# Problem Set 1

Nicholas Wu

Fall 2020

**Note:** I use bold symbols to denote vectors and nonbolded symbols to denote scalars. I primarily use vector notation to shorthand some of the sums, since many of the sums are dot products.

## Problem 1

**(1)** The maximization FOCs give us:

$$\beta^t c_t^{-\theta} = \lambda_t$$

$$\lambda_t = \lambda_{t+1}(1 - \delta + \alpha A k_{t+1}^{\alpha-1})$$

$$\beta^t c_t^{-\theta} = \beta^{t+1} c_{t+1}^{-\theta}(1 - \delta + \alpha A k_{t+1}^{\alpha-1})$$

At steady state,

$$1 = \beta(1 - \delta + \alpha A(k^*)^{\alpha-1})$$

$$\beta^{-1} - (1 - \delta) = \alpha A(k^*)^{\alpha-1}$$

$$\left(\frac{\beta^{-1} - (1 - \delta)}{\alpha A}\right)^{1/(\alpha-1)} = k^*$$

$$c^* = A(k^*)^\alpha - \delta k^*$$

$$c^* = A\left(\frac{\delta}{\alpha A}\right)^{\alpha/(\alpha-1)} - \delta\left(\frac{\delta}{\alpha A}\right)^{1/(\alpha-1)}$$

**(2)** See separate files.

**(3)** See figures.

**(4)**

# Problem 2

Since $\bar{w} > (1-\alpha)(k^*)^\alpha$, the minimum wage is higher than the firms marginal product of labor even at full employment in the steady state. Hence, the minimum wage binds, and is the wage at each time. The employment is then

$$(1-\alpha)\left(\frac{k_t}{E_t}\right)^\alpha = \bar{w}$$

$$E_t = k_t\left(\frac{1-\alpha}{\bar{w}}\right)^{1/\alpha}$$

So output is

$$Y_t = k_t\left(\frac{1-\alpha}{\bar{w}}\right)^{(1-\alpha)/\alpha}$$

And the capital change:

$$k_{t+1} - k_t = -\delta k_t + s k_t\left(\frac{1-\alpha}{\bar{w}}\right)^{(1-\alpha)/\alpha}$$

$$k_{t+1} - k_t = \delta k_t\left(\frac{s}{\delta}\left(\frac{1-\alpha}{\bar{w}}\right)^{(1-\alpha)/\alpha} - 1\right)$$

Using the fact that $k = (s/\delta)^{1/(1-\alpha)}$

$$k_{t+1} - k_t = \delta k_t\left(\left(\frac{(1-\alpha)(k^*)^\alpha}{\bar{w}}\right)^{(1-\alpha)/\alpha} - 1\right)$$

Now, since $(1-\alpha)/\alpha$ is positive and $\bar{w} > (1-\alpha)(k^*)^\alpha$, the parenthesized expression is negative. Further, the rate of decrease is

$$\rho = \delta\left(\left(1 - \frac{(1-\alpha)(k^*)^\alpha}{\bar{w}}\right)^{(1-\alpha)/\alpha}\right)$$

Hence at every period, $k_{t+1} = (1-\rho)k_t$. Since $\rho$ is independent of $k_t$, we have $k_t \to 0$, which jointly then implies output and employment both drop to 0.

## Problem 3

**(1)** The consumer problem is

$$\max \sum_{t=0}^{\infty} \beta^t (\log c_t + g_t)$$

subject to

$$c_t + i_t \leq r_t k_t + w_t$$

$$k_{t+1} = (1-\tau) i_t + (1-\delta) k_t$$

We can write the Bellman equation:

$$V(k) = \max_{k'} \left[ \log \left( rk + w - \frac{k' - (1-\delta)k}{1-\tau} \right) + g + \beta V(k') \right]$$

The Euler equation is

$$\beta (r_{t+1} + \frac{1-\delta}{1-\tau_{t+1}}) \frac{1}{c_{t+1}} = \frac{1/(1-\tau_t)}{c_t}$$

Letting $f(k) = F(k,1)$, we have $r_t = f'(k_t)$, so

$$\beta ((1-\tau_{t+1}) f'(k_{t+1}) + 1 - \delta) \frac{1-\tau_t}{1-\tau_{t+1}} = \frac{c_{t+1}}{c_t}$$

**(2)** At steady state, we get

$$\beta ((1-\bar{\tau}) f'(k^*) + 1 - \delta) = 1$$

$$f'(k^*) = \frac{1 - \beta(1-\delta)}{\beta(1-\bar{\tau})}$$

$$i^* = \frac{\delta k^*}{1-\bar{\tau}}$$

$$c^* = f(k^*) - i^*$$

**(3)** Steady state utility is given by:

$$\log(c^*) + g^* = \log \left( f(k^*) - \frac{\delta k^*}{1-\bar{\tau}} \right) + \bar{\tau} \frac{\delta k^*}{1-\bar{\tau}}$$

Taking the FOC:

$$\frac{f'(k^*) - \frac{\delta}{1-\bar{\tau}}}{f(k^*) - \frac{\delta k^*}{1-\bar{\tau}}} + \bar{\tau} \frac{\delta k^*}{1-\bar{\tau}} = 0$$