ECON501 Problem Set 2

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Problem 2

First, we note that B is strictly dominated by A for the row player ($(10, -4, 5, 0) \gg (9, -6, 3, -2)$). Eliminating B, we have that a is dominated by b for the column player, since $(5, 6, -2, 3) \gg (2, 0, -3, 1)$. Among the remaining strategies, A and C are both dominated by E since $(-4, 5, 0) \ll (-2, 6, 2)$ and $(-4, 5, 0.1) \ll (-2, 6, 2)$.

Now, the remaining strategies for the row player are D and E, and the remaining column strategies are b, c, d. However, we note that the strategy $0.4b + 0.6d$ dominates c since $(-0.8, 1.2) \gg (-1, 1)$. Hence c is strictly dominated.

The remaining strategies D, E and b, d cannot be eliminated further, so these remain.

Problem 3

We first argue playing 0 is weakly dominated by playing 1. Playing 0 guarantees payoff 0 in any situation. Playing 1 guarantees payoff 1 in any situation (if the other person plays less than 100, then you get 1, and if the other person plays 100, you still get 1). Hence 0 is a weakly dominated strategy, so we eliminate it.

We now argue that the strategy of playing 100 is weakly dominated by the strategy of playing 99. If the other player also plays 100, then 99 achieves a better payoff ($99 > 50$). Suppose the other player played 99. Then again playing 99 gives a better payoff than 100, because $50 > 1$. If the other player played $k \in [2, 98]$, then the player gets $100 - k$, so playing 99 does equally as well as 100 here. Finally, if the other player played 1, playing 100 would get payoff 99, and playing 99 also ensures payoff 99, so for $k \in [1, 98]$ the player does the same playing 99 as 100, and for $k \in [99, 100]$ the player does better playing 99 than playing 100. Hence 100 is weakly dominated.

We now can iterate this argument. Suppose we have not yet eliminated the strategies from $[50-x, 50+x]$, where x is a positive integer. Consider playing $50 - x + 1$ instead of $50 - x$. This guarantees a payoff of at least $50 - x + 1$, since if the other person plays any number less than $50 + x - 1$ the sum is ≤ 100, and the player gets $50 - x + 1$, and if the other person plays $50 + x$, $50 - x + 1$ is the lower bid and hence the player always gets $50 - x + 1$. This is strictly better than playing $50 - x$, which guarantees payoff $50 - x$, so $50 - x$ is weakly dominated by $50 - x + 1$.

Now, after eliminating $50 - x$, we can show $50 + x$ is weakly dominated by $50 + x - 1$. Suppose the other player plays $50 + x$. Playing $50 + x - 1$ guarantees payoff $50 + x - 1 \geq 50$, so playing $50 + x - 1$ is weakly better in this case. If the other player plays $50 + x - 1$, then playing $50 + x - 1$ guarantees payoff $50 \geq 50 - x + 1$, so $50 + x - 1$ is also weakly better. If the other player plays anything else from $50 - x + 2$ to $50 + x - 2$, they will have the lower bid and the sum in either case will be greater than 100, so there is no change in payoff from switching to $50 + x - 1$ from $50 + x$. Lastly, if the other player plays $50 - x + 1$, switching to $50 + x - 1$ makes the sum 100, but in either case the payoff is $50 + x - 1$, so there is no change in payoff still here. Thus, $50 - x$ is weakly dominated by $50 - x + 1$, and now the range of remaining strategies is $[50 - x + 1, 50 + x - 1]$

We can apply our argument iteratively; the only strategy remaining will be playing 50.

Problem 4

Suppose player 1 plays $pU + (1 - p)D$, and player 2 plays $qL + (1 - q)R$. The payoffs of player 3's actions are then:

A	$9pq$
B	$9p(1-q) + 9(1-p)q = 9p + 9q - 18pq$
C	$9(1-p)(1-q) = 9 - 9p - 9q + 9pq$
D	$6pq + 6(1-p)(1-q) = 6 - 6p - 6q + 12pq$

Suppose, for sake of contradiction, D is the best reply. For D to be better than A, we have

$$6 - 6p - 6q + 12pq > 9pq$$

$$2 - 2p - 2q + pq > 0$$

$$\frac{1}{2}pq + 1 > p + q$$

$$p - \frac{1}{2}pq < 1 - q$$

$$p\left(1 - \frac{1}{2}q\right) < 1 - q$$

$$p < \frac{1-q}{1-0.5q}$$

For D to be better than C,

$$6 - 6p - 6q + 12pq > 9 - 9p - 9q + 9pq$$

$$p + q > 1 - pq$$

$$p + pq > 1 - q$$

$$p(1 + q) > 1 - q$$

$$p > \frac{1-q}{1+q}$$

For D to be better than B,

$$6 - 6p - 6q + 12pq > 9p + 9q - 18pq$$

$$6 - 15p - 15q + 30pq > 0$$

$$(2/5) - p - q + 2pq > 0$$

$$\frac{2}{5} + 2pq > p + q$$

$$p - 2pq < \frac{2}{5} - q$$

$$p(1 - 2q) < \frac{2}{5} - q$$

For $q < 1/2$, this implies

$$p < \frac{0.4 - q}{1 - 2q}$$

But from a prior condition,

$$p > \frac{1 - q}{1 + q}$$

But on the region $q \in (0, 1/2)$,

$$\frac{1 - q}{1 + q} > \frac{0.4 - q}{1 - 2q}$$

So we have a contradiction.

Similarly, if $q > 1/2$,

$$p(1 - 2q) < \frac{2}{5} - q$$

$$p > \frac{0.4 - q}{1 - 2q}$$

But

$$p < \frac{1 - q}{1 - 0.5q}$$

And on $q \in (1/2, 1)$

$$\frac{0.4 - q}{1 - 2q} > \frac{1 - q}{1 - 0.5q}$$

Hence we have a contradiction.

Lastly, if $q = 1/2$, then the payoffs become

A	$4.5p$
B	4.5
C	$9(1-p)(1-q) = 4.5 - 4.5p$
D	$6pq + 6(1-p)(1-q) = 3$

And B is strictly better than D, a contradiction again.

Therefore, D cannot be a best reply, no matter what values q takes. However, D is not dominated. Consider any arbitrary other strategy $pA + qB + rC + (1 - p - q - r)D$. In order for this to dominate D, it must be better than D in every situation. The payoffs of this strategy for (U, L) are:

$$9p + 6(1 - p - q - r) = 6 + 3p - 6q - 6r$$

For (U, R) and (D, L), the payoffs are just

$$9q$$

For (D, R) the payoff is

$$9r + 6(1 - p - q - r) = 6 - 6p - 6q + 3r$$

We need these to be better than D in each case, so

$$6 + 3p - 6q - 6r \geq 6$$

$$9q \geq 0$$

$$6 - 6p - 6q + 3r \geq 6$$

From the first and third inequalities, we get

$$p \geq 2q + 2r$$

$$r \geq 2p + 2q$$

$$p + r \geq 2(p + r) + 4q$$

But then we get

$$p + r + 4q \leq 0$$

which is only possible if $p = r = q = 0$, which is just D itself. Hence no other strategy dominates D.

Problem 5

(a) If either player has a strategy guaranteeing a win, then that player i by definition has a maximin strategy s_i such that $u_i(s_i, s_j) \geq 1$ for all s_j. Since the utility can only be 0 or ± 1, this implies $u_i(s_i, s_j) = 1$, for any s_j. Hence s_i weakly dominates any other strategy, and so $D(N)$ has u_i constant and equal to 1, and hence u_j is constant and equal to -1, so this is dominance solvable in one step.

(b) Now, assume that neither player can guarantee a win, and hence each player can enforce 0.

(i) Fix the strategy profile (x_1, x_2) and define x_{h^*} as in the problem. By definition, $v_1(x_{h^*}) \neq 0$, so it is either 1 or -1. Suppose, for sake of contradiction, that it is 1. Then for the subgame $\Gamma(x_{h^*})$, there exists some strategy s_1' within this subgame that is winning, i.e. achieves 1, since $v_1(x_{h^*}) = 1$ by our assumption. Consider the alternative strategy

$$s_1^*(x) = \begin{cases} s_1(x) & x \notin X(x_{h^*}) \\ s_1'(x) & x \in X(x_{h^*}) \end{cases}$$

Then clearly s_1^* does equally as good as s_1 if the subtree x_{h^*} is never reached, and since it achieves 1 in that subtree, it does weakly better in that subtree. Hence s_1^* dominates s_1 in N, a contradiction. Therefore $v_1(x_{h^*}) = -1$.

(ii) If it were player 2's turn at x_{h^*-1}, then player 2 could take an action that would force 1 to x_{h^*}, and we know that since $v_1(x_h^*) = -1$, this is optimal for player 2. That implies that $v_1(x_{h^*-1}) = -1$, which is false, since we know by construction that $v_1(x_{h^*-1}) = 0$. Hence it must be player 1's turn at x_{h^*-1}.

(iii) (A) Suppose not, that some $s_2' \in D(N)$ is such that $u_1(s_1, s_2') > u_1(s_1', s_2')$. Note the paths dictated by s_1, s_2' and s_1', s_2' must reach x_{h^*-1}, since s_1' is equal to s_1 outside $\Gamma(x_{h^*-1})$ and $u_1(s_1', s_2') \neq u_1(s_1, s_2')$. Since s_1' is maximin on $\Gamma(x_{h^*-1})$, and since $v_1(x_{h^*-1}) = 0$, we have $u_1(s_1', s_2') \geq 0$, which means that $u_1(s_1, s_2') = 1$. Recall that $v_1(x_{h^*}) = -1$, as proved before, so $v_2(x_{h^*}) = 1$. Pick a winning strategy s_2^* in $\Gamma(x_{h^*})$. Consider s_2^* such that

$$s_2''(x) = \begin{cases} s_2^*(x) & x \notin X(x_{h^*}) \\ s_2'(x) & x \in X(x_{h^*}) \end{cases}$$

Clearly, since s_2'' equals s_2' outside of $\Gamma(x_{h^*})$, it does equally well outside $\Gamma(x_{h^*})$. However, since s_2^* is winning in $\Gamma(x_{h^*})$, and $u_2(s_1, s_2') = -1$, $u_2(s_1, s_2'') > u_2(s_1, s_2')$, and hence s_2' is dominated by s_2'' in N, so we have a contradiction that $s_2' \in D(N)$. Hence s_1' is at least as good as s_1 against any strategy in $D(N)$.

(B) When player 2 plays s_2, we note that s_1' and s_1 will both reach $\Gamma(x_{h^*-1})$. This implies by construction of s_1', $u_1(s_1', s_2) = 0$, since s_1' plays maximin in $\Gamma(x_{h^*-1})$ and $v_1(x_{h^*-1}) = 0$. However, $u_1(s_1, s_2) = -1$, so s_1 is dominated by s_1', and hence is dominated in $D(N)$.