

# Problem Set 3

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**Note:** I use bold symbols to denote vectors and nonbolded symbols to denote scalars. I primarily use vector notation to shorthand some of the sums, since many of the sums are dot products.

## Problem 1

(1)

- (a) We note that the absolute value is trivially nonnegative, and  $|x - y| = 0$  implies  $x = y$ . Further, we have that the absolute value is symmetric, so  $|x - y| = |y - x|$ . Finally, we need to show the triangle inequality. Consider  $x, y, z$ . Then (using the facts that  $|x| \geq x$  and  $|x| \geq -x$ )

$$|x - y| + |y - z| \geq (x - y) + (y - z) \geq x - z$$

and

$$|x - y| + |y - z| \geq (y - x) + (z - y) \geq z - x$$

Lastly, since  $|x| \leq \max(x, -x)$ , we have that

$$|x - y| + |y - z| \geq \max(z - x, x - z) \geq |x - z|$$

and so we have the triangle inequality. Hence  $\rho$  is a metric space.

- (b) By definition,  $\rho$  is nonnegative, and  $\rho(x, y) = 0$  only when  $x = y$ .  $\rho$  is also trivially symmetric:  $\rho(x, y) = 1 = \rho(y, x)$  for  $x \neq y$ , and  $\rho(x, x) = \rho(x, x)$ . Finally, we have: If  $x \neq y \neq z$ ,

$$\rho(x, y) + \rho(y, z) = 2 > \rho(x, z)$$

If  $x = y \neq z$ ,

$$\rho(x, y) + \rho(y, z) = 1 \geq \rho(x, z)$$

And if  $x = y = z$ :

$$\rho(x, y) + \rho(y, z) = 0 \geq \rho(x, z)$$

Hence in all cases we still have the triangle inequality. Thus  $\rho$  is a metric.

- (c) We note that since  $|x(t) - y(t)|$  is nonnegative, the metric is nonnegative. We note that if  $\rho(x, y) = 0$ , then by definition of  $\rho$ ,  $\max |x(t) - y(t)| = 0$ , implying  $x(t) - y(t) = 0$  everywhere, which means  $x(t) = y(t)$  everywhere.

Now we argue symmetry. This follows due to symmetry of the absolute value:

$$\rho(x, y) = \max |x(t) - y(t)| = \max |y(t) - x(t)| = \rho(y, x)$$

Lastly, we argue for the triangle inequality. Consider  $\rho(x, y) + \rho(y, z)$ . We have

$$\begin{aligned} \rho(x, y) + \rho(y, z) &= \max |x(t) - y(t)| + \max |y(t) - z(t)| \\ &\geq \max (|x(t) - y(t)| + |y(t) - z(t)|) \\ &\geq \max |x(t) - z(t)| = \rho(x, z) \end{aligned}$$

And hence we have triangle inequality. So  $\rho$  is a metric.

- (d) We know that since  $|x(t) - y(t)| \geq 0$ ,  $\rho(x, y) \geq 0$ . Additionally, the only way for the integral  $\int |x(t) - y(t)| = 0$  is if the integrand is 0 everywhere, since the integrand cannot be negative. Hence, if  $\rho(x, y) = 0$ , we have  $|x(t) - y(t)| = 0 \implies x(t) = y(t)$ .

For symmetry, we have another easy argument from symmetry of the absolute value difference:

$$\rho(x, y) = \int |x(t) - y(t)| = \int |y(t) - x(t)| = \rho(y, x)$$

Lastly, using linearity of integration and triangle inequality on absolute value we argued for earlier,

$$\begin{aligned} \rho(x, y) + \rho(y, z) &= \int |x(t) - y(t)| + \int |y(t) - z(t)| = \int (|x(t) - y(t)| + |y(t) - z(t)|) \\ &\geq \int |x(t) - z(t)| = \rho(x, z) \end{aligned}$$

And hence  $\rho$  is a metric.

- (e) The argument that  $\rho$  is a metric proceeds exactly as in part a. We note that expanding the domain of  $S$  from integers to rational numbers does not change the behavior of the metric.
- (f) We have nonnegativity since  $f$  is increasing,  $f(0) = 0$ , and the absolute value is always nonnegative. Hence  $f(|x - y|)$  will always be nonnegative. Further, since  $f$  is strictly increasing, we have that for any  $a > 0$ ,  $f(a) > f(0) = 0$ . Hence, if  $f(|x - y|) = 0$ , we must have  $|x - y| = 0$  and therefore  $x = y$ .
- Symmetry follows from symmetry of differences under absolute value:

$$\rho(x, y) = f(|x - y|) = f(|y - x|) = \rho(y, x)$$

Finally, we argue for triangle inequality. By strict concavity of  $f$  and the fact that  $f(0) = 0$ , for

$a, b \geq 0$ ,

$$\begin{aligned} f(a) + f(b) &= f\left(\frac{a}{a+b}(a+b)\right) + f\left(\frac{b}{a+b}(a+b)\right) \\ &\geq \frac{a}{a+b}f(a+b) + \frac{b}{a+b}f(0) + \frac{b}{a+b}f(a+b) + \frac{a}{a+b}f(0) \\ &= f(a+b) \end{aligned}$$

Therefore, by the identity above and by the fact that  $f$  is increasing and the triangle inequality of absolute value we showed earlier,

$$\begin{aligned} \rho(x, y) + \rho(y, z) &= f(|x - y|) + f(|y - z|) \\ &\geq f(|x - y| + |y - z|) \\ &\geq f(|x - z|) = \rho(x, z) \end{aligned}$$

and we are done. Hence  $\rho$  is a metric on  $\mathbb{R}$ .

**(2)** Statement: If  $(S, \rho)$  is a complete metric space, and  $T : S \rightarrow S$  is a contraction mapping with modulus  $\beta$ , then

(a)  $T$  has exactly one fixed point  $v$  in  $S$

(b) for any  $v_0 \in S$ ,  $\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$ .

Proof: Pick an arbitrary  $x \in S$ , and define the sequence  $v_n = T^n x$ , where  $T^0 x = x$ . We first argue that  $\{v_n\}$  is Cauchy. We first note that

$$\rho(v_1, v_0) \leq \beta^0 \rho(v_1, v_0)$$

Inductively, now suppose that for  $n - 1$ ,  $\rho(v_n, v_{n-1}) \leq \beta^{n-1} \rho(v_1, v_0)$ . Then by contraction mapping,

$$\rho(v_{n+1}, v_n) = \rho(Tv_n, Tv_{n-1}) \leq \beta \rho(v_n, v_{n-1}) \leq \beta^n \rho(v_1, v_0)$$

Hence we know that by induction, for any arbitrary  $n$ ,  $\rho(v_{n+1}, v_n) \leq \beta^n \rho(v_1, v_0)$ . Now, for any  $m > n$ , we have by triangle inequality,

$$\begin{aligned} \rho(v^m, v^n) &\leq \sum_{i=n}^{m-1} \rho(v_i, v_{i+1}) \\ &\leq \sum_{i=n}^{m-1} \beta^i \rho(v_1, v_0) \\ &\leq \sum_{i=n}^{\infty} \beta^i \rho(v_1, v_0) \\ &= \frac{\beta^n}{1 - \beta} \rho(v_1, v_0) \end{aligned}$$

Hence, for any  $\epsilon$ , we can pick an  $n$  such that  $\beta^n \leq (1 - \beta)\epsilon/(2\rho(v_1, v_0))$ , and then for all  $m, m' \geq n$ , by the triangle inequality,

$$\begin{aligned}\rho(v^m, v^{m'}) &\leq \rho(v^m, v^n) + \rho(v^n, v^{m'}) \\ &\leq 2\frac{\beta^n}{1-\beta}\rho(v_1, v_0) \\ &\leq \epsilon\end{aligned}$$

And hence  $v_n$  is a Cauchy sequence.

Now, because  $\{v_n\}$  is Cauchy, and  $S$  is complete, the sequence converges:  $v_n \rightarrow v$  for some  $v \in S$ . We claim  $v$  is our fixed point. By the triangle inequality,

$$\rho(Tv, v) \leq \rho(Tv, T^n x) + \rho(T^n x, v)$$

By the contraction mapping property,

$$\rho(Tv, v) \leq \beta\rho(v, v_{n-1}) + \rho(v_n, v)$$

Since  $v_n \rightarrow v$ , we have that as  $n \rightarrow \infty$ ,  $\rho(v, v_{n+1}) \rightarrow 0$  and  $\rho(v_n, v) \rightarrow 0$ . Hence, taking  $n \rightarrow \infty$  we get

$$\rho(Tv, v) \leq \beta(0) + 0 = 0$$

Then since  $\rho$  is nonnegative, we must have  $\rho(Tv, v) = 0$ , so  $v = Tv$ . Hence  $v$  is a fixed point.

To finish (a), we now argue that  $v$  is unique. Pick some fixed point  $v'$ . Then

$$\rho(Tv', Tv) = \rho(v', v)$$

But by contraction mapping property,  $\rho(Tv', Tv) \leq \beta\rho(v', v)$ , so we have

$$\rho(v', v) = \rho(Tv', Tv) \leq \beta\rho(v', v)$$

$$(\beta - 1)\rho(v', v) \geq 0$$

But we know  $\beta < 1$ , so in order for this to be true, we must have

$$\rho(v', v) \leq 0$$

But  $\rho$  is nonnegative, so we must have  $\rho(v', v) = 0$  and hence  $v' = v$ . Hence the only fixed point is  $v$ .

For part (b), we proceed by induction. We can trivially confirm that for  $n = 0$ ,  $\rho(v_0, v) \leq \beta^0\rho(v_0, v)$ . Suppose the inductive hypothesis holds for  $n - 1$ . Then by the contraction mapping property, since  $Tv = v$ , we get

$$\rho(T^n v_0, v) = \rho(T^n v_0, Tv) \leq \beta\rho(T^{n-1} v_0, v) \leq \beta(\beta^{n-1}\rho(v_0, v))$$

where we used the inductive hypothesis in the last step. This implies

$$\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$$

and we are done.

Now, we prove Theorem 3.3:

**Statement:** Let  $X \subseteq \mathbb{R}^l$ , and let  $B(X)$  be the space of bounded functions  $f : X \rightarrow \mathbb{R}$  under the sup norm. Let  $T : B(X) \rightarrow B(X)$  be an operator satisfying:

- (a)  $\forall f, g \in B(X)$  such that  $f(x) \leq g(x) \forall x \in X$ ,  $(Tf)(x) \leq (Tg)(x) \forall x \in X$ .
- (b)  $\exists \beta \in (0, 1)$  such that  $\forall f \in B(X), a \geq 0, x \in X$ ,

$$(T(f + a))(x) \leq (Tf)(x) + \beta a$$

Then  $T$  is a contraction with modulus  $\beta$ .

**Proof:** Since

$$\rho(f, g) = \sup_x |f(x) - g(x)|$$

$$g(x) + \rho(f, g) = g(x) + \sup_x |f(x) - g(x)| \geq g(x) + \sup_x f(x) - g(x) \geq g(x) + (f(x) - g(x)) = f(x)$$

Symmetrically,

$$f(x) + \rho(f, g) = f(x) + \sup_x |f(x) - g(x)| \geq f(x) + \sup_x g(x) - f(x) \geq f(x) + (g(x) - f(x)) = g(x)$$

Since this holds for all  $x$ , we can apply condition (a) on  $T$  to get:

$$(T(g + \rho(f, g)))(x) \geq (Tf)(x)$$

$$(T(f + \rho(f, g)))(x) \geq (Tg)(x)$$

Applying condition 2, we have

$$(Tg)(x) + \beta \rho(f, g) \geq (T(g + \rho(f, g)))(x) \geq (Tf)(x)$$

$$(Tf)(x) + \beta \rho(f, g) \geq (T(f + \rho(f, g)))(x) \geq (Tg)(x)$$

Rearranging, we have

$$(Tf)(x) - (Tg)(x) \leq \beta \rho(f, g)$$

$$(Tg)(x) - (Tf)(x) \leq \beta \rho(f, g)$$

Then

$$\sup_x |(Tf)(x) - (Tg)(x)| \leq \beta \rho(f, g)$$

$$\rho(Tf, Tg) \leq \beta \rho(f, g)$$

and hence  $T$  is a contraction mapping with modulus  $\beta$ .

**(3)**

**(4.6) Statement:** Let  $X \subseteq \mathbb{R}^l$  be convex. Let the correspondence  $\Gamma : X \rightarrow X$  be nonempty, compact-valued, and continuous. Define  $A = \{(x, y) \in X \times X : y \in \Gamma(x)\}$ . Let  $F : A \rightarrow \mathbb{R}$  be continuous and bounded, and let  $\beta \in (0, 1)$ . Let  $C(X)$  be the space of continuous, bounded functions  $X \rightarrow \mathbb{R}$  under the sup norm.

Then the operator  $T$  defined as  $Tf(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta f(y)$  maps  $C(X)$  into itself, has a unique fixed point  $v \in C(X)$ , and for all  $v_0 \in C(X)$ ,

$$\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|$$

Further, the optimal policy correspondence  $G_v : X \rightarrow X$  defined by  $G_v(x) = \{y \in \Gamma(x) : v(x) = F(x, y) + \beta v(y)\}$  is compact-valued and continuous.

**Proof:** We first show that for any  $f \in C(X)$ ,  $Tf$  is bounded and continuous. Note that since  $F$  and  $f$  are both bounded and  $\Gamma$  is compact valued (and hence is bounded-valued), we have that  $Tf$  must also be bounded. Also, since  $F$  and  $f$  are both continuous, and  $\Gamma$  is compact-valued and continuous, by Berge's theorem of the maximum we know  $Tf$  is continuous. Therefore,  $Tf \in C(X)$  since it is bounded and continuous.

We now show  $T$  satisfies the conditions for Theorem 3.3 that we proved in the previous problem part. We first check monotonicity. Suppose  $f(x) \leq g(x) \forall x$ :

$$(Tf)(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta f(y) \leq \max_{y \in \Gamma(x)} F(x, y) + \beta g(y) = (Tg)(x)$$

Now we show discounting:

$$\begin{aligned} (T(f + a))(x) &= \max_{y \in \Gamma(x)} F(x, y) + \beta(f(y) + a) = \max_{y \in \Gamma(x)} (F(x, y) + \beta f(y)) + \beta a \\ &\leq (Tf)(x) + \beta a \end{aligned}$$

Therefore, we know by Theorem 3.3 that  $T$  is a contraction mapping with modulus  $\beta$ . By theorem 3.2 we proved in the previous problem, we have that  $T$  has a unique fixed point  $v \in C(X)$ , and further that

$$\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|$$

for all  $v_0 \in C(X)$ .

Lastly, by Berge's theorem of the maximum, the maximizer correspondence  $G$  is compact-valued and continuous.

**(4.7) Statement:** Let  $X \subseteq \mathbb{R}^l$  be convex. Let the correspondence  $\Gamma : X \rightarrow X$  be nonempty, compact-valued, continuous and monotone; for  $x \leq x'$ ,  $\Gamma(x) \subseteq \Gamma(x')$ . Define  $A = \{(x, y) \in X \times X : y \in \Gamma(x)\}$ . Let  $F : A \rightarrow \mathbb{R}$  be continuous, bounded, and strictly increasing in its first  $l$  arguments, and let  $\beta \in (0, 1)$ . Then the solution to

$$v(x) = \max_{y \in \Gamma(x)} (F(x, y) + \beta v(y))$$

is strictly increasing.

**Proof:** We know from theorem 4.6 we proved previously that  $v$  is the unique fixed point of  $T$ , which takes

$$(Tf)(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta f(y)$$

Suppose  $f$  is a nondecreasing function. Then if  $x < x'$ , since  $F$  is strictly increasing in the first  $l$  arguments,

$$(Tf)(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta f(y) < \max_{y \in \Gamma(x)} F(x', y) + \beta f(y)$$

Since  $\Gamma$  is monotone,

$$\max_{y \in \Gamma(x)} F(x', y) + \beta f(y) \leq \max_{y \in \Gamma(x')} F(x', y) + \beta f(y) = (Tf)(x')$$

Hence  $(Tf)(x) < (Tf)(x')$ , so  $(Tf)$  is strictly increasing. Hence, if we pick  $v_0$  to be a strictly increasing function, we have the sequence  $\{T^n v_0\}$  consists of nondecreasing functions, which is a closed set. Hence by theorem 3.2 we showed, the sequence converges to  $v$ , and by closure of the set of nondecreasing functions, we know  $v$  is a nondecreasing function. However,  $v = Tv$ , so by what we showed,  $Tv = v$  must be strictly increasing. Hence we are done.

**(4.8) Statement:** Let  $X \subseteq \mathbb{R}^l$  be convex. Let the correspondence  $\Gamma : X \rightarrow X$  be nonempty, compact-valued, and continuous. Define  $A = \{(x, y) \in X \times X : y \in \Gamma(x)\}$ . Let  $F : A \rightarrow \mathbb{R}$  be continuous, bounded, and strictly concave, and let  $\beta \in (0, 1)$ . Finally, let  $\Gamma(x)$  be such that  $\forall y \in \Gamma(x), y' \in \Gamma(x'), \theta y + (1 - \theta)y' \in \Gamma(\theta x + (1 - \theta)x')$ . Then the solution to

$$v(x) = \max_{y \in \Gamma(x)} (F(x, y) + \beta v(y))$$

and the corresponding maximizer

$$G(x) = \{y \in \Gamma(x) : v(x) = F(x, y) + \beta v(y)\}$$

are such that  $v$  is strictly concave and  $G$  is continuous and single-valued.

**Proof:** We know from theorem 4.6 we proved previously that  $v$  is the unique fixed point of  $T$ , which takes

$$(Tf)(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta f(y)$$

Suppose  $f$  is a weakly concave function. Let  $y$  be such that  $Tf(x) = F(x, y) + \beta f(y)$ ,  $y'$  such that  $Tf(x') = F(x', y') + \beta f(y')$ . Then by concavity of  $F$ , since  $\lambda y + (1 - \lambda)y' \in \Gamma(\lambda x + (1 - \lambda)x')$ , we get

$$\begin{aligned} (Tf)(\lambda x + (1 - \lambda)x') &\geq F(\lambda x + (1 - \lambda)x', \lambda y + (1 - \lambda)y') + \beta f(\lambda y + (1 - \lambda)y') \\ &> \lambda F(x, \lambda y + (1 - \lambda)y') + (1 - \lambda)F(x', \lambda y + (1 - \lambda)y') + \beta f(\lambda y + (1 - \lambda)y') \end{aligned}$$

By weak concavity of  $f$ ,

$$\geq \lambda F(x, y) + \lambda \beta f(y) + (1 - \lambda)F(x', y) + (1 - \lambda)\beta f(y)$$

$$= \lambda(Tf)(x) + (1 - \lambda)(Tf)(x')$$

Hence  $Tf$  is strictly concave. By the same logic in theorem 4.8, if we pick a strictly concave  $v_0$ , we have the sequence  $\{T^n v_0\}$  consists of weakly concave functions, which is a closed set. Hence by theorem 3.2 we showed, the sequence converges to  $v$ , and by closure of the set of weakly concave functions, we know  $v$  is a weakly concave function. However,  $v = Tv$ , so by what we showed,  $Tv = v$  must be strictly concave.

Finally, we must show  $G$  is single valued. Suppose  $y \neq y' \in G(x)$ . Then

$$v(x) = F(x, y) + \beta v(y) = F(x, y') + \beta v(y')$$

Then by strict concavity of  $F$  and  $v$ ,  $y'' = (y' + y)/2$  must satisfy

$$F(x, y'') + \beta v(y'') \geq \frac{1}{2}(F(x, y) + \beta v(y)) + \frac{1}{2}(F(x, y') + \beta v(y')) = v(x)$$

which contradicts the maximization of  $v$ . Hence, no such pair  $y, y'$  exist, and therefore  $G$  is single-valued. By the theorem of the maximum,  $G$  is upper hemicontinuous, and since any upper hemicontinuous, single-valued correspondence is continuous,  $G$  is continuous.

(4)

(a) Let  $f$  be bounded. We need to show  $Tf$  is also bounded. Then

$$Tf(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta f(y)$$

Now, since  $F$  is bounded,  $f$  is bounded, and  $\Gamma(x)$  is compact and hence also bounded, we must have  $Tf(x)$  is also bounded for all  $x$ , so  $Tf$  is bounded. Hence  $T : B(X) \rightarrow B(X)$ . We then confirm that if  $f(x) \leq g(x) \forall x$ , then

$$Tf(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta f(y) \leq \max_{y \in \Gamma(x)} F(x, y) + \beta g(y) = Tg(x)$$

so we have monotonicity. We then check discounting:

$$\begin{aligned} (T(f + a))(x) &= \max_{y \in \Gamma(x)} F(x, y) + \beta(f(y) + a) = \max_{y \in \Gamma(x)} (F(x, y) + \beta f(y)) + \beta a \\ &\leq (Tf)(x) + \beta a \end{aligned}$$

So we have both discounting and monotonicity, so we satisfy the Blackwell conditions, so by Theorem 3.3,  $T$  is a contraction mapping, and by theorem 3.2,  $T$  has a unique fixed point  $v$ , and for any  $v_0 \in B(X)$ ,

$$\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|$$

Lastly, we see by the theorem of the maximum that the maximizer correspondence  $G$  is nonempty, since  $\Gamma$  is finite-valued and nonempty.

(b) It suffices to show  $T_h$  is a contraction mapping. We do this by using the Blackwell condition and



theorem 3.3. We first check monotonicity. Suppose  $f(x) \leq g(x) \forall x$ . Then

$$(T_h f)(x) = F(x, h(x)) + \beta f(h(x)) \leq F(x, h(x)) + \beta g(h(x)) = (T_h g)(x)$$

Now, we check discounting:

$$(T_h(f + a))(x) = F(x, h(x)) + \beta(f(h(x)) + a) = F(x, h(x)) + \beta f(h(x)) + \beta a \leq (T_h f)(x) + \beta a$$

Hence  $T_h$  is a contraction mapping by theorem 3.3, and so it has a unique fixed point  $w$ .

(c) Consider  $w_n$ . We have,  $\forall x$ ,

$$w_n(x) = F(x, h_n(x)) + \beta w_n(h_n(x)) \leq \max_{y \in \Gamma(x)} F(x, y) + \beta w_n(y) = (Tw_n)(x)$$

Further, since  $h_{n+1}(x) \in \arg \max_{y \in \Gamma(x)} F(x, y) + \beta w_n(y)$ , we have that

$$(Tw_n)(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta w_n(y) = F(x, h_{n+1}(x)) + \beta w_n(h_{n+1}(x)) = (T_{h_{n+1}} w_n)(x)$$

But  $w_{n+1}(x) = F(x, h_{n+1}(x)) + \beta w_{n+1}(h_{n+1}(x))$ , so

$$(Tw_n)(x) = (T_{h_{n+1}} w_n)(x)$$

Since  $w_n \leq Tw_n$ , we have by monotonicity,

$$T_{h_{n+1}} w_n \leq T_{h_{n+1}}(Tw_n) = T_{h_{n+1}}^2 w_n$$

Repeating, we get

$$T_{h_{n+1}} w_n \leq T_{h_{n+1}}^2 w_n \leq T_{h_{n+1}}^3 w_n \dots \leq T_{h_{n+1}}^N w_n$$

Using the fact that we showed  $T_{h_{n+1}} w_n = Tw_n$ , we have

$$Tw_n \leq T_{h_{n+1}}^N w_n$$

for all  $N$ . But as  $N \rightarrow \infty$ , by contraction mapping theorem, the RHS approaches  $w_{n+1}$ . Hence we have

$$Tw_n \leq w_{n+1}$$

Now, we note that  $T^N w_n \geq T^{N-1} w_n \geq \dots \geq w_n$ . Note that as  $N \rightarrow \infty$ , the LHS approaches  $v$ , by contraction mapping. Hence,  $v \geq w_n$  for all  $n$ . Now, by the monotonicity conditions we showed ( $Tw_n \leq w_{n+1}$  and  $w_n \leq Tw_n$ ),

$$\begin{aligned} \|v - w_n\| &\leq \|v - Tw_{n-1}\| \\ &\leq \|v - T^2 w_{n-2}\| \\ &\leq \|v - T^n w_0\| \end{aligned}$$

But by the contraction mapping theorem,  $\|v - T^n w_0\| \leq \beta^n \|v - w_0\|$ . So

$$\|v - w_n\| \leq \beta^n \|v - w_0\|$$

Hence, for any  $\epsilon$ , we can always pick an  $N$  such that  $\beta^N \|v - w_0\| \leq \epsilon$ , and then for all  $n \geq N$ ,  $\|v - w_n\| \leq \epsilon$ . Hence  $w_n$  converges to  $v$ , the unique fixed point of  $T$  by contraction mapping.

## Problem 2

(1) Let us guess the value function has form:

$$V(k) = m \log k + n$$

Then the maximization problem is

$$\begin{aligned} \max \log(\theta k^\alpha - k') + \beta V(k') \\ \max \log(\theta k^\alpha - k') + \beta m \log k' + \beta n \end{aligned}$$

This problem has the FOC:

$$\begin{aligned} \frac{\beta m}{k'} - \frac{1}{\theta k^\alpha - k'} &= 0 \\ \frac{\theta k^\alpha - k'}{k'} &= \frac{1}{\beta m} \\ \frac{\theta k^\alpha}{k'} &= \frac{\beta m + 1}{\beta m} \\ k' &= \frac{\beta m \theta k^\alpha}{\beta m + 1} \end{aligned}$$

Plugging this in, we get

$$\begin{aligned} m \log k + n &= \log \left( \theta k^\alpha - \frac{\beta m \theta k^\alpha}{\beta m + 1} \right) + \beta m \log \left( \frac{\beta m \theta k^\alpha}{\beta m + 1} \right) + \beta n \\ m \log k + n &= \log \left( \frac{\theta k^\alpha}{\beta m + 1} \right) + \beta m \log(\beta m) + \beta m \log \left( \frac{\theta k^\alpha}{\beta m + 1} \right) + \beta n \\ m \log k + n &= (\beta m + 1) \log \left( \frac{\theta k^\alpha}{\beta m + 1} \right) + \beta m \log(\beta m) + \beta n \\ m \log k + n &= \alpha(\beta m + 1) \log k + (\beta m + 1) \log \left( \frac{\theta}{\beta m + 1} \right) + \beta m \log(\beta m) + \beta n \end{aligned}$$

Matching the coefficient of  $\log k$ , we get

$$\begin{aligned} m &= \alpha(\beta m + 1) \\ m - \alpha\beta m &= \alpha \\ m &= \frac{\alpha}{1 - \alpha\beta} \end{aligned}$$

Last, we find  $n$  by matching constant terms and using our expression for  $m$ :

$$\begin{aligned}
n &= (\beta m + 1) \log \left( \frac{\theta}{\beta m + 1} \right) + \beta m \log(\beta m) + \beta n \\
(1 - \beta)n &= \left( \frac{1}{1 - \alpha\beta} \right) \log(\theta(1 - \alpha\beta)) + \frac{\alpha\beta}{1 - \alpha\beta} \log \left( \frac{\alpha\beta}{1 - \alpha\beta} \right) \\
(1 - \beta)(1 - \alpha\beta)n &= \log \theta + \log(1 - \alpha\beta) + \alpha\beta \log \alpha\beta - \alpha\beta \log(1 - \alpha\beta) \\
(1 - \beta)(1 - \alpha\beta)n &= \log \theta + (1 - \alpha\beta) \log(1 - \alpha\beta) + \alpha\beta \log \alpha\beta \\
n &= \frac{\log(\theta(1 - \alpha\beta)^{1 - \alpha\beta}(\alpha\beta)^{\alpha\beta})}{(1 - \beta)(1 - \alpha\beta)}
\end{aligned}$$

All together, the value function is:

$$V(k) = m \log k + n = \frac{\alpha}{1 - \alpha\beta} \log k + \frac{\log(\theta(1 - \alpha\beta)^{1 - \alpha\beta}(\alpha\beta)^{\alpha\beta})}{(1 - \beta)(1 - \alpha\beta)}$$

And the policy function is

$$g(k) = k' = \frac{\beta m \theta k^\alpha}{\beta m + 1} = \alpha \beta \theta k^\alpha$$

**(2)** We could go ahead and prove contraction mapping (which guarantees unique steady state and convergence) but we present a simpler argument. Let  $k_n$  be the sequence of capital choices, where  $k_n = g(k_{n-1})$ . We know from the previous part that the steady state is given by

$$g(k_{ss}) = \alpha \beta \theta k_{ss}^\alpha = k_{ss}$$

We then note that

$$\frac{k_n}{k_{ss}} = \frac{\alpha \beta \theta k_{n-1}^\alpha}{\alpha \beta \theta k_{ss}^\alpha} = \left( \frac{k_{n-1}}{k_{ss}} \right)^\alpha$$

Chaining this, we get

$$\frac{k_n}{k_{ss}} = \left( \frac{k_0}{k_{ss}} \right)^{\alpha^n}$$

Since  $\alpha < 1$ , as  $n \rightarrow \infty$ ,  $\alpha^n \rightarrow 0$ , and hence  $k_n/k_{ss} \rightarrow (k_0/k_{ss})^0 = 1$ . Therefore, as  $n \rightarrow \infty$ , the sequence  $k_n$  approaches the steady state.

**(3)** Rewriting, we get

$$V(k) = \max_{l, k'} \log(\theta k^\alpha l^{1 - \alpha} - k') + \log(1 - l) + \beta V(k')$$

We once again try  $V(k) = m \log k + n$ . We get

$$\max_{l, k'} \log(\theta k^\alpha l^{1 - \alpha} - k') + \log(1 - l) + \beta m \log k' + \beta n$$

The FOCs are:

$$\frac{\beta m}{k'} - \frac{1}{(\theta k^\alpha l^{1-\alpha} - k')} = 0$$

$$\frac{(1-\alpha)\theta k^\alpha l^{-\alpha}}{(\theta k^\alpha l^{1-\alpha} - k')} - \frac{1}{1-l} = 0$$

Solving, the first one gives us

$$\frac{(\theta k^\alpha l^{1-\alpha} - k')}{k'} = \frac{1}{\beta m}$$

$$\frac{k'}{\theta k^\alpha l^{1-\alpha}} = \frac{\beta m}{1 + \beta m}$$

$$k' = \frac{\beta m \theta k^\alpha l^{1-\alpha}}{1 + \beta m}$$

The second gives

$$\frac{(1-\alpha)\theta k^\alpha}{(\theta k^\alpha l - k' l^\alpha)} = \frac{1}{1-l}$$

Plugging in  $k'$ , we get

$$\frac{(1-\alpha)\theta k^\alpha}{\theta k^\alpha l - \frac{\beta m \theta k^\alpha l^{1-\alpha}}{1+\beta m} l^\alpha} = \frac{1}{1-l}$$

$$\frac{(1-\alpha)(1+\beta m)}{l} = \frac{1}{1-l}$$

$$\frac{1}{(1-\alpha)(1+\beta m)} = \frac{1-l}{l} = \frac{1}{l} - 1$$

$$\frac{1 + (1-\alpha)(1+\beta m)}{(1-\alpha)(1+\beta m)} = \frac{1}{l}$$

$$l = \frac{(1-\alpha)(1+\beta m)}{1 + (1-\alpha)(1+\beta m)}$$

$$k' = \frac{\beta m \theta k^\alpha l^{1-\alpha}}{1 + \beta m}$$

Plugging into the overall expression for  $V(k)$ , we get

$$V(k) = \log \left( \theta k^\alpha l^{1-\alpha} - \frac{\beta m \theta k^\alpha l^{1-\alpha}}{1 + \beta m} \right) + \log(1-l) + \beta m \log \left( \frac{\beta m \theta k^\alpha l^{1-\alpha}}{1 + \beta m} \right) + \beta n$$

$$m \log k + n = \log k^\alpha + \log \left( \frac{\theta l^{1-\alpha}}{1 + \beta m} \right) + \log(1-l) + \beta m \log \beta m + \beta m \log k^\alpha + \beta m \log \left( \frac{\theta l^{1-\alpha}}{1 + \beta m} \right) + \beta n$$

Matching the  $\log k$  terms, we get

$$m \log k = \log k^\alpha + \beta m \log k^\alpha$$

$$m = \alpha + \alpha \beta m$$

$$m = \frac{\alpha}{1 - \alpha \beta}$$

$$1 + \beta m = \frac{1}{1 - \alpha \beta}$$

Then our expression for  $l$  becomes

$$l = \frac{(1-\alpha)(1+\beta m)}{1+(1-\alpha)(1+\beta m)}$$

$$= \frac{(1-\alpha)}{(1-\alpha\beta) + (1-\alpha)}$$

We note that optimal labor choice is independent of capital. Our policy function is then

$$g(k) = \frac{\beta m \theta k^\alpha l^{1-\alpha}}{1+\beta m} = \beta \alpha \theta k^\alpha \left( \frac{(1-\alpha)}{(1-\alpha\beta) + (1-\alpha)} \right)^{1-\alpha}$$

Lastly, we solve for the constant term in the value function.

$$n = \log \left( \frac{\theta l^{1-\alpha}}{1+\beta m} \right) + \log(1-l) + \beta m \log \beta m + \beta m \log \left( \frac{\theta l^{1-\alpha}}{1+\beta m} \right) + \beta n$$

$$n(1-\beta) = \log(\theta l^{1-\alpha}(1-\alpha\beta)) + \log(1-l) + \beta m \log \beta m + \beta m \log(\theta l^{1-\alpha}(1-\alpha\beta))$$

$$n(1-\beta)(1-\alpha\beta) = \log(\theta l^{1-\alpha}(1-\alpha\beta)) + (1-\alpha\beta) \log(1-l) + \alpha\beta \log \alpha\beta - \alpha\beta \log(1-\alpha\beta)$$

$$n(1-\beta)(1-\alpha\beta) = \log \theta + (1-\alpha) \log l + (1-\alpha\beta) \log(1-l) + \alpha\beta \log \alpha\beta + (1-\alpha\beta) \log(1-\alpha\beta)$$

$$= \log \theta + (1-\alpha) \log \left( \frac{(1-\alpha)}{(1-\alpha\beta) + (1-\alpha)} \right) + (1-\alpha\beta) \log \left( \frac{(1-\alpha\beta)}{(1-\alpha\beta) + (1-\alpha)} \right) + \alpha\beta \log \alpha\beta + (1-\alpha\beta) \log(1-\alpha\beta)$$

$$= \log \theta + (1-\alpha) \log(1-\alpha) - (2-\alpha-\alpha\beta) \log(2-\alpha-\alpha\beta) + \alpha\beta \log \alpha\beta + 2(1-\alpha\beta) \log(1-\alpha\beta)$$

$$n(1-\beta)(1-\alpha\beta) = \log \left( \frac{\theta(1-\alpha)^{1-\alpha}(\alpha\beta)^{\alpha\beta}(1-\alpha\beta)^{2-2\alpha\beta}}{(2-\alpha-\alpha\beta)^{2-\alpha-\alpha\beta}} \right)$$

$$n = \frac{1}{(1-\beta)(1-\alpha\beta)} \log \left( \frac{\theta(1-\alpha)^{1-\alpha}(\alpha\beta)^{\alpha\beta}(1-\alpha\beta)^{2-2\alpha\beta}}{(2-\alpha-\alpha\beta)^{2-\alpha-\alpha\beta}} \right)$$

And so our value function is

$$V(k) = \frac{1}{1-\alpha\beta} \log k + \frac{1}{(1-\beta)(1-\alpha\beta)} \log \left( \frac{\theta(1-\alpha)^{1-\alpha}(\alpha\beta)^{\alpha\beta}(1-\alpha\beta)^{2-2\alpha\beta}}{(2-\alpha-\alpha\beta)^{2-\alpha-\alpha\beta}} \right)$$

(4) See separate file for code. The analytical result in 3 matches the numerical result from running the code, as we can see in the figures. The value function iteration is straightforward; for the policy function, we need to derive the optimization conditions for  $l$  given  $c(k)$ ,  $k$  and the Euler condition. From our FOCs we have :

$$\frac{(1-\alpha)\theta k^\alpha l^{-\alpha}}{(\theta k^\alpha l^{1-\alpha} - k')} - \frac{1}{1-l} = 0$$

Our optimization condition for  $l$  is then:

$$(1-\alpha)\theta k^\alpha l^{-\alpha}(1-l) = c(k)$$

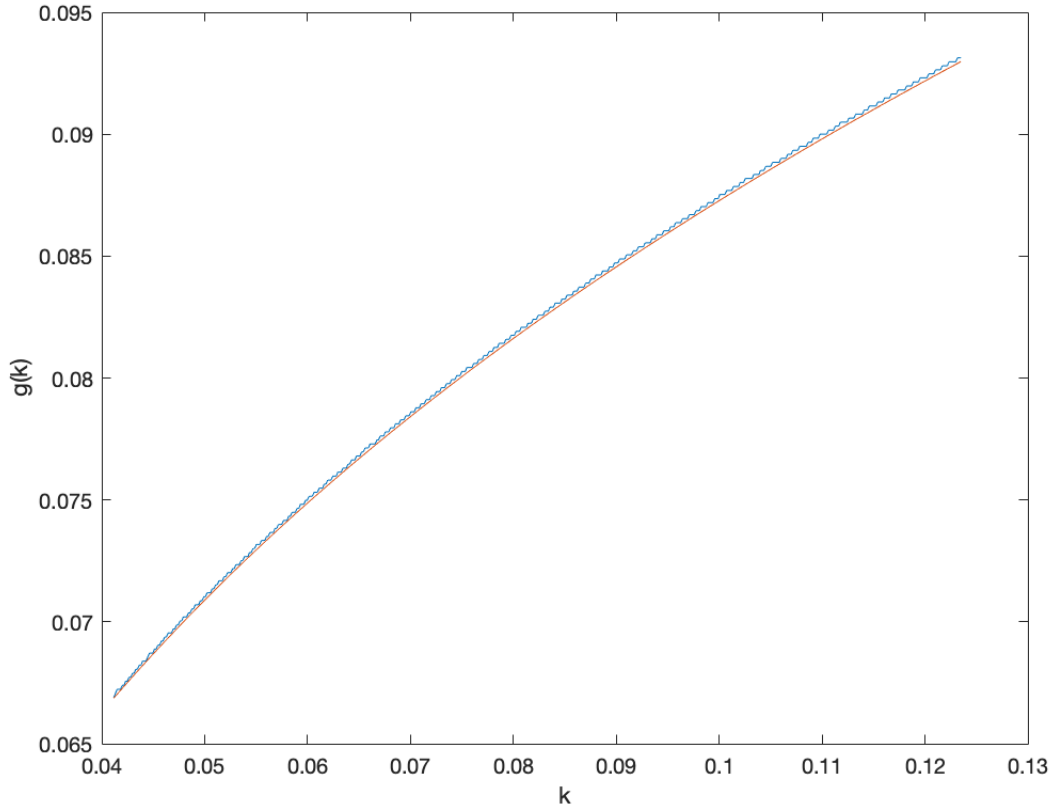


Figure 1: Value function iteration. Analytical result in red, numerical result in blue.

For the Euler condition, we derive from the envelope theorem:

$$\frac{\partial V}{\partial k} = \frac{\alpha \theta k^{\alpha-1} l^{1-\alpha}}{c(k)}$$

And hence the maximization FOC is:

$$\frac{-1}{c(k)} + \beta \frac{\alpha \theta (k')^{\alpha-1} l^{1-\alpha}}{\theta (k')^{\alpha} l^{1-\alpha} - k'} = 0$$

So for our optimization,

$$\frac{1}{c^{\text{new}}(k)} = \frac{\beta \alpha \theta (k')^{\alpha-1} l^{1-\alpha}}{c^{\text{old}}(k')}$$

$$c^{\text{new}}(k) = \frac{c^{\text{old}}(k')}{\beta \alpha \theta (k')^{\alpha-1} l^{1-\alpha}}$$

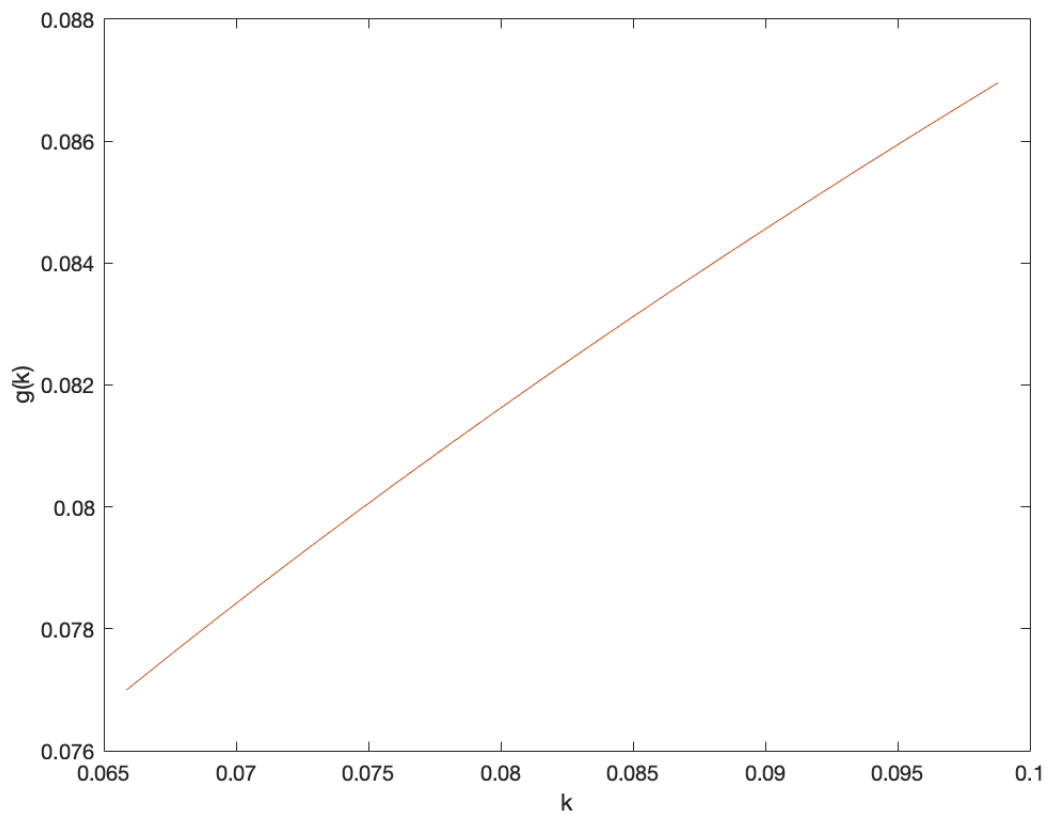


Figure 2: Policy function iteration. Analytical result in red, numerical result in blue. (These are super close, so they basically coincide)

### Problem 3

(1) By the Envelope theorem,

$$\frac{\partial V}{\partial k_1} = \frac{\theta \alpha k_1^{\alpha-1} l_1^{1-\alpha} + (1 - \delta_1)}{(\theta k_1^\alpha l_1^{1-\alpha} + (1 - \delta_1)k_1 - k'_1)^\sigma}$$

$$\frac{\partial V}{\partial k_2} = \frac{\theta \mu k_2^{\mu-1} l_2^{1-\mu} + (1 - \delta_2)}{(\theta k_2^\mu l_2^{1-\mu} + (1 - \delta_2)k_2 - k'_2)^\sigma}$$

Note we can substitute out  $c_1$  and  $c_2$  in terms of the respective  $k_i, l_i, k'_i$ , so we only have to worry about the constraint  $l_1 + l_2 = 1$ . Let  $\lambda$  be the Lagrange multiplier on that constraint. Now, we have the maximization FOCs are, by taking partial derivatives with respect to  $k'_1, k'_2, l_1, l_2$ :

$$-\frac{1}{(\theta k_1^\alpha l_1^{1-\alpha} + (1 - \delta_1)k_1 - k'_1)^\sigma} + \beta \frac{\partial V}{\partial k_1} = 0$$

$$-\frac{1}{(\theta k_2^\mu l_2^{1-\mu} + (1 - \delta_2)k_2 - k'_2)^\sigma} + \beta \frac{\partial V}{\partial k_2} = 0$$

$$\frac{(1 - \alpha)\theta k_1^\alpha l_1^{-\alpha}}{(\theta k_1^\alpha l_1^{1-\alpha} + (1 - \delta_1)k_1 - k'_1)^\sigma} - \lambda = 0$$

$$\frac{(1 - \mu)\theta k_2^\mu l_2^{-\mu}}{(\theta k_2^\mu l_2^{1-\mu} + (1 - \delta_2)k_2 - k'_2)^\sigma} - \lambda = 0$$

We can use the envelope theorem to plug in for the first two, and unify the last two to eliminate  $\lambda$ . Hence we get

$$-\frac{1}{(\theta k_1^\alpha l_1^{1-\alpha} + (1 - \delta_1)k_1 - k'_1)^\sigma} + \beta \frac{\theta \alpha k_1^{\alpha-1} l_1^{1-\alpha} + (1 - \delta_1)}{(\theta k_1^\alpha l_1^{1-\alpha} + (1 - \delta_1)k_1 - k'_1)^\sigma} = 0$$

$$-\frac{1}{(\theta k_2^\mu l_2^{1-\mu} + (1 - \delta_2)k_2 - k'_2)^\sigma} + \beta \frac{\theta \mu k_2^{\mu-1} l_2^{1-\mu} + (1 - \delta_2)}{(\theta k_2^\mu l_2^{1-\mu} + (1 - \delta_2)k_2 - k'_2)^\sigma} = 0$$

$$\frac{(1 - \alpha)\theta k_1^\alpha l_1^{-\alpha}}{(\theta k_1^\alpha l_1^{1-\alpha} + (1 - \delta_1)k_1 - k'_1)^\sigma} = \frac{(1 - \mu)\theta k_2^\mu l_2^{-\mu}}{(\theta k_2^\mu l_2^{1-\mu} + (1 - \delta_2)k_2 - k'_2)^\sigma}$$

Making some simplifications, we get

$$\frac{\beta \theta \alpha k_1^{\alpha-1} l_1^{1-\alpha} + \beta(1 - \delta_1) - 1}{(\theta k_1^\alpha l_1^{1-\alpha} + (1 - \delta_1)k_1 - k'_1)^\sigma} = 0$$

$$\frac{\beta \theta \mu k_2^{\mu-1} l_2^{1-\mu} + \beta(1 - \delta_2) - 1}{(\theta k_2^\mu l_2^{1-\mu} + (1 - \delta_2)k_2 - k'_2)^\sigma} = 0$$

$$\frac{(1 - \alpha)k_1^\alpha l_1^{-\alpha}}{(\theta k_1^\alpha l_1^{1-\alpha} + (1 - \delta_1)k_1 - k'_1)^\sigma} = \frac{(1 - \mu)k_2^\mu l_2^{-\mu}}{(\theta k_2^\mu l_2^{1-\mu} + (1 - \delta_2)k_2 - k'_2)^\sigma}$$

(2) See separate file.



(3) To compute the steady-states, we know  $(k_1^*)' = k_1^*$ ,  $(k_2^*)' = k_2^*$ . Then using the given parameter values, our FOCs can be written as

$$\begin{aligned}\frac{\beta\alpha k_1^{\alpha-1} l_1^{1-\alpha} + \beta(1-\delta_1) - 1}{(k_1^\alpha l_1^{1-\alpha} - \delta_1 k_1)^2} &= 0 \\ \frac{\beta\mu k_2^{\mu-1} l_2^{1-\mu} + \beta(1-\delta_2) - 1}{(k_2^\mu l_2^{1-\mu} - \delta_2 k_2)^2} &= 0 \\ \frac{(1-\alpha)k_1^\alpha l_1^{-\alpha}}{(k_1^\alpha l_1^{1-\alpha} - \delta_1 k_1)^2} &= \frac{(1-\mu)k_2^\mu l_2^{-\mu}}{(k_2^\mu l_2^{1-\mu} - \delta_2 k_2)^2}\end{aligned}$$

From the first two, we get

$$\begin{aligned}\beta\alpha k_1^{\alpha-1} l_1^{1-\alpha} + \beta(1-\delta_1) &= 1 \\ k_1^{\alpha-1} l_1^{1-\alpha} &= \frac{1 - \beta(1-\delta_1)}{\beta\alpha} \\ \frac{l_1}{k_1} &= \left( \frac{1 - \beta(1-\delta_1)}{\beta\alpha} \right)^{\frac{1}{1-\alpha}} \\ l_1 &= k_1 \left( \frac{1 - \beta(1-\delta_1)}{\beta\alpha} \right)^{\frac{1}{1-\alpha}} \\ \beta\mu k_2^{\mu-1} l_2^{1-\mu} + \beta(1-\delta_2) &= 1 \\ k_2^{\mu-1} l_2^{1-\mu} &= \frac{1 - \beta(1-\delta_2)}{\beta\mu} \\ l_2 &= k_2 \left( \frac{1 - \beta(1-\delta_2)}{\beta\mu} \right)^{\frac{1}{1-\mu}}\end{aligned}$$

Before we finish, we make two notes that will simplify our calculations:

$$\begin{aligned}k_1^\alpha l_1^{1-\alpha} &= k_1 \left( \frac{1 - \beta(1-\delta_1)}{\beta\alpha} \right) \\ k_2^\mu l_2^{1-\mu} &= k_2 \left( \frac{1 - \beta(1-\delta_2)}{\beta\mu} \right)\end{aligned}$$

so

$$\begin{aligned}k_1^\alpha l_1^{1-\alpha} - \delta_1 k_1 &= k_1 \left( \frac{1 - \beta(1-\delta_1)}{\beta\alpha} - \delta_1 \right) \\ k_2^\mu l_2^{1-\mu} - \delta_2 k_2 &= k_2 \left( \frac{1 - \beta(1-\delta_2)}{\beta\mu} - \delta_2 \right)\end{aligned}$$

Now, we use the third FOC:

$$\begin{aligned}\frac{(1-\alpha)k_1^\alpha l_1^{-\alpha}}{(k_1^\alpha l_1^{1-\alpha} - \delta_1 k_1)^2} &= \frac{(1-\mu)k_2^\mu l_2^{-\mu}}{(k_2^\mu l_2^{1-\mu} - \delta_2 k_2)^2} \\ \frac{(1-\alpha) \left( \frac{1-\beta(1-\delta_1)}{\beta\alpha} \right)^{\frac{-\alpha}{1-\alpha}}}{k_1^2 \left( \frac{1-\beta(1-\delta_1)}{\beta\alpha} - \delta_1 \right)^2} &= \frac{(1-\mu) \left( \frac{1-\beta(1-\delta_2)}{\beta\mu} \right)^{\frac{-\mu}{1-\mu}}}{k_2^2 \left( \frac{1-\beta(1-\delta_2)}{\beta\mu} - \delta_2 \right)^2}\end{aligned}$$

$$\begin{aligned}
& \sqrt{\frac{(1-\alpha) \left( \frac{1-\beta(1-\delta_2)}{\beta\mu} \right)^{\frac{\mu}{1-\mu}}}{(1-\mu) \left( \frac{1-\beta(1-\delta_1)}{\beta\alpha} \right)^{\frac{\alpha}{1-\alpha}}}} = \frac{k_1 \left( \frac{1-\beta(1-\delta_1)}{\beta\alpha} - \delta_1 \right)}{k_2 \left( \frac{1-\beta(1-\delta_2)}{\beta\mu} - \delta_2 \right)} \\
& \frac{k_1}{k_2} = \frac{\left( \frac{1-\beta(1-\delta_2)}{\beta\mu} - \delta_2 \right)}{\left( \frac{1-\beta(1-\delta_1)}{\beta\alpha} - \delta_1 \right)} \sqrt{\frac{(1-\alpha) \left( \frac{1-\beta(1-\delta_2)}{\beta\mu} \right)^{\frac{\mu}{1-\mu}}}{(1-\mu) \left( \frac{1-\beta(1-\delta_1)}{\beta\alpha} \right)^{\frac{\alpha}{1-\alpha}}}} \\
& \approx 0.653567
\end{aligned}$$

Noting the labor constraint:

$$\begin{aligned}
& l_1 + l_2 = 1 \\
& k_1 \left( \frac{1-\beta(1-\delta_1)}{\beta\alpha} \right)^{\frac{1}{1-\alpha}} + k_2 \left( \frac{1-\beta(1-\delta_2)}{\beta\mu} \right)^{\frac{1}{1-\mu}} = 1 \\
& k_2 \left( (k_1/k_2) \left( \frac{1-\beta(1-\delta_1)}{\beta\alpha} \right)^{\frac{1}{1-\alpha}} + \left( \frac{1-\beta(1-\delta_2)}{\beta\mu} \right)^{\frac{1}{1-\mu}} \right) = 1 \\
& k_2 = \frac{1}{(k_1/k_2) \left( \frac{1-\beta(1-\delta_1)}{\beta\alpha} \right)^{\frac{1}{1-\alpha}} + \left( \frac{1-\beta(1-\delta_2)}{\beta\mu} \right)^{\frac{1}{1-\mu}}} \\
& = \frac{1}{\frac{\left( \frac{1-\beta(1-\delta_2)}{\beta\mu} - \delta_2 \right)}{\left( \frac{1-\beta(1-\delta_1)}{\beta\alpha} - \delta_1 \right)} \sqrt{\frac{(1-\alpha) \left( \frac{1-\beta(1-\delta_2)}{\beta\mu} \right)^{\frac{\mu}{1-\mu}}}{(1-\mu) \left( \frac{1-\beta(1-\delta_1)}{\beta\alpha} \right)^{\frac{\alpha}{1-\alpha}}}} \left( \frac{1-\beta(1-\delta_1)}{\beta\alpha} \right)^{\frac{1}{1-\alpha}} + \left( \frac{1-\beta(1-\delta_2)}{\beta\mu} \right)^{\frac{1}{1-\mu}}} \\
& = \frac{1}{\frac{\left( \frac{1-0.95(1-0.08)}{0.95 \cdot 0.5} - 0.08 \right)}{\left( \frac{1-0.95(1-0.05)}{0.95 \cdot 0.36} - 0.05 \right)} \sqrt{\frac{(1-0.36) \left( \frac{1-0.95(1-0.08)}{0.95 \cdot 0.5} \right)^{\frac{0.5}{1-0.36}}}{(1-0.5) \left( \frac{1-0.95(1-0.05)}{0.95 \cdot 0.36} \right)^{\frac{0.36}{1-0.36}}}} \left( \frac{1-0.95(1-0.05)}{0.95 \cdot 0.36} \right)^{\frac{1}{1-0.36}} + \left( \frac{1-0.95(1-0.08)}{0.95 \cdot 0.5} \right)^{\frac{1}{1-0.5}}} \\
& k_2 \approx 6.1597 \\
& k_1 = (k_1/k_2)k_2 \approx 4.0258
\end{aligned}$$

See figures for the sequences of capital, labor, and consumption.

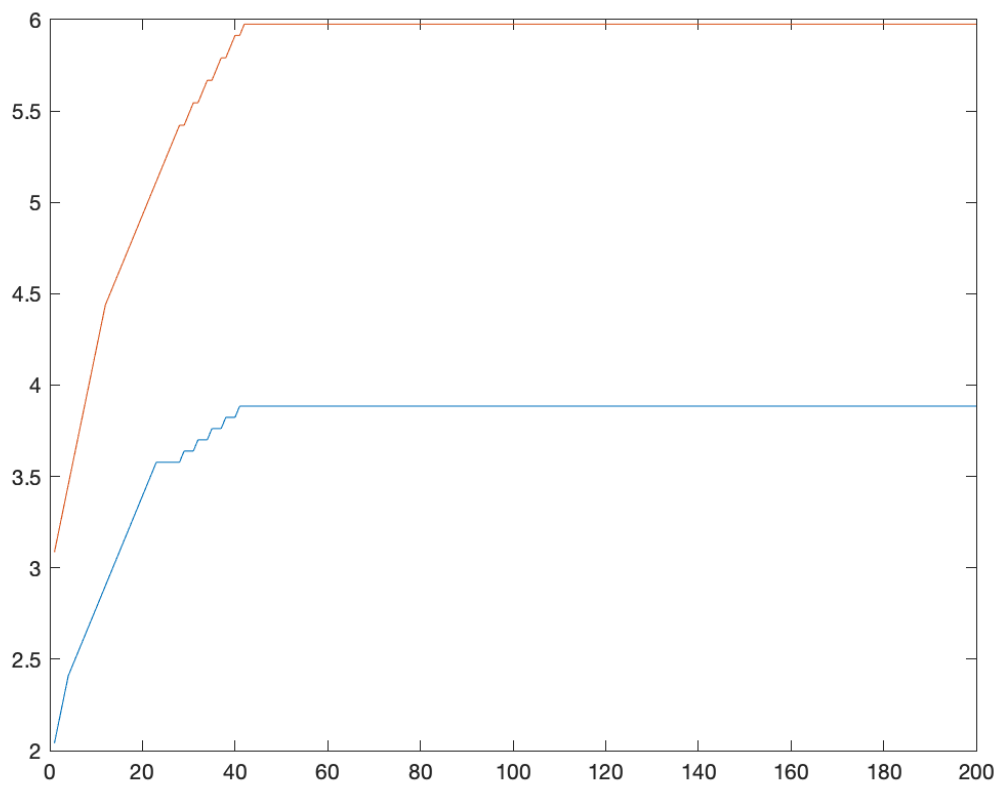


Figure 3: Two-sector economy, sequences of capital

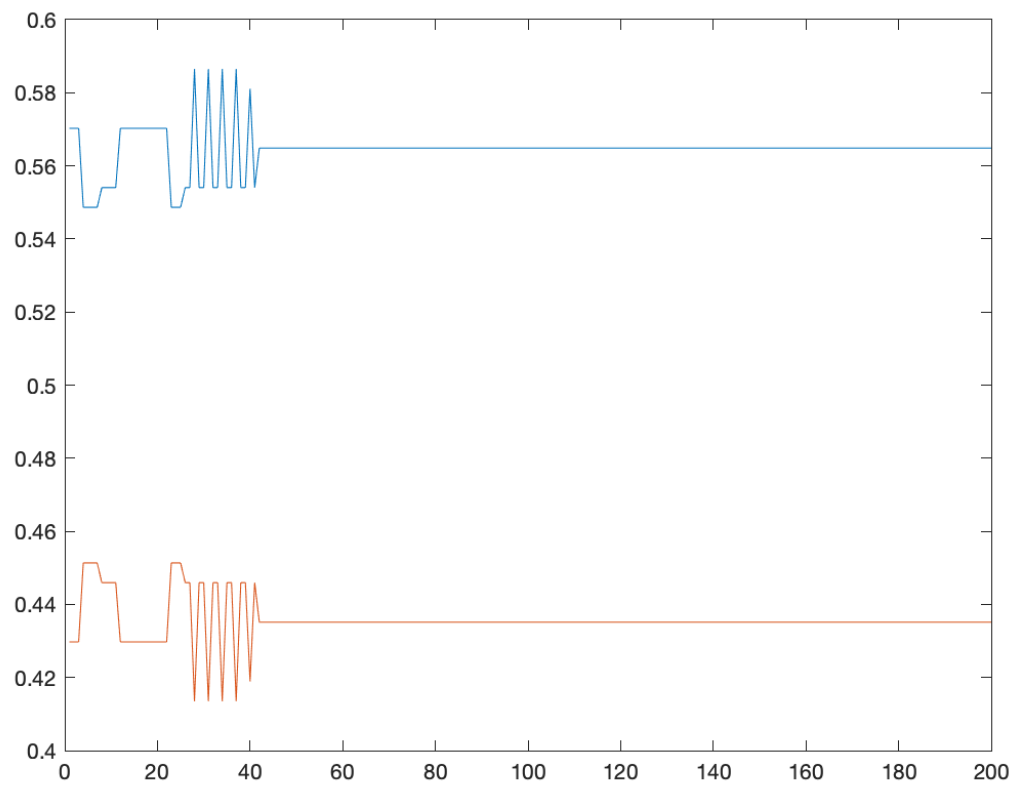


Figure 4: Two-sector economy, sequences of labor

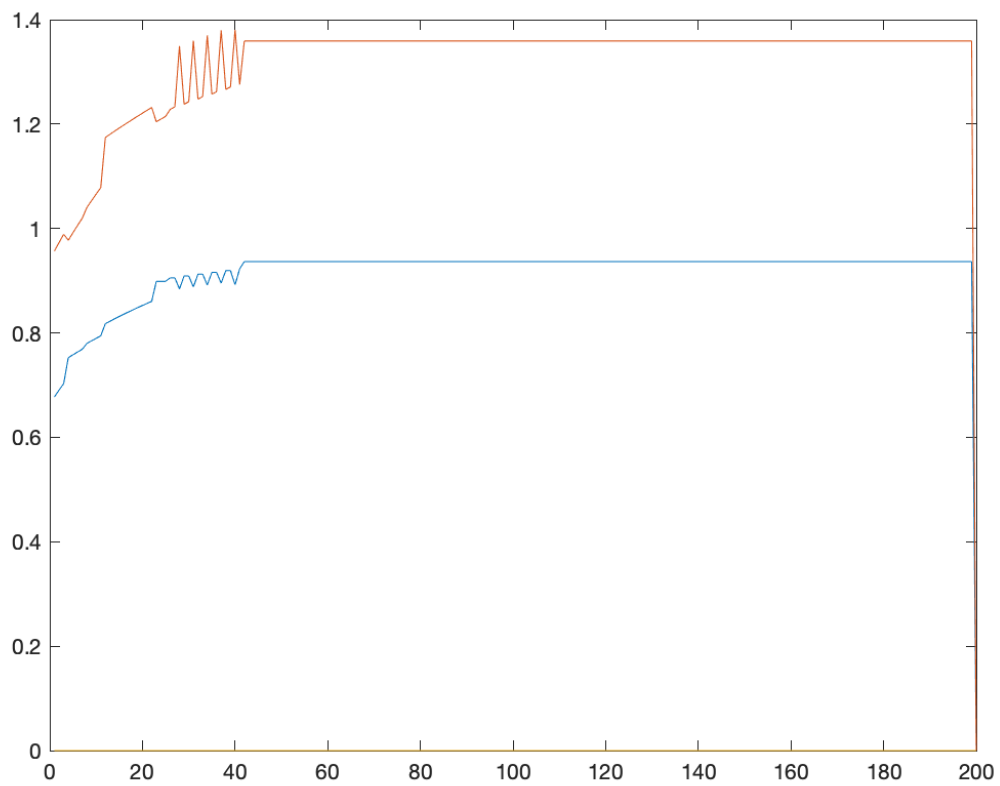


Figure 5: Two-sector economy, sequences of consumption