ECON500: Problem Set 2

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Part 1

(1.1) Homogeneity follows from the nature of the budget constraint; given that the maximization problem is given by

$$\max u(x)$$

$$p \cdot x \leq w$$

But the constraint $p \cdot x \leq w$ holds if and only if $\alpha p \cdot x \leq \alpha w$. Hence the maximizer $x^*(p, w) = x^*(\alpha p, \alpha w)$, since the maximization problems are equivalent. So the demand is homogeneous of degree 0.

Now, we suppose the utility is weakly monotonic. To show Walras' law, suppose $x^*(p, w)$ is a maximizer for the utility maximization problem, but suppose $p \cdot x^* < w$.

(1.2)

(1.3) We take the general form:

$$u(x) = \left(\sum_{i=1}^{N} \alpha_i x_i^\rho \right)^{1/\rho}$$

The first order conditions are given by:

$$\frac{\partial u}{\partial x_k} = \alpha_k \left(\sum_{i=1}^{N} \alpha_i x_i^\rho \right)^{(1/\rho)-1} x_k^{\rho-1} = \lambda p_k$$

Equivalently,

$$\alpha_k \left(\sum_{i=1}^{N} \alpha_i x_i^\rho \right)^{(1/\rho)-1} x_k^{\rho} = \lambda p_k x_k$$

Summing over all k, we get

$$\left(\sum_{i=1}^{N} \alpha_i x_i^\rho \right)^{(1/\rho)-1} \sum_{k=1}^{N} \alpha_k x_k^{\rho} = \lambda w$$

$$\left(\sum_{i=1}^{N} \alpha_i x_i^\rho \right)^{(1/\rho)} = \lambda w$$

$$\lambda = \frac{1}{w}\left(\sum_{i=1}^{N}\alpha_i x_i^{\rho}\right)^{(1/\rho)}$$

Plugging in our Lagrange multiplier to the FOC, we get

$$\alpha_k\left(\sum_{i=1}^{N}\alpha_i x_i^{\rho}\right)^{(1/\rho)-1} x_k^{\rho-1} = \frac{p_k}{w}\left(\sum_{i=1}^{N}\alpha_i x_i^{\rho}\right)^{(1/\rho)}$$

$$\alpha_k x_k^{\rho-1}\left(\sum_{i=1}^{N}\alpha_i x_i^{\rho}\right)^{-1} = \frac{p_k}{w}$$

$$\alpha_k x_k^{\rho-1} = \frac{p_k}{w}\left(\sum_{i=1}^{N}\alpha_i x_i^{\rho}\right)$$

$$x_k^{\rho-1} = \frac{p_k}{\alpha_k}\left(\frac{1}{w}\sum_{i=1}^{N}\alpha_i x_i^{\rho}\right)$$

$$x_k = \left(\frac{p_k}{\alpha_k}\right)^{\frac{1}{\rho-1}}\left(\frac{1}{w}\sum_{i=1}^{N}\alpha_i x_i^{\rho}\right)^{\frac{1}{\rho-1}}$$

Note the sum term is the same for all k. Working to eliminate this, we get

$$x_k^{\rho} = \left(\frac{p_k}{\alpha_k}\right)^{\frac{\rho}{\rho-1}}\left(\frac{1}{w}\sum_{i=1}^{N}\alpha_i x_i^{\rho}\right)^{\frac{\rho}{\rho-1}}$$

$$\alpha_k x_k^{\rho} = \frac{p_k^{\frac{\rho}{\rho-1}}}{\alpha_k^{\frac{1}{\rho-1}}}\left(\frac{1}{w}\sum_{i=1}^{N}\alpha_i x_i^{\rho}\right)^{\frac{\rho}{\rho-1}}$$

$$\sum \alpha_k x_k^{\rho} = \left(\sum \frac{p_k^{\frac{\rho}{\rho-1}}}{\alpha_k^{\frac{1}{\rho-1}}}\right)\left(\frac{1}{w}\sum_{i=1}^{N}\alpha_i x_i^{\rho}\right)^{\frac{\rho}{\rho-1}}$$

$$\sum_{k=1}^{N}\alpha_k x_k^{\rho} = \left(\sum_{k=1}^{N}\left(\frac{p_k^{\rho}}{\alpha_k w^{\rho}}\right)^{\frac{1}{\rho-1}}\right)\left(\sum_{i=1}^{N}\alpha_i x_i^{\rho}\right)^{\frac{\rho}{\rho-1}}$$

$$\left(\sum_{k=1}^{N}\alpha_k x_k^{\rho}\right)^{\frac{1}{\rho-1}} = \left(\sum_{k=1}^{N}\left(\frac{p_k^{\rho}}{\alpha_k w^{\rho}}\right)^{\frac{1}{\rho-1}}\right)^{-1}$$

Plugging back in, we get

$$x_k = \left(\frac{p_k}{\alpha_k}\right)^{\frac{1}{\rho-1}}\left(\frac{1}{w}\sum_{i=1}^{N}\alpha_i x_i^{\rho}\right)^{\frac{1}{\rho-1}}$$

$$x_k = \left(\frac{p_k}{\alpha_k w}\right)^{\frac{1}{\rho-1}}\left(\sum_{k=1}^{N}\left(\frac{p_k^{\rho}}{\alpha_k w^{\rho}}\right)^{\frac{1}{\rho-1}}\right)^{-1}$$

$$x_k = w\left(\frac{p_k}{\alpha_k}\right)^{\frac{1}{\rho-1}}\left(\sum_{k=1}^{N}\left(\frac{p_k^\rho}{\alpha_k}\right)^{\frac{1}{\rho-1}}\right)^{-1}$$

$$x_k = w\left(\frac{\alpha_k}{p_k}\right)^{\frac{1}{1-\rho}}\left(\sum_{i=1}^{N}\left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-1}$$

(1.4) We first compute the partial derivative:

$$\frac{\partial x_l}{\partial p_k} = \frac{\rho}{1-\rho} w\left(\frac{\alpha_l}{p_l}\right)^{\frac{1}{1-\rho}}\left(\sum_{i=1}^{N}\left(\frac{\alpha_i}{p_i^\rho}\right)^{\frac{1}{1-\rho}}\right)^{-2}\left(\alpha_k^{\frac{1}{1-\rho}} p_k^{\frac{-\rho}{1-\rho}-1}\right)$$

$$= \frac{\rho}{1-\rho} w\left(\frac{\alpha_l\alpha_k}{p_l p_k}\right)^{\frac{1}{1-\rho}}\left(\sum_{i=1}^{N}\left(\frac{\alpha_i}{p_i^\rho}\right)^{\frac{1}{1-\rho}}\right)^{-2}$$

For an off-diagonal entry:

$$S_{lk} = \frac{\partial x_l}{\partial p_k} + \frac{\partial x_l}{\partial w}x_k$$

$$= \frac{\rho}{1-\rho} w\left(\frac{\alpha_l\alpha_k}{p_l p_k}\right)^{\frac{1}{1-\rho}}\left(\sum_{i=1}^{N}\left(\frac{\alpha_i}{p_i^\rho}\right)^{\frac{1}{1-\rho}}\right)^{-2} + w\left(\frac{\alpha_k\alpha_l}{p_k p_l}\right)^{\frac{1}{1-\rho}}\left(\sum_{i=1}^{N}\left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-2}$$

$$= \frac{w}{1-\rho}\left(\frac{\alpha_l\alpha_k}{p_l p_k}\right)^{\frac{1}{1-\rho}}\left(\sum_{i=1}^{N}\left(\frac{\alpha_i}{p_i^\rho}\right)^{\frac{1}{1-\rho}}\right)^{-2}$$

For the symmetric case, we first compute:

$$\frac{\partial x_k}{\partial p_k} = w\left(-\frac{1}{1-\rho}\left(\alpha_k p_k^{\rho-2}\right)^{\frac{1}{1-\rho}}\left(\sum_{i=1}^{N}\left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-1} + \frac{\rho}{1-\rho}\left(\frac{\alpha_k^2}{p_k^2}\right)^{\frac{1}{1-\rho}}\left(\sum_{i=1}^{N}\left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-2}\right)$$

The Slutsky matrix entry on the diagonal is then

$$S_{kk} = w\left(-\frac{1}{1-\rho}\left(\alpha_k p_k^{\rho-2}\right)^{\frac{1}{1-\rho}}\left(\sum_{i=1}^{N}\left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-1} + \frac{1}{1-\rho}\left(\frac{\alpha_k^2}{p_k^2}\right)^{\frac{1}{1-\rho}}\left(\sum_{i=1}^{N}\left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-2}\right)$$

$$= -\frac{w}{1-\rho}\left(\left(\alpha_k p_k^{\rho-2}\right)^{\frac{1}{1-\rho}}\left(\sum_{i=1}^{N}\left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-1} - \left(\frac{\alpha_k^2}{p_k^2}\right)^{\frac{1}{1-\rho}}\left(\sum_{i=1}^{N}\left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-2}\right)$$

$$= -\frac{w}{1-\rho}\left(\frac{\alpha_k}{p_k^2}\right)^{\frac{1}{1-\rho}}\left(p_k^{\frac{\rho}{1-\rho}}\left(\sum_{i=1}^{N}\left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right) - \alpha_k^{\frac{1}{1-\rho}}\right)\left(\sum_{i=1}^{N}\left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-2}$$

$$= -\frac{w}{1-\rho}\left(\frac{\alpha_k}{p_k^2}\right)^{\frac{1}{1-\rho}}\left(\left(\sum_{i=1}^{N}\left(\frac{\alpha_i p_k^\rho}{p_i^\rho}\right)^{\frac{1}{1-\rho}}\right) - \alpha_k^{\frac{1}{1-\rho}}\right)\left(\sum_{i=1}^{N}\left(\frac{\alpha_i}{p_i^\rho}\right)^{\frac{1}{1-\rho}}\right)^{-2}$$

(1.5) We let

$$w(p) = p \cdot x(\tilde{p}, \tilde{w})$$
$$= \sum_{k=1}^{N} p_k \tilde{w} \left(\frac{\alpha_k}{\tilde{p_k}}\right)^{\frac{1}{1-\rho}} \left(\sum_{i=1}^{N} \left(\alpha_i \tilde{p_i}^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-1}$$

We now consider

$$x(p, w(p))_k = w(p) \left(\frac{\alpha_k}{p_k}\right)^{\frac{1}{1-\rho}} \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-1}$$

Then

$$\frac{\partial x(p, w(p))_l}{\partial p_k} = \frac{\partial w(p)}{\partial p_k} \left(\frac{\alpha_l}{p_l}\right)^{\frac{1}{1-\rho}} \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-1} + \frac{\rho}{1-\rho} w(p) \left(\frac{\alpha_l \alpha_k}{p_l p_k}\right)^{\frac{1}{1-\rho}} \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-2}$$
$$= w \left(\frac{\alpha_l \alpha_k}{p_l p_k}\right)^{\frac{1}{1-\rho}} \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-2} + \frac{\rho}{1-\rho} w(p) \left(\frac{\alpha_l \alpha_k}{p_l p_k}\right)^{\frac{1}{1-\rho}} \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-2}$$
$$= \frac{w}{1-\rho} \left(\frac{\alpha_l \alpha_k}{p_l p_k}\right)^{\frac{1}{1-\rho}} \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-2}$$
$$= S_{lk}$$

In the symmetric case,

$$\frac{\partial x(p, w(p))_k}{\partial p_k} = \frac{\partial w(p)}{\partial p_k} \left(\frac{\alpha_k}{p_k}\right)^{\frac{1}{1-\rho}} \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-1} - \frac{1}{1-\rho} w(p) \left(\frac{\alpha_k}{p_k^{2-\rho}}\right)^{\frac{1}{1-\rho}} \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-1}$$
$$+ \frac{\rho}{1-\rho} w(p) \left(\frac{\alpha_k^2}{p_k^2}\right)^{\frac{1}{1-\rho}} \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-2}$$
$$= -\frac{1}{1-\rho} w(p) \left(\frac{\alpha_k}{p_k^{2-\rho}}\right)^{\frac{1}{1-\rho}} \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-1} + \frac{1}{1-\rho} w(p) \left(\frac{\alpha_k^2}{p_k^2}\right)^{\frac{1}{1-\rho}} \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-2}$$
$$= -\frac{w}{1-\rho} \left(\left(\alpha_k p_k^{\rho-2}\right)^{\frac{1}{1-\rho}} \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-1} - \left(\frac{\alpha_k^2}{p_k^2}\right)^{\frac{1}{1-\rho}} \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho}\right)^{\frac{1}{1-\rho}}\right)^{-2} \right)$$
$$= -\frac{w}{1-\rho} \left(\frac{\alpha_k}{p_k^2}\right)^{\frac{1}{1-\rho}} \left(\left(\sum_{i=1}^{N} \left(\frac{\alpha_i p_k^{\rho}}{p_i^{\rho}}\right)^{\frac{1}{1-\rho}} \right) - \alpha_k^{\frac{1}{1-\rho}} \right) \left(\sum_{i=1}^{N} \left(\frac{\alpha_i}{p_i^{\rho}}\right)^{\frac{1}{1-\rho}}\right)^{-2}$$
$$= S_{kk}$$

(1.6) A matrix M is negative semidefinite if for all vectors x,

$$x^T M x \leq 0$$

M is positive definite if for all vectors $x \neq 0$,

$$x^T M x > 0$$

By Sylvester's criterion, we know that the determinant of all the principal minors must be positive for a matrix to be positive definite. Similarly, the determinant of all the principal minors must be nonpositive for a matrix to be negative semidefinite.

If a matrix is negative definite, we show the diagonal entries must be negative. Consider the entry M_{ii}. Consider $x = (0, 0, 0, ...1, ...)$ where $x_i = 1$ and $x_j = 0$ for $j \neq i$. Then $x^T M x = M_{ii} < 0$ by negative definiteness. Hence $M_{ii} < 0$ for all i.

Similarly, if M is positive definite, consider M_{ii}. Consider the $x = (0, 0, 0, ...1, ...)$ where $x_i = 1$ and $x_j = 0$ for $j \neq i$. Once again, $x^T M x = M_{ii}$, so by positive definiteness, $M_{ii} > 0$.

(1.7) At (p, w) the utility of $x(p, w)$ is

$$
\begin{aligned}
u(x) &= \left(\sum_{i=1}^{N} \alpha_i \left(w \left(\frac{\alpha_k}{p_k} \right)^{\frac{1}{1-\rho}} \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho} \right)^{\frac{1}{1-\rho}} \right)^{-1} \right)^{\rho} \right)^{1/\rho} \\
&= \left(\sum_{k=1}^{N} \alpha_k w^{\rho} \left(\frac{\alpha_k}{p_k} \right)^{\frac{\rho}{1-\rho}} \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho} \right)^{\frac{1}{1-\rho}} \right)^{-\rho} \right)^{1/\rho} \\
&= \left(\sum_{k=1}^{N} w^{\rho} \left(\frac{\alpha_k}{p_k^{\rho}} \right)^{\frac{1}{1-\rho}} \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho} \right)^{\frac{1}{1-\rho}} \right)^{-\rho} \right)^{1/\rho} \\
&= \left(w^{\rho} \left(\sum_{k=1}^{N} \left(\frac{\alpha_k}{p_k^{\rho}} \right)^{\frac{1}{1-\rho}} \right) \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho} \right)^{\frac{1}{1-\rho}} \right)^{-\rho} \right)^{1/\rho} \\
&= \left(w^{\rho} \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho} \right)^{\frac{1}{1-\rho}} \right)^{1-\rho} \right)^{1/\rho} \\
&= w \left(\sum_{i=1}^{N} \left(\alpha_i p_i^{-\rho} \right)^{\frac{1}{1-\rho}} \right)^{\frac{1-\rho}{\rho}}
\end{aligned}
$$

(1.8)

(1.9)

(1.10)

Part 2

(2.1)

(2.2)

(2.3)

(2.4)

(2.5)

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