Problem Set 5

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Note: I use bold symbols to denote vectors and nonbolded symbols to denote scalars. I primarily use vector notation to shorthand some of the sums, since many of the sums are dot products.

Problem 1

(1) Let $s_0 = (3, 1)$, and $s_1 = (1, 3)$. Define the set $S = \{s_0, s_1\}$. Then a history is given by a sequence:

$$s_t = \{s_j\}_{i=1}^t$$

for $j \in \{0, 1\}$. Specifically, the set of histories S^t is given by

$$S^t = S \times S^{t-1}$$

where $S^0 = S$. Then the probability

$$\pi(s^t) = p^k(1-p)^{t-k}$$

where k is the number of s_0's in the sequence s_t.

(2) In the AD equilibrium, players trade the good at all time periods and states. An equilibrium thus consists of consumption sequences $\{\{c^1(s^t), c^2(s^t)\}_{s^t}\}_{t=0}^\infty$, prices $\{\{p(s^t)\}_{s^t}\}_{t=0}^\infty$ such that

- Each consumer $i \in \{A, B\}$'s consumption solves the utility maximization at the prices:

$$\max \sum_{t=0}^\infty \beta^t \sum_{s^t \in S^t} \pi(s^t) \log(c^i(s^t))$$

  subject to

$$\sum_{t=0}^\infty \sum_{s^t \in S^t} p(s^t) c^i(s^t) \leq \sum_{t=0}^\infty \sum_{s^t \in S^t} p(s^t) \omega^i(s^t)$$

- Markets clear: for all t, s^t,

$$c^A(s^t) + c^B(s^t) = 4$$

(3) Solving the FOC for the consumer, we have

$$\beta^t \frac{\pi(s^t)}{c^i(s^t)} - \lambda p(s^t) = 0$$

$$\lambda = \beta^t \frac{\pi(s^t)}{p(s^t)c^i(s^t)}$$

At $t = 0$, there is a fixed start state history, so we have

$$\lambda = \frac{1}{c^i(s^0)}$$

Then we have

$$c^i(s^t) = \beta^t \pi(s^t) \frac{c^i(s_0)}{p(s^t)}$$

Summing i and using the market clearing condition, we get

$$4 = \beta^t \pi(s^t) \frac{4}{p(s^t)}$$

$$p(s^t) = \beta^t \pi(s^t)$$

(4) Using the prices derived in the previous part, we get

$$c^i(s^t) = c^i(s_0)$$

And hence consumption is constant over time. Now, note that since the endowment probability is independent of history:

$$\sum_{s^t} p(s^t)\omega^A(s^t) = \sum_{s^t} \beta^t \pi(s^t)\omega^A(s^t) = \beta^t(p(3) + (1-p)1)$$

$$= \beta^t(1 + 2p)$$

Likewise

$$\sum_{s^t} p(s^t)\omega^B(s^t) = \sum_{s^t} \beta^t \pi(s^t)\omega^B(s^t) = \beta^t(p(1) + (1-p)(3))$$

$$= \beta^t(3 - 2p)$$

Then from the budget constraint:

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} p(s^t)c^A(s^t) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} p(s^t)\omega^A(s^t)$$

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t)c^A(s^0) = 3 + \sum_{t=1}^{\infty} \beta^t(1 + 2p)$$

$$c^A(s^0) \sum_{t=0}^{\infty} \beta^t \sum_{s^t \in S^t} \pi(s^t) = 3 + (1 + 2p) \sum_{t=1}^{\infty} \beta^t$$

$$\frac{c^A(s^0)}{1-\beta} = 3 + \frac{\beta(1+2p)}{1-\beta}$$

$$c^A(s^t) = c^A(s^0) = 3(1-\beta) + \beta(1+2p) = 3 - 2\beta(1-p)$$

and

$$c^B(s^t) = 4 - c^A(s^t) = 1 + 2\beta(1-p)$$

Problem 2

(1) We first define a history as an element $s^t \in S^t$, where $S^0 = \{H, L\}$ and

$$S^t = S^{t-1} \times \{H, L\}$$

The probability of a given history is then

$$\pi(s^t) = p\pi(s^{t-1})\mathbb{1}_{s^t_{t-1}=s^t_t} + (1-p)\pi(s^{t-1})\mathbb{1}_{s^t_{t-1}\neq s^t_t}$$

where s^t_k denotes the state at time k in history s^t, and s^{t-1} denotes the truncation of history s^t at time $t-1$, and we use $\mathbb{1}_E$ as an indicator for event E.

Now, we can define an AD equilibrium. An AD equilibrium consists of consumption and capital savings sequences $\{\{c(s^t), k(s^t)\}_{s^t}\}_{t=0}^{\infty}$, prices $\{\{p(s^t), w(s^t), r^k(s^t)\}_{s^t}\}_{t=0}^{\infty}$, firm allocations $\{\{l^f(s^t), k^f(s^t)\}_{s^t}\}_{t=0}^{\infty}$ such that

- Consumers maximize utility:

$$\max_{c,k} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) u(c(s^t))$$

subject to

$$\sum_{t=0}^{\infty} \sum_{s^t} p(s^t)(c(s^t) + k(s^t)) \leq \sum_{t=0}^{\infty} \sum_{s^t} p(s^t)(w(s^t) + (1 + r^k(s^t) - \delta)k(s^{t-1}))$$

$$k(s^{-1}) = \frac{\bar{k}}{1-\delta}$$

(dividing such that the initially available amount of consumption at the initial state is $\bar{k}$).

- Firms maximize profit: l^f, k^f satisfy:

$$\max_{k,l} z(s^t)k^f(s^t)^{\alpha} l^f(s^t)^{1-\alpha} - r^k(s^t)k^f(s^t) - w(s^t)l(s^t)$$

- Markets clear:

$$k^f(s^t) = k(s,^t)$$

$$l^f(s^t) = 1$$

$$c(s^t) + k(s^t) = z(s^t)k(s^{t-1})^{\alpha} + (1-\delta)k(s^{t-1})$$

(2) The social planner problem is to choose capital investment levels $k(s^t)$ such that

$$\max \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) u(z(s^t)k(s^{t-1})^{\alpha} + (1-\delta)k(s^{t-1}) - k(s^t))$$

We start with the AD equilibrium conditions. Taking the FOC of the firm's problem and applying market clearing, we get

$$r^k(s^t) = \alpha z(s^t)k(s^t)^{\alpha-1}$$

$$w(s^t) = (1-\alpha)z(s^t)k(s^t)^{\alpha}$$

Now, we take the FOCs for the consumer problem.

$$\beta^t \pi(s^t)u'(c(s^t)) = p(s^t)$$

For each state s^t, we can take the FOC for $k(s^t)$. Fix z as the state at the end of s^t, and let $z' \neq z$. To clean up notation, let us also define s_z^{t+1} as the history that ends with state z after s^t, and $s_{z'}^{t+1}$ as the history ending with z' after s^t.

$$p(s^t) = p(s_z^{t+1})(r^k(s_z^{t+1}) + 1 - \delta) + p(s_{z'}^{t+1})(r^k(s_{z'}^{t+1}) + 1 - \delta)$$

Plugging in the other FOC, we get

$$\beta^t \pi(s^t)u'(c(s^t)) = \beta^{t+1}\pi(s_z^{t+1})u'(c(s_z^{t+1}))(r^k(s_z^{t+1}) + 1 - \delta) + \beta^{t+1}\pi(s_{z'}^{t+1})u'(c(s_{z'}^{t+1}))(r^k(s_{z'}^{t+1}) + 1 - \delta)$$

Using the recursion on π, we get

$$\frac{1}{\beta}u'(c(s^t)) = pu'(c(s_z^{t+1}))(\alpha z k(s^t)^{\alpha-1} + 1 - \delta) + (1-p)u'(c(s_{z'}^{t+1}))(\alpha z' k(s^t)^{\alpha-1} + 1 - \delta)$$

Now we investigate the social planner problem. The FOCs for the social planner problem are:

$$\beta^{t+1}\pi(s_z^{t+1})u'(zk(s^t)^{\alpha} + (1-\delta)k(s^t) - k(s^{t+1}))(\alpha z k(s^t)^{\alpha-1} + (1-\delta))$$

$$+\beta^{t+1}\pi(s_{z'}^{t+1})u'(z'k(s^t)^{\alpha} + (1-\delta)k(s^t) - k(s^{t+1}))(\alpha z' k(s^t)^{\alpha-1} + (1-\delta))$$

$$-\beta^t \pi(s^t)u'(zk(s^{t-1})^{\alpha} + (1-\delta)k(s^{t-1}) - k(s^t)) = 0$$

Using the recursion on π, we can simplify this

$$\beta p u'(zk(s^t)^{\alpha} + (1-\delta)k(s^t) - k(s^{t+1}))(\alpha z k(s^t)^{\alpha-1} + (1-\delta))$$

$$+\beta(1-p)u'(z'k(s^t)^{\alpha} + (1-\delta)k(s^t) - k(s^{t+1}))(\alpha z' k(s^t)^{\alpha-1} + (1-\delta))$$

$$= u'(zk(s^{t-1})^{\alpha} + (1-\delta)k(s^{t-1}) - k(s^t))$$

Substituting $c(s^t)$ for sake of cleanliness, we get

$$\frac{u'(c(s^t))}{\beta} = pu'(c(s_z^{t+1}))(\alpha z k(s^t)^{\alpha-1} + (1-\delta)) + (1-p)u'(c(s_{z'}^{t+1}))(\alpha z' k(s^t)^{\alpha-1} + (1-\delta))$$

This is exactly the same condition we found for the AD equilibrium. Hence the AD equilibrium allocation is efficient.

(3) The value function needs to know the previous period capital investment and the current state.

$$V(k, z) = \max_{k'} u(zk^\alpha + (1-\delta)k - k') + \beta p V(k', z) + \beta(1-p)V(k', z')$$

subject to

$$0 \leq k' \leq zk^\alpha + (1-\delta)k$$

where $z \neq z'$, $z, z' \in \{z_H, z_L\}$.

(4)

Problem 3

(1) The value function needs to know the current state in the Markov chain:

$$V(k,z) = \max_{k'} u(z + (1+r)k - k') + p\beta V(k', z) + (1-p)\beta V(k', z')$$

where

$$k, k' \geq \underline{b}$$

and $z \neq z', z, z' \in \{z^H, z^L\}$.

(2) See attached code and figures.

(3) See attached code.