ECON500: Problem Set 4

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Problem 2

(17.D.1) Normalize the price of good 2 to be 1, and let the price of good 1 be p. The FOCs of consumer 1 are:

$$p = \frac{2x_{11}^{\rho-1}}{(2x_{11}^{\rho} + x_{21}^{\rho})^{(\rho-1)/\rho}}$$

$$1 = \frac{x_{21}^{\rho-1}}{(2x_{11}^{\rho} + x_{21}^{\rho})^{(\rho-1)/\rho}}$$

So

$$\left(\frac{p}{2}\right)^{1/(\rho-1)} = \frac{x_{11}}{x_{21}}$$

Plugging in

$$px_{11} + x_{21} = p$$

$$p\left(\frac{p}{2}\right)^{1/(\rho-1)} x_{21} + x_{21} = p$$

$$x_{21}(p) = \frac{p}{p\left(\frac{p}{2}\right)^{1/(\rho-1)} + 1}$$

$$x_{11}(p) = \frac{p\left(\frac{p}{2}\right)^{1/(\rho-1)}}{p\left(\frac{p}{2}\right)^{1/(\rho-1)} + 1}$$

Doing the same for consumer 2:

$$p = \frac{x_{12}^{\rho-1}}{(x_{12}^{\rho} + 2x_{22}^{\rho})^{(\rho-1)/\rho}}$$

$$1 = \frac{2x_{22}^{\rho-1}}{(x_{12}^{\rho} + 2x_{22}^{\rho})^{(\rho-1)/\rho}}$$

$$(2p)^{1/(\rho-1)} = \frac{x_{12}}{x_{22}}$$

$$px_{12} + x_{22} = 1$$

$$x_{22} = \frac{1}{p(2p)^{1/(\rho-1)} + 1}$$

$$x_{21} = \frac{(2p)^{1/(\rho-1)}}{p(2p)^{1/(\rho-1)} + 1}$$

So the economy-wide excess demand functions are:

$$z_1(p) = \frac{p\left(\frac{p}{2}\right)^{1/(\rho-1)}}{p\left(\frac{p}{2}\right)^{1/(\rho-1)} + 1} + \frac{(2p)^{1/(\rho-1)}}{p(2p)^{1/(\rho-1)} + 1} - 1$$

$$= \frac{(2p)^{1/(\rho-1)}}{p(2p)^{1/(\rho-1)} + 1} - \frac{2^{1/(\rho-1)}}{p^{\rho/(\rho-1)} + 2^{1/(\rho-1)}}$$

$$z_2(p) = \frac{p}{p\left(\frac{p}{2}\right)^{1/(\rho-1)} + 1} + \frac{1}{p(2p)^{1/(\rho-1)} + 1} - 1$$

$$= \frac{2^{1/(\rho-1)}p}{p^{\rho/(\rho-1)} + 2^{1/(\rho-1)}} - \frac{p^{\rho/(\rho-1)}2^{1/(\rho-1)}}{p^{\rho/(\rho-1)}2^{1/(\rho-1)} + 1}$$

I plotted the excess demand, and it doesn't appear that there are multiple equilibria. However, for sake of the exercise, one way to show that there are multiple equilibria is to compute one equilibrium (the obvious case is $p = 1$), and show that the index of this equilibrium is negative and hence at least two other equilibria must exist. (I've struggled with this algebra for a bit, but it appears that that $z_1'(p) < 0$, which means that $p = 1$ is not the equilibrium that has negative index, assuming one exists in this economy).

(17.D.3) Consider $D_{\omega_1} z_1(p)$. Since $z_1(p) = x_1(p, p \cdot \omega_1) - \omega_1$, if we define I^* as the truncated $(L-1)xL$ identity matrix (with the last row removed), we have that

$$D_{\omega_1} z_1(p) = \frac{\partial x_1(p, p \cdot \omega_1)}{\partial w} p^T - I^*$$

To show this matrix is full rank, consider any arbitrary vector $v \in \mathbb{R}^{L-1}$. Consider

$$D_{\omega_1} z_1(p) \begin{bmatrix} -v \\ \sum_{l=1}^{L-1} p_l v_l \end{bmatrix} = \frac{\partial x_1(p, p \cdot \omega_1)}{\partial w} p^T \begin{bmatrix} -v \\ \sum_{l=1}^{L-1} p_l v_l \end{bmatrix} - I^* \begin{bmatrix} -v \\ \sum_{l=1}^{L-1} p_l v_l \end{bmatrix}$$

$$= 0 + v = v$$

Hence all of $\mathbb{R}^{L-1}$ is in the image of the operator $D_{\omega_1} z_1(p)$, so this operator has full rank.

(17.D.8) Let the consumer utility be

$$u(x) = \prod_l x_l^{\alpha_l}$$

where $\sum_l \alpha_l = 1$. Then the consumer demand is

$$x_l(p) = \frac{\alpha_l (p \cdot \omega)}{p_l}$$

So excess demand in good l is

$$z_l(p) = \frac{\alpha_l (p \cdot \omega)}{p_l} - \omega_l$$

The partial derivative matrix is for $k \neq l$

$$\frac{\partial z_l(p)}{\partial p_k} = \frac{\alpha_l \omega_k}{p_l}$$

and

$$\frac{\partial z_l(p)}{\partial p_l} = \frac{\alpha_l \omega_l}{p_l} - \frac{\alpha_l (p \cdot \omega)}{p_l^2}$$

Let Δ denote the diagonal matrix with $-\frac{\alpha_l (p \cdot \omega)}{p_l^2}$ as the lth entry, and let q_l denote the vector $\{\alpha_l / p_l\}$. Then we have

$$D\hat{z}(p) = \Delta + q\omega^T$$

By the matrix determinant lemma,

$$|D\hat{z}(p)| = (1 + \omega^T \Delta^{-1} q)|\Delta|$$

Note that Δ is conveniently diagonal, so Δ^{-1} is the diagonal matrix with the lth entry given by $-\frac{p_l^2}{\alpha_l (p \cdot \omega)}$. So we get

$$\omega^T \Delta^{-1} q = -\sum_{l=1}^{L-1} \frac{p_l \omega_l}{p \cdot \omega} = 1 - \frac{p_L \omega_L}{p \cdot \omega}$$

So using this, we get

$$\begin{aligned} |D\hat{z}(p)| &= (1 + \omega^T \Delta^{-1} q)|\Delta| \\ &= \frac{p_L \omega_L}{p \cdot \omega} |\Delta| \end{aligned}$$

But Δ is diagonal, so

$$\begin{aligned} |\Delta| &= \prod_{l=1}^{L-1} \left(-\frac{\alpha_l (p \cdot \omega)}{p_l^2} \right) \\ &= (-1)^{L-1} \prod_{l=1}^{L-1} \left(\frac{\alpha_l (p \cdot \omega)}{p_l^2} \right) \end{aligned}$$

So

$$\begin{aligned} |D\hat{z}(p)| &= \frac{p_L \omega_L}{p \cdot \omega} |\Delta| \\ &= (-1)^{L-1} \frac{p_L \omega_L}{p \cdot \omega} \prod_{l=1}^{L-1} \left(\frac{\alpha_l (p \cdot \omega)}{p_l^2} \right) \end{aligned}$$

Hence

$$\text{sign}(|D\hat{z}(p)|) = (-1)^{L-1}$$

So the index is $+1$.

(17.E.1) By homogeneity of z, we get $z(\alpha p) = z(p)$. Differentiating both sides wrt α, and evaluating at $\alpha = 1$, we get

$$\sum_k p_k \frac{\partial z_l(p)}{\partial p_k} = 0$$

which is 17.E.1. Now, by Walras' law, we have

$$p \cdot z(p) = 0$$

Differentiating both sides wrt p_l, we get

$$0 = z_l(p) + p_l \frac{\partial z_l(p)}{\partial p_l} + \sum_{k \neq l} p_k \frac{\partial z_k(p)}{\partial p_l}$$

$$-z_l(p) = \sum_k p_k \frac{\partial z_k(p)}{\partial p_l}$$

which is 17.E.2.

(17.E.2) We know

$$z_i(p) = x_i(p, p \cdot \omega_i) - \omega_i$$

Taking the derivative matrix wrt p, we get

$$Dz_i(p) = D_1 x_i(p, p \cdot \omega_i) + D_2 x_i(p, p \cdot \omega_i)\omega_i^T$$

By the Slutsky equation,

$$S_i(p, p \cdot \omega_i) = D_1 x_i(p, p \cdot \omega_i) + D_2 x_i(p, p \cdot \omega_i) x_i(p, p \cdot \omega_i)^T$$

or

$$S_i(p, p \cdot \omega_i) - D_2 x_i(p, p \cdot \omega_i) x_i(p, p \cdot \omega_i)^T = D_1 x_i(p, p \cdot \omega_i)$$

Plugging this in, we get

$$Dz_i(p) = S_i(p, p \cdot \omega_i) - D_2 x_i(p, p \cdot \omega_i) x_i(p, p \cdot \omega_i)^T + D_2 x_i(p, p \cdot \omega_i)\omega_i^T$$

$$Dz_i(p) = S_i(p, p \cdot \omega_i) - D_2 x_i(p, p \cdot \omega_i) z_i(p)^T$$

Since $z(p) = \sum_i z_i(p)$,

$$Dz(p) = \sum_i Dz_i(p) = \sum_i S_i(p, p \cdot \omega_i) - D_2 x_i(p, p \cdot \omega_i) z_i(p)^T$$

as desired.

(17.E.3) Taking after the diagram in the proof of proposition 17.E.2, we try to construct a utility function such that the indifference curve at x_i has a kink at x_i and x_i minimizes the dot product in the direction of the vector p (equivalently, the indifference curve lies entirely within the half-plane defined by $\{\lambda\ : p \cdot \lambda \geq p \cdot x_i\}$). To construct the kink, fix some $\alpha \in (0, 1)$, and consider the set of vector directions $\{\lambda^l = p - \alpha p_l e^l\}$. Define the renormalized vectors $\{\mu_i^l = \lambda^l/\lambda_i^l\}$ Note that by construction

$$\sum_l \mu_i^l = Lp/\lambda_i^l - \alpha p/\lambda_i^l = (L-\alpha)/\lambda_i^l p$$

and hence p is a linear combination of the vectors μ_i^l for each i. We can then define the utility of agent i as

$$u_i(y_i) = \min_l(\mu_i^l \cdot y_i - \mu_i^l \cdot x_i)$$

Note that the indifference curve containing x_i also contains $x_i + v$ for any v that is orthogonal to some μ_i^l (and also λ^l), which gives us the desired kink at x_i. Now, note that $\mu_i^l \cdot e^i = 1$, so $D_w x_i(p, p \cdot \omega_i) = \frac{1}{p_i} e^i$ and hence this satisfies the desired condition in the proof of proposition 17.E.2. We can then select the right endowments ω_i such that $x_i = \omega_i - e^i(a^i)^T$ and then give us the desired excess demand properties, so this utility works.

(17.E.6) Suppose $p \neq p'$, and $||p|| = ||p'|| = 1$. Note that this implies $p \cdot p' < 1$. It suffices to show that $z_i(p)$ is not proportional to $z_i(p')$. We know by Walras's law that $p \cdot z_i(p) = 0$. WLOG, suppose $p_i \geq p_i'$. Then we have $p \cdot z_i(p') = p_i - p_i(p \cdot p') > p_i - p_i = 0$. Since $p' \cdot z_i(p') = 0$ and $p \cdot z_i(p') > 0$, we have that $z_i(p)$ is not proportional to $z_i(p')$ and hence this is proportionally one-to-one.

Problem 3

Suppose, for sake of contradiction, a Walrasian equilibrium exists. Fix the price of good 1 to be 1, and suppose the equilibrium price of good 2 is p. (It is quick to see that if the price of good 1 is 0, then each consumer will just demand infinite amounds of good 1, so this cannot happen in an equilibrium).

If $p < 1$, then each consumer's optimal bundle will demand their entire budget's worth of good 2. Then, the excess demand of good 2 will be positive, and markets cannot clear at these prices and consumer optimal bundles, a contradiction.

If $p > 1$, each consumer will demand their entire budget in good 1. Once again, the excess demand of good 1 will be positive, and markets cannot clear.

If $p = 1$, then the only demanded bundles for consumer 1 are $(12, 0)$ and $(0, 12)$. Similarly, for conumer 2, the demanded bundles are $(20, 0)$ and $(0, 20)$. Likewise, for consumer 3, the optimal bundles are $(16, 0)$ and $(0, 16)$. By manual inspection of each of the 8 potential combinations (it is symmetric so we only really have to inspect 4 of them), we can see that in no case can each consumer consume an optimal bundle and have markets clear for both goods.

Hence, all together, it is impossible for a Walrasian equilibrium to exist for this economy.

The convexity of preferences assumption fails; for an illustrative example, $(8, 0) \succ (5, 0)$ and $(0, 8) \succ (5, 0)$ but $(4, 4) \not\succ (5, 0)$.