Problem Set 3

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Note: I use bold symbols to denote vectors and nonbolded symbols to denote scalars. I primarily use vector notation to shorthand some of the sums, since many of the sums are dot products.

Problem 1

(1) The optimal time length of schooling maximizes:

$$\max_S \int_S^\infty e^{-rt} w(t) h(t)$$

$$\max_S \int_S^\infty e^{-(r-g)t} S^\alpha$$

$$\max_S \frac{S^\alpha w(0) e^{-(r-g)S}}{r-g}$$

$$\frac{\alpha S^{\alpha-1}}{S^\alpha} = r - g$$

$$S = \frac{\alpha}{r-g} = 11.25$$

(2) The present value of consumption is

$$\int_S^\infty e^{-rt} c_t = \int_S^\infty w(0) e^{-(r-g)t} S^\alpha$$

$$= \frac{w(0) e^{-(r-g)S} S^\alpha}{r-g}$$

(3) We need

$$\max \int e^{-(\rho+\nu)t} \log c(t)$$

subject to

$$\int e^{-rt} c(t) = \frac{w(0) e^{-(r-g)S} S^\alpha}{r-g}$$

we have

$$e^{-(\rho+\nu)t} \frac{1}{c(t)} = \lambda e^{-rt}$$

$$\frac{1}{\lambda}e^{-(\rho+\nu-r)t} = c(t)$$

and we get the EE:

$$\frac{\dot{c}(t)}{c(t)} = r - \rho - \nu$$

which implies

$$c(t) = c(0)e^{(r-\rho-\nu)t}$$

Plugging into budget constraint:

$$\frac{c(0)}{\rho+\nu} = \frac{w(0)e^{-(r-g)S}S^{\alpha}}{r-g}$$

$$c(0) = \frac{w(0)(\rho+\nu)e^{-(r-g)S}S^{\alpha}}{r-g}$$

(4) Intuitively, since consumption is smoothed across time, individuals will go into debt before they reach their optimal level of schooling. Specifically, at time $T < S$, we have the present value of an individual's asset holding is given by

$$-e^{rT}\int_0^T e^{-rt}c(t)$$

which is clearly negative. However, after the individual finishes schooling, the present value of asset holding is

$$e^{rT}\left(\int_S^T e^{-rt}W_t dt - \int_0^T e^{-rt}c_t\right)$$

$$= e^{rT}\left(w(0)S^{\alpha}\int_S^T e^{-(r-g)t}dt - c(0)\int_0^T e^{-(\rho+\nu)t}\right)$$

$$= e^{rT}\left(\frac{w(0)S^{\alpha}}{r-g}\left(e^{-(r-g)T} - e^{-(r-g)S}\right) - \frac{w(0)(\rho+\nu)e^{-(r-g)S}S^{\alpha}}{r-g}\left(\frac{e^{-(\rho+\nu)T}-1}{\rho+\nu}\right)\right)$$

$$= \frac{w(0)S^{\alpha}e^{rT}}{r-g}\left(\left(e^{-(r-g)T} - e^{-(r-g)S}\right) - e^{-(r-g)S}\left(e^{-(\rho+\nu)T}-1\right)\right)$$

$$= \frac{w(0)S^{\alpha}e^{rT}}{r-g}\left(e^{-(r-g)T} - e^{-(r-g)S}\left(e^{-(\rho+\nu)T}\right)\right)$$

$$= \frac{w(0)S^{\alpha}e^{rT}e^{-(r-g)S}}{r-g}\left(e^{-(r-g)(T-S)} - e^{-(\rho+\nu)T}\right)$$

Since $r - g > \rho + \nu$, this will eventually turn positive at

$$e^{-(r-g)(T-S)+(\rho+\nu)T} = 1$$

$$(r-g)(T-S) = (\rho+\nu)T$$

$$(r-g-\rho-\nu)T = (r-g)S$$

$$T = \frac{(r-g)S}{(r-g-\rho-\nu)}$$

so after that point, the individual paid of their student loans and will begin accummulating wealth.

(5) If we take the log of the wage rate, we find that

$$\log W(t) = \log w(0) + \alpha \log S + gt$$

which we can use to empirically measure the schooling effectivness parameter α.

(6) Imposing a borrowing constraint can induce people to take less schooling than they would otherwise, since due to a borrowing constraint they would have to reduce consumption as a result. Alternatively, individuals might also have to instead work first, and take schooling later in their life. One potential way to alleviate this issue is to provide students a stipend, funded from wages. That way, consumers may be able to still achieve the same welfare since they receive some income during their schooling.

Problem 2

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Problem 3

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Problem 4

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Problem 5

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