ECON500: Problem Set 4

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Problem 2

(17.D.1) Normalize the price of good 2 to be 1, and let the price of good 1 be p. The FOCs of consumer 1 are:

$$p = \frac{2x_{11}^{\rho-1}}{(2x_{11}^{\rho} + x_{21}^{\rho})^{(\rho-1)/\rho}}$$

$$1 = \frac{x_{21}^{\rho-1}}{(2x_{11}^{\rho} + x_{21}^{\rho})^{(\rho-1)/\rho}}$$

So

$$\left(\frac{p}{2}\right)^{1/(\rho-1)} = \frac{x_{11}}{x_{21}}$$

Plugging in

$$px_{11} + x_{21} = p$$

$$p\left(\frac{p}{2}\right)^{1/(\rho-1)} x_{21} + x_{21} = p$$

$$x_{21}(p) = \frac{p}{p\left(\frac{p}{2}\right)^{1/(\rho-1)} + 1}$$

$$x_{11}(p) = \frac{p\left(\frac{p}{2}\right)^{1/(\rho-1)}}{p\left(\frac{p}{2}\right)^{1/(\rho-1)} + 1}$$

Doing the same for consumer 2:

$$p = \frac{x_{12}^{\rho-1}}{(x_{12}^{\rho} + 2x_{22}^{\rho})^{(\rho-1)/\rho}}$$

$$1 = \frac{2x_{22}^{\rho-1}}{(x_{12}^{\rho} + 2x_{22}^{\rho})^{(\rho-1)/\rho}}$$

$$(2p)^{1/(\rho-1)} = \frac{x_{12}}{x_{22}}$$

$$px_{12} + x_{22} = 1$$

$$x_{22} = \frac{1}{p(2p)^{1/(\rho-1)} + 1}$$

$$x_{21} = \frac{(2p)^{1/(\rho-1)}}{p(2p)^{1/(\rho-1)} + 1}$$

So the economy-wide excess demand functions are:

$$z_1(p) = \frac{p\left(\frac{p}{2}\right)^{1/(\rho-1)}}{p\left(\frac{p}{2}\right)^{1/(\rho-1)}+1} + \frac{(2p)^{1/(\rho-1)}}{p(2p)^{1/(\rho-1)}+1} - 1$$

$$= \frac{(2p)^{1/(\rho-1)}}{p(2p)^{1/(\rho-1)}+1} - \frac{2^{1/(\rho-1)}}{p^{\rho/(\rho-1)}+2^{1/(\rho-1)}}$$

$$z_2(p) = \frac{p}{p\left(\frac{p}{2}\right)^{1/(\rho-1)}+1} + \frac{1}{p(2p)^{1/(\rho-1)}+1} - 1$$

$$= \frac{2^{1/(\rho-1)}p}{p^{\rho/(\rho-1)}+2^{1/(\rho-1)}} - \frac{p^{\rho/(\rho-1)}2^{1/(\rho-1)}}{p^{\rho/(\rho-1)}2^{1/(\rho-1)}+1}$$

We can clearly see that these are 0 at $p = 1$, and $z_1'(p) > 0$, which implies by the index theorem that there are more zeros since z_1, z_2 are continuous. Hence there are multiple equilibria.

(17.D.3) Consider $D_{\omega_1} z_1(p)$. Since $z_1(p) = x_1(p, p\cdot\omega_1) - \omega_1$, if we define I^* as the truncated $(L-1)xL$ identity matrix (with the last row removed), we have that

$$D_{\omega_1} z_1(p) = \frac{\partial x_1(p, p\cdot\omega_1)}{\partial w} p^T - I^*$$

To show this matrix is full rank, consider any arbitrary vector $v \in \mathbb{R}^{L-1}$. Consider

$$D_{\omega_1} z_1(p) \begin{bmatrix} -v \\ \sum_{l=1}^{L-1} p_l v_l \end{bmatrix} = \frac{\partial x_1(p, p\cdot\omega_1)}{\partial w} p^T \begin{bmatrix} -v \\ \sum_{l=1}^{L-1} p_l v_l \end{bmatrix} - I^* \begin{bmatrix} -v \\ \sum_{l=1}^{L-1} p_l v_l \end{bmatrix}$$

$$= 0 + v = v$$

Hence all of $\mathbb{R}^{L-1}$ is in the image of the operator $D_{\omega_1} z_1(p)$, so this operator has full rank.

(17.D.8) Let the consumer utility be

$$u(x) = \prod_l x_l^{\alpha_l}$$

where $\sum_l \alpha_l = 1$. Then the consumer demand is

$$x_l(p) = \frac{\alpha_l(p\cdot\omega)}{p}$$

So excess demand in good l is

$$z_l(p) = \frac{\alpha_l(p\cdot\omega)}{p} - \omega_l$$

TODO finish

(17.E.1) By homogeneity of z, we get $z(\alpha p) = z(p)$. Differentiating both sides wrt α, and evaluating at $\alpha = 1$, we get

$$\sum_k p_k \frac{\partial z_l(p)}{\partial p_k} = 0$$

which is 17.E.1. Now, by Walras' law, we have

$$p \cdot z(p) = 0$$

Differentiating both sides wrt p_l, we get

$$0 = z_l(p) + p_l \frac{\partial z_l(p)}{\partial p_l} + \sum_{k \neq l} p_k \frac{\partial z_k(p)}{\partial p_l}$$

$$-z_l(p) = \sum_k p_k \frac{\partial z_k(p)}{\partial p_l}$$

which is 17.E.2.

(17.E.2) We know

$$z_i(p) = x_i(p, p \cdot \omega_i) - \omega_i$$

Taking the derivative matrix wrt p, we get

$$Dz_i(p) = D_1 x_i(p, p \cdot \omega_i) + D_2 x_i(p, p \cdot \omega_i)\omega_i^T$$

By the Slutsky equation,

$$S_i(p, p \cdot \omega_i) = D_1 x_i(p, p \cdot \omega_i) + D_2 x_i(p, p \cdot \omega_i) x_i(p, p \cdot \omega_i)^T$$

or

$$S_i(p, p \cdot \omega_i) - D_2 x_i(p, p \cdot \omega_i) x_i(p, p \cdot \omega_i)^T = D_1 x_i(p, p \cdot \omega_i)$$

Plugging this in, we get

$$Dz_i(p) = S_i(p, p \cdot \omega_i) - D_2 x_i(p, p \cdot \omega_i) x_i(p, p \cdot \omega_i)^T + D_2 x_i(p, p \cdot \omega_i)\omega_i^T$$

$$Dz_i(p) = S_i(p, p \cdot \omega_i) - D_2 x_i(p, p \cdot \omega_i) z_i(p)^T$$

Since $z(p) = \sum_i z_i(p)$,

$$Dz(p) = \sum_i Dz_i(p) = \sum_i S_i(p, p \cdot \omega_i) - D_2 x_i(p, p \cdot \omega_i) z_i(p)^T$$

as desired.

(17.E.3) TODO

(17.E.6) Suppose $p \neq p'$, and $||p|| = ||p'|| = 1$. Note that this implies $p \cdot p' < 1$. It suffices to show that $z_i(p)$ is not proportional to $z_i(p')$. We know by Walras's law that $p \cdot z_i(p) = 0$. WLOG, suppose $p_i \geq p'_i$. Then we have $p \cdot z_i(p') = p_i - p_i(p \cdot p') > p_i - p_i = 0$. Since $p' \cdot z_i(p') = 0$ and $p \cdot z_i(p') > 0$, we have that $z_i(p)$ is not proportional to $z_i(p')$ and hence this is proportionally one-to-one.

Problem 3

Suppose, for sake of contradiction, a Walrasian equilibrium exists. Fix the price of good 1 to be 1, and suppose the equilibrium price of good 2 is p. (It is quick to see that if the price of good 1 is 0, then each consumer will just demand infinite amounds of good 1, so this cannot happen in an equilibrium).

If $p < 1$, then each consumer's optimal bundle will demand their entire budget's worth of good 2. Then, the excess demand of good 2 will be positive, and markets cannot clear at these prices and consumer optimal bundles, a contradiction.

If $p > 1$, each consumer will demand their entire budget in good 1. Once again, the excess demand of good 1 will be positive, and markets cannot clear.

If $p = 1$, then the only demanded bundles for consumer 1 are $(12, 0)$ and $(0, 12)$. Similarly, for conumer 2, the demanded bundles are $(20, 0)$ and $(0, 20)$. Likewise, for consumer 3, the optimal bundles are $(16, 0)$ and $(0, 16)$. By manual inspection of each of the 8 potential combinations (it is symmetric so we only really have to inspect 4 of them), we can see that in no case can each consumer consume an optimal bundle and have markets clear for both goods.

Hence, all together, it is impossible for a Walrasian equilibrium to exist for this economy.

The convexity of preferences assumption fails; for an illustrative example, $(8, 0) \succ (5, 0)$ and $(0, 8) \succ (5, 0)$ but $(4, 4) \not\succ (5, 0)$.