ECON500: Problem Set 6

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Part 1

(1.1) Free disposal implies that if $y_i' \leq y_i \ \forall i$ and $y \in Y$, then $y' \in Y$.

Suppose Y is closed, convex, and contains the negative orthant. Suppose $y_i' \leq y_i$ and $y \in Y$. Define the vector Δ as $\Delta_i = y_i' - y_i$. Then by our supposition, $\Delta_i \leq 0$, so Δ lies in the negative orthant, and therefore there exists $\lambda > 1$ such that $y + \lambda\Delta$ lies in the negative orthant. This implies $y + \lambda\Delta \in Y$. Then by convexity:

$$y' = \Delta + y = \frac{1}{\lambda}(y + \lambda\Delta) + \frac{\lambda - 1}{\lambda} y \in Y$$

Hence $y' \in Y$, so Y satisfies free disposal.

(1.2) Decreasing returns to scale:

$$y \in Y \implies \alpha y \in Y \ \alpha \in [0, 1]$$

Increasing returns to scale:

$$y \in Y \implies \alpha y \in Y \ \alpha \in [1, \infty)$$

Constant returns to scale:

$$y \in Y \implies \alpha y \in Y \ \alpha \in [0, \infty)$$

Suppose Y exhibits constant returns to scale. Then for $y \in Y$, $\alpha \in [0, 1]$, by definition of constant returns $\alpha y \in Y$, so Y exhibits decreasing returns to scale. Similarly, if $\alpha \in [1, \infty)$, $\alpha y \in Y$, so Y also exhibits increasing returns to scale. In the reverse, if Y exhibits increasing and decreasing returns to scale, then for $y \in Y$, for any $\alpha \in [0, 1] \cup [1, \infty) = [0, \infty)$, $\alpha y \in Y$, so Y exhibits constant returns to scale.

Decreasing returns is not equivalent to convexity of Y. Consider $Y = \{(x, y) \ : \ xy < 1 \text{ or } x < 0 \text{ or } y < 0\}$. This exhibits decreasing returns: if $x, y \in Y$, $\alpha \in [0, 1]$, then $(\alpha x)(\alpha y) = \alpha^2 xy < \alpha^2 \leq 1$, so $\alpha(x, y) \in Y$. However, this is not convex: $(0, 3), (3, 0) \in Y$, but $(1.5, 1.5) \notin Y$.

Convexity of Y does imply decreasing returns, however. If Y is convex, then for any $y \in Y$, $\alpha \in [0, 1]$, $\alpha y + (1 - \alpha)0 = \alpha y \in Y$.

Increasing returns is not equivalent to the convexity of Y^c. Consider $Y = \{(x, y) \ : \ xy < 1 \text{ or } x < 0 \text{ or } y < 0\}$. $Y^c = \{(x, y) \ : \ x \geq 0, y \geq 0, xy \geq 1\}$ is convex. However, Y does not exhibit increasing returns: $(0.5, 0.5) \in Y$, but $(2, 2) \notin Y$.

Increasing returns does not even imply the convexity of Y^c. Consider $Y = \{(x, y) \ : x < 0, y < 0\}$ (the negative orthant). Then clearly Y satisfies increasing returns: if $y \in Y$, then $\alpha y \in Y$. However, Y^c is not convex: $(-3, 1)$ and $(1, -3)$ are not in Y, but their midpoint $(-1, -1)$ is in Y.

Now, suppose there are L goods and the production set Y exhibits constant returns to scale. Fix one of the inputs at $y^* < 0$, and consider the production set Y' over the remaining $L - 1$ goods. Suppose $(y_1, y_2, ...y_{L-1}) \in Y'$, and $\alpha \in [0, 1]$. This implies that

$$(y_1, y_2, ...y_{L-1}, y^*) \in Y$$

By constant returns to scale on Y,

$$(\alpha y_1, \alpha y_2, ...\alpha y_{L-1}, \alpha y^*) \in Y$$

By free disposal, $\alpha y^* > y^*$, so

$$(\alpha y_1, \alpha y_2, ...\alpha y_{L-1}, y^*) \in Y$$

Then

$$(\alpha y_1, \alpha y_2, ...\alpha y_{L-1}) \in Y'$$

and hence Y' exhibits decreasing returns.

(1.3) Increasing returns: for $\alpha \geq 1$,

$$F(y) \leq 0 \implies F(\alpha y) \leq 0$$

Decreasing returns: for $\alpha \in [0, 1]$,

$$F(y) \leq 0 \implies F(\alpha y) \leq 0$$

(1.4) We first show that increasing returns implies failure of the SOCs for profit maximization. Suppose that Y satisfies increasing returns. Let $F(y) = 0$ denote the frontier of Y. Then the maximization problem is

$$\max p \cdot y$$

$$F(y) = 0$$

The Lagrangian is

$$\mathcal{L} = p \cdot y - \lambda F(y)$$

Consider an arbitrary critical point, y^*. By the FOCs,

$$p = \lambda \nabla F(y^*)$$

$$F(y^*) = 0$$

Consider $y' = (1 + \epsilon)y^*$ for some small $\epsilon > 0$. By increasing returns, $F(y') \leq 0$. However, $p \cdot y' - p \cdot y^* = \epsilon(p \cdot y^*) > 0$ since $p \cdot y^* \geq 0$ as $0 \in Y$. This implies that y^* cannot maximize the objective, and hence the second order conditions must fail.

Now, consider the production set $Y = \{(x, y) \ : \ x + y \leq 0\}$. Clearly, this set exhibits decreasing returns. Suppose prices are $(1, 2)$. There is no solution to the profit maximization: for $k > 0$, $(-k, k) \in Y$, and $p \cdot (-k, k) = k$ is arbitrarily increasing in k.

(1.5) Suppose $Y = \{(-z, q) \ : f(z) \geq q\}$. Let the prices be given by w, and the cost function is the minimum value of

$$\min w \cdot z$$

subject to

$$q \leq f(z)$$

Let $z(w, q)$ be the minimizer, $c(w, q)$ be the minimum value. Increasing costs implies that for $\alpha \geq 1$,

$$\alpha c(w, q) \leq c(w, \alpha q)$$

Decreasing costs implies that for $\alpha \geq 1$

$$\alpha c(w, q) \geq c(w, \alpha q)$$

And constant costs implies that for $\alpha \geq 1$

$$\alpha c(w, q) = c(w, \alpha q)$$

We now relate this cost function to the features of the set Y in terms of increasing/decreasing/constant returns. Suppose Y satisfies increasing returns. Pick a w, and consider $(-z(w, q), q) \in Y$. Let $\alpha \geq 1$. By increasing returns $(-\alpha z(w, q), \alpha q) \in Y$. Then we know $f(\alpha z(w, q)) \geq \alpha q$. So $c(w, \alpha q) \leq w\alpha z(w, q) = \alpha c(w, q)$. Hence increasing returns implies decreasing costs.

Similarly, we show that decreasing returns implies increasing costs. Suppose Y satisfies decreasing returns. Take $\alpha \geq 1$. Pick a w, and consider $(-z(w, \alpha q), \alpha q) \in Y$. By decreasing returns $(-\frac{1}{\alpha} z(w, \alpha q), q) \in Y$. Therefore

$$\frac{1}{\alpha} w \cdot z(w, \alpha q) \geq c(w, q)$$

$$\frac{1}{\alpha} c(w, \alpha q) \geq c(w, q)$$

$$c(w, \alpha q) \geq \alpha c(w, q)$$

so we have increasing costs.

Clearly, we can observe that if we have constant returns, we have both increasing and decreasing returns, and hence increasing and decreasing costs. This implies we have constant costs.

TODO: if i don't feel too lazy I'll try some reverse implications.

(1.6) Let the firm have production function f, and face prices p_1, p_2, q. The maximization problem is

$$\max qy - p_1 x_1 - p_2 x_2$$

$$y \leq f(x_1, x_2)$$

From FOCs:

$$q - \lambda = 0$$

$$\lambda f_1(x_1, x_2) = p_1$$

$$\lambda f_2(x_1, x_2) = p_2$$

Differentiating in p_1, we have

$$qf_{11}(x_1, x_2)\frac{\partial x_1}{\partial p_1} + qf_{12}(x_1, x_2)\frac{\partial x_2}{\partial p_1} = 1$$

$$qf_{21}(x_1, x_2)\frac{\partial x_1}{\partial p_1} + qf_{22}(x_1, x_2)\frac{\partial x_2}{\partial p_1} = 0$$

Together, in matrix form, we have

$$\begin{bmatrix} qf_{11}(x_1, x_2) & qf_{12}(x_1, x_2) \\ qf_{12}(x_1, x_2) & qf_{22}(x_1, x_2) \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial p_1} \\ \frac{\partial x_2}{\partial p_1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Applying Cramer's rule,

$$\frac{\partial x_1}{\partial p_1} = \frac{1}{|H|}(qf_{22}(x_1, x_2))$$

$$\frac{\partial x_2}{\partial p_1} = -\frac{1}{|H|}(qf_{12}(x_1, x_2))$$

Then since at maximization, $y = f(x_1, x_2)$,

$$\frac{\partial y}{\partial p_1} = f_1(x_1, x_2)\frac{\partial x_1}{\partial p_1} + f_2(x_1, x_2)\frac{\partial x_2}{\partial p_1} = \frac{q}{|H|}\left(f_1(x_1, x_2)f_{22}(x_1, x_2) - f_2(x_1, x_2)f_{12}(x_1, x_2)\right)$$

In order for this sign to be negative, we need

$$f_1(x_1, x_2)f_{22}(x_1, x_2) < f_2(x_1, x_2)f_{12}(x_1, x_2))$$

For this sign to be positive, we need

$$f_1(x_1, x_2)f_{22}(x_1, x_2) > f_2(x_1, x_2)f_{12}(x_1, x_2))$$

TODO relate to giffen

(1.7) TODO

(1.8)

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Part 2

(2.1) Consider the lotteries δ_x, $x \in X$, which give outcome x with probability 1. Since there are finite δ_x and preferences are transitive and complete, there exists some y, z, such that among these δ_x, δ_y is best and δ_z is worst. Now, we argue that δ_y is best among all lotteries, and δ_z is worst among all lotteries.

Consider an arbitrary lottery p, which assigns probabilities $p(x)$ to outcome x, and $\sum_{x \in X} p(x) = 1$. Then since for all $x \in X$,

$$\delta_y \succeq \delta_x$$

Let us enumerate the xs, and WLOG we set $y = x_n$, $z = x_1$. By repeatedly applying independence,

$$\delta_{x_n} = p(x_1)\delta_{x_n} + (1 - p(x_1))\delta_{x_n} \succeq p(x_1)\delta_{x_1} + (1 - p(x_1))\delta_{x_n}$$

$$\succeq p(x_1)\delta_{x_1} + (1 - p(x_1)) \left(\frac{p(x_2)}{1 - p(x_1)} \delta_{x_2} + \frac{1 - p(x_1) - p(x_2)}{1 - p(x_1)} \delta_{x_n} \right)$$

$$= p(x_1)\delta_{x_1} + p(x_2)\delta_{x_2} + (1 - p(x_1) - p(x_2))\delta_{x_n}$$

$$... \succeq \sum_{i=1}^{k} p(x_i)\delta_{x_i} + \left(1 - \sum_{i=1}^{k} p(x_i) \right) \delta_{x_n}$$

$$... \succeq \sum_{i=1}^{k} p(x_i)\delta_{x_i} + \left(1 - \sum_{i=1}^{k} p(x_i) \right) \left(\frac{p(x_{k+1})}{1 - \sum_{i=1}^{k} p(x_i)} \delta_{x_k} + \frac{1 - \sum_{i=1}^{k+1} p(x_i)}{1 - \sum_{i=1}^{k} p(x_i)} \delta_{x_n} \right)$$

and hence from induction we find

$$\delta_{x_n} \succeq \sum_{i=1}^{n} p(x_i)\delta_{x_i} = p$$

Thus, δ_{x_n} is the best lottery. Similarly, since $\delta_x \succeq \delta_z$ for all $x \in X$, enumerating such that $x_1 = z$, we can repeatedly apply independence:

$$\delta_{x_1} = p(x_n)\delta_{x_1} + (1 - p(x_n))\delta_{x_1} \preceq p(x_n)\delta_{x_n} + (1 - p(x_1))\delta_{x_1}$$

$$\preceq p(x_n)\delta_{x_n} + (1 - p(x_n)) \left(\frac{p(x_{n-1})}{1 - p(x_n)} \delta_{x_{n-1}} + \frac{1 - p(x_n) - p(x_{n-1})}{1 - p(x_n)} \delta_{x_1} \right)$$

$$= p(x_n)\delta_{x_n} + p(x_{n-1})\delta_{x_{n-1}} + (1 - p(x_n) - p(x_{n-1}))\delta_{x_1}$$

$$... \preceq \sum_{i=0}^{k} p(x_{n-i})\delta_{x_{n-i}} + \left(1 - \sum_{i=0}^{k} p(x_{n-i}) \right) \delta_{x_1}$$

$$... \preceq \sum_{i=0}^{k} p(x_{n-i})\delta_{x_{n-i}} + \left(1 - \sum_{i=0}^{k} p(x_{n-i}) \right) \left(\frac{p(x_{n-k-1})}{1 - \sum_{i=0}^{k} p(x_{n-i})} \delta_{x_k} + \frac{1 - \sum_{i=0}^{k+1} p(x_{n-i})}{1 - \sum_{i=0}^{k} p(x_{n-i})} \delta_{x_1} \right)$$

and hence from induction we find

$$\delta_{x_1} \preceq \sum_{i=1}^{n} p(x_i)\delta_{x_i} = p$$

So δ_{x_1} is the worst lottery.

(2.2) We first show independence implies betweenness. Suppose we have independence, and $p \sim q$. Then for $\alpha \in [0,1]$

$$p = \alpha p + (1-\alpha)p \sim \alpha p + (1-\alpha)q$$

where the second part follows from independence (taking $r = p$). Hence betweenness is implied by independence.

To argue that betweenness does not necessitate independence, consider the Allais paradox. Consider a preference over lotteries $(p, q, 1-p-q)$ on $(5, 1, 0)$. Suppose a consumer's utility over a lottery (p, q) is given by

$$u(p,q) = \frac{1.01 + p}{1.01 - p - q}$$

Note then, for any α if $u = u(p,q) = u(p',q')$, the lottery $(\alpha p + (1-\alpha)p', \alpha q + (1-\alpha)q')$ satisfies

$$\frac{1.01 + (\alpha p + (1-\alpha)p')}{1.01 - (\alpha p + (1-\alpha)p') - (\alpha q + (1-\alpha)q')} = \frac{1.01 + (\alpha p + (1-\alpha)p')}{\alpha(1.01 - p - q) + (1-\alpha)(1.01 - p - 1)}$$

$$= \frac{1.01 + (\alpha p + (1-\alpha)p')}{\alpha(1.01 + p)/u + (1-\alpha)(1.01 + p')/u} = u\frac{1.01 + (\alpha p + (1-\alpha)p')}{1.01 + \alpha p + (1-\alpha)p'} = u$$

and hence these preferences satisfy betweenness. We now note that this can accommodate Allais. Specifically, if we let p be the probability of 5, q the probability of 1, then our preferences say that

$$u(0,1) = \frac{1.01}{.01} = 101$$

$$u(.1, .89) = \frac{1.11}{.02} = 55.5$$

And hence $A \succ B$. Additionally,

$$u(0, .11) = \frac{1.01}{0.9} = 1.122...$$

$$u(.1, 0) = \frac{1.11}{.91} = 1.219...$$

So $D \succ C$. Hence, these preferences, which satisfy betweenness, accommodate Allais, while independence cannot accommodate Allais. Therefore, there exist situations where betweenness holds, but not independence.

(2.3)

- The utility maximization problem is:

$$\max_x \sum_\theta p(\theta) u(\theta, x)$$

- The utility maximization given the state revelation of θ is

$$\max_x u(\theta, x)$$

Suppose the respective argmax is given by $x^*(\theta)$, and the argmax of the previous problem is x^*. Then

the expected utility assuming actions occur after state revelation is

$$\sum_{\theta} p(\theta)u(\theta, x^*(\theta)) \geq \sum_{\theta} p(\theta)u(\theta, x^*)$$

where we have just used that $u(\theta, x^*(\theta)) \geq u(\theta, x^*)$, which follows from the properties of $x^*(\theta)$. But the RHS of this is the expected utility of acting before state revelation. Hence, we have the expected utility is greater if we can act after state revelation rather than acting before.

We construct an example where the inequality is strict. Let $\theta \in \{0, 1\}$ with equal probability, and suppose the actions are $x \in [0, 1]$. Define $u(\theta, x) = -(x - \theta)^2$. If we can pick x after observing θ, we clearly just take $x = \theta$, and our expected utility is 0. However, if we must commit to x before observing θ, our utility maximization is

$$\max -\frac{1}{2}x^2 - \frac{1}{2}(x-1)^2 = \max -x^2 + x - 1/2$$

This is maximized at $x = 1/2$, which gives us expected utility $-1/4 < 0$.

- The action after state revelation allows us to always achieve better utility, sometimes strictly. Hence, we can interpret information about the state as increasing our expected utility, and we might claim in this situation that more information is better.

(2.4)

- Your expected salary is

$$\int_{\underline{z}}^{\overline{z}} zf(z)\ dz$$

and the expected utility is

$$\int_{\underline{z}}^{\overline{z}} u(z)f(z)\ dz$$

- By Jensen's inequality, since u is strictly concave,

$$u(E[z]) > E[u(z)]$$

Hence, your utility from simply taking the expected salary is better than your expected utility if you take the test and reveal your score. Hence, you wouldn't want to take the exam.

- If you know your score, then you clearly just take the exam if $z \geq E[z]$, because as long as u is increasing, your subsequent utility satisfies $u(z) \geq u(E[z])$. Therefore, the people who don't take the exam all must have $z < E[z]$, so the average score among all people not taking the exam must be less than $E[z]$.

Now, suppose the salary given to people who don't take the exam is the average score of people who don't take the exam. We prove that the only people who won't take the exam have $z = \underline{z}$. Suppose the average score of people who don't take the exam is μ_z (i.e. someone with $z > \underline{z}$ doesn't take the exam). Everyone with score greater than μ_z will want to take the exam, but this implies that the average of

people who don't take the exam must be at most μ_z. This implies that everyone who takes the exam has score $z = \mu_z$, and the only way for that to be possible is if $\mu_z = \underline{z}$. Hence, the only people who don't take the exam have score $\underline{z}$ (to them, it does not matter whether they take the exam or not), and everyone with a score higher than that will take the exam. Anyone not taking the exam receives a salary of $\underline{z}$.

- The outcome of this vote depends on the distribution f. Let z' denote the median score. If $z' > E[z]$, then a majority of students will want to take their exam, because they would get a score higher than $E[z]$ and hence a higher salary. If $z' < E[z]$, then a majority of students will not want to take the exam, since half of the students would expect to receive a higher salary $E[z]$ than what they would get if they took the exam.

- The students **on aggregate** do achieve a higher utility from receiving information about their score. If they all know that most of them would be better off from not taking the exam, then they will vote to not take the exam, and more than half the class would be better off. If the exam is taken, then this is the same result as the scenario previously where only the students with the lowest possible score do not take the exam, and no welfare is lost

(2.5) I'd not want to know. You can modify the model by taking the actor as being risk-loving in some sense, either by assigning a utility value to the uncertainty or taking a transformed expected utility rather than a vNM utility.