

# Problem Set 4

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Fall 2020

**Note:** I use bold symbols to denote vectors and nonbolded symbols to denote scalars. I primarily use vector notation to shorthand some of the sums, since many of the sums are dot products.

## Problem 1

(1) The F-firms are the normal type we have seen:

$$R = \alpha k_F^{\alpha-1} n_F^{1-\alpha}$$

$$w = (1 - \alpha) k_F^\alpha n_F^{-\alpha}$$

Since the measure of the F firms is 1

$$\frac{K_F}{N_F} = \left( \frac{\alpha}{R} \right)^{1/(1-\alpha)}$$

$$\frac{K_F}{Y} = \frac{k_f}{k_F^\alpha n_F^{1-\alpha}} = \frac{1}{k_F^{\alpha-1} n_F^{1-\alpha}} = \frac{\alpha}{R}$$

(2) We already showed the equilibrium wage for the F-firm above:

$$w = (1 - \alpha) k_F^\alpha n_F^{-\alpha} = (1 - \alpha) \left( \frac{\alpha}{R} \right)^{\alpha/(1-\alpha)}$$

Since this wage is competitive, this also must satisfy the wage FOC for maximization of the E-firm:

$$\begin{aligned} w &= (1 - \alpha)(1 - \psi) \chi^{1-\alpha} k_E^\alpha n_E^{-\alpha} \\ &= (1 - \alpha)(1 - \psi) \chi^{1-\alpha} (\kappa s_E)^\alpha (\kappa n_E)^{-\alpha} \\ &= (1 - \alpha)(1 - \psi) \chi^{1-\alpha} S_E^\alpha N_E^{-\alpha} \end{aligned}$$

Using the wage expression, we get

$$N_E^\alpha = \frac{(1 - \alpha)(1 - \psi) \chi^{1-\alpha} S_E^\alpha}{(1 - \alpha) \left( \frac{\alpha}{R} \right)^{\alpha/(1-\alpha)}} = \frac{(1 - \psi) \chi^{1-\alpha} S_E^\alpha}{(\alpha/R)^{\alpha/(1-\alpha)}}$$

So

$$\begin{aligned}
N_E &= \frac{(1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha} S_E}{(\alpha/R)^{1/(1-\alpha)}} \\
N_F &= 1 - \frac{(1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha} S_E}{(\alpha/R)^{1/(1-\alpha)}} \\
m &= \psi S_E^\alpha \chi^{1-\alpha} N_E^{1-\alpha} \kappa^{-1} \\
&= \psi S_E^\alpha \chi^{1-\alpha} \frac{R(1-\psi)^{(1-\alpha)/\alpha} \chi^{(1-\alpha)^2/\alpha} S_E^{1-\alpha}}{\alpha \kappa} \\
&= \frac{R\psi(1-\psi)^{(1-\alpha)/\alpha} \chi^{(1-\alpha)/\alpha} S_E}{\alpha \kappa}
\end{aligned}$$

(3) The profit is

$$\begin{aligned}
&(1-\psi)\chi^{1-\alpha} s_E^\alpha n_E^{1-\alpha} - w n_E \\
&= (1-\psi)\chi^{1-\alpha} S_E^\alpha N_E^{1-\alpha} \kappa^{-1} - w \kappa^{-1} N_E \\
&= (1-\psi)\chi^{1-\alpha} S_E^\alpha \frac{(1-\psi)^{(1-\alpha)/\alpha} \chi^{(1-\alpha)^2/\alpha} S_E^{1-\alpha}}{\alpha/R} \kappa^{-1} - (1-\alpha) \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)} \kappa^{-1} \frac{(1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha} S_E}{(\alpha/R)^{1/(1-\alpha)}} \\
&= R S_E \frac{(1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha}}{\alpha \kappa} - (1-\alpha) R \frac{(1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha} S_E}{\alpha \kappa} \\
&= R S_E \frac{(1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha}}{\kappa} \\
&= R s_E (1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha}
\end{aligned}$$

So the return rate on savings  $s_E$  is

$$R(1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha}$$

(4) The utility is equivalent to Cobb-Douglas between the two period goods, so each type of consumer saves  $\beta/(1+\beta)$  fraction of their income.

$$s_W = \frac{\beta}{1+\beta} w$$

$$s_E = \frac{\beta}{1+\beta} m$$

(5) We need the coefficient of  $R$  for the return rate on saavings from part (3) to be greater than 1. That is,

$$(1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha} > 1$$

taking the  $\alpha$  power of both sides

$$(1-\psi)\chi^{1-\alpha} > 1$$

$$\chi^{1-\alpha} > \frac{1}{1-\psi}$$

The LHS is the productivity factor of the E-firm, so we need this productivity to offset the cost of paying the manager using the firm profit. If this condition fails, no one will invest their savings into E-firms, and hence no E-firms will operate after the next period because they will have no capital.

(6)

(i) We first find  $N_{Et}, N_{Ft}$ :

$$N_{Et} = \frac{(1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha} S_E}{(\alpha/R)^{1/(1-\alpha)}} = \frac{(1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha} K_{Et}}{(\alpha/R)^{1/(1-\alpha)}}$$

$$N_{Ft} = 1 - \frac{(1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha} K_{Et}}{(\alpha/R)^{1/(1-\alpha)}}$$

Then

$$Y_{Et} = \chi^{1-\alpha} K_{Et}^\alpha N_{Et}^{1-\alpha}$$

$$= (R/\alpha)(1-\psi)^{(1-\alpha)/\alpha} \chi^{(1-\alpha)/\alpha} K_{Et}$$

And from part (1):

$$\frac{K_{Ft}}{N_{Ft}} = \left(\frac{\alpha}{R}\right)^{1/(1-\alpha)}$$

$$K_{Ft} = \left(\frac{\alpha}{R}\right)^{1/(1-\alpha)} N_{Ft}$$

$$Y_{Ft} = \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)} N_{Ft}$$

$$= \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)} \left(1 - \frac{(1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha} K_{Et}}{(\alpha/R)^{1/(1-\alpha)}}\right)$$

So total output

$$Y_t = Y_{Et} + Y_{Ft} = (R/\alpha)(1-\psi)^{(1-\alpha)/\alpha} \chi^{(1-\alpha)/\alpha} K_{Et} + \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)} \left(1 - \frac{(1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha} K_{Et}}{(\alpha/R)^{1/(1-\alpha)}}\right)$$

$$= (R/\alpha)(1-\psi)^{(1-\alpha)/\alpha} \chi^{(1-\alpha)/\alpha} K_{Et} + \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)} - (R/\alpha)(1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha} K_{Et}$$

$$= (R/\alpha) \chi^{(1-\alpha)/\alpha} K_{Et} \left( (1-\psi)^{(1-\alpha)/\alpha} - (1-\psi)^{1/\alpha} \right) + \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)}$$

$$= (R/\alpha) \psi (1-\psi)^{(1-\alpha)/\alpha} \chi^{(1-\alpha)/\alpha} K_{Et} + \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)}$$

(ii) The law of motion for  $K_{Et}$ :

$$K_{E,t+1} = \kappa s_E = \kappa \frac{\beta}{1+\beta} m$$

$$= \frac{\beta}{1+\beta} \psi K_{Et}^\alpha \chi^{1-\alpha} N_{Et}^{1-\alpha}$$

$$= \frac{\beta}{1+\beta} \psi(R/\alpha)(1-\psi)^{(1-\alpha)/\alpha} \chi^{(1-\alpha)/\alpha} K_{Et}$$

(iii) Finally, we can slog through the algebra to compute out  $\rho_t$ :

$$\begin{aligned} \rho_t &= \frac{K_{Et}}{K_{Et} + K_{Ft}} \rho_E + \frac{K_{Ft}}{K_{Et} + K_{Ft}} \rho_F \\ &= \frac{K_{Et}}{K_{Et} + K_{Ft}} (\alpha \chi^{1-\alpha} K_{Et}^{\alpha-1} N_{Et}^{1-\alpha}) + \frac{K_{Ft}}{K_{Et} + K_{Ft}} (\alpha K_{Ft}^{\alpha-1} N_{Ft}^{1-\alpha}) \\ &= \frac{\alpha Y_{Et}}{K_{Et} + K_{Ft}} + \frac{\alpha Y_{Ft}}{K_{Et} + K_{Ft}} \\ &= \frac{\alpha Y_t}{K_{Et} + K_{Ft}} \\ &= \frac{R\psi(1-\psi)^{(1-\alpha)/\alpha} \chi^{(1-\alpha)/\alpha} K_{Et} + \alpha \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)}}{K_{Et} + \left(\frac{\alpha}{R}\right)^{1/(1-\alpha)} N_{Ft}} \\ &= \frac{R\psi(1-\psi)^{(1-\alpha)/\alpha} \chi^{(1-\alpha)/\alpha} K_{Et} + \alpha \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)}}{K_{Et} + \left(\frac{\alpha}{R}\right)^{1/(1-\alpha)} \left(1 - \frac{(1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha} K_{Et}}{(\alpha/R)^{1/(1-\alpha)}}\right)} \\ &= \frac{R\psi(1-\psi)^{(1-\alpha)/\alpha} \chi^{(1-\alpha)/\alpha} K_{Et} + \alpha \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)}}{K_{Et} + \left(\frac{\alpha}{R}\right)^{1/(1-\alpha)} - (1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha} K_{Et}} \\ &= \frac{R\psi(1-\psi)^{(1-\alpha)/\alpha} \chi^{(1-\alpha)/\alpha} K_{Et} + \alpha \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)}}{K_{Et} + \left(\frac{\alpha}{R}\right)^{1/(1-\alpha)} - (1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha} K_{Et}} \end{aligned}$$

(7) The wages for normal workers is

$$w = (1-\alpha) \left(\frac{\alpha}{R}\right)^{\alpha/(1-\alpha)}$$

from the firm FOC. This is thus constant over time, since the FOCs for the F-firms fully determines this wage.

(8) From part (6), we see the law of motion for  $K_{Et}$  can be rewritten:

$$\frac{K_{E,t+1}}{K_{E,t}} = \frac{\beta}{1+\beta} \psi(R/\alpha)(1-\psi)^{(1-\alpha)/\alpha} \chi^{(1-\alpha)/\alpha}$$

Thus  $K_E$  has constant growth rate. If the RHS is  $< 1$ , then the stock of E-firm capital must decrease to 0, and in the long run no E-firms will operate. If the RHS is  $> 1$ , then the stock of E-firm capital is growing, and since

$$N_{Et} = \frac{(1-\psi)^{1/\alpha} \chi^{(1-\alpha)/\alpha}}{(\alpha/R)^{1/(1-\alpha)}} K_{Et}$$

Since  $N_{Et}$  is linear in  $K_{Et}$  which grows at a constant rate, eventually  $N_{Et} = 1$  due to the labor constraint. This implies  $N_{Ft} = 0$  eventually, and since no workers work for F-firms, all the F-firms stop operating.

Therefore, depending on whether

$$\frac{\beta}{1+\beta}\psi(R/\alpha)(1-\psi)^{(1-\alpha)/\alpha}\chi^{(1-\alpha)/\alpha}$$

Is larger than or smaller than 1, either E-firms or F-firms will stop operating in the long run (respectively).

(9) All labor goes to the E-firms after the F-firms stop operating. Hence

$$Y_t = Y_{Et} = \chi^{1-\alpha} K_{Et}^\alpha N_{Et}^{1-\alpha} = \chi^{1-\alpha} K_{Et}^\alpha$$

So the law of motion of  $K_{Et}$  is

$$\begin{aligned} K_{E,t+1} &= \kappa s_{E,t} = \frac{\beta}{1+\beta} \kappa m_t \\ &= \frac{\beta}{1+\beta} \psi Y_t = \frac{\beta}{1+\beta} \psi \chi^{1-\alpha} K_{Et}^\alpha \end{aligned}$$

Note that since  $\alpha < 1$ , this no longer enjoys sustained growth, and will converge to a steady state at a higher level of capital, and wages are:

$$w = (1-\alpha)\chi^{1-\alpha} K_{Et}^\alpha$$

and hence also converge to a higher level.

(10) In 6(ii) we showed the growth rate of  $K_{Et}$  is

$$\frac{\beta}{1+\beta}\psi(R/\alpha)(1-\psi)^{(1-\alpha)/\alpha}\chi^{(1-\alpha)/\alpha}$$

which is linearly proportional to  $R$ . Since  $Y_t$  is an affine function of  $K_{Et}$ , and the coefficient of  $K_{Et}$  in  $Y_t$  is linearly proportional to  $R$ ,  $Y_t$  will grow faster at higher  $R$  (even though the constant term is slightly decreased at higher  $R$ ). From the expression from part (1) the normal worker's wages are pinned down by the F-firm's FOC, and are inversely proportional to  $R$ . So wages are lower at higher  $R$ , but growth rate is higher at higher  $R$ .

(11) If E-firms can borrow at rental rate  $R$ , then they aren't constrained by their supply of capital. Therefore, because  $\chi^{1-\alpha} > 1$ , the E-firms immediately offer a higher wage than any F-firm can match, so all the workers just work for the E-firms, no F-firms operate, and we reach steady state.

## Problem 2

- (1) From the firm FOC: wage is

$$w = \sqrt{k}$$

Utility is Cobb-Douglas variant, so the consumer saves  $(1/3)/(4/3) = 1/4$  of his/her income, so savings is

$$s = \sqrt{k}/4$$

- (2) Noting the probability  $p$  given in the problem is  $1/2$ , we have the investment decision is to maximize:

$$\begin{aligned} & \frac{1}{2} \log(4s_R + (s - s_R)) + \frac{1}{2} \log(s - s_R) \\ & \frac{1}{2} \log(3s_R + s) + \frac{1}{2} \log(s - s_R) \end{aligned}$$

where  $s_R$  is the savings in the risky asset. The FOCs above are

$$\frac{3}{2(3s_R + s)} = \frac{1}{2(s - s_R)}$$

$$3(s - s_R) = 3s_R + s$$

$$2s = 6s_R$$

$$s_R = \frac{1}{3}s$$

So the consumer puts  $1/3$  of their savings into the risky asset,  $2/3$  into the safe asset.

- (3) If the risky investment is successful,

$$k_{t+1} = 4((1/3)\sqrt{k_t}/4) + (2/3)(\sqrt{k_t}/4) = \frac{\sqrt{k_t}}{2}$$

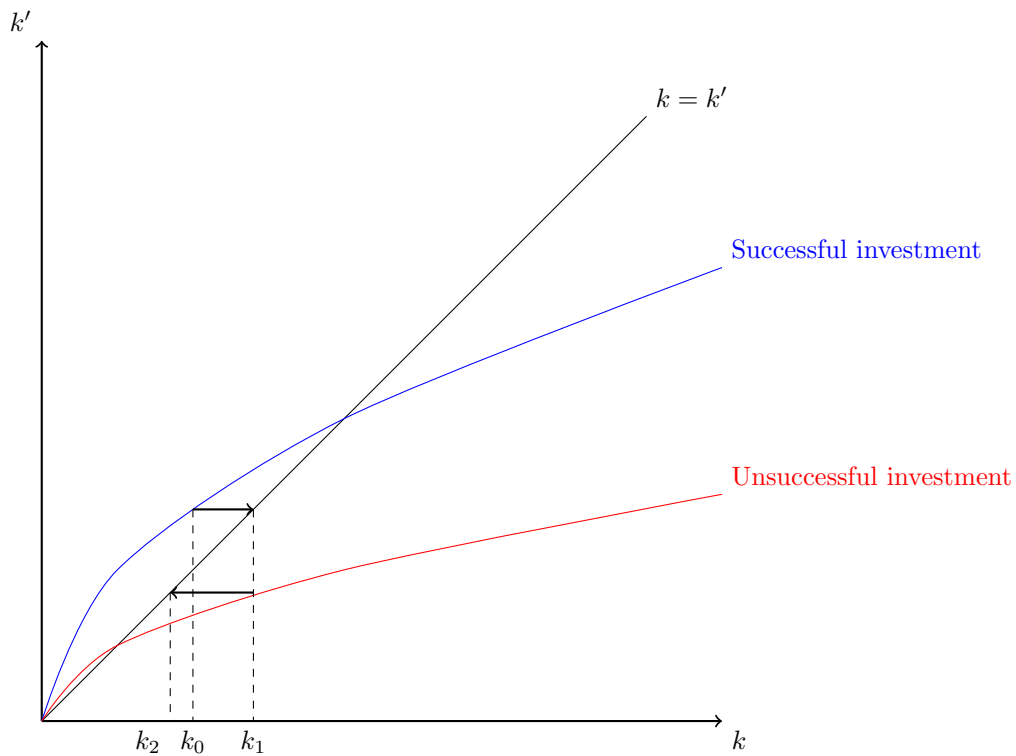
In the risky investment is not successful,

$$k_{t+1} = (2/3)(\sqrt{k_t}/4) = \frac{\sqrt{k_t}}{6}$$

- (4) The stochastic steady state is in between the steady-states if the investment is always successful and if the investment is always failed: that is,

$$k^* \in \left[ \frac{1}{36}, \frac{1}{4} \right]$$

For a geometric representation, we plot the phase diagram, and demonstrate an example of the evolution of capital after a period of success and then a period without investment success. Capital initially increases towards the high steady state after the success, but falls towards the low steady state after failure.



(5) In the good state,  $k_1^H = 0.5\sqrt{0.1}$ . Consumption in period 1 is then  $3/4$

(6)

(7)

### **Problem 3**

(1)

(2)



(3)

### Problem 3

(1)

(2)

(3)

(4)