Problem Set 3

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Note: I use bold symbols to denote vectors and nonbolded symbols to denote scalars. I primarily use vector notation to shorthand some of the sums, since many of the sums are dot products.

Problem 1

(1)

(a) We note that the absolute value is trivially nonnegative, and $|x - y| = 0$ implies $x = y$. Further, we have that the absolute value is symmetric, so $|x - y| = |y - x|$. Finally, we need to show the triangle inequality. Consider x, y, z. Then (using the facts that $|x| \geq x$ and $|x| \geq -x$)

$$|x - y| + |y - z| \geq (x - y) + (y - z) \geq x - z$$

and

$$|x - y| + |y - z| \geq (y - x) + (z - y) \geq z - x$$

Lastly, since $|x| \leq \max(x, -x)$, we have that

$$|x - y| + |y - z| \geq \max(z - x, x - z) \geq |x - z|$$

and so we have the triangle inequality. Hence ρ is a metric space.

(b) By definition, ρ is nonnegative, and $\rho(x, y) = 0$ only when $x = y$. ρ is also trivially symmetric: $\rho(x, y) = 1 = \rho(y, x)$ for $x \neq y$, and $\rho(x, x) = \rho(x, x)$. Finally, we have: If $x \neq y \neq z$,

$$\rho(x, y) + \rho(y, z) = 2 > \rho(x, z)$$

If $x = y \neq z$,

$$\rho(x, y) + \rho(y, z) = 1 \geq \rho(x, z)$$

And if $x = y = z$:

$$\rho(x, y) + \rho(y, z) = 0 \geq \rho(x, z)$$

Hence in all cases we still have the triangle inequality. Thus ρ is a metric.

(c) We note that since $|x(t) - y(t)|$ is nonnegative, the metric is nonnegative. We note that if $\rho(x,y) = 0$, then by definition of ρ, $\max|x(t) - y(t)| = 0$, implying $x(t) - y(t) = 0$ everywhere, which means $x(t) = y(t)$ everywhere.

Now we argue symmetry. This follows due to symmetry of the absolute value:

$$\rho(x,y) = \max|x(t) - y(t)| = \max|y(t) - x(t)| = \rho(y,x)$$

Lastly, we argue for the triangle inequality. Consider $\rho(x,y) + \rho(y,z)$. We have

$$\rho(x,y) + \rho(y,z) = \max|x(t) - y(t)| + \max|y(t) - z(t)|$$

$$\geq \max\left(|x(t) - y(t)| + |y(t) - z(t)|\right)$$

$$\geq \max|x(t) - z(t)| = \rho(x,z)$$

And hence we have triangle inequality. So ρ is a metric.

(d) We know that since $|x(t) - y(t)| \geq 0$, $\rho(x,y) \geq 0$. Additionally, the only way for the integral $\int |x(t) - y(t)| = 0$ is if the integrand is 0 everywhere, since the integrand cannot be negative. Hence , if $\rho(x,y) = 0$, we have $|x(t) - y(t)| = 0 \implies x(t) = y(t)$.

For symmetry, we have another easy argument from symmetry of the absolute value difference:

$$\rho(x,y) = \int |x(t) - y(t)| = \int |y(t) - x(t)| = \rho(y,x)$$

Lastly, using linearity of integration and triangle inequality on absolute value we argued for earlier,

$$\rho(x,y) + \rho(y,z) = \int |x(t) - y(t)| + \int |y(t) - z(t)| = \int \left(|x(t) - y(t)| + |y(t) - z(t)|\right)$$

$$\geq \int |x(t) - z(t)| = \rho(x,z)$$

And hence ρ is a metric.

(e) The argument that ρ is a metric proceeds exactly as in part a. We note that expanding the domain of S from integers to rational numbers does not change the behavior of the metric.

(f) We have nonnegativity since f is increasing, $f(0) = 0$, and the absolute value is always nonnegative. Hence $f(|x - y|)$ will always be nonnegative. Further, since f is strictly increasing, we have that for any $a > 0$, $f(a) > f(0) = 0$. Hence, if $f(|x - y|) = 0$, we must have $|x - y| = 0$ and therefore $x = y$.

Symmetry follows from symmetry of differences under absolute value:

$$\rho(x,y) = f(|x - y|) = f(|y - x|) = \rho(y,x)$$

Finally, we argue for triangle inequality. By strict concavity of f and the fact that $f(0) = 0$, for

$a, b \geq 0$,

$$f(a) + f(b) = f\left(\frac{a}{a+b}(a+b)\right) + f\left(\frac{b}{a+b}(a+b)\right)$$

$$\geq \frac{a}{a+b}f(a+b) + \frac{b}{a+b}f(0) + \frac{b}{a+b}f(a+b) + \frac{a}{a+b}f(0)$$

$$= f(a+b)$$

Therefore, by the identity above and by the fact that f is increasing and the triangle inequality of absolute value we showed earlier,

$$\rho(x, y) + \rho(y, z) = f(|x-y|) + f(|y-z|)$$

$$\geq f(|x-y| + |y-z|)$$

$$\geq f(|x-z|) = \rho(x, z)$$

and we are done. Hence ρ is a metric on $\mathbb{R}$.

(2) Statement: If (S, ρ) is a complete metric space, and $T : S \to S$ is a contraction mapping with modulus β, then

(a) T has exactly one fixed point v in S

(b) for any $v_0 \in S$, $\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$.

Proof: Pick an arbitrary $x \in S$, and define the sequence $v_n = T^n x$, where $T^0 x = x$. We first argue that $\{v_n\}$ is Cauchy. We first note that

$$\rho(v_1, v_0) \leq \beta^0 \rho(v_1, v_0)$$

Inductively, now suppose that for $n-1$, $\rho(v_n, v_{n-1}) \leq \beta^{n-1}\rho(v_1, v_0)$. Then by contraction mapping,

$$\rho(v_{n+1}, vn) = \rho(Tv_n, Tv_{n-1}) \leq \beta\rho(v_n, v_{n-1}) \leq \beta^n \rho(v_1, v_0)$$

Hence we know that by induction, for any arbitrary n, $\rho(v_{n+1}, v_n) \leq \beta^n \rho(v_1, v_0)$. Now, for any $m > n$, we have by triangle inequality,

$$\rho(v^m, v^n) \leq \sum_{i=n}^{m-1} \rho(v_i, v_{i+1})$$

$$\leq \sum_{i=n}^{m-1} \beta^i \rho(v_1, v_0)$$

$$\leq \sum_{i=n}^{\infty} \beta^i \rho(v_1, v_0)$$

$$= \frac{\beta^n}{1-\beta}\rho(v_1, v_0)$$

Hence, for any ϵ, we can pick an n such that $\beta^n \leq (1-\beta)\epsilon/(2\rho(v_1, v_0))$, and then for all $m, m' \geq n$, by the triangle inequality,

$$\begin{aligned}\rho(v^m, v^{m'}) &\leq \rho(v^m, v^n) + \rho(v^n, v^{m'}) \\ &\leq 2\frac{\beta^n}{1-\beta}\rho(v_1, v_0) \\ &\leq \epsilon\end{aligned}$$

And hence v_n is a Cauchy sequence.

Now, because $\{v_n\}$ is Cauchy, and S is complete, the sequence converges: $v_n \to v$ for some $v \in S$. We claim v is our fixed point. By the triangle inequality,

$$\rho(Tv, v) \leq \rho(Tv, T^n x) + \rho(T^n x, v)$$

By the contraction mapping property,

$$\rho(Tv, v) \leq \beta\rho(v, v_{n-1}) + \rho(v_n, v)$$

Since $v_n \to v$, we have that as $n \to \infty$, $\rho(v, v_{n+1}) \to 0$ and $\rho(v_n, v) \to 0$. Hence, taking $n \to \infty$ we get

$$\rho(Tv, v) \leq \beta(0) + 0 = 0$$

Then since ρ is nonnegative, we must have $\rho(Tv, v) = 0$, so $v = Tv$. Hence v is a fixed point.

To finish (a), we now argue that v is unique. Pick some fixed point v'. Then

$$\rho(Tv', Tv) = \rho(v', v)$$

But by contraction mapping property, $\rho(Tv', Tv) \leq \beta\rho(v', v)$, so we have

$$\rho(v', v) = \rho(Tv', Tv) \leq \beta\rho(v', v)$$

$$(\beta - 1)\rho(v', v) \geq 0$$

But we know $\beta < 1$, so in order for this to be true, we must have

$$\rho(v', v) \leq 0$$

But ρ is nonnegative, so we must have $\rho(v', v) = 0$ and hence $v' = v$. Hence the only fixed point is v.

For part (b), we proceed by induction. We can trivially confirm that for $n = 0$, $\rho(v_0, v) \leq \beta^0\rho(v_0, v)$. Suppose the inductive hypothesis holds for $n-1$. Then by the contraction mapping property, since $Tv = v$, we get

$$\rho(T^n v_0, v) = \rho(T^n v_0, Tv) \leq \beta\rho(T^{n-1}v_0, v) \leq \beta(\beta^{n-1}\rho(v_0, v))$$

where we used the inductive hypothesis in the last step. This implies

$$\rho(T^n v_0, v) \leq \beta^n\rho(v_0, v)$$

and we are done.

Now, we prove Theorem 3.3:

Statement: Let $X \subseteq \mathbb{R}^l$, and let $B(X)$ be the space of bounded functions $f : X \to \mathbb{R}$ under the sup norm. Let $T : B(X) \to B(X)$ be an operator satisfying:

(a) $\forall f, g \in B(X)$ such that $f(x) \leq g(x) \forall x \in X$, $(Tf)(x) \leq (Tg)(x) \forall x \in X$.

(b) $\exists \beta \in (0, 1)$ such that $\forall f \in B(X), a \geq 0, x \in X$,

$$(T(f + a))(x) \leq (Tf)(x) + \beta a$$

Then T is a contraction with modulus β.

Proof: Since

$$\rho(f, g) = \sup_x |f(x) - g(x)|$$

$$g(x) + \rho(f, g) = g(x) + \sup_x |f(x) - g(x)| \geq g(x) + \sup_x f(x) - g(x) \geq g(x) + (f(x) - g(x)) = f(x)$$

Symmetrically,

$$f(x) + \rho(f, g) = f(x) + \sup_x |f(x) - g(x)| \geq f(x) + \sup_x g(x) - f(x) \geq f(x) + (g(x) - f(x)) = g(x)$$

Since this holds for all x, we can apply condition (a) on T to get:

$$(T(g + \rho(f, g)))(x) \geq (Tf)(x)$$

$$(T(f + \rho(f, g)))(x) \geq (Tg)(x)$$

Applying condition 2, we have

$$(Tg)(x) + \beta\rho(f, g) \geq (T(g + \rho(f, g)))(x) \geq (Tf)(x)$$

$$(Tf)(x) + \beta\rho(f, g) \geq (T(f + \rho(f, g)))(x) \geq (Tg)(x)$$

Rearranging, we have

$$(Tf)(x) - (Tg)(x) \leq \beta\rho(f, g)$$

$$(Tg)(x) - (Tf)(x) \leq \beta\rho(f, g)$$

Then

$$\sup_x |(Tf)(x) - (Tg)(x)| \leq \beta\rho(f, g)$$

$$\rho(Tf, Tg) \leq \beta\rho(f, g)$$

and hence T is a contraction mapping with modulus β.

(3)

(4.6) Statement: Let $X \subseteq \mathbb{R}^l$ be convex. Let the correspondence $\Gamma : X \to X$ be nonempty, compact-valued, and continuous. Define $A = \{(x,y) \in X \times X : y \in \Gamma(x)\}$. Let $F : A \to \mathbb{R}$ be continuous and bounded, and let $\beta \in (0,1)$. Let $C(X)$ be the space of continuous, bounded functions $X \to \mathbb{R}$ under the sup norm.

Then the operator T defined as $Tf(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta f(y)$ maps $C(X)$ into itself, has a unique fixed point $v \in C(X)$, and for all $v_0 \in C(X)$,

$$||T^n v_0 - v|| \leq \beta^n ||v_0 - v||$$

Further, the optimal policy correspondence $G_v : X \to X$ defined by $G(x) = \{y \in \Gamma(x) : \; v(x) = F(x,y) + \beta v(y)\}$ is compact-valued and continuous.

Proof: We first show that for any $f \in C(X)$, Tf is bounded and continuous. Note that since F and f are both bounded and Γ is compact valued (and hence is bounded-valued), we have that Tf must also be bounded. Also, since F and f are both continuous, and Γ is compact-valued and continuous, by Berge's theorem of the maximum we know Tf is continuous. Therefore, $Tf \in C(X)$ since it is bounded and continuous.

We now show T satisfies the conditions for Theorem 3.3 that we proved in the previous problem part. We first check monotonicity. Suppose $f(x) \leq g(x) \forall x$:

$$(Tf)(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta f(y) \leq \max_{y \in \Gamma(x)} F(x,y) + \beta g(y) = (Tg)(x)$$

Now we show discounting:

$$(T(f+a))(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta(f(y) + a) = \max_{y \in \Gamma(x)} (F(x,y) + \beta f(y)) + \beta a$$

$$\leq (Tf)(x) + \beta a$$

Therefore, we know by Theorem 3.3 that T is a contraction mapping with modulus β. By theorem 3.2 we proved in the previous problem, we have that T has a unique fixed point $v \in C(X)$, and further that

$$||T^n v_0 - v|| \leq \beta^n ||v_0 - v||$$

for all $v_0 \in C(X)$.

Lastly, by Berge's theorem of the maximum, the maximizer correspondence G is compact-valued and continuous.

(4.7) Statement: Let $X \subseteq \mathbb{R}^l$ be convex. Let the correspondence $\Gamma : X \to X$ be nonempty, compact-valued, continuous and monotone; for $x \leq x'$, $\Gamma(x) \subseteq \Gamma(x')$. Define $A = \{(x,y) \in X \times X : y \in \Gamma(x)\}$. Let $F : A \to \mathbb{R}$ be continuous, bounded, and strictly increasing in its first l arguments, and let $\beta \in (0,1)$. Then the solution to

$$v(x) = \max_{y \in \Gamma(x)} (F(x,y) + \beta v(y))$$

is strictly increasing.

Proof: We know from theorem 4.6 we proved previously that v is the unique fixed point of T, which takes

$$(Tf)(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta f(y)$$

Suppose f is a nondecreasing function. Then if $x < x'$, since F is strictly increasing in the first l arguments,

$$(Tf)(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta f(y) < \max_{y \in \Gamma(x)} F(x',y) + \beta f(y)$$

Since Γ is monotone,

$$\max_{y \in \Gamma(x)} F(x',y) + \beta f(y) \leq \max_{y \in \Gamma(x')} F(x',y) + \beta f(y) = (Tf)(x')$$

Hence $(Tf)(x) < (Tf)(x')$, so (Tf) is strictly increasing. Hence, if we pick v_0 to be a strictly increasing function, we have the sequence $\{T^n v_0\}$ consists of nondecreasing functions, which is a closed set. Hence by theorem 3.2 we showed, the sequence converges to v, and by closure of the set of nondecreasing functions, we know v is a nondecreasing function. However, $v = Tv$, so by what we showed, $Tv = v$ must be strictly increasing. Hence we are done.

(4.8) **Statement**: Let $X \subseteq \mathbb{R}^l$ be convex. Let the correspondence $\Gamma : X \to X$ be nonempty, compact-valued, and continuous. Define $A = \{(x,y) \in X \times X : y \in \Gamma(x)\}$. Let $F : A \to \mathbb{R}$ be continuous, bounded, and strictly concave, and let $\beta \in (0,1)$. Finally, let $\Gamma(x)$ be such that $\forall y \in \Gamma(x)$, $y' \in \Gamma x'$, $\theta y + (1-\theta)y' \in \Gamma(\theta x + (1-\theta)x')$. Then the solution to

$$v(x) = \max_{y \in \Gamma(x)} (F(x,y) + \beta v(y))$$

and the corresponding maximizer

$$G(x) = \{y \in \Gamma(x) : \ v(x) = F(x,y) + \beta v(y)\}$$

are such that v is strictly concave and G is continuous and single-valued.

Proof: We know from theorem 4.6 we proved previously that v is the unique fixed point of T, which takes

$$(Tf)(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta f(y)$$

Suppose f is a weakly concave function. Let y be such that $Tf(x) = F(x,y) + \beta f(y)$, y' such that $Tf(x') = F(x',y'), \beta f(y')$ Then by concavity of F, since $\lambda y + (1-\lambda)y' \in \Gamma(\lambda x + (1-\lambda)x')$, we get

$$(Tf)(\lambda x + (1-\lambda)x') \geq F(\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y') + \beta f(\lambda y + (1-\lambda)y')$$

$$> \lambda F(x, \lambda y + (1-\lambda)y') + (1-\lambda)F(x', \lambda y + (1-\lambda)y') + \beta f(\lambda y + (1-\lambda)y')$$

By weak concavity of f,

$$\geq \lambda F(x,y) + \lambda \beta f(y) + (1-\lambda)F(x',y) + (1-\lambda)\beta f(y)$$

$$= \lambda(Tf)(x) + (1-\lambda)(Tf)(x')$$

Hence Tf is strictly concave. By the same logic in theorem 4.8, if we pick a strictly concave v_0, we have the sequence $\{T^n v_0\}$ consists of weakly concave functions, which is a closed set. Hence by theorem 3.2 we showed, the sequence converges to v, and by closure of the set of weakly concave functions, we know v is a weakly concave function. However, $v = Tv$, so by what we showed, $Tv = v$ must be strictly concave.

Finally, we must show G is single valued. Suppose $y \neq y' \in G(x)$. Then

$$v(x) = F(x,y) + \beta v(y) = F(x,y') + \beta v(y')$$

Then by strict concavity of F and v, $y'' = (y'+y)/2$ must satisfy

$$F(x,y'') + \beta v(y'') \geq \frac{1}{2}(F(x,y) + \beta v(y)) + \frac{1}{2}(F(x,y') + \beta v(y')) = v(x)$$

which contradicts the maximization of v. Hence, no such pair y, y' exist, and therefore G is single-valued. By the theorem of the maximum, G is upper hemicontinuous, and since any upper hemicontinuous, single-valued correspondence is continuous, G is continuous.

(4)

(a) Let f be bounded. We need to show Tf is also bounded. Then

$$Tf(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta f(y)$$

Now, since F is bounded, f is bounded, and $\Gamma(x)$ is compact and hence also bounded, we must have $Tf(x)$ is also bounded for all x, so Tf is bounded. Hence $T : B(X) \to B(X)$. We then confirm that if $f(x) \leq g(x) \forall x$, then

$$Tf(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta f(y) \leq \max_{y \in \Gamma(x)} F(x,y) + \beta g(y) = Tg(x)$$

so we have monotonicity. We then check discounting:

$$(T(f+a))(x) = \max_{y \in \Gamma(x)} F(x,y) + \beta(f(y)+a) = \max_{y \in \Gamma(x)} (F(x,y) + \beta f(y)) + \beta a$$

$$\leq (Tf)(x) + \beta a$$

So we have both discounting and monotonicity, so we satisfy the Blackwell conditions, so by Theorem 3.3, T is a contraction mapping, and by theorem 3.2, T has a unique fixed point v, and for any $v_0 \in B(X)$,

$$||T^n v_0 - v|| \leq \beta^n ||v_0 - v||$$

Lastly, we see by the theorem of the maximum that the maximizer correspondence G is nonempty, since Γ is finite-valued and nonempty.

(b) It suffices to show T_h is a contraction mapping. We do this by using the Blackwell condition and theorem 3.3. We first check monotonicity. Suppose $f(x) \leq g(x) \forall x$. Then

$$(T_h f)(x) = F(x, h(x)) + \beta f(h(x)) \leq F(x, h(x)) + \beta g(h(x)) = (T_h g)(x)$$

Now, we check discounting:

$$(T_h(f+a))(x) = F(x, h(x)) + \beta(f(h(x)) + a) = F(x, h(x)) + \beta f(h(x)) + \beta a \leq (T_h f)(x) + \beta a$$

Hence T_h is a contraction mapping by theorem 3.3, and so it has a unique fixed point w.

(c) Consider w_n. We have, $\forall x$,

$$w_n(x) = F(x, h_n(x)) + \beta w_n(h_n(x)) \leq \max_{y \in \Gamma(x)} F(x, y) + \beta w_n(y) = (T w_n)(x)$$

Further, since $h_{n+1}(x) \in \arg\max_{y \in \Gamma(x)} F(x, y) + \beta w_n(y)$, we have that

$$(T w_n)(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta w_n(y) = F(x, h_{n+1}(x)) + \beta w_n(h_{n+1}(x)) = (T_{h_{n+1}} w_n)(x)$$

But $w_{n+1}(x) = F(x, h_{n+1}(x)) + \beta w_{n+1}(h_{n+1}(x))$, so

$$(T w_n)(x) = (T_{h_{n+1}} w_n)(x)$$

Since $w_n \leq T w_n$, we have by monotonicity,

$$T_{h_{n+1}} w_n \leq T_{h_{n+1}}(T w_n) = T_{h_{n+1}}^2 w_n$$

Repeating, we get

$$T_{h_{n+1}} w_n \leq T_{h_{n+1}}^2 w_n \leq T_{h_{n+1}}^3 w_n ... \leq T_{h_{n+1}}^N w_n$$

Using the fact that we showed $T_{h_{n+1}} w_n = T w_n$, we have

$$T w_n \leq T_{h_{n+1}}^N w_n$$

for all N. But as $N \to \infty$, by contraction mapping theorem, the RHS approaces w_{n+1}. Hence we have

$$T w_n \leq w_{n+1}$$

Now, we note that $T^n w_0 \geq T^{n-1} w_1 \geq ... \geq w_n$. Note that as $n \to \infty$, the LHS approaches v, by contraction mapping. Hence, $v \geq w_n$ as $n \to \infty$, so by our monotonicity demonstrations, $v \geq w_n$ for all n. Now, by the monotonicity conditions we showed ($T w_n \leq w_{n+1}$ and $w_n \leq T w_n$),

$$||v - w_n|| \leq ||v - T w_{n-1}||$$

$$\leq ||v - T^2 w_{n-2}||$$

$$\leq ||v - T^n w_0||$$

But by the contraction mapping theorem, $||v - T^n w_0|| \leq \beta^n ||v - w_0||$. So

$$||v - w_n|| \leq \beta^n ||v - w_0||$$

Hence, for any ϵ, we can always pick an N such that $\beta^N ||v - w_0|| \leq \epsilon$, and then for all $n \geq N$, $||v - w_n|| \leq \epsilon$. Hence w_n converges to v, the unique fixed point of T by contraction mapping.

Problem 2

(1) Let us guess the value function has form:

$$V(k) = m \log k + n$$

Then the maximization problem is

$$\max \log(\theta k^\alpha - k') + \beta V(k')$$

$$\max \log(\theta k^\alpha - k') + \beta m \log k' + \beta n$$

This problem has the FOC:

$$\frac{\beta m}{k'} - \frac{1}{\theta k^\alpha - k'} = 0$$

$$\frac{\theta k^\alpha - k'}{k'} = \frac{1}{\beta m}$$

$$\frac{\theta k^\alpha}{k'} = \frac{\beta m + 1}{\beta m}$$

$$k' = \frac{\beta m \theta k^\alpha}{\beta m + 1}$$

Plugging this in, we get

$$m \log k + n = \log\left(\theta k^\alpha - \frac{\beta m \theta k^\alpha}{\beta m + 1}\right) + \beta m \log\left(\frac{\beta m \theta k^\alpha}{\beta m + 1}\right) + \beta n$$

$$m \log k + n = \log\left(\frac{\theta k^\alpha}{\beta m + 1}\right) + \beta m \log(\beta m) + \beta m \log\left(\frac{\theta k^\alpha}{\beta m + 1}\right) + \beta n$$

$$m \log k + n = (\beta m + 1) \log\left(\frac{\theta k^\alpha}{\beta m + 1}\right) + \beta m \log(\beta m) + \beta n$$

$$m \log k + n = \alpha(\beta m + 1) \log k + (\beta m + 1) \log\left(\frac{\theta}{\beta m + 1}\right) + \beta m \log(\beta m) + \beta n$$

Matching the coefficient of $\log k$, we get

$$m = \alpha(\beta m + 1)$$

$$m - \alpha\beta m = \alpha$$

$$m = \frac{\alpha}{1-\alpha\beta}$$

Last, we find n by matching constant terms and using our expression for m:

$$n = (\beta m + 1)\log\left(\frac{\theta}{\beta m + 1}\right) + \beta m \log(\beta m) + \beta n$$

$$(1-\beta)n = \left(\frac{1}{1-\alpha\beta}\right)\log\left(\theta(1-\alpha\beta)\right) + \frac{\alpha\beta}{1-\alpha\beta}\log\left(\frac{\alpha\beta}{1-\alpha\beta}\right)$$

$$(1-\beta)(1-\alpha\beta)n = \log\theta + \log(1-\alpha\beta) + \alpha\beta\log\alpha\beta - \alpha\beta\log(1-\alpha\beta)$$

$$(1-\beta)(1-\alpha\beta)n = \log\theta + (1-\alpha\beta)\log(1-\alpha\beta) + \alpha\beta\log\alpha\beta$$

$$n = \frac{\log\left(\theta(1-\alpha\beta)^{1-\alpha\beta}(\alpha\beta)^{\alpha\beta}\right)}{(1-\beta)(1-\alpha\beta)}$$

All together, the value function is:

$$V(k) = m\log k + n = \frac{\alpha}{1-\alpha\beta}\log k + \frac{\log\left(\theta(1-\alpha\beta)^{1-\alpha\beta}(\alpha\beta)^{\alpha\beta}\right)}{(1-\beta)(1-\alpha\beta)}$$

And the policy function is

$$g(k) = k' = \frac{\beta m\theta k^{\alpha}}{\beta m + 1} = \alpha\beta\theta k^{\alpha}$$

(2) We could go ahead and prove contraction mapping (which guarantees unique steady state and convergence) but we present a simpler argument. Let k_n be the sequence of capital choices, where $k_n = g(k_{n-1})$. We know from the previous part that the steady state is given by

$$g(k_{ss}) = \alpha\beta\theta k_{ss}^{\alpha} = k_{ss}$$

We then note that

$$\frac{k_n}{k_{ss}} = \frac{\alpha\beta\theta k_{n-1}^{\alpha}}{\alpha\beta\theta k_{ss}^{\alpha}} = \left(\frac{k_{n-1}}{k_{ss}}\right)^{\alpha}$$

Chaining this, we get

$$\frac{k_n}{k_{ss}} = \left(\frac{k_0}{k_{ss}}\right)^{\alpha^n}$$

Since $\alpha < 1$, as $n \to \infty$, $\alpha^n \to 0$, and hence $k_n/k_{ss} \to (k_0/k_{ss})^0 = 1$. Therefore, as $n \to \infty$, the sequence k_n approaches the steady state.

(3) Rewriting, we get

$$V(k) = \max_{l,k'} \log(\theta k^{\alpha} l^{1-\alpha} - k') + \log(1-l) + \beta V(k')$$

We once again try $V(k) = m\log k + n$. We get

$$\max_{l,k'} \log(\theta k^{\alpha} l^{1-\alpha} - k') + \log(1-l) + \beta m \log k' + \beta n$$

The FOCs are:

$$\frac{\beta m}{k'} - \frac{1}{(\theta k^{\alpha} l^{1-\alpha} - k')} = 0$$

$$\frac{(1-\alpha)\theta k^{\alpha} l^{-\alpha}}{(\theta k^{\alpha} l^{1-\alpha} - k')} - \frac{1}{1-l} = 0$$

Solving, the first one gives us

$$\frac{(\theta k^{\alpha} l^{1-\alpha} - k')}{k'} = \frac{1}{\beta m}$$

$$\frac{k'}{\theta k^{\alpha} l^{1-\alpha}} = \frac{\beta m}{1+\beta m}$$

$$k' = \frac{\beta m \theta k^{\alpha} l^{1-\alpha}}{1+\beta m}$$

The second gives

$$\frac{(1-\alpha)\theta k^{\alpha}}{(\theta k^{\alpha} l - k' l^{\alpha})} = \frac{1}{1-l}$$

Plugging in k', we get

$$\frac{(1-\alpha)\theta k^{\alpha}}{\theta k^{\alpha} l - \frac{\beta m \theta k^{\alpha} l^{1-\alpha}}{1+\beta m} l^{\alpha}} = \frac{1}{1-l}$$

$$\frac{(1-\alpha)(1+\beta m)}{l} = \frac{1}{1-l}$$

$$\frac{1}{(1-\alpha)(1+\beta m)} = \frac{1-l}{l} = \frac{1}{l} - 1$$

$$\frac{1+(1-\alpha)(1+\beta m)}{(1-\alpha)(1+\beta m)} = \frac{1}{l}$$

$$l = \frac{(1-\alpha)(1+\beta m)}{1+(1-\alpha)(1+\beta m)}$$

$$k' = \frac{\beta m \theta k^{\alpha} l^{1-\alpha}}{1+\beta m}$$

Plugging into the overall expression for $V(k)$, we get

$$V(k) = \log\left(\theta k^{\alpha} l^{1-\alpha} - \frac{\beta m \theta k^{\alpha} l^{1-\alpha}}{1+\beta m}\right) + \log(1-l) + \beta m \log\left(\frac{\beta m \theta k^{\alpha} l^{1-\alpha}}{1+\beta m}\right) + \beta n$$

$$m \log k + n = \log k^{\alpha} + \log\left(\frac{\theta l^{1-\alpha}}{1+\beta m}\right) + \log(1-l) + \beta m \log \beta m + \beta m \log k^{\alpha} + \beta m \log\left(\frac{\theta l^{1-\alpha}}{1+\beta m}\right) + \beta n$$

Matching the $\log k$ terms, we get

$$m \log k = \log k^{\alpha} + \beta m \log k^{\alpha}$$

$$m = \alpha + \alpha \beta m$$

$$m = \frac{\alpha}{1-\alpha\beta}$$

$$1 + \beta m = \frac{1}{1-\alpha\beta}$$

Then our expression for l becomes

$$l = \frac{(1-\alpha)(1+\beta m)}{1+(1-\alpha)(1+\beta m)}$$
$$= \frac{(1-\alpha)}{(1-\alpha\beta)+(1-\alpha)}$$

We note that optimal labor choice is independent of capital. Our policy function is then

$$g(k) = \frac{\beta m \theta k^\alpha l^{1-\alpha}}{1+\beta m} = \beta\alpha\theta k^\alpha \left(\frac{(1-\alpha)}{(1-\alpha\beta)+(1-\alpha)}\right)^{1-\alpha}$$

Lastly, we solve for the constant term in the value function.

$$n = \log\left(\frac{\theta l^{1-\alpha}}{1+\beta m}\right) + \log(1-l) + \beta m \log \beta m + \beta m \log\left(\frac{\theta l^{1-\alpha}}{1+\beta m}\right) + \beta n$$

$$n(1-\beta) = \log\left(\theta l^{1-\alpha}(1-\alpha\beta)\right) + \log(1-l) + \beta m \log \beta m + \beta m \log\left(\theta l^{1-\alpha}(1-\alpha\beta)\right)$$

$$n(1-\beta)(1-\alpha\beta) = \log\left(\theta l^{1-\alpha}(1-\alpha\beta)\right) + (1-\alpha\beta)\log(1-l) + \alpha\beta\log\alpha\beta - \alpha\beta\log(1-\alpha\beta)$$

$$n(1-\beta)(1-\alpha\beta) = \log\theta + (1-\alpha)\log l + (1-\alpha\beta)\log(1-l) + \alpha\beta\log\alpha\beta + (1-\alpha\beta)\log(1-\alpha\beta)$$

$$= \log\theta + (1-\alpha)\log\left(\frac{(1-\alpha)}{(1-\alpha\beta)+(1-\alpha)}\right) + (1-\alpha\beta)\log\left(\frac{(1-\alpha\beta)}{(1-\alpha\beta)+(1-\alpha)}\right) + \alpha\beta\log\alpha\beta + (1-\alpha\beta)\log(1-\alpha\beta)$$

$$= \log\theta + (1-\alpha)\log(1-\alpha) - (2-\alpha-\alpha\beta)\log(2-\alpha-\alpha\beta) + \alpha\beta\log\alpha\beta + 2(1-\alpha\beta)\log(1-\alpha\beta)$$

$$n(1-\beta)(1-\alpha\beta) = \log\left(\frac{\theta(1-\alpha)^{1-\alpha}(\alpha\beta)^{\alpha\beta}(1-\alpha\beta)^{2-2\alpha\beta}}{(2-\alpha-\alpha\beta)^{2-\alpha-\alpha\beta}}\right)$$
$$n = \frac{1}{(1-\beta)(1-\alpha\beta)}\log\left(\frac{\theta(1-\alpha)^{1-\alpha}(\alpha\beta)^{\alpha\beta}(1-\alpha\beta)^{2-2\alpha\beta}}{(2-\alpha-\alpha\beta)^{2-\alpha-\alpha\beta}}\right)$$

And so our value function is

$$V(k) = \frac{1}{1-\alpha\beta}\log k + \frac{1}{(1-\beta)(1-\alpha\beta)}\log\left(\frac{\theta(1-\alpha)^{1-\alpha}(\alpha\beta)^{\alpha\beta}(1-\alpha\beta)^{2-2\alpha\beta}}{(2-\alpha-\alpha\beta)^{2-\alpha-\alpha\beta}}\right)$$

(4) See separate file for code. The analytical result in 3 matches the numerical result from running the code.