Problem Set 4

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Note: I use bold symbols to denote vectors and nonbolded symbols to denote scalars. I primarily use vector notation to shorthand some of the sums, since many of the sums are dot products.

Problem 1

(1) The F-firms are the normal type we have seen:

$$R = \alpha k_F^{\alpha-1} n_F^{1-\alpha}$$

$$w = (1-\alpha) k_F^{\alpha} n_F^{-\alpha}$$

Since the measure of the F firms is 1

$$\frac{K_F}{N_F} = \left(\frac{\alpha}{R}\right)^{1/(1-\alpha)}$$

$$\frac{K_F}{Y} = \frac{k_f}{k_F^{\alpha} n_F^{1-\alpha}} = \frac{1}{k_F^{\alpha-1} n_F^{1-\alpha}} = \frac{\alpha}{R}$$

(2) We already showed the equilibrium wage for the F-firm above:

$$w = (1-\alpha) k_F^{\alpha} n_F^{-\alpha} = (1-\alpha)\left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)}$$

Since this wage is competitive, this also must satisfy the wage FOC for maximization of the E-firm:

$$w = (1-\alpha)(1-\psi)\chi^{1-\alpha} k_E^{\alpha} n_E^{-\alpha}$$

$$= (1-\alpha)(1-\psi)\chi^{1-\alpha} (\kappa s_E)^{\alpha} (\kappa n_E)^{-\alpha}$$

$$= (1-\alpha)(1-\psi)\chi^{1-\alpha} S_E^{\alpha} N_E^{-\alpha}$$

Using the wage expression, we get

$$N_E^{\alpha} = \frac{(1-\alpha)(1-\psi)\chi^{1-\alpha} S_E^{\alpha}}{(1-\alpha)\left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)}} = \frac{(1-\psi)\chi^{1-\alpha} S_E^{\alpha}}{(\alpha/R)^{\alpha/(1-\alpha)}}$$

So

$$N_E = \frac{(1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}S_E}{(\alpha/R)^{1/(1-\alpha)}}$$

$$N_F = 1 - \frac{(1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}S_E}{(\alpha/R)^{1/(1-\alpha)}}$$

$$\begin{aligned}
m &= \psi S_E^{\alpha}\chi^{1-\alpha}N_E^{1-\alpha}\kappa^{-1} \\
&= \psi S_E^{\alpha}\chi^{1-\alpha}\frac{R(1-\psi)^{(1-\alpha)/\alpha}\chi^{(1-\alpha)^2/\alpha}S_E^{1-\alpha}}{\alpha\kappa} \\
&= \frac{R\psi(1-\psi)^{(1-\alpha)/\alpha}\chi^{(1-\alpha)/\alpha}S_E}{\alpha\kappa}
\end{aligned}$$

(3) The profit is

$$\begin{aligned}
&(1-\psi)\chi^{1-\alpha}s_E^{\alpha}n_E^{1-\alpha} - wn_E \\
&= (1-\psi)\chi^{1-\alpha}S_E^{\alpha}N_E^{1-\alpha}\kappa^{-1} - w\kappa^{-1}N_E \\
&= (1-\psi)\chi^{1-\alpha}S_E^{\alpha}\frac{(1-\psi)^{(1-\alpha)/\alpha}\chi^{(1-\alpha)^2/\alpha}S_E^{1-\alpha}}{\alpha/R}\kappa^{-1} - (1-\alpha)\left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)}\kappa^{-1}\frac{(1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}S_E}{(\alpha/R)^{1/(1-\alpha)}} \\
&= RS_E\frac{(1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}}{\alpha\kappa} - (1-\alpha)R\frac{(1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}S_E}{\alpha\kappa} \\
&= RS_E\frac{(1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}}{\kappa} \\
&= Rs_E(1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}
\end{aligned}$$

So the return rate on savings s_E is

$$R(1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}$$

(4) The utility is equivalent to Cobb-Douglas between the two period goods, so each type of consumer saves $\beta/(1+\beta)$ fraction of their income.

$$s_W = \frac{\beta}{1+\beta}w$$

$$s_E = \frac{\beta}{1+\beta}m$$

(5) We need the coefficient of R for the return rate on saavings from part (3) to be greater than 1. That is,

$$(1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha} > 1$$

taking the α power of both sides

$$(1-\psi)\chi^{1-\alpha} > 1$$

$$\chi^{1-\alpha} > \frac{1}{1-\psi}$$

The LHS is the productivity factor of the E-firm, so we need this productivity to offset the cost of paying the manager using the firm profit. If this condition fails, no one will invest their savings into E-firms, and hence no E-firms will operate after the next period because they will have no capital.

(6)

(i) We first find N_{Et}, N_{Ft}:

$$N_{Et} = \frac{(1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}S_E}{(\alpha/R)^{1/(1-\alpha)}} = \frac{(1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}K_{Et}}{(\alpha/R)^{1/(1-\alpha)}}$$

$$N_{Ft} = 1 - \frac{(1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}K_{Et}}{(\alpha/R)^{1/(1-\alpha)}}$$

Then

$$\begin{aligned} Y_{Et} &= \chi^{1-\alpha}K_{Et}^{\alpha}N_{Et}^{1-\alpha} \\ &= (R/\alpha)(1-\psi)^{(1-\alpha)/\alpha}\chi^{(1-\alpha)/\alpha}K_{Et} \end{aligned}$$

And from part (1):

$$\begin{aligned} \frac{K_{Ft}}{N_{Ft}} &= \left(\frac{\alpha}{R}\right)^{1/(1-\alpha)} \\ K_{Ft} &= \left(\frac{\alpha}{R}\right)^{1/(1-\alpha)} N_{Ft} \\ Y_{Ft} &= \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)} N_{Ft} \\ &= \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)} \left(1 - \frac{(1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}K_{Et}}{(\alpha/R)^{1/(1-\alpha)}}\right) \end{aligned}$$

So total output

$$Y_t = Y_{Et} + Y_{Ft} = (R/\alpha)(1-\psi)^{(1-\alpha)/\alpha}\chi^{(1-\alpha)/\alpha}K_{Et} + \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)} \left(1 - \frac{(1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}K_{Et}}{(\alpha/R)^{1/(1-\alpha)}}\right)$$

$$\begin{aligned} &= (R/\alpha)(1-\psi)^{(1-\alpha)/\alpha}\chi^{(1-\alpha)/\alpha}K_{Et} + \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)} - (R/\alpha)(1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}K_{Et} \\ &= (R/\alpha)\chi^{(1-\alpha)/\alpha}K_{Et}\left((1-\psi)^{(1-\alpha)/\alpha} - (1-\psi)^{1/\alpha}\right) + \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)} \\ &= (R/\alpha)\psi(1-\psi)^{(1-\alpha)/\alpha}\chi^{(1-\alpha)/\alpha}K_{Et} + \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)} \end{aligned}$$

(ii) The law of motion for K_{Et}:

$$\begin{aligned} K_{E,t+1} &= \kappa s_E = \kappa\frac{\beta}{1+\beta}m \\ &= \frac{\beta}{1+\beta}\psi K_{Et}^{\alpha}\chi^{1-\alpha}N_{Et}^{1-\alpha} \end{aligned}$$

$$= \frac{\beta}{1+\beta}\psi(R/\alpha)(1-\psi)^{(1-\alpha)/\alpha}\chi^{(1-\alpha)/\alpha}K_{Et}$$

(iii) Finally, we can slog through the algebra to compute out ρ_t:

$$\rho_t = \frac{K_{Et}}{K_{Et}+K_{Ft}}\rho_E + \frac{K_{Ft}}{K_{Et}+K_{Ft}}\rho_F$$

$$= \frac{K_{Et}}{K_{Et}+K_{Ft}}(\alpha\chi^{1-\alpha}K_{Et}^{\alpha-1}N_{Et}^{1-\alpha}) + \frac{K_{Ft}}{K_{Et}+K_{Ft}}(\alpha K_{Ft}^{\alpha-1}N_{Ft}^{1-\alpha})$$

$$= \frac{\alpha Y_{Et}}{K_{Et}+K_{Ft}} + \frac{\alpha Y_{Ft}}{K_{Et}+K_{Ft}}$$

$$= \frac{\alpha Y_t}{K_{Et}+K_{Ft}}$$

$$= \frac{R\psi(1-\psi)^{(1-\alpha)/\alpha}\chi^{(1-\alpha)/\alpha}K_{Et} + \alpha\left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)}}{K_{Et} + \left(\frac{\alpha}{R}\right)^{1/(1-\alpha)}N_{Ft}}$$

$$= \frac{R\psi(1-\psi)^{(1-\alpha)/\alpha}\chi^{(1-\alpha)/\alpha}K_{Et} + \alpha\left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)}}{K_{Et} + \left(\frac{\alpha}{R}\right)^{1/(1-\alpha)}\left(1 - \frac{(1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}K_{Et}}{(\alpha/R)^{1/(1-\alpha)}}\right)}$$

$$= \frac{R\psi(1-\psi)^{(1-\alpha)/\alpha}\chi^{(1-\alpha)/\alpha}K_{Et} + \alpha\left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)}}{K_{Et} + \left(\frac{\alpha}{R}\right)^{1/(1-\alpha)} - (1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}K_{Et}}$$

$$= \frac{R\psi(1-\psi)^{(1-\alpha)/\alpha}\chi^{(1-\alpha)/\alpha}K_{Et} + \alpha\left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)}}{K_{Et} + \left(\frac{\alpha}{R}\right)^{1/(1-\alpha)} - (1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}K_{Et}}$$

(7) The wages for normal workers is

$$w = (1-\alpha)\left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)}$$

from the firm FOC. This is thus constant over time, since the FOCs for the F-firms fully determines this wage.

(8) From part (6), we see the law of motion for K_{Et} can be rewritten:

$$\frac{K_{E,t+1}}{K_{E,t}} = \frac{\beta}{1+\beta}\psi(R/\alpha)(1-\psi)^{(1-\alpha)/\alpha}\chi^{(1-\alpha)/\alpha}$$

Thus K_E has constant growth rate. If the RHS is < 1, then the stock of E-firm capital must decrease to 0, and in the long run no E-firms will operate. If the RHS is > 1, then the stock of E-firm capital is growing, and since

$$N_{Et} = \frac{(1-\psi)^{1/\alpha}\chi^{(1-\alpha)/\alpha}}{(\alpha/R)^{1/(1-\alpha)}}K_{Et}$$

Since N_{Et} is linear in K_{Et} which grows at a constant rate, eventually $N_{Et} = 1$ due to the labor constraint. This implies $N_{Ft} = 0$ eventually, and since no workers work for F-firms, all the F-firms stop operating.

Therefore, depending on whether

$$\frac{\beta}{1+\beta}\psi(R/\alpha)(1-\psi)^{(1-\alpha)/\alpha}\chi^{(1-\alpha)/\alpha}$$

Is larger than or smaller than 1, either E-firms or F-firms will stop operating in the long run (respectively).

(9) All labor goes to the E-firms after the F-firms stop operating. Hence

$$Y_t = Y_{Et} = \chi^{1-\alpha}K_{Et}^{\alpha}N_{Et}^{1-\alpha} = \chi^{1-\alpha}K_{Et}^{\alpha}$$

So the law of motion of K_{Et} is

$$K_{E,t+1} = \kappa s_{E,t} = \frac{\beta}{1+\beta}\kappa m_t$$

$$= \frac{\beta}{1+\beta}\psi Y_t = \frac{\beta}{1+\beta}\psi\chi^{1-\alpha}K_{Et}^{\alpha}$$

Note that since $\alpha < 1$, this no longer enjoys sustained growth, and will converge to a steady state at a higher level of capital, and wages are:

$$w = (1-\alpha)\chi^{1-\alpha}K_{Et}^{\alpha}$$

and hence also converge to a higher level.

(10) In 6(ii) we showed the growth rate of K_{Et} is

$$\frac{\beta}{1+\beta}\psi(R/\alpha)(1-\psi)^{(1-\alpha)/\alpha}\chi^{(1-\alpha)/\alpha}$$

which is linearly proportional to R. Since Y_t is an affine function of K_{Et}, and the coefficient of K_{Et} in Y_t is linearly proportional to R, Y_t will grow faster at higher R (even though the constant term is slightly decreased at higher R). From the expression from part (1) the normal worker's wages are pinned down by the F-firm's FOC, and are inversely proportional to R. So wages are lower at higher R, but growth rate is higher at higher R.

(11) If E-firms can borrow at rental rate R, then they aren't constrained by their supply of capital. Therefore, because $\chi^{1-\alpha} > 1$, the E-firms immediately offer a higher wage than any F-firm can match, so all the workers just work for the E-firms, no F-firms operate, and we reach steady state.

Problem 2

(1) From the firm FOC: wage is

$$w = \sqrt{k}$$

Utility is Cobb-Douglas variant, so the consumer saves $(1/3)/(4/3) = 1/4$ of his/her income, so savings is

$$s = \sqrt{k}/4$$

(2) Noting the probability p given in the problem is $1/2$, we have the investment decision is to maximize:

$$\frac{1}{2}\log(4s_R + (s - s_R)) + \frac{1}{2}\log(s - s_R)$$

$$\frac{1}{2}\log(3s_R + s)) + \frac{1}{2}\log(s - s_R)$$

where s_R is the savings in the risky asset. The FOCs above are

$$\frac{3}{2(3s_R + s)} = \frac{1}{2(s - s_R)}$$

$$3(s - s_R) = 3s_R + s$$

$$2s = 6s_R$$

$$s_R = \frac{1}{3}s$$

So the consumer puts 1/3 of their savings into the risky asset, 2/3 into the safe asset.

(3) If the risky investment is successful,

$$k_{t+1} = 4((1/3)\sqrt{k_t}/4) + (2/3)(\sqrt{k_t}/4) = \frac{\sqrt{k_t}}{2}$$

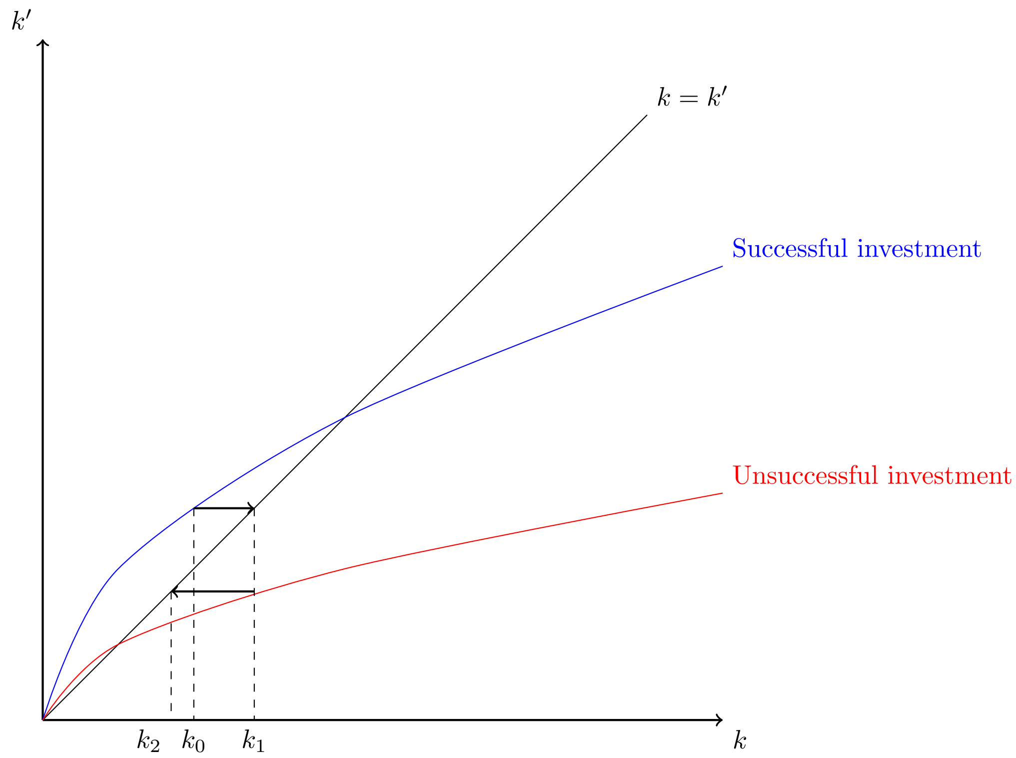
In the risky investment is not successful,

$$k_{t+1} = (2/3)(\sqrt{k_t}/4) = \frac{\sqrt{k_t}}{6}$$

(4) The stochastic steady state is in between the steady-states if the investment is always successful and if the investment is always failed: that is,

$$k^* \in \left[\frac{1}{36}, \frac{1}{4}\right]$$

For a geometric representation, we plot the phase diagram, and demonstrate an example of the evolution of capital after a period of success and then a period without investment success. Capital initially increases towards the high steady state after the success, but falls towards the low steady state after failure.



(5) The $t = 0$ generation will have received wage $w = \sqrt{k_0}$. They invest $\sqrt{k_0}/4$, 1/3 of which goes into the risky asset, 2/3 of which stay in the safe asset. So their consumption in the good state will be

$$4(1/3)\frac{\sqrt{k_0}}{4} + (2/3)\frac{\sqrt{k_0}}{4} = \sqrt{k_0}/2 \approx 0.1581$$

and in the bad state:

$$(2/3)\frac{\sqrt{k_0}}{4} = \sqrt{k_0}/6 \approx 0.0527$$

(6) The tax falls on the consumers, so the equilibrium wage does not change since it is determined by the firm's FOC. The consumer's income is then halved:

$$I = \sqrt{k}/2$$

The utility is still a Cobb-Douglas variant, so the consumer still invests 1/4 of their income.

$$s = \sqrt{k}/8$$

The assets have not changed, so the consumer still puts 1/3 of their savings into the risky asset. So the laws of motion can be determined. If the risky investment is successful,

$$k_{t+1} = 4((1/3)\sqrt{k_t}/8) + (2/3)(\sqrt{k_t}/8) = \frac{\sqrt{k_t}}{4}$$

In the risky investment is not successful,

$$k_{t+1} = (2/3)(\sqrt{k_t}/8) = \frac{\sqrt{k_t}}{12}$$

Then the new range of steaady states is:

$$(k^*)' = \left[\frac{1}{144}, \frac{1}{16}\right]$$

The steady state capital levels fall by a factor of 4, so steady state output is halved.

(7) This is not a great policy. The tax drives investment down, and reduces the payment to the elderly as time passes. However, it is not Pareto inferior- the first generation, whose tax payments were harvested at higher capital levels, will be better off, so at least one generation is better off under this policy.