ECON500: Problem Set 1

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Problem 2

(16.AA.1)

(a)

Problem 4

Problem 6

We provide a counterexample. Take $K = [0, 1]$, and let $\gamma(0) = (0, 1)$ and $\gamma(k) = \{0.5\}$ for any $k \in (0, 1]$. It is clear that γ is a (constant) continuous function on $(0, 1]$, so we just need to check upper hemicontinuity at 0. Take a sequence $k_n \in K$ converging to 0. Then $\gamma(k_n)$ only contains 0.5 if $k_n \neq 0$, so we can always pick out a subsequence k'_n such that 0.5 is always in the image, and hence the image sequence converges to 0.5, and $0.5 \in \gamma(0)$. So γ is upper hemicontinuous. It is trivial to see that γ is not closed; $\gamma(0)$ is not closed.

Problem 7

Problem 8

We first prove a lemma.

Lemma: Any compact, convex set $K \subseteq \mathbb{R}^d$ is diffeomorphic to a n-simplex for some n.

Proof: TODO

Now, the generalization of Brouwer's follows from this lemma. Suppose $f : K \to K$ is continuous. Since K is diffeomorphic to a d-simplex, let $g : K \to S$ be such a diffeomorphism to some n-simplex S. Consider the composition $h = g \circ f \circ g^{-1}$. Then h takes $S \to S$, and since composition of continuous functions is continuous, h must be continuous. So by Brouwer's theorem on simplices, we have that h has some fixed point $p \in S$. This implies

$$(g \circ f \circ g^{-1})(p) = p$$

$$g^{-1}((g \circ f \circ g^{-1})(p)) = g^{-1}(p)$$

$$f(g^{-1}(p)) = g^{-1}(p)$$

Then since $g^{-1}(p) \in K$, we have that $g^{-1}(p)$ is a fixed point of f. Hence we have extended Brouwer to an arbitrary compact convex domain K.

Problem 9

Collaborator: Jingyi Cui, but solutions independently written

Take an arbitrary labelling of the vertices $v_0, v_1, ...v_n$ of the simplex S. Define ϕ as the shift-by-one mapping, or

$$\phi(i) = i + 1 \mod n + 1$$

We construct a continuous mapping φ as follows. Consider an arbitrary $s \in S$. Define the subdivision carrier $\chi^*(s)$ as the vertices of the smallest simplex in the subdivision $\{S_i\}$ that contains s, which exists yy definition of a subdivision. Let L_i denote the labelling map of the vertices of S_i, that is it maps each vertex v of S_i to a number $0, ...n$. Let the vector of barycentric coordinates $\vec{\lambda}_s^{S_i}$ of s with respect to the vertices of the subdivision S_i; specifically,

$$s = \sum_{v \in \chi^*(s)} \lambda_s^{S_i}(v)v$$

Then define

$$\varphi(s) = \sum_{w \in \chi^*(s)} \lambda^{S_i}(w)v_{\phi(L(w))}$$

We need a couple of important properties of φ, that we will prove.

Lemma 1: φ has no fixed points on the boundary of the simplex S. **Proof**:

Lemma 2: φ is continuous. **Proof**:

Now, by Brouwer, we have that since φ is a continuous mapping $S \to S$ by Lemma 2, we know that there exists a fixed point p of φ. Now, by Lemma 1, p must be in the interior of S. Let λ_p^S be the barycentric coordinates of p with respect to the vertices of S. We then note that since p is interior, $\lambda_p^S > 0$. Further, since p is a fixed point of φ, we have

$$\sum_{w \in \chi^*(s)} \lambda^{S_i}(w)v_{\phi(L(w))} = \varphi(p) = p = \sum_{j=0}^{n} \lambda^S(v_j)v_j$$

Then since each of the v_i are linearly independent and form a basis for the simplex S, we can match sum terms in the projection in the direction of each v_j. Specifically,

$$\lambda^S(v_j) = \lambda^{S_i}(w_j)$$

where w_j is such that $\phi(L(w_j)) = j$, or $L(w_j) = j - 1 \mod (n + 1)$. Since p exists, each w_j must exist, and therefore $(L(w_1), L(w_2), ...L(w_n), L(w_0)) = (0, 1, ...n)$ in that order. Hence, this implies that the subdivision S_i containing p is fully labeled, and thus Sperner's lemma holds.