Problem Set 2

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Note: I use bold symbols to denote vectors and nonbolded symbols to denote scalars. I primarily use vector notation to shorthand some of the sums, since many of the sums are dot products.

Problem 1

(1) The Bellman equation is given by

$$V(k,z) = \max_{h,k'} \left(u(c, 1-h) + \beta \sum_{z'} \pi(z'|z) V(k', z') \right)$$

subject to

$$c + k' \leq f(k,z,h) + (1-\delta)k$$

(2) Taking FOCs, we find:

$$u_1(c, 1-h) = \beta \sum_{z'} \pi(z'|z) V_1(k', z')$$

$$-u_2(c, 1-h) + u_1(c, 1-h) f_3(k,z,h) z = 0$$

The envelope theorem gives

$$V_1(k,z) = u_1(c, 1-h) \left(1 - \delta + f_1(k,z,h)\right)$$

so we can rewrite the first FOC as

$$u_1(c, 1-h) = \beta \sum_{z'} \pi(z'|z) u_1(c', 1-h') \left(1 - \delta + f_1(k', z', h')\right)$$

(3) No. The FOC determining labor is

$$-u_2(c, 1-h) + u_1(c, 1-h) f_3(k,z,h) = 0$$

and is fully categorized by the current state and capital levels. Hence there is no state uncertainty in determining labor supply.

(4) We guess that the labor law of motion is constant for some h^* and the capital supply follows the functional form:

$$k' = Czk^\alpha h^{1-\alpha}$$

then

$$c = zk^\alpha h^{1-\alpha}(1 - C)$$

The Euler equation gives

$$1/c = \beta \sum_{z'} \pi(z'|z) \frac{\alpha z'(k')^{\alpha-1}(h')^{1-\alpha}}{(1-C)z'(k')^\alpha (h')^{1-\alpha}}$$

$$\frac{1}{zk^\alpha h^{1-\alpha}} = \beta \frac{\alpha}{k'}$$

$$\frac{1}{zk^\alpha h^{1-\alpha}} = \beta \frac{\alpha}{Czk^\alpha h^{1-\alpha}}$$

$$C = \beta\alpha$$

So capital evolves as

$$k' = \alpha\beta zk^\alpha h^{1-\alpha}$$

The labor condition is

$$\gamma g'(h) = \frac{1}{c}(1-\alpha)zk^\alpha h^{-\alpha}$$

$$\gamma g'(h) = \frac{1}{zk^\alpha h^{-\alpha}(1-\alpha\beta)}(1-\alpha)zk^\alpha h^{1-\alpha}$$

$$\gamma g'(h) = \frac{1}{zk^\alpha h^{-\alpha}(1-\alpha\beta)}(1-\alpha)zk^\alpha h^{1-\alpha}$$

$$h^* g'(h^*) = \frac{1-\alpha}{\gamma(1-\alpha\beta)}$$

which should pin down h^*.

(5) There is no labor supply fluctuation; in this model, labor is constant. The change in wages as a result of increases/decreases in technology does not affect the amount of labor supplied.

Problem 2

(1) The Bellman equations are

$$V_H(k) = \max_{k'} [u(z_H k^\alpha + (1-\delta)k - k') + \beta\pi_{HH}V_H(k') + \beta\pi_{HL}V_L(k')]$$

$$V_L(k) = \max_{k'} [u(z_L k^\alpha + (1-\delta)k - k') + \beta\pi_{LH}V_H(k') + \beta\pi_{LL}V_L(k')]$$

(2) We guess

$$V_H(k) = a_H + b_H \log k$$

$$V_L(k) = a_L + b_L \log k$$

Pluggin into the Bellman equations:

$$a_H + b_H \log k = \max_{k'} [\log(z_H k^\alpha + (1-\delta)k - k') + \beta\pi_{HH}(a_H + b_H \log k') + \beta\pi_{HL}(a_L + b_L \log k')]$$

$$a_L + b_L \log k = \max_{k'} [\log(z_L k^\alpha + (1-\delta)k - k') + \beta\pi_{LH}(a_H + b_H \log k') + \beta\pi_{LL}(a_L + b_L \log k')]$$

Taking the maximiation FOC of the first:

$$\frac{1}{z_H k^\alpha + (1-\delta)k - k'} = \beta\pi_{HH}b_H\frac{1}{k'} + \beta\pi_{HL}b_L\frac{1}{k'}$$

$$\frac{k'}{z_H k^\alpha + (1-\delta)k - k'} = \beta(\pi_{HH}b_H + \pi_{HL}b_L)$$

$$k' = \beta(\pi_{HH}b_H + \pi_{HL}b_L)(z_H k^\alpha + (1-\delta)k - k')$$

$$k'(1 + \beta(\pi_{HH}b_H + \pi_{HL}b_L)) = \beta(\pi_{HH}b_H + \pi_{HL}b_L)(z_H k^\alpha + (1-\delta)k)$$

$$k' = \frac{\beta(\pi_{HH}b_H + \pi_{HL}b_L)(z_H k^\alpha + (1-\delta)k)}{1 + \beta(\pi_{HH}b_H + \pi_{HL}b_L)}$$

Plugging into the first FOC, using $\delta = 1$, and isolating only the k terms, we get

$$b_H \log k = \alpha \log k + \alpha\beta(\pi_{HH}b_H + \pi_{HL}b_L)\log k$$

$$b_H = \alpha(1 + \beta(\pi_{HH}b_H + \pi_{HL}b_L))$$

Doing the same the second FOC, we get

$$b_L = \alpha(1 + \beta(\pi_{LH}b_H + \pi_{LL}b_L))$$

Solving, we get

$$b_L = b_H = \frac{\alpha}{1 - \alpha\beta}$$

We can also solve for a_H, a_L, but we can determine the optimal policies and the path of capital from just b. The optimal policies are

$$k'_H = \frac{\beta b_H z_H k^\alpha}{1 + \beta b_H} = \alpha\beta z_H k^\alpha$$

$$k'_L = \frac{\beta b_L z_L k^\alpha}{1 + \beta b_L} = \alpha\beta z_L k^\alpha$$

(3) If the state is always z_L, then capital just converges to the steady state at z_L, which is

$$k^*_L = (\alpha\beta z_L)^{1/(1-\alpha)}$$

(4) We recall from our observation from the last problem set that log utility is the limit case of the CRRA as $\theta = 1$, and the income/substitution effects balance out, and the savings rate is constant. We thus note then that changing π_{HH} and π_{LL} has no effect on the law of motion of capital.

Problem 3

(1)

Problem 4

(1)