Problem Set 1

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Note: I use bold symbols to denote vectors and nonbolded symbols to denote scalars. I primarily use vector notation to shorthand some of the sums, since many of the sums are dot products.

Problem 1

(1) The maximization FOCs give us:

$$\beta^t c_t^{-\theta} = \lambda_t$$

$$\lambda_t = \lambda_{t+1}(1 - \delta + \alpha A k_{t+1}^{\alpha-1})$$

$$\beta^t c_t^{-\theta} = \beta^{t+1} c_{t+1}^{-\theta}(1 - \delta + \alpha A k_{t+1}^{\alpha-1})$$

At steady state,

$$1 = \beta(1 - \delta + \alpha A (k^*)^{\alpha-1})$$

$$\beta^{-1} - (1 - \delta) = \alpha A (k^*)^{\alpha-1}$$

$$\left(\frac{\beta^{-1} - (1 - \delta)}{\alpha A}\right)^{1/(\alpha-1)} = k^*$$

$$c^* = A(k^*)^\alpha - \delta k^*$$

$$c^* = A\left(\frac{\delta}{\alpha A}\right)^{\alpha/(\alpha-1)} - \delta\left(\frac{\delta}{\alpha A}\right)^{1/(\alpha-1)}$$

(2) See separate file for code.

(3) Capital is in figure 1, returns in figure 2, wages in figure 3.

(4) See figures 4 and 5. As we can see in the graphs, the smaller θ is, the faster the sequences of capital converge to the new steady state. Increasing A increases net economy production, and hence there will be both an income effect and substitution effect affecting the investment-to-output and consumption-to-output ratios. Since θ dictates the substitution effect of investment for consumption, the savings rate adjustment depends on θ. When $\theta = 1$, these two effects are equal and in opposite directions, so savings rate remains constant. However, when $\theta < 1$, the substitution effect dominates, so the rate of savings goes down. When $\theta > 1$, the income effect dominates, and the rate of savings increases.

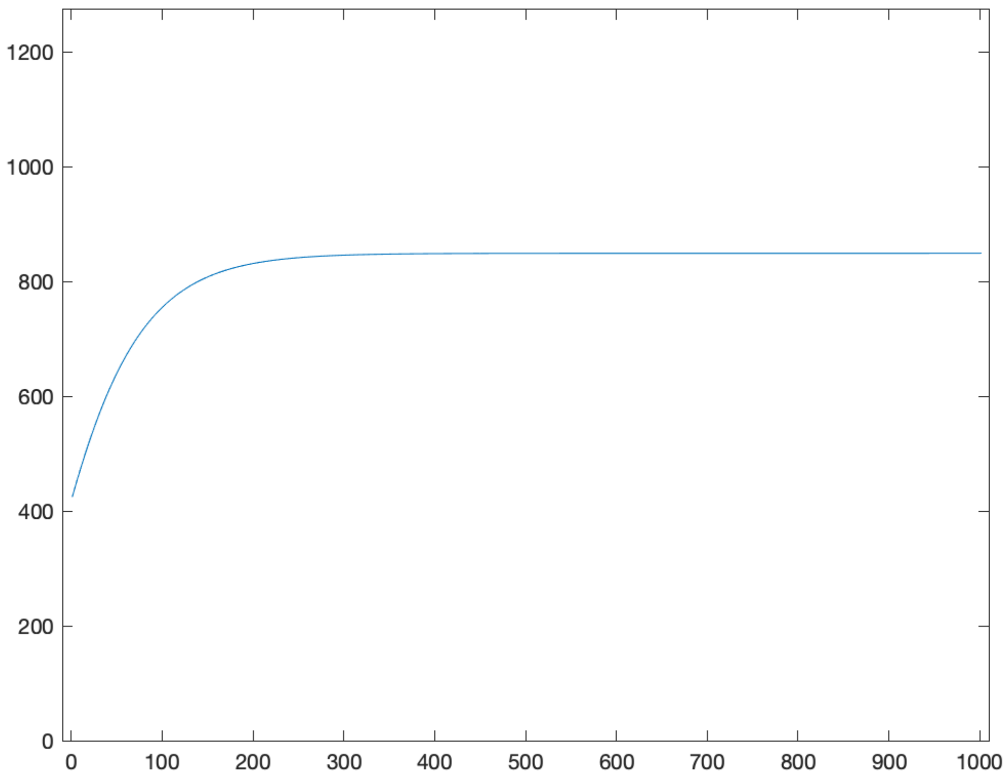

Figure 1: Question 1, part 3. Sequence of capital.

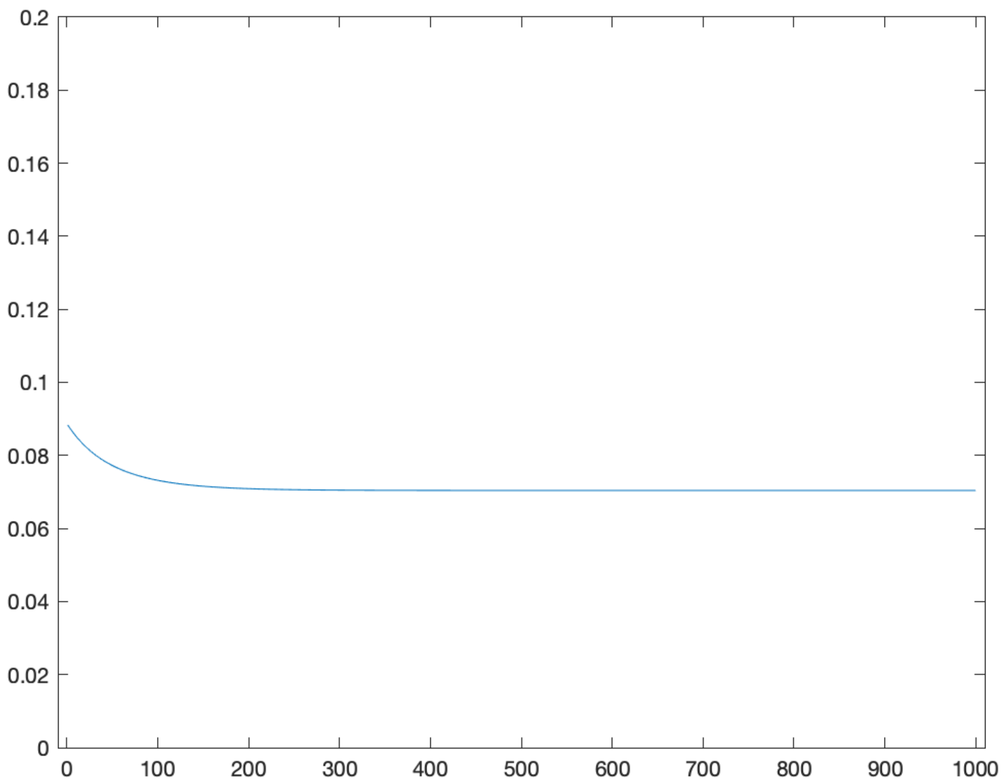

Figure 2: Question 1, part 3. Sequence of returns to capital.

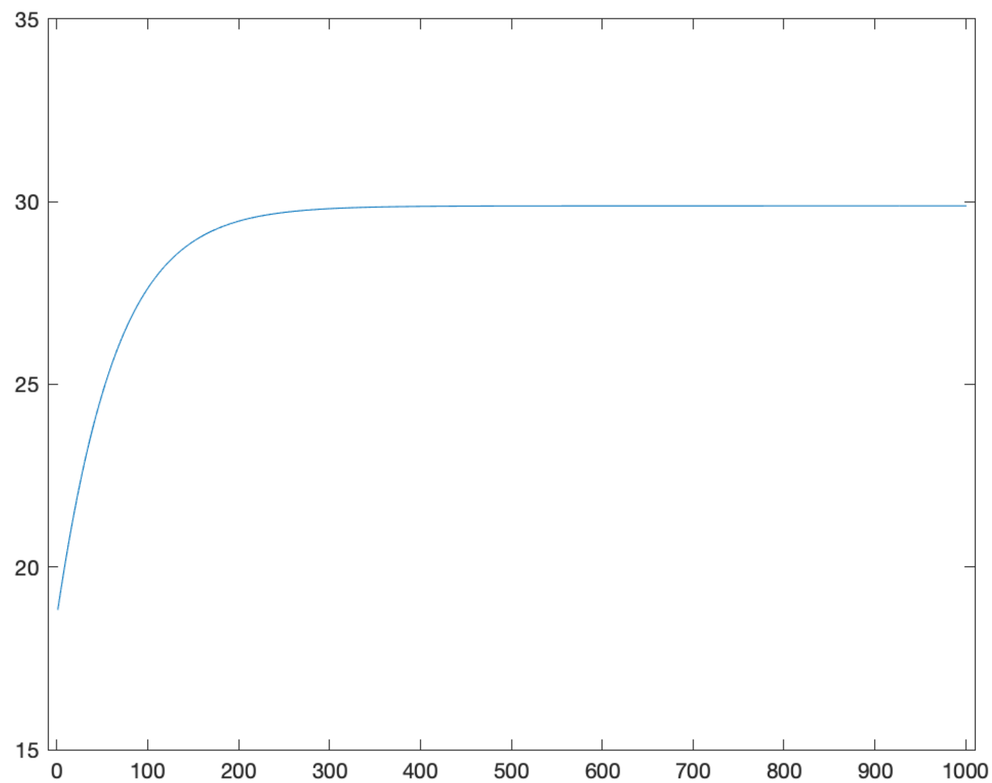

Figure 3: Question 1, part 3. Sequence of wages.

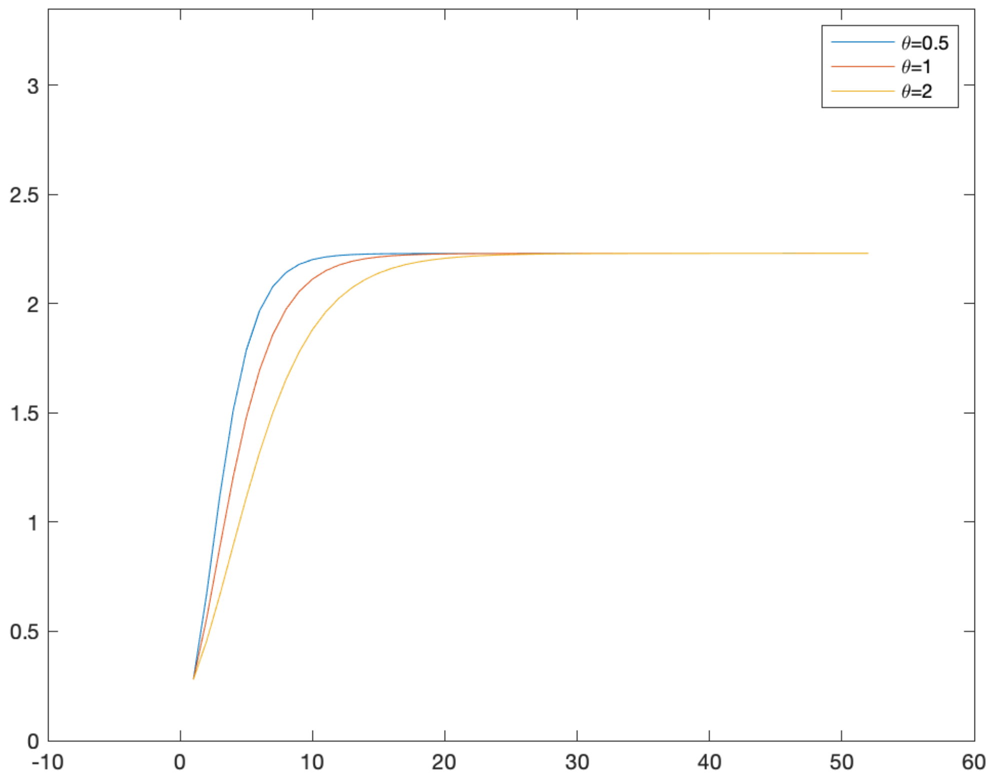

Figure 4: Question 1, part 4. Sequence of capital.

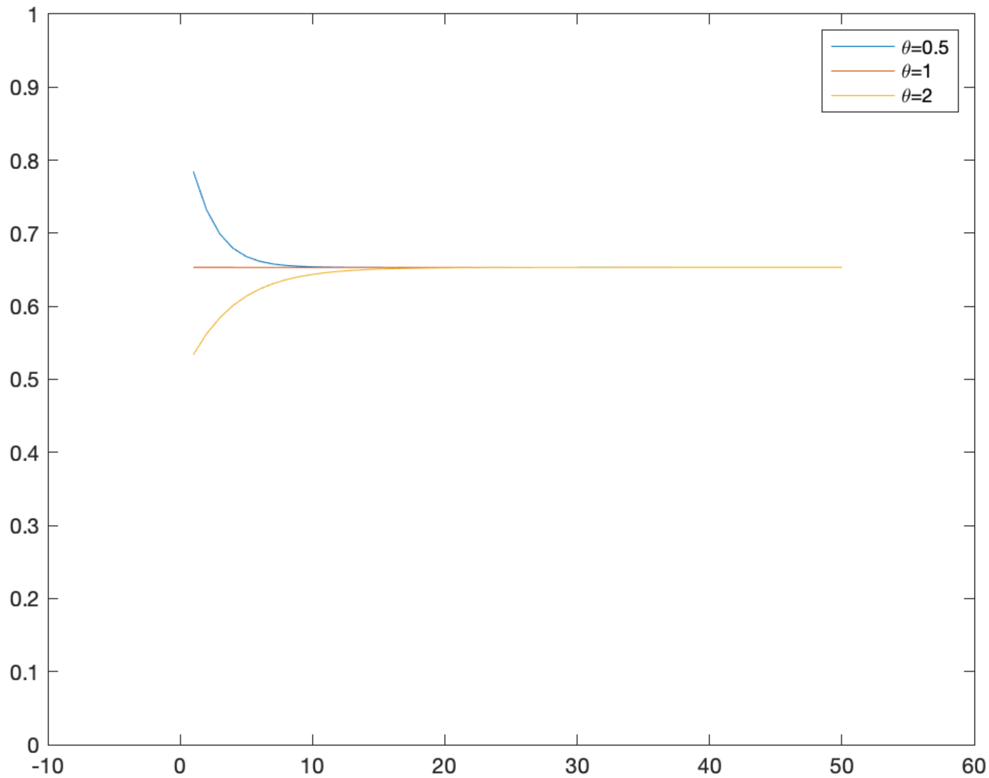

Figure 5: Question 1, part 4. Sequence of savings rates.

Problem 2

Since $\bar{w} > (1-\alpha)(k^*)^\alpha$, the minimum wage is higher than the firms marginal product of labor even at full employment in the steady state. Hence, the minimum wage binds, and is the wage at each time. The employment is then

$$(1-\alpha)\left(\frac{k_t}{E_t}\right)^\alpha = \bar{w}$$

$$E_t = k_t\left(\frac{1-\alpha}{\bar{w}}\right)^{1/\alpha}$$

So output is

$$Y_t = k_t\left(\frac{1-\alpha}{\bar{w}}\right)^{(1-\alpha)/\alpha}$$

And the capital change:

$$k_{t+1} - k_t = -\delta k_t + s k_t\left(\frac{1-\alpha}{\bar{w}}\right)^{(1-\alpha)/\alpha}$$

$$k_{t+1} - k_t = \delta k_t\left(\frac{s}{\delta}\left(\frac{1-\alpha}{\bar{w}}\right)^{(1-\alpha)/\alpha} - 1\right)$$

Using the fact that $k = (s/\delta)^{1/(1-\alpha)}$

$$k_{t+1} - k_t = \delta k_t\left(\left(\frac{(1-\alpha)(k^*)^\alpha}{\bar{w}}\right)^{(1-\alpha)/\alpha} - 1\right)$$

Now, since $(1-\alpha)/\alpha$ is positive and $\bar{w} > (1-\alpha)(k^*)^\alpha$, the parenthesized expression is negative. Further, the rate of decrease is

$$\rho = \delta\left(\left(1 - \frac{(1-\alpha)(k^*)^\alpha}{\bar{w}}\right)^{(1-\alpha)/\alpha}\right)$$

Hence at every period, $k_{t+1} = (1-\rho)k_t$. Since ρ is independent of k_t, we have $k_t \to 0$, which jointly then implies output and employment both drop to 0.

Problem 3

(1) The consumer problem is

$$\max \sum_{t=0}^{\infty} \beta^t (\log c_t + g_t)$$

subject to

$$c_t + i_t \leq r_t k_t + w_t$$

$$k_{t+1} = (1-\tau) i_t + (1-\delta) k_t$$

We can write the Bellman equation:

$$V(k) = \max_{k'} \left[\log \left(rk + w - \frac{k' - (1-\delta)k}{1-\tau} \right) + g + \beta V(k') \right]$$

The Euler equation is

$$\beta (r_{t+1} + \frac{1-\delta}{1-\tau_{t+1}}) \frac{1}{c_{t+1}} = \frac{1/(1-\tau_t)}{c_t}$$

Letting $f(k) = F(k, 1)$, we have $r_t = f'(k_t)$, so

$$\beta((1-\tau_{t+1}) f'(k_{t+1}) + 1 - \delta) \frac{1-\tau_t}{1-\tau_{t+1}} = \frac{c_{t+1}}{c_t}$$

(2) At steady state, we get

$$\beta((1-\bar{\tau}) f'(k^*) + 1 - \delta) = 1$$

$$f'(k^*) = \frac{1-\beta(1-\delta)}{\beta(1-\bar{\tau})}$$

$$i^* = \frac{\delta k^*}{1-\bar{\tau}}$$

$$c^* = f(k^*) - i^*$$

(3) Steady state utility is given by:

$$\log(c^*) + g^* = \log \left(f(k^*) - \frac{\delta k^*}{1-\bar{\tau}} \right) + \bar{\tau} \frac{\delta k^*}{1-\bar{\tau}}$$

We first note that since

$$f'(k^*) = \frac{1-\beta(1-\delta)}{\beta(1-\bar{\tau})}$$

$$f''(k^*) \frac{\partial k^*}{\partial \bar{\tau}} = \frac{1-\beta(1-\delta)}{\beta(1-\bar{\tau})^2}$$

$$\frac{\partial k^*}{\partial \bar{\tau}} = \frac{1-\beta(1-\delta)}{f''(k^*)\beta(1-\bar{\tau})^2}$$

Taking the FOC from the utility expression, we get:

$$\frac{f'(k^*)\frac{\partial k^*}{\partial \bar{\tau}} - \frac{\delta}{1-\bar{\tau}}\frac{\partial k^*}{\partial \bar{\tau}} - \frac{\delta k^*}{(1-\bar{\tau})^2}}{f(k^*) - \frac{\delta k^*}{1-\bar{\tau}}} + \frac{\delta k^*}{1-\bar{\tau}} + \delta\bar{\tau}\frac{(1-\bar{\tau})\frac{\partial k^*}{\partial \bar{\tau}} + k^*}{(1-\bar{\tau})^2} = 0$$

$$\frac{\frac{1-\beta(1-\delta)}{\beta(1-\bar{\tau})}\frac{1-\beta(1-\delta)}{f''(k^*)\beta(1-\bar{\tau})^2} - \frac{\delta}{1-\bar{\tau}}\frac{1-\beta(1-\delta)}{f''(k^*)\beta(1-\bar{\tau})^2} - \frac{\delta k^*}{(1-\bar{\tau})^2}}{c^*} + \frac{\delta k^*}{1-\bar{\tau}} + \delta\bar{\tau}\frac{(1-\bar{\tau})\frac{1-\beta(1-\delta)}{f''(k^*)\beta(1-\bar{\tau})^2} + k^*}{(1-\bar{\tau})^2} = 0$$

$$\frac{1}{c^*}\left(\frac{(1-\beta(1-\delta))^2}{f''(k^*)\beta^2(1-\bar{\tau})^3} - \frac{\beta\delta(1-\beta(1-\delta))}{f''(k^*)\beta^2(1-\bar{\tau})^3} - \frac{\delta k^*}{(1-\bar{\tau})^2}\right) + \frac{\delta k^*}{1-\bar{\tau}} + \delta\bar{\tau}\frac{1-\beta(1-\delta)}{f''(k^*)\beta(1-\bar{\tau})^3} + \frac{\delta\bar{\tau}k^*}{(1-\bar{\tau})^2} = 0$$

$$\frac{1}{c^*}\left(\frac{(1-\beta(1-\delta))(1-\beta)}{f''(k^*)\beta^2(1-\bar{\tau})^3} - \frac{\delta k^*}{(1-\bar{\tau})^2}\right) + \delta\bar{\tau}\frac{1-\beta(1-\delta)}{f''(k^*)\beta(1-\bar{\tau})^3} + \frac{\delta k^*}{(1-\bar{\tau})^2} = 0$$

$$\frac{1}{c^*}\left(\frac{(1-\beta(1-\delta))(1-\beta)}{f''(k^*)k^*\beta^2(1-\bar{\tau})} - \delta\right) + \delta\bar{\tau}\frac{1-\beta(1-\delta)}{f''(k^*)k^*\beta(1-\bar{\tau})} + \delta = 0$$

This together with the expressions for c^*, k^*, i^* will determine the optimal $\bar{\tau}$. In general, the solution will **not** maximize utility away from the steady state; since taxes influence the consumer's household investment choices (and therefore affect the capital stock, consumption behavior, and rate of steady state convergence), the maximization choice of $\bar{\tau}$ only looking at the steady state will generally not optimize utility if starting away from steady state.