Problem Set 1

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Note: I use bold symbols to denote vectors and nonbolded symbols to denote scalars. I primarily use vector notation to shorthand some of the sums, since many of the sums are dot products.

Problem 1

(1) The consumer's problem is given by:

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to

$$c_t - b_{t+1} + k_{t+1} - (1-\delta)k_t \leq r_t^k k_t + w_t - (1+r_t^b)b_t$$

$$b_0 = 0$$

$$k_0 \leq \bar{k}_0$$

$$c_t, k_t \geq 0$$

$$b_{t+1} \leq B$$

where B is the borrowing limit. We have to include a borrowing limit else a consumer can achieve any allocation they want by borrowing forward; i.e., for any c^*, we can set

$$k_t = 0$$

$$b_1 = c_0^* - w_0$$

$$b_2 = (c_1^* - w_1) + (1+r_1^b)(w_0 - c_0^*)$$

and so on

$$b_{t+1} = c_t^* - w_t + b_t(1+r_t^b)$$

The form of b_t is then

$$b_t = \sum_{k=1}^{t} (c_{k-1}^* - w_{k-1}) \prod_{i=k}^{t-1} (1+r_i^b)$$

The consumer is borrowing infinitely against the future, so the borrowing limit prevents these outcomes. Using the FOCs derived in lecture, the Euler condition is given by

$$\frac{1}{c_t} = \beta(1-\delta+r^k_{t+1})\frac{1}{c_{t+1}}$$

$$c_{t+1} = c_t\beta(1-\delta+r^k_{t+1})$$

and the transversality condition is given by

$$\lim_{t\to\infty} \beta^t \frac{(1+r^b_t)b_t}{c_t} = 0$$

(2) The sequential equilibrium consists of initial savings and capital $\bar{b}_0$, $\bar{k}_0$, a consumer allocation $\{(c_t, k_t, b_t)\}_{t=0}^{\infty}$, a producer allocation $\{k^f_t, l^f_t\}_{t=0}^{\infty}$, and prices/wages $\{w_t, r^b_t, r^k_t\}_{t=0}^{\infty}$, that satisfy the following conditions:

- The consumer alllocation solves the consumer maximization problem for the given $\bar{b}_0$, $\bar{k}_0$, $\{w_t, r^b_t, r^k_t\}_{t=0}^{\infty}$. We stated the problem in the previous part, but here it is again:

  $$\max \sum_{t=0}^{\infty} \beta^t \log c_t$$

  subject to

  $$c_t - b_{t+1} + k_{t+1} - (1-\delta)k_t \leq r^k_t k_t + w_t - (1+r^b_t)b_t$$

  $$b_0 = \bar{b}_0,\ k_0 \leq \bar{k}_0$$

  $$c_t, k_t \geq 0,\ b_{t+1} \geq -B$$

- The producer allocation maximizes profits for the firm, or (k^f_t, l^f_t) maximizes:

  $$\max F(k^f_t, l^f_t) - r^k_t k^f_t - w_t l^f_t$$

  subject to

  $$k^f_t, l^f_t \geq 0$$

- All markets clear:

  $$k_t = k^f_t$$

  $$l^f_t = 1$$

  $$b_t = 0$$

  $$c_t + k_{t+1} - (1-\delta)k_t = F(k^f_t, l^f_t)$$

In terms of relating the interest rates, we have the no-arbitrage condition, that

$$1 - \delta + r^k_{t+1} = 1 + r^b_t$$

$$r^k_{t+1} - \delta = r^b_t$$

Intuitively, this makes sense; if a consumer can achieve higher returns by investing in bonds rather than in capital, they would invest no capital, and no production could occur. Conversely, if a consumer can achieve higher returns in capital rather than bonds, the consumer would then choose to always borrow using bonds to buy capital, and will guarantee a profit, which violates the equilibrium assumption of $b_t = 0$.

(3) In this case, an allocation is governed by $\{k^f_t, c_t, l^f_t\}$, and our feasibility constraints are:

$$k^f_0 \leq \bar{k}_0$$

$$c_t \geq 0,\ k^f_t \geq 0,\ l^f_t \in [0, 1]$$

$$c_t \leq F(k^f_t, l^f_t) + (1-\delta)k^f_t - k^f_{t+1}$$

(4) An Arrow-Debreu equilibrium consists of allocation $\{k^f_t, c_t, l^f_t\}$, wages/good prices $\{w_t, p_t\}$, and an initial price of capital p^k such that:

- The consumer maximizes utility subject to his/her budget constraint. That is, the consumer allocation $\{c_t\}$ is a maximizer for:

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t$$

  subject to

$$\sum_{t=0}^{\infty} p_t c_t \leq p_0 p^k \bar{k}_0 + \sum_{t=0}^{\infty} p_t w_t$$

$$c_t \geq 0$$

- The producer maximizes profits. This means $\{k^f_t, l^f_t\}$ maximizes:

$$\max \left(-p_0 p^k \bar{k}_0 + \sum_{t=0}^{\infty} p_t F(k^f_t, l^f_t) - p_t w_t l^f_t - p_t k^f_{t+1} + p_t(1-\delta)k^f_t \right)$$

  subject to

$$k^f_t \geq 0,\ l^f_t \geq 0$$

- Markets clear. That is,

$$c_t + k^f_{t+1} = F(k^f_t, l^f_t)$$

$$l^f_t = 1$$

$$k^f_0 = \bar{k}_0$$

(5) In this case, an equilibrium consists of firm allocation $\{k^f_t, l^f_t\}$, consumer allocation $\{c_t, k_t\}$, prices $\{p_t\}$, wages $\{w_t\}$, and rental rate $\{r_t\}$. The equilibrium must satisfy:

- Consumers maximize their own utility: that is, $\{c_t, k_t\}$ is an optimizer of

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to

$$\sum_{t=0}^{\infty} p_t c_t + \sum_{t=0}^{\infty} p_t(k_{t+1} - (1-\delta)k_t) \leq \sum_{t=0}^{\infty}(p_t w_t + p_t r_t k_t)$$

$$c_t \geq 0,\ k_t \geq 0$$

- Firms maximize profits: that is, k_t^f, l_t^f maximizes

$$\max p_t(F(k_t^f, l_t^f) - w_t l_t^f - r_t k_t^f)$$

subject to

$$k_t^f \geq 0,\ l_t^f \geq 0$$

- Markets clear:

$$k_t = k_t^f$$

$$l_t^f = 1$$

$$c_t + k_{t+1}^f = F(k_t^f, l_t^f)$$

$$k_0 = \bar{k}_0$$

(6)

Problem 2

(1) Using the constraint binding, we get

$$c_t = \theta k_t^\alpha + (1-\delta)k_t - k_{t+1}$$

Then we can take:

$$F(k_t, k_{t+1}) = \frac{(\theta k_t^\alpha + (1-\delta)k_t - k_{t+1})^{1-\sigma} - 1}{1-\sigma}$$

and

$$\Gamma(k_t) = [0, \theta k_t^\alpha + (1-\delta)k_t]$$

The Lagrangian is given by

$$\beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} - \lambda_t(c_t + k_{t+1} - (1-\delta)k_t - \theta k_t^\alpha)$$

The FOCs are:

$$\frac{\beta^t}{c_t^\sigma} = \lambda_t$$

$$\lambda_t = \lambda_{t+1}((1-\delta) + \alpha\theta k_t^{\alpha-1})$$

The Euler condition is

$$\frac{1}{(\theta k_t^\alpha + (1-\delta)k_t - k_{t+1})^\sigma} = \frac{\beta(1-\delta+\alpha\theta k_t^{\alpha-1})}{(\theta k_{t+1}^\alpha + (1-\delta)k_{t+1} - k_{t+2})^\sigma}$$

$$\theta k_{t+1}^\alpha + (1-\delta)k_{t+1} - k_{t+2} = (\theta k_t^\alpha + (1-\delta)k_t - k_{t+1})\beta^{1/\sigma}(1-\delta+\alpha\theta k_t^{\alpha-1})^{1/\sigma}$$

The transversality condition is

$$\lim_{t\to\infty} \frac{\beta^t\,(\theta\alpha k_t^\alpha + (1-\delta)k_t)}{(\theta k_t^\alpha + (1-\delta)k_t - k_{t+1})^\sigma} = 0$$

(2) Suppose the sequence $\{k_t\}_{t=0}^\infty$ satisfies the Euler condition and transversality condition. To show $\{k_t\}_{t=0}^\infty$ indeed optimizes the objective, consider some other feasible $\{k_t'\}_{t=0}^\infty$. We claim

$$\Delta = \sum_{t=0}^\infty \beta^T F(k_t) - \sum_{t=0}^\infty \beta^T F(k_t') \geq 0$$

Using the fact that F is concave, continuous, and differentiable, we have

$$F(k_t,k_{t+1}) - F(k_t',k_{t+1}') \geq F_1(k_t,k_{t+1})(k_t-k_t') + F_2(k_t,k_{t+1})(k_{t+1}-k_{t+1}')$$

Multiplying both sides by β^t, we get

$$\Delta = \lim_{T\to\infty}\left(\sum_{t=0}^T \beta^t(F(k_t,k_{t+1}) - F(k_t',k_{t+1}'))\right)$$

$$\geq \lim_{T\to\infty}\left(\sum_{t=0}^T \beta^t(F_1(k_t,k_{t+1})(k_t-k_t') + F_2(k_t,k_{t+1})(k_{t+1}-k_{t+1}'))\right)$$

$$= \lim_{T\to\infty}\left(\sum_{t=0}^T \beta^t F_1(k_t,k_{t+1})(k_t-k_t') + \sum_{t=0}^T \beta^t F_2(k_t,k_{t+1})(k_{t+1}-k_{t+1}')\right)$$

$$= \lim_{T\to\infty}\left(\sum_{t=0}^T \beta^t F_1(k_t,k_{t+1})(k_t-k_t') + \sum_{t=0}^{T-1} \beta^t F_2(k_t,k_{t+1})(k_{t+1}-k_{t+1}') + \beta^T F_2(k_T,k_{T+1})(k_{T+1}-k_{T+1}')\right)$$

Note $k_0 = k_0'$, so we have

$$\Delta \geq \lim_{T\to\infty}\left(\sum_{t=0}^T \beta^t F_1(k_t,k_{t+1})(k_t-k_t') + \sum_{t=0}^{T-1} \beta^t F_2(k_t,k_{t+1})(k_{t+1}-k_{t+1}') + \beta^T F_2(k_T,k_{T+1})(k_{T+1}-k_{T+1}')\right)$$

$$= \lim_{T\to\infty}\left(\sum_{t=1}^T \beta^t F_1(k_t,k_{t+1})(k_t-k_t') + \sum_{t=0}^{T-1} \beta^t F_2(k_t,k_{t+1})(k_{t+1}-k_{t+1}') + \beta^T F_2(k_T,k_{T+1})(k_{T+1}-k_{T+1}')\right)$$

$$= \lim_{T\to\infty} \left(\sum_{t=0}^{T-1} \beta^{t+1} F_1(k_{t+1}, k_{t+2})(k_{t+1} - k'_{t+1}) + \sum_{t=0}^{T-1} \beta^t F_2(k_t, k_{t+1})(k_{t+1} - k'_{t+1}) + \beta^T F_2(k_T, k_{T+1})(k_{T+1} - k'_{T+1}) \right)$$

$$= \lim_{T\to\infty} \left(\sum_{t=0}^{T-1} \beta^t \left(\beta F_1(k_{t+1}, k_{t+2}) + F_2(k_t, k_{t+1}) \right) (k_{t+1} - k'_{t+1}) + \beta^T F_2(k_T, k_{T+1})(k_{T+1} - k'_{T+1}) \right)$$

By the Euler condition, the sum is 0 (because the summand is 0), so

$$\Delta \geq \lim_{T\to\infty} \left(\beta^T F_2(k_T, k_{T+1})(k_{T+1} - k'_{T+1}) \right)$$

Also by the Euler condition, $F_2(k_T, k_{T+1}) = -\beta F_1(k_{T+1}, k_{T+2})$, so this becomes

$$\Delta \geq \lim_{T\to\infty} \left(-\beta^{T+1} F_1(k_{T+1}, k_{T+2})(k_{T+1} - k'_{T+1}) \right)$$

$$\Delta \geq \lim_{T\to\infty} \left(\beta^{T+1} F_1(k_{T+1}, k_{T+2})(k'_{T+1} - k_{T+1}) \right)$$

Since k'_{T+1} is nonnegative,

$$\Delta \geq - \lim_{T\to\infty} \left(\beta^{T+1} F_1(k_{T+1}, k_{T+2}) k_{T+1} \right)$$

But by transversality condition, the limit on the RHS is 0, so $\Delta \geq 0$. Hence $\{k_t\}_{t=0}^{\infty}$ is optimal.

(3) If we examine the original social planner problem, we see that the budget constraint must bind: if not, we can allocate more c_t for some period, and this gives a strictly larger value for the objective. Hence, since the constraint must bind, given an optimal sequence $\{k_t\}$ solving our problem that we phrased in part (1) of this question, we can easily backsolve for c_t:

$$c_t = \theta k_t^\alpha + (1-\delta)k_t - k_{t+1}$$

Hence, we can derive the second-order equation from the Euler condition:

$$\frac{1}{(\theta k_t^\alpha + (1-\delta)k_t - k_{t+1})^\sigma} = \frac{\beta(1 - \delta + \alpha\theta k_t^{\alpha-1})}{(\theta k_{t+1}^\alpha + (1-\delta)k_{t+1} - k_{t+2})^\sigma}$$

$$\frac{1}{(\theta k_t^\alpha + (1-\delta)k_t - k_{t+1})^\sigma} - \frac{\beta(1 - \delta + \alpha\theta k_t^{\alpha-1})}{(\theta k_{t+1}^\alpha + (1-\delta)k_{t+1} - k_{t+2})^\sigma} = 0$$

Then taking:

$$S(k_t, k_{t+1}, k_{t+2}) = \frac{1}{(\theta k_t^\alpha + (1-\delta)k_t - k_{t+1})^\sigma} - \frac{\beta(1 - \delta + \alpha\theta k_t^{\alpha-1})}{(\theta k_{t+1}^\alpha + (1-\delta)k_{t+1} - k_{t+2})^\sigma}$$

gives the desired condition.

(4)

Problem 3

(1) At steady state, $\bar{k}$, we have $S(\bar{k}, \bar{k}, \bar{k}) = 0$, or

$$\frac{1}{(\theta\bar{k}^\alpha + (1-\delta)\bar{k} - \bar{k})^\sigma} - \frac{\beta(1-\delta+\alpha\theta\bar{k}^{\alpha-1})}{(\theta\bar{k}^\alpha + (1-\delta)\bar{k} - \bar{k})^\sigma} = 0$$

$$1 = \beta(1-\delta+\alpha\theta\bar{k}^{\alpha-1})$$

$$(1/\beta) - (1-\delta) = \alpha\theta\bar{k}^{\alpha-1}$$

$$\frac{(1/\beta) - (1-\delta)}{\alpha\theta} = \bar{k}^{\alpha-1}$$

$$\bar{k} = \left(\frac{(1/\beta) - (1-\delta)}{\alpha\theta}\right)^{1/(\alpha-1)}$$

Consumption is then

$$\bar{c} = \theta\bar{k}^\alpha - \delta\bar{k}$$

$$= \theta\left(\frac{(1/\beta) - (1-\delta)}{\alpha\theta}\right)^{\alpha/(\alpha-1)} - \delta\left(\frac{(1/\beta) - (1-\delta)}{\alpha\theta}\right)^{1/(\alpha-1)}$$

(2)

(3)

(4)