) + s(\theta)$$

$$s(\theta) = \int_{\theta}^{\overline{\theta}} q(x)\ dx - p(q(\theta))q(\theta) + K + \theta q(\theta) = \int_{\theta}^{\overline{\theta}} q(x)\ dx - p(q(\theta))q(\theta) + C(q,\theta)$$

Then the maximization problem becomes:

$$\max \int_{\underline{\theta}}^{\overline{\theta}} \left(V(q(\theta)) + \alpha\Pi(q(\theta)) - s(\theta)\right) f(\theta)\ d\theta$$

$$= \max \int_{\underline{\theta}}^{\overline{\theta}} \left( \int_0^{q(\theta)} p(x)\ dx - p(q(\theta))q(\theta) + \alpha \left( \int_{\theta}^{\overline{\theta}} q(x)\ dx \right) - \left( \int_{\theta}^{\overline{\theta}} q(x)\ dx - p(q(\theta))q(\theta) + K + \theta q(\theta) \right) \right) f(\theta)\ d\theta$$

$$= \max_q \int_{\underline{\theta}}^{\overline{\theta}} \left( \int_0^{q(\theta)} p(x)\ dx - (1-\alpha) \left( \int_{\theta}^{\overline{\theta}} q(x)\ dx \right) - K - \theta q(\theta) \right) f(\theta)\ d\theta$$

$$= \max_q \int_{\underline{\theta}}^{\overline{\theta}} \left( \int_0^{q(\theta)} p(x)\ dx - K - \theta q(\theta) \right) f(\theta)\ d\theta - (1-\alpha) \int_{\underline{\theta}}^{\overline{\theta}} \left( \int_{\theta}^{\overline{\theta}} q(x)\ dx \right) f(\theta) d\theta$$

$$= \max_q \int_{\underline{\theta}}^{\overline{\theta}} \left( \int_0^{q(\theta)} p(x)\ dx - K - \theta q(\theta) \right) f(\theta)\ d\theta - (1-\alpha) \int_{\underline{\theta}}^{\overline{\theta}} \int_{\theta}^{\overline{\theta}} q(x) f(\theta)\ dx\ d\theta$$

$$= \max_q \int_{\underline{\theta}}^{\overline{\theta}} \left( \int_0^{q(\theta)} p(x)\ dx - K - \theta q(\theta) \right) f(\theta)\ d\theta - (1-\alpha) \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{x} q(x) f(\theta)\ d\theta\ dx$$

$$= \max_q \int_{\underline{\theta}}^{\overline{\theta}} \left( \int_0^{q(\theta)} p(x)\ dx - K - \theta q(\theta) \right) f(\theta)\ d\theta - (1-\alpha) \int_{\underline{\theta}}^{\overline{\theta}} q(x) F(x)\ dx$$

$$= \max_q \int_{\underline{\theta}}^{\overline{\theta}} \left( \int_0^{q(\theta)} p(x)\ dx - K - \theta q(\theta) \right) f(\theta)\ d\theta - (1-\alpha) \int_{\underline{\theta}}^{\overline{\theta}} q(x) \frac{F(x)}{f(x)} f(x0)\ dx$$

$$= \max_q \int_{\underline{\theta}}^{\overline{\theta}} \left( \int_0^{q(\theta)} p(x)\ dx - K - \theta q(\theta) - (1-\alpha) \frac{F(\theta)}{f(\theta)} q(\theta) \right) f(\theta)\ d\theta$$

$$= \max_q \int_{\underline{\theta}}^{\overline{\theta}} \left( \int_0^{q(\theta)} p(x)\ dx - K - \psi(\theta) q(\theta) \right) f(\theta)\ d\theta$$

where

$$\psi(\theta) = \theta + (1-\alpha)\frac{F(\theta)}{f(\theta)}$$

Since $\theta + F(\theta)/f(\theta)$ is increasing, we have

$$1 + \frac{\partial}{\partial\theta}\frac{F(\theta)}{f(\theta)} > 0$$

$$\frac{\partial}{\partial\theta}\frac{F(\theta)}{f(\theta)} > -1$$

$$(1-\alpha)\frac{\partial}{\partial\theta}\frac{F(\theta)}{f(\theta)} > -(1-\alpha)$$

$$1 + (1-\alpha)\frac{\partial}{\partial\theta}\frac{F(\theta)}{f(\theta)} > \alpha > 0$$

Hence $\theta + (1-\alpha)F(\theta)/f(\theta)$ is also increasing. Now, we just have to maximize:

$$\int_0^{q(\theta)} p(x)\ dx - K - \psi(\theta) q(\theta)$$

at each $\theta$. The FOC gives

$$p(q(\theta)) - \psi(\theta) = 0$$

$$1 - 2q(\theta) - \psi(\theta) = 0$$

$$q(\theta) = \frac{1-\psi(\theta)}{2}$$

This interior solution is only valid as long as the integrand is positive, or

$$\int_0^{q(\theta)} p(x)\ dx - K - \psi(\theta) q(\theta) \geq 0$$

$$K \leq \int_0^{q(\theta)} p(x)\ dx - \psi(\theta)q(\theta) = q(\theta) - q(\theta)^2 - \psi(\theta)q(\theta) = \frac{(1-\psi(\theta))^2}{4}$$

But by assumption, $K \leq \frac{(1-\underline{\theta})^2}{4} = \frac{(1-\psi(\underline{\theta}))^2}{4}$. Since $\psi$ is increasing, there exists some $\theta^*$ such that $K = (1-\psi(\theta^*))^2/4$. For $\theta \geq \theta^*$, the regulator sets $q = 0$, and for $\theta < \theta^*$, the regulator sets $q = (1-\psi(\theta))/2$. This is almost the case of first best, but has $\psi(\theta)$ instead of $\theta$; that is, the regulator is forced to decrease quantity in order to properly incentivize truthful reporting of $\theta$ by the monopoly.

# Problem 5

(a) Suppose agent $i$ realizes type $\theta_i$. By truthful reporting, the expected payout is given by the probability of winning the good times the expected payout given the good was won:

$$\theta_i\left(\theta_i - 2*(\theta_i/2)\right) = \theta_i(0) = 0$$

However, by reporting some $\theta_i - \epsilon$, the expected payout is then

$$(\theta_i - \epsilon)\left(\theta_i - 2*((\theta_i-\epsilon)/2)\right) = (\theta_i - \epsilon)(\epsilon) > 0$$

Hence truthful reporting cannot be an equilibrium, since both players gain strictly higher expected payoff by underreporting.

(b) Fix the player 2 report as $b_2$. Player 1 only wants the good iff $\theta_1 \geq 2b_2$, or $\theta_1/2 \geq b_2$. Hence, it is optimal to bid $\theta_1/2$. Symmetrically, it is optimal for player 2 to bid $\theta_2/2$. Hence, each player bids half of his/her own true value. Note it is ex-post efficient, since the player with the highest type gets the good. The direct mechanism then has the alloction rule:

$$\bar{q}_i(\theta) = \int_0^{2\theta} 1\ dt = 2\theta$$

with transfers

$$\begin{aligned}\bar{t}_i(\theta) &= \theta\bar{q}_i(\theta) - \int_0^{\theta} \bar{q}_i(t)\ dt \\ &= 2\theta^2 - \theta^2 = \theta^2\end{aligned}$$

# Problem 6

(a) In the direct mechanism, each player reports his/her type. We need to specify the allocation rule that induces maximal revenue. We sell to the agent with the highest virtual type. The virtual type of player 1 is given by:

$$\psi_1(\theta_1) = \theta_1 - \frac{1-(\theta_1-1)}{1} = 2\theta_1 - 2$$

The virtual type of 2 is

$$\psi_2(\theta_2) = \theta_2 - \frac{1-(1/2)(\theta_2-1)}{1/2} = 2\theta_2 - 3$$

Note that $\psi_1(\theta_1) \geq 0$ always, and $\psi_2(\theta) > 0$ for $\theta_2 > 1.5$. Hence we want to give to player 2 if $\psi_2 > \psi_1$, else give to player 1. This condition is also rewriteable as:

$$2\theta_2 - 3 > 2\theta_1 - 2$$

$$\theta_2 > \theta_1 + \frac{1}{2}$$

The direct mechanism is then

$$\bar{q}_1(\theta) = \frac{\theta - 1/2}{2} = \max\left(\frac{2\theta - 1}{4}, 0\right)$$

$$\bar{q}_2(\theta) = \theta - 1/2 - 1 = \min\left(\theta - \frac{3}{2}, 1\right)$$

$$\begin{aligned}
\bar{t}_1(\theta) &= \theta\bar{q}_1(\theta) - \int_1^\theta \bar{q}_1(t)\ dt \\
&= \theta\frac{2\theta - 1}{4} - \int_1^\theta \frac{2t-1}{4}\ dt \\
&= \frac{2\theta^2 - \theta}{4} - \frac{\theta^2 - \theta}{4} \\
&= \frac{\theta^2}{4}
\end{aligned}$$

$$\begin{aligned}
\bar{t}_2(\theta) &= \theta\bar{q}_2(\theta) - \int_1^\theta \bar{q}_2(t)\ dt \\
&= \theta(\theta - 3/2) - \int_{3/2}^\theta t - 3/2\ dt \\
&= \theta^2 - 3\theta/2 - (\theta^2/2 - 3\theta/2 - (9/8 - 9/4)) \\
&= \theta^2/2 - 9/8
\end{aligned}$$

(b) We present a BNE such that the auction implements the same allocation rule. Consider the strategy for player 1, with $b_2$ fixed. Player 1 wants the good iff $\theta_1 > b_2 - \frac{1}{2}$, or $\theta_1 + (1/2) \geq b_2$. Hence, it is optimal to bid $\theta_1 + 1/2$. Now, for player 2, fixing $b_1$, player 2 wants the good iff $\theta_2 \geq b_1$, and hence it is optimal to bid $\theta_2$. So player 1 bids $\theta_1 + 1/2$, and player 2 bids $\theta_2$, and player 2 gets the good iff $\theta_2 > \theta_1 + (1/2)$, exactly the revenue maximizing result we derived in the previous part.

## Problem 7

Let $G(x)$ denote the CDF of the the highest bid of $N-1$ players. That is,

$$G(x) = (F(x))^{N-1}$$

We denote the pdf assicated with $G$ as $g$. Then we know by the revenue equivalence theorem, the interim expected payoff is

$$\int_{\underline{v}}^{v_i} G(x)\ dx = G(v_i)v_i + (1 - G(v_i))(-b(v_i))$$

Solving for $b(v_i)$, we get

$$(1 - G(v_i))b(v_i) = G(v_i)v_i - \int_{\underline{v}}^{v_i} G(x)\ dx$$

$$b(v_i) = \frac{G(v_i)}{1 - G(v_i)}v_i - \frac{1}{1 - G(v_i)} \int_{\underline{v}}^{v_i} G(x)\ dx$$

Integrating by parts, we get

$$b(v_i) = \frac{G(v_i)}{1 - G(v_i)}v_i - \frac{1}{1 - G(v_i)} \int_{\underline{v}}^{v_i} G(x)\ dx$$

$$= \frac{G(v_i)}{1 - G(v_i)}v_i - \frac{1}{1 - G(v_i)} \left(v_i G(v_i) - \int_{\underline{v}}^{v_i} xg(x)\ dx\right)$$

$$= \int_{\underline{v}}^{v_i} xg(x)\ dx$$

Hence players bidding according to this function gives a symmetric equilibrium.

# Problem 8

Once again, let $G(x)$ denote the CDF of the highest bid of $N - 1$ players as in the previous problem, and let the pdf associated with $G$ be $g$. By revenue equivalence, the interim expected payoff of player with value $v_i$ is

$$\int_0^{v_i} G(x) = G(v_i)\left(v_i - E(\alpha s(v_i) + (1 - \alpha)s(\max v_{-i})|v_i = \max v)\right)$$

$$\int_0^{v_i} G(x) = G(v_i)\left(v_i - \alpha s(v_i) - (1 - \alpha)E(s(\max v_{-i})|v_i = \max v)\right)$$

$$\int_0^{v_i} G(x) = (v_i - \alpha s(v_i))G(v_i) - (1 - \alpha) \int_0^{v_i} s(x)g(x)\ dx$$

Differentiating both sides wrt $v_i$, we get

$$G(v_i) = (v_i - \alpha s(v_i))g(v_i) + (1 - \alpha s'(v_i))G(v_i) - (1 - \alpha)s(v_i)g(v_i)$$

$$G(v_i) = v_i g(v_i) - \alpha s(v_i)g(v_i) + G(v_i) - \alpha G(v_i)s'(v_i) - (1 - \alpha)s(v_i)g(v_i)$$

$$0 = v_i g(v_i) - \alpha G(v_i)s'(v_i) - s(v_i)g(v_i)$$

$$\alpha G(v_i)s'(v_i) + s(v_i)g(v_i) = v_i g(v_i)$$

$$s'(v_i) + s(v_i)\frac{g(v_i)}{\alpha G(v_i)} = v_i \frac{g(v_i)}{\alpha G(v_i)}$$

Let

$$\varphi(x) = e^{\int_0^x \frac{g(t)}{\alpha G(t)}\ dt}$$

Then

$$\varphi'(x) = \frac{g(t)}{\alpha G(t)}\varphi(x)$$

Then

$$s'(v_i)\varphi(v_i) + s(v_i)\frac{g(v_i)}{\alpha G(v_i)}\varphi(v_i) = v_i\frac{g(v_i)}{\alpha G(v_i)}\varphi(v_i)$$

$$s'(v_i)\varphi(v_i) + s(v_i)\varphi'(v_i) = v_i\varphi'(v_i)$$

$$\frac{\partial}{\partial v_i}s(v_i)\varphi(v_i) = v_i\varphi'(v_i)$$

Integrating, we get

$$s(v_i)\varphi(v_i) = 0(\varphi'(0)) + \int_0^{v_i} x\varphi'(x)\ dx$$

$$s(v_i)\varphi(v_i) = \int_0^{v_i} x\varphi'(x)\ dx$$

$$s(v_i)\varphi(v_i) = x\varphi(x)\Big|_0^{v_i} - \int_0^{v_i} \varphi(x)\ dx$$

$$s(v_i)\varphi(v_i) = v_i\varphi(v_i) - \int_0^{v_i} \varphi(x)\ dx$$

$$s(v_i) = v_i - \frac{1}{\varphi(v_i)}\int_0^{v_i} \varphi(x)\ dx$$

where we integrated by parts. Now, we notice that we can rewrite $\varphi:$

$$\varphi(x) = e^{\int_0^x \frac{g(t)}{\alpha G(t)}\ dt} = e^{\frac{1}{\alpha}(\log G(x) - \log G(0))} = e^{\log(G(x))/\alpha} = G(x)^{1/\alpha} = F(x)^{(N-1)/\alpha}$$

So we can plug in

$$s(v_i) = v_i - \frac{1}{F(v_i)^{(N-1)/\alpha}}\int_0^{v_i} F(x)^{(N-1)/\alpha}\ dx$$