ECON500: Problem Set 1

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Problem 2

(15.B.4)

(a)

(b)

(c)

(d)

(e)

(f)

(15.B.8) Let the two utility functions be

$$u_1(x, y) = x + f(y)$$

$$u_2(x, y) = x + g(y)$$

By the second welfare theorem, any Pareto optimal allocation is supported by some prices and endowment choices. Normalize the numeraire price to 1. Let some interior Pareto optimal allocation be supported by the price vector $(1, p)$. By the optimization FOCs, since the allocation is interior, we require

$$1 - \lambda = 0$$

$$f'(y_1) - \lambda p = 0$$

or

$$f'(y_1) = p$$

Similarly,

$$g'(y_2) = p$$

Let the aggregate endowment of y be ω_y. Then at any Pareto optimal allocation, we must have

$$f'(y_1) = g'(\omega_y - y_1) = p$$

$$f'(y_1) - g'(\omega_y - y_1) = 0$$

Define

$$h(y) = f'(y) - g'(\omega_y - y)$$

Then

$$h'(y) = f''(y) + g''(\omega_y - y)$$

Due to continuity and strictly convex preferences, both f'' and g'' are nonzero and have the same sign, and hence h is a strictly monotonic function. Therefore, $h(y) = 0$ for at most one unique value of y, and since y_1 satisfies $h(y_1) = 0$, we know that since any interior Pareto optimal allocation must satisfy $h(y) = 0$, any interior Pareto optimal allocation always assigns y_1 to consumer 1 and $\omega_y - y_1$ to consumer 2.

Problem 3

.... TODO smoothness ....

To show x_i is a diffeomorphism, it suffices to argue the preimage map x_i^{-1} is a well-defined function and smooth. Suppose that x^* is some consumption bundle for player i, where x^* is part of a feasible allocation (inside the Edgeworth box). For well-definedness, it suffices to show that there exists a unique price p and wealth w such that $x_i(p, w) = x^*$. The utility maximization problem is

$$x_i(p, w) = \arg\max_{p \cdot x \leq w \cdot} u(x)$$

Let

$$p = \frac{\left.\frac{\partial u}{\partial x_1}\right|_{x_1 = x_1^*}}{\left.\frac{\partial u}{\partial x_2}\right|_{x_1 = x_2^*}}$$

and

$$w = p x_1^* + x_2^*$$

We note that by our assumptions on the utility function (i.e. differentiable and increasing), our expression for p is always positive and well defined, since the denominator is never zero.

Then we note first order conditions for utility maximization are: (assuming (i) - (iii) and strictly convex preferences)

$$\left.\frac{\partial u}{\partial x_1}\right|_{x_1 = x_1^*} = \lambda p$$

$$\left.\frac{\partial u}{\partial x_2}\right|_{x_1 = x_2^*} = \lambda$$

$$p \cdot x^* = w$$

Hence, p solves these at x^*, so by our assumptions, and x^* is feasible and binding under the budget constraint at (p, w). Hence we have that $x_i(p, w) = x^*$, and so x_i is invertible.

Further, we can examine our expression for p:

$$p(x^*) = \frac{\left.\frac{\partial u}{\partial x_1}\right|_{x=x^*}}{\left.\frac{\partial u}{\partial x_2}\right|_{x=x^*}}$$

Since u is assumed to be smooth, we know that the partial derivatives of u with respect to x_1 and x_2 are also smooth. Now, we note the transformation $f(x, y) = x/y$ is smooth for $x, y > 0$, and $u_1, u_2 > 0$, we know that since a composition of smooth functions is smooth, we have that $p(x^*)$ is smooth. Similarly, since $w(x^*) = p(x^*)x_1^* + x_2^*$ is also a composition of a smooth function and p is smooth, w is also smooth. Therefore, we have the preimage of demand under fixation of the numeraire price of good 2 is well-defined and smooth, so x_i is a diffeomorphism between the 2-manifolds of the consumption space and the price/wealth (p, w) space (where p is the price of good 2).

Problem 4

The offer curve is $\mathcal{O} =$