Problem Set 4

Nicholas Wu

Fall 2020

Note: I use bold symbols to denote vectors and nonbolded symbols to denote scalars. I primarily use vector notation to shorthand some of the sums, since many of the sums are dot products.

Problem 1

(1) A rational expectations recursive equilibrium for this economy consists of pairs of functions for groups 1 and 2 describing the laws of motion $\mathcal{G}^A$ and $\mathcal{G}^B$, household value functions V^A and V^B, household savings policy functions g^A and g^B (where we denote the first two arguments as the given sequences of capital and the last argument as the previous capital realization), interest rate function R, wage function W, such that the following conditions hold:

- Household optimization for A: V^A, g^A, solve the following household optimization:

$$V^A(K^A, K^B, a; \mathcal{G}^A, \mathcal{G}^B) = \max_{a'} u(c) + \beta^A V^A(K'^A, K'^B, a'; \mathcal{G}^A, \mathcal{G}^B)$$

  subject to

$$c = R(K)a + W(K) - a'$$

$$a, a' \geq \underline{A}$$

$$K = \mu K^A + (1-\mu)K^B$$

$$K'^A = \mathcal{G}^A(K^A, K^B)$$

$$K'^B = \mathcal{G}^B(K^A, K^B)$$

  where the optimal policy is

$$a' = g^A(K^A, K^B, a; \mathcal{G}^A, \mathcal{G}^B)$$

- Household optimization for B: V^B, g^B, solve the following household optimization:

$$V^B(K^A, K^B, a; \mathcal{G}^A, \mathcal{G}^B) = \max_{a'} u(c) + \beta^B V^B(K'^A, K'^B, a'; \mathcal{G}^A, \mathcal{G}^B)$$

  subject to

$$c = R(K)a + W(K) - a'$$

$$a, a' \geq \underline{A}$$

$$K = \mu K^A + (1-\mu)K^B$$

$$K'^A = \mathcal{G}^A(K^A, K^B)$$

$$K'^B = \mathcal{G}^B(K^A, K^B)$$

where the optimal policy is

$$a' = g^B(K^A, K^B, a; \mathcal{G}^A, \mathcal{G}^B)$$

- Consistency with expectations:

$$g^A(K^A, K^B, K^A; \mathcal{G}^A, \mathcal{G}^b) = \mathcal{G}^A(K^A, K^B)$$

$$g^B(K^A, K^B, K^B; \mathcal{G}^A, \mathcal{G}^b) = \mathcal{G}^A(K^B, K^B)$$

- Market clearing: (where $K = \mu K^1 + (1-\mu)K^2$)

$$W(K) = (1-\alpha)\theta K^\alpha$$

$$R(K) = \alpha\theta K^{\alpha-1} + 1 - \delta$$

(2) Due to the asset holding constraint $\underline{A}$, the Euler conditions are:

$$u'(c^A) \geq \beta^A R(K) u'(c'^A)$$

$$u'(c^B) \geq \beta^B R(K) u'(c'^B)$$

At steady state, $c^A = c'^A$ and $c^B = c'^B$, so we get

$$1 \geq \beta^A R(K)$$

$$1 \geq \beta^B R(K)$$

But we know $\beta^A > \beta^B$, and we cannot have both be strict inequality (else both consumers go to $\underline{A}$ for their asset holdings), so we must have

$$1 = \beta^A R(K)$$

$$1 > \beta^B R(K)$$

This implies that the steady state asset holding of consumers in group B is: $A^B_{ss} = \underline{A}$. For group A, we have

$$\frac{1}{\beta^A} = \alpha\theta K^{\alpha-1} + 1 - \delta$$

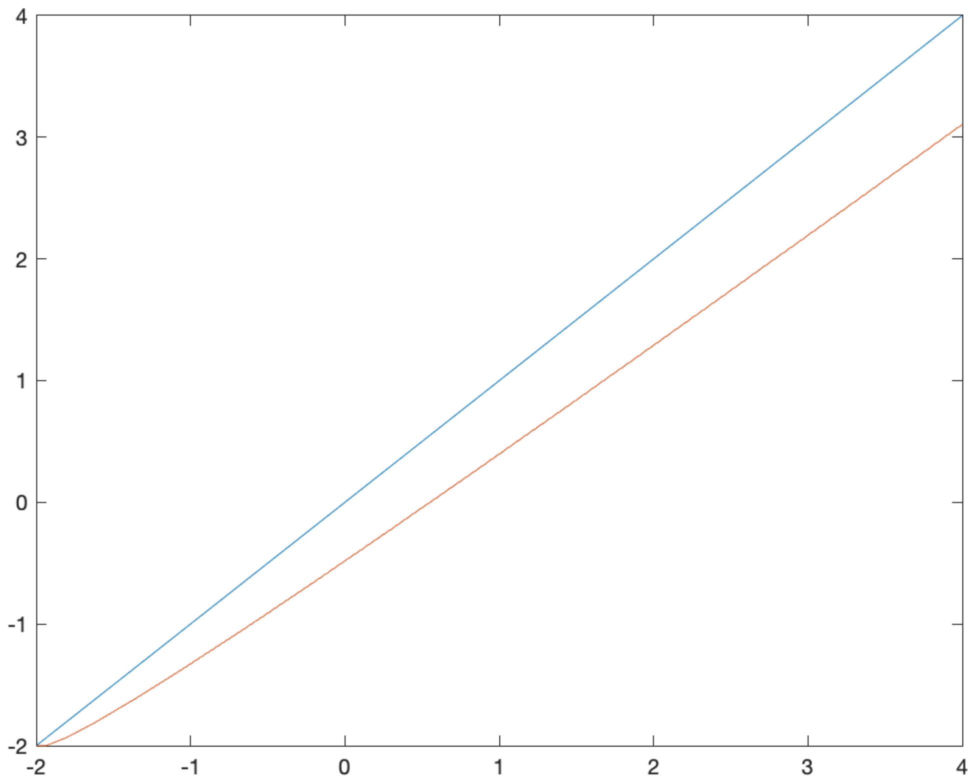

Figure 1: Heterogenous preferences: Blue line shows g_A, red line shows g_B. Note blue line essentially coincides with the 45 degree line.

So we can solve for the aggregate capital level:

$$K_{ss} = \left(\frac{1}{\alpha\theta\beta^A} - \frac{1-\delta}{\alpha\theta} \right)^{1/(\alpha-1)}$$

Then since

$$\mu A^A_{ss} + (1-\mu)A^B_{ss} = K_{ss}$$

$$A^A_{ss} = \frac{1}{\mu} \left(\frac{1}{\alpha\theta\beta^A} - \frac{1-\delta}{\alpha\theta} \right)^{1/(\alpha-1)} - \frac{1-\mu}{\mu}\underline{A}$$

(3) See attached file for code, and Figure 1 for the plot. Note that since g_A essentially coincides with the 45 degree line, consumers in group A essentially will consume the same consumption as the previous period at steady state, while the consumers in group B will always consume less and less until they consume the minimum $\underline{A}$.

Problem 2

(1) A rational expectations recursive equilibrium for this economy consists of pairs of functions for groups A and B describing the laws of motion for capital $\mathcal{G}^A$ and $\mathcal{G}^B$, law of motion for supply $\mathcal{H}$, household value functions V^A and V^B, household savings policy functions g^A and g^B (where we denote the first two arguments as the given sequences of capital and the last argument as the previous capital realization), labor policy functions h^A and h^B interest rate function R, wage function W, such that the following conditions hold:

- Consumers of type A optimize: g^A, h^A, V^A solve:

$$V^A(K^A, K^B, a; \mathcal{G}^A, \mathcal{G}^B, \mathcal{H}) = \max_{a', n} u(c, n) + \beta V^A((K^A)', (K^B)', a'; \mathcal{G}^A, \mathcal{G}^B, \mathcal{H})$$

  subject to:

$$c = W(K, N)\epsilon_A n + R(K, N)a - a'$$

$$K = \mu K^A + (1-\mu)K^B$$

$$N = \mathcal{H}(K^A, K^B)$$

$$(K^A)' = \mathcal{G}^A(K^A, K^B)$$

$$(K^B)' = \mathcal{G}^B(K^A, K^B)$$

  where h^A defines the optimal policy for n and g^A describes the optimal policy for a'.

- Consumers of type B optimize: g^B, h^B, V^B solve:

$$V^B(K^A, K^B, a; \mathcal{G}^A, \mathcal{G}^B, \mathcal{H}) = \max_{a', n} u(c, n) + \beta V^B((K^A)', (K^B)', a'; \mathcal{G}^A, \mathcal{G}^B, \mathcal{H})$$

  subject to:

$$c = W(K, N)\epsilon_B n + R(K, N)a - a'$$

$$K = \mu K^A + (1-\mu)K^B$$

$$N = \mathcal{H}(K^A, K^B)$$

$$(K^A)' = \mathcal{G}^A(K^A, K^B)$$

$$(K^B)' = \mathcal{G}^B(K^A, K^B)$$

  where h^B defines the optimal policy for n and g^B describes the optimal policy for a'.

- Consistency with expectations for capital and for labor:

$$g^A(K^A, K^B, K^A; \mathcal{G}^A, \mathcal{G}^B, \mathcal{H}) = \mathcal{G}^A(K^A, K^B)$$

$$g^B(K^A, K^B, K^B; \mathcal{G}^A, \mathcal{G}^B, \mathcal{H}) = \mathcal{G}^B(K^A, K^B)$$

$$N = \mu\epsilon_A h^A(K^A, K^B, K^A; \mathcal{G}^A, \mathcal{G}^B, \mathcal{H}) + (1-\mu)\epsilon_B h^B(K^A, K^B, K^B; \mathcal{G}^A, \mathcal{G}^B, \mathcal{H}) = \mathcal{H}(K^A, K^B)$$

- Market clearing prices:

$$W(K,N) = (1-\alpha)\theta K^{\alpha} N^{-\alpha}$$

$$R(K,N) = \alpha\theta K^{\alpha-1} N^{1-\alpha} + (1-\delta)$$

where

$$K = \mu K^A + (1-\mu)K^B$$

(2) From steady state in the Euler condition, after using the envelope theorem, $1 = \beta R(K,N)$ so we have:

$$1 = \beta\left(\alpha\theta K_{ss}^{\alpha-1} N_{ss}^{1-\alpha} + (1-\delta)\right)$$

We take the FOC of the maximizations with respect to n

$$(1-\alpha)\theta K_{ss}^{\alpha} N_{ss}^{-\alpha}\epsilon_A u_1(c_{ss}^A, n_{ss}^A) + u_2(c_{ss}^A, n_{ss}^A) = 0$$

$$(1-\alpha)\theta K_{ss}^{\alpha} N_{ss}^{-\alpha}\epsilon_B u_1(c_{ss}^B, n_{ss}^B) + u_2(c_{ss}^B, n_{ss}^B) = 0$$

From expectation consistency:

$$K_{ss} = \mu k_{ss}^A + (1-\mu)k_{ss}^B$$

$$N_{ss} = \mu\epsilon_A n_{ss}^A + (1-\mu)\epsilon_B n_{ss}^B$$

Lastly, from the budget constraints, we get

$$c_{ss}^A = (1-\alpha)\theta K_{ss}^{\alpha} N_{ss}^{-\alpha}\epsilon_A n_{ss}^A + \left(\alpha\theta K_{ss}^{\alpha-1} N_{ss}^{1-\alpha} - \delta)\right) k_{ss}^A$$

$$c_{ss}^B = (1-\alpha)\theta K_{ss}^{\alpha} N_{ss}^{-\alpha}\epsilon_B n_{ss}^B + \left(\alpha\theta K_{ss}^{\alpha-1} N_{ss}^{1-\alpha} - \delta)\right) k_{ss}^B$$

The unknowns are $c_{ss}^A, c_{ss}^A, k_{ss}^A, k_{ss}^A, n_{ss}^A, n_{ss}^A, K_{ss}$, and N_{ss}, but we only have 7 equations. Hence this system has a spectrum of steady states. The aggregate capital level thus can vary, and there is no unique steady state level of aggregate capital.

(3) A rational expectations recursive equilibrium for this economy consists of pairs of functions for groups A and B describing the laws of motion for capital $\mathcal{G}^A$ and $\mathcal{G}^B$, law of motion for aggregate labor supply $\mathcal{H}$, law of motion for labor supply of group A $\mathcal{H}^A$, household value functions V^A and V^B, household savings policy functions g^A and g^B (where we denote the first two arguments as the given sequences of capital and the last argument as the previous capital realization), labor policy functions h^A and h^B interest rate function R, wage function W, such that the following conditions hold:

- Consumers of type A optimize: g^A, h^A, V^A solve:

$$V^A(K^A, K^B, N^A, a, n; \mathcal{G}^A, \mathcal{G}^B, \mathcal{H}, \mathcal{H}^A) = \max_{a', n'} u(c, n', n) + \beta V^A((K^A)', (K^B)', (N^A)', a', n'; \mathcal{G}^A, \mathcal{G}^B, \mathcal{H}, \mathcal{H}^A)$$

subject to:

$$c = W(K,N)\epsilon_A n' + R(K,N)a - a'$$

$$K = \mu K^A + (1-\mu) K^B$$

$$N = \mathcal{H}(K^A, K^B, N^A)$$

$$(K^A)' = \mathcal{G}^A(K^A, K^B, N^A)$$

$$(K^B)' = \mathcal{G}^B(K^A, K^B, N^A)$$

$$(N^A)' = \mathcal{H}^A(K^A, K^B, N^A)$$

where h^A defines the optimal policy for n and g^A describes the optimal policy for a'.

- Consumers of type B optimize: g^B, h^B, V^B solve:

$$V^B(K^A, K^B, N^A, a; \mathcal{G}^A, \mathcal{G}^B, \mathcal{H}, \mathcal{H}^A) = \max_{a', n} u(c, n) + \beta V^B((K^A)', (K^B)', (N^A)', a'; \mathcal{G}^A, \mathcal{G}^B, \mathcal{H}, \mathcal{H}^A)$$

subject to:

$$c = W(K, N)\epsilon_B n + R(K, N)a - a'$$

$$K = \mu K^A + (1-\mu) K^B$$

$$N = \mathcal{H}(K^A, K^B, N^A)$$

$$(K^A)' = \mathcal{G}^A(K^A, K^B, N^A)$$

$$(K^B)' = \mathcal{G}^B(K^A, K^B, N^A)$$

$$(N^A)' = \mathcal{H}^A(K^A, K^B, N^A)$$

where h^B defines the optimal policy for n and g^B describes the optimal policy for a'.

- Consistency with expectations for capital and for labor:

$$g^A(K^A, K^B, N^A, K^A; \mathcal{G}^A, \mathcal{G}^B, \mathcal{H}, \mathcal{H}^A) = \mathcal{G}^A(K^A, K^B, N^A)$$

$$g^B(K^A, K^B, N^A, K^B; \mathcal{G}^A, \mathcal{G}^B, \mathcal{H}, \mathcal{H}^A) = \mathcal{G}^B(K^A, K^B, N^A)$$

$$h^A(K^A, K^B, N^A, K^A; \mathcal{G}^A, \mathcal{G}^B, \mathcal{H}, \mathcal{H}^A) = \mathcal{H}^A(K^A, K^B, N^A)$$

$$N = \mu\epsilon_A h^A(K^A, K^B, N^A, K^A; \mathcal{G}^A, \mathcal{G}^B, \mathcal{H}, \mathcal{H}^A) + (1-\mu)\epsilon_B h^B(K^A, K^B, N^A, K^B; \mathcal{G}^A, \mathcal{G}^B, \mathcal{H}, \mathcal{H}^A) = \mathcal{H}(K^A, K^B)$$

- Market clearing prices:

$$W(K, N) = (1-\alpha)\theta K^\alpha N^{-\alpha}$$

$$R(K, N) = \alpha\theta K^{\alpha-1} N^{1-\alpha} + (1-\delta)$$

where

$$K = \mu K^A + (1-\mu) K^B$$

To argue efficiency, we consider the social planner problem:

$$\Upsilon(K, N_A) = \max_{K', N'_A, N'_B} \gamma u(C_A, N'_A, N_A) + (1-\gamma)u(C_B, N'_B) + \beta \Upsilon(K', N'_A)$$

subject to:

$$\mu C_A + (1-\mu)C_B + K' = \theta K^\alpha (\mu \epsilon_A N'_A + (1-\mu)\epsilon_B N'_B)^{1-\alpha} + (1-\delta)K$$

We take $\gamma = \mu$, and then write the FOCs of this problem (applying the envelope theorem when necessary), where λ is the Lagrange multiplier on the constraint:

$$\mu u_1(C_A, N'_A, N_A) - \mu\lambda = 0$$

$$(1-\mu)u_1(C_B, N_B) - (1-\mu)\lambda = 0$$

$$\beta \Upsilon_1(K', N'_A) - \lambda = \beta\lambda(\alpha\theta K^{\alpha-1}(\mu\epsilon_A N'_A + (1-\mu)\epsilon_B N'_B)^{1-\alpha} + 1 - \delta) - \lambda = 0$$

$$\mu u_2(C_A, N'_A, N_A) + \beta \Upsilon_2(K', N'_A) + \lambda(1-\alpha)\theta K^\alpha(\mu\epsilon_A N'_A + (1-\mu)\epsilon_B N'_B)^{-\alpha}\mu\epsilon_A =$$

$$\mu u_2(C_A, N'_A, N_A) + \beta\mu u_3(C_A, N'_A, N_A) + \lambda(1-\alpha)\theta K^\alpha(\mu\epsilon_A N'_A + (1-\mu)\epsilon_B N'_B)^{-\alpha}\mu\epsilon_A = 0$$

$$(1-\mu)u_2(C_B, N_B) + \lambda(1-\alpha)\theta K^\alpha(\mu\epsilon_A N'_A + (1-\mu)\epsilon_B N'_B)^{-\alpha}(1-\mu)\epsilon_B = 0$$

If we sub in the terms $R(K, N)$ and $W(K, N)$ determined by the market clearing conditions, we get

$$u_1(C_A, N'_A, N_A) = \beta R(K', N')u_1(C'_A, N''_A, N'_A)$$

$$u_1(C_B, N_B) = \beta R(K', N')u_1(C'_B, N'_B)$$

$$u_2(C_A, N'_A, N_A) + \beta u_3(C_A, N'_A, N_A) + W(K, N)\epsilon_A u_1(C_A, N'_A, N_A) = 0$$

$$u_2(C_B, N_B) + W(K, N)\epsilon_B u_1(C_B, N_B) = 0$$

But note that the first and third of these are exactly the FOC conditions for the maximization problem for consumer group A, and the second and fourth are exactly the FOCs for consumer group B. Hence, an equilibrium allocation also solves the social planner problem for $\gamma = \mu$, and hence is efficient.

Problem 3

(1) A rational expectations recursive equilibrium for this economy consists of pairs of functions for groups A and B describing the laws of motion for capital $\mathcal{G}^A$ and $\mathcal{G}^B$, household value function V, household savings policy function g, interest rate function R, wage function W, such that the following conditions hold:

- Consumers maximize: that is, V, g solve the maximization problem

$$V(K^A, K^B, a; \mathcal{G}^A, \mathcal{G}^b) = \max_{a'} \log(c) + \beta V((K^A)', (K^B)', a'; \mathcal{G}^A, \mathcal{G}^B)$$

  where

$$c = R(K)a + W(K) - a'$$

$$a, a' \geq \underline{A}$$

$$K = \mu K^A + (1-\mu) K^B$$

$$(K^A)' = \mathcal{G}^A(K^A, K^B)$$

$$(K^B)' = \mathcal{G}^B(K^A, K^B)$$

- Consistency with expectations:

$$g(K^A, K^B, K^A) = \mathcal{G}^A(K^A, K^B)$$

$$g(K^A, K^B, K^B) = \mathcal{G}^B(K^A, K^B)$$

- Market clearing:

$$R(K) = \alpha\theta K^{\alpha-1} + (1-\delta)$$

$$W(K) = (1-\alpha)\theta K^{\alpha}$$

  To characterize the steady state, we have from the Euler condition:

$$\beta(\alpha\theta K_{ss}^{\alpha-1} + (1-\delta)) = 1$$

$$\alpha\theta K_{ss}^{\alpha-1} = \frac{1}{\beta} - (1-\delta)$$

$$K^* = \left(\frac{\beta\alpha\theta}{1-\beta(1-\delta)}\right)^{1/(1-\alpha)}$$

(2) To do this, we start by solving for V, g using value function iteration, interpolating in the maximization term $\beta V((K^A)', (K^B)', a')$. We note that since the consumers all have homogeneous preferences, we only have to solve for a single policy function g. We further note that the dependence on K^A and K^B is only through K. That is, let us examine the maximization problem:

$$V(K^A, K^B, a; \mathcal{G}^A, \mathcal{G}^b) = \max_{a'} \log(c) + \beta V((K^A)', (K^B)', a'; \mathcal{G}^A, \mathcal{G}^B)$$

where

$$c = R(K)a + W(K) - a'$$

$$a, a' \geq \underline{A}$$

$$K = \mu K^A + (1-\mu)K^B$$

$$(K^A)' = \mathcal{G}^A(K^A, K^B)$$

$$(K^B)' = \mathcal{G}^B(K^A, K^B)$$

We will now prove a lemma: the policy function g only depends on k, K, and g is linear in k.

Proof: Take $g(k,K) = a(K)k + b(K)$. Then the aggregation:

$$K' = \mu g(K^A, K) + (1-\mu)g(K^B, K) = \mu(a(K)K^A + b(K)) + (1-\mu)(a(K)K^B + b(K)) = b(k) + a(K)(\mu K^A + (1-\mu)K^B)$$

$$= a(K)K + b(K) = g(K,K) = \mathcal{G}(K)$$

It suffices then to check that something of the form $g(k,K)$ satisfies the Euler condition.

$$\frac{1}{c} = \beta R(K')\frac{1}{c'}$$

$$c' = \beta R(K')c$$

$$R(K')k' + W(K') - k'' = (\beta R(K'))(R(K)k + W(K) - k')$$

$$(1-\beta)R(K')k' + W(K') - k'' = \beta R(K')R(K)k + \beta R(K')W(K)$$

$$(1-\beta)R(G(K))(a(K)k + b(K)) + W(G(K)) - a(K')k' - b(K') = \beta R(G(K))R(K)k + \beta R(G(K))W(K)$$

$$(1-\beta)R(G(K))(a(K)k + b(K)) + W(G(K)) - a(G(K))(a(K)k + b(K)) - b(G(K)) = \beta R(G(K))R(K)k + \beta R(G(K))W(K)$$

$$a(K)k((1-\beta)R(G(K)) - a(G(K))) + (1-\beta)R(G(K))b(K) - a(G(K))b(K) + W(G(K)) - b(G(K)) =$$

$$\beta R(G(K))R(K)k + \beta R(G(K))W(K)$$

Hence we can pick a, b such that

$$a(K)((1-\beta)R(G(K)) - a(G(K))) = \beta R(G(K))R(K)$$

$$b(K)((1-\beta)R(G(K)) - a(G(K))) - b(G(K)) = \beta R(G(K))W(K) - W(G(K))$$

and we will satisfy the Euler condition. Hence we have finished the proof.

Now, under this, the maximization problem becomes:

$$V(K, a; \mathcal{G}) = \max_{a'} \log(c) + \beta V(K', a'; \mathcal{G})$$

where

$$c = R(K)a + W(K) - a'$$

$$a, a' \geq \underline{A}$$

$$K' = \mathcal{G}(K)$$

Therefore, we can solve for $g(k, K)$ under no heterogeneity, and then we can simply use the different initial conditions to determine $\mathcal{G}^A$, $\mathcal{G}^B$. To solve for $g(k, K)$, we can use value function iteration repeatedly and the result of exercise 4.4 from SLP that we did last week. Specifically, we can start from an arbitrary $g_0(k, K)$, and use VFI to approximate the fixed point V_0 such that

$$V_0(k, K) = \max_{k'} u(R(K)a + W(K) - k') + \beta V_0(k', g_0(K, K))$$

Defining g_1 then as the optimal policy of V_0, and subsequently g_i as the optimal policy for V_{i-1} and V_i as the fixed point under g_i, then we exactly have the proposition of SLP exercise 4.4 part c. Hence we know V_i will converge to the true fixed point. We can then use the following algorithm to find V, g:

1. Guess a $\mathcal{G}(K)$.
2. Use PFI to find the optimal policy g for k' such that

   $$V(k, K) = \max_{k'} u(R(K)a + W(K) - k') + \beta V(k', \mathcal{G}(K))$$

   where we interpolate V to find $V(k', \mathcal{G}(K))$.
3. Update $\mathcal{G}(K) = g(K, K)$.
4. Go back to 2 and repeat until desired precision.

We show our results for $\alpha = 0.4$, $\theta = 0.8$, $\mu = 0.5$, $\delta = 0.3$, $\beta = 0.9$.

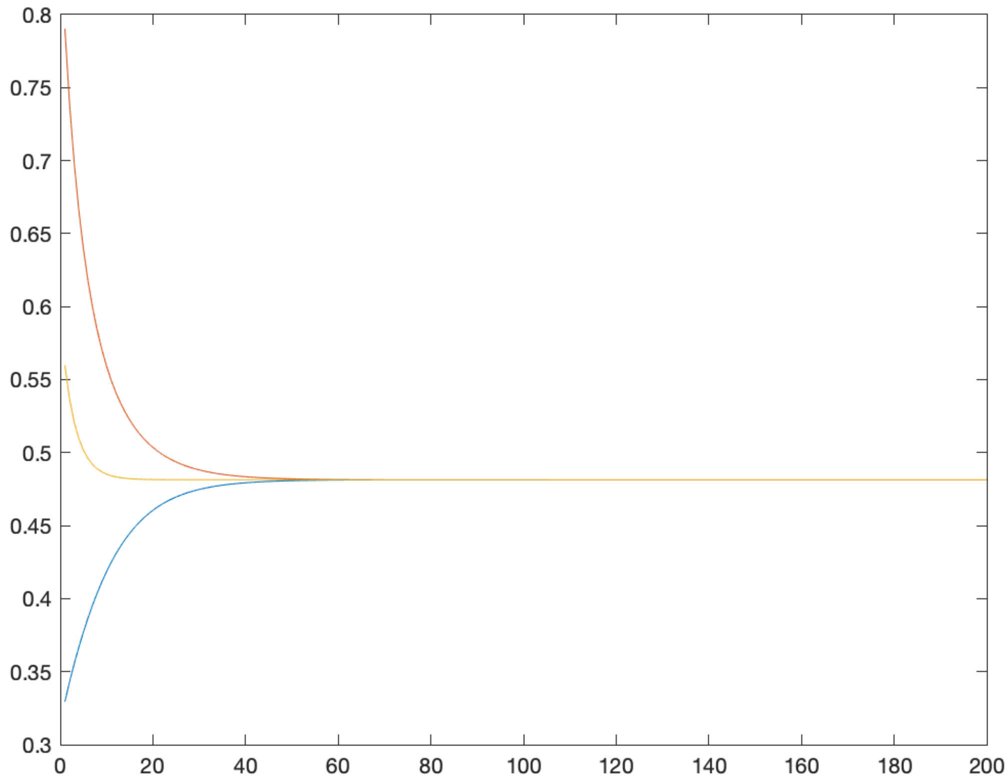

Figure 2: Heterogenous wealth. The richer type's capital sequence is in red, the poorer type in blue, and the aggregate capital in yellow. For these parameters, the capital sequences converge to steady state.