ECON 511 Problem Set 2

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Problem 1

The maximization Hamiltonian is

$$H = \pi(k_t, x_t) - p_t^k I_t - p_t^k I_t D(I_t, k_t) + \lambda_t(I_t - \delta k_t)$$

The maximization conditions are:

$$\lambda_t = p_t^k + p_t^k I_t D_I(I_t, k_t) + p_t^k D(I_t, k_t)$$

$$\pi_k(k_t, x_t) - p_t^k I_t D_k(I_t, k_t) - \delta\lambda_t = -\dot{\lambda}_t + r\lambda_t$$

$$\lim_{t\to\infty} \lambda_t e^{-rt} k_t = 0$$

Rearranging, we have

$$\dot{\lambda}_t - r\lambda_t = \delta\lambda_t - \pi_k(k_t, x_t) + p_t^k I_t D_k(I_t, k_t)$$

Using the hint, we consider

$$e^{rt}\frac{d}{dt}[\lambda_t k_t e^{-rt}]$$

By the product rule

$$\begin{aligned} e^{rt}\frac{d}{dt}[\lambda_t k_t e^{-rt}] &= e^{rt}\left(\dot{\lambda}_t k_t e^{-rt} + \lambda_t \dot{k}_t e^{-rt} - r\lambda_t k_t e^{-rt}\right) \\ &= \dot{\lambda}_t k_t + \lambda_t \dot{k}_t - r\lambda_t k_t \\ &= (\dot{\lambda}_t - r\lambda_t)k_t + \lambda_t \dot{k}_t \end{aligned}$$

Plugging in what we know from maximization problem, we have

$$= (\delta\lambda_t - \pi_k(k_t, x_t) + p_t^k I_t D_k(I_t, k_t))k_t + \lambda_t(I_t - \delta k_t)$$

$$= (-\pi_k(k_t, x_t) + p_t^k I_t D_k(I_t, k_t))k_t + \lambda_t I_t$$

$$= -k_t\pi_k(k_t, x_t) + k_t p_t^k I_t D_k(I_t, k_t) + (p_t^k + p_t^k I_t D_I(I_t, k_t) + p_t^k D(I_t, k_t))I_t$$

$$= -k_t\pi_k(k_t, x_t) + p_t^k I_t(k_t D_k(I_t, k_t) + 1 + I_t D_I(I_t, k_t) + D(I_t, k_t))$$

Multiplying both sides by e^{-rt} we get

$$\frac{d}{dt}[\lambda_t k_t e^{-rt}] = \left(-k_t \pi_k(k_t, x_t) + p_t^k I_t(k_t D_k(I_t, k_t) + 1 + I_t D_I(I_t, k_t) + D(I_t, k_t))\right) e^{-rt}$$

Integrating from 0 to ∞ and using transversality, we get

$$-\lambda_0 k_0 = \int_0^\infty \left(-k_t \pi_k(k_t, x_t) + p_t^k I_t(k_t D_k(I_t, k_t) + 1 + I_t D_I(I_t, k_t) + D(I_t, k_t))\right) e^{-rt}\ dt$$

$$\lambda_0 k_0 = \int_0^\infty \left(k_t \pi_k(k_t, x_t) - p_t^k I_t(k_t D_k(I_t, k_t) + 1 + I_t D_I(I_t, k_t) + D(I_t, k_t))\right) e^{-rt}\ dt$$

Then we have marginal q at time zero is

$$q_0 = \frac{\lambda_0}{p_0^k} = \frac{\int_0^\infty \left(k_t \pi_k(k_t, x_t) - p_t^k I_t(k_t D_k(I_t, k_t) + 1 + I_t D_I(I_t, k_t) + D(I_t, k_t))\right) e^{-rt}\ dt}{p_0^k k_0}$$

Average Q is given by

$$Q_0 = \frac{V_0}{p_0^k k_0} = \frac{\int_0^\infty (\pi(k_t, x_t) - p_t^k I_t - p_t^k I_t D(I_t, k_t)) e^{-rt}\ dt}{p_0^k k_0}$$

We want to find the conditions when $q_0 = Q_0$. Obviously, if the integrands are equal, $q_0 = Q_0$. Hence, the forward direction is clear. If the firm satisfies (i), (ii), and (iii), then the integrand of q_0 is

$$\left(k_t \pi_k(k_t, x_t) - p_t^k I_t(k_t D_k(I_t, k_t) + 1 + I_t D_I(I_t, k_t) + D(I_t, k_t))\right) e^{-rt}$$

From (i) and (ii), we get $k_t \pi_k(k_t, x_t) = \pi(k_t, x_t)$,

$$= \left(\pi(k_t, x_t) - p_t^k I_t(k_t D_k(I_t, k_t) + 1 + I_t D_I(I_t, k_t) + D(I_t, k_t))\right) e^{-rt}$$

$$= \left(\pi(k_t, x_t) - p_t^k I_t - p_t^k I_t(k_t D_k(I_t, k_t) + I_t D_I(I_t, k_t) + D(I_t, k_t))\right) e^{-rt}$$

By (iii) and Euler's theorem, we get that $k_t D_k(I_t, k_t) + I_t D_I(I_t, k_t) = 0$, so

$$= \left(\pi(k_t, x_t) - p_t^k I_t - p_t^k I_t D(I_t, k_t)\right) e^{-rt}$$

Hence (i), (ii), and (iii) imply $q_0 = Q_0$.

For the converse, we need to show that if $q_0 = Q_0$, then the conditions (i)-(iii) must hold. Since $q_0 = Q_0$, we must have that this holds regardless of choice of k_0, p_0. Consider π_k. Suppose, for sake of contradiction, that (i) and (ii) do not both hold, and hence π_k is not constant. If $d\pi_k/dk > 0$ there are increasing returns to scale and the value of the firm is unbounded. If $d\pi_k/dk < 0$, then we can find some steady state k^* at price p^* and set $k_0 = k^*$ and $p_0 = p^*$. Then investment will be 0, so the maximization conditions require:

$$\lambda_t = p^*$$

This implies λ_t is constant, so $\dot{\lambda}_t = 0$, and hence

$$\pi_k(k_t, x_t) - \delta\lambda_t = r\lambda_t$$

$$\pi_k(k_t, x_t) = (r + \delta)p^*$$

which is a contradiction of $d\pi_k/dk < 0$. Hence, we need (i) and (ii) to hold.

Finally, we have to show that $q_0 = Q_0$ implies (iii). We know that (i) and (ii) hold as we just showed. This implies $\pi_k(k_t, x_t)k_t = \pi(k_t, x_t)$. Then consider $p_0 k_0(q_0 - Q_0) = 0$. Simplifying the expression, we get

$$0 = \int_0^\infty \left(p_t^k I_t(1 + D(I_t, k_t)) - p_t^k I_t(k_t D_k(I_t, k_t) + 1 + I_t D_I(I_t, k_t) + D(I_t, k_t))\right) e^{-rt}\, dt$$

$$= \int_0^\infty \left(-p_t^k I_t(k_t D_k(I_t, k_t) + I_t D_I(I_t, k_t))\right) e^{-rt}\, dt$$

$$0 = \int_0^\infty p_t^k I_t(k_t D_k(I_t, k_t) + I_t D_I(I_t, k_t)) e^{-rt}\, dt$$

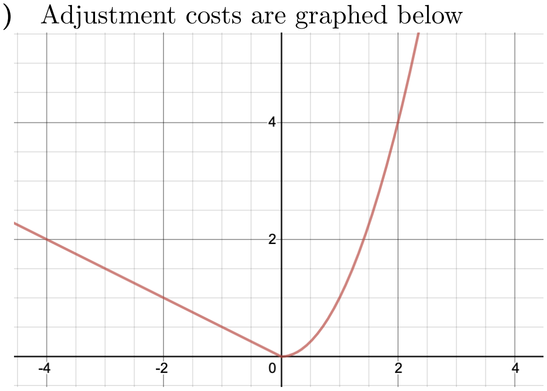
We can pick k_0, p_0 such that investment embarks on a saddle path and $I_t \geq 0$ always as it converges to steady state, and $I_0 > 0$, and we can consider k_0', p_0' such that $I_0 < 0$ and $I_t \leq 0$. The only way that both of the resulting integrals for these two paths to both be zero requires

$$k_t D_k(I_t, k_t) + I_t D_I(I_t, k_t) = 0$$

By Euler's theorem, this implies D is homogeneous of degree 0. Hence (iii) holds and we are done.

Problem 2

(1) Adjustment costs are graphed below



When investment is negative, the adjustment costs are still positive, and hence the sign on the ωI is negative. The firm's optimization problem is given by

$$\max \int_0^\infty \left(\pi(K_t) - p^K I_t - p^K C(I_t)\right) e^{-rt}\, dt$$

subject to

$$\dot{K}_t = I_t$$

(2) The present-value Hamiltonian is given by

$$H = \pi(K_t) - p^K I_t - p^K C(I_t) + \lambda_t I_t$$

The FOC for $I_t > 0$ are then

$$\lambda_t = p^K + 2p^K I_t$$

$$q_t = 1 + 2I_t$$

$$I_t = \frac{q_t - 1}{2}$$

where we have required $q_t > 1$ for this to be positive as assumed. For $I_t < 0$, we have

$$\lambda_t - (p^K - \omega p^K)$$

Since λ_t is constant in this case, if $q_t < (1 - \omega)$, capital will immediately jump.

Finally, for $I_t = 0$, we need $1 - \omega \leq q_t \leq 1$. The firm then does not want to invest further, but also does not disinvest due to adjustment costs.

All together, the investment policy is

$$I(q_t) = \begin{cases} \frac{q_t - 1}{2} & q_t > 1 \\ 0 & 1 - \omega \leq q_t \leq 1 \\ -x & q_t < 1 - \omega \end{cases}$$

where $-x$ is selling investment up to $q_t = 1 - \omega$.

(3) The other FOC:

$$a - K_t = r\lambda_t - \dot{\lambda}_t$$

$$\frac{a - k}{p^K} = rq - \dot{q}$$

$$\dot{k} = I(q)$$

(4) The steady state condition

$$\dot{k} = 0$$

implies $I(q) = 0$ which means $q \in [1 - \omega, 1]$. The steady state

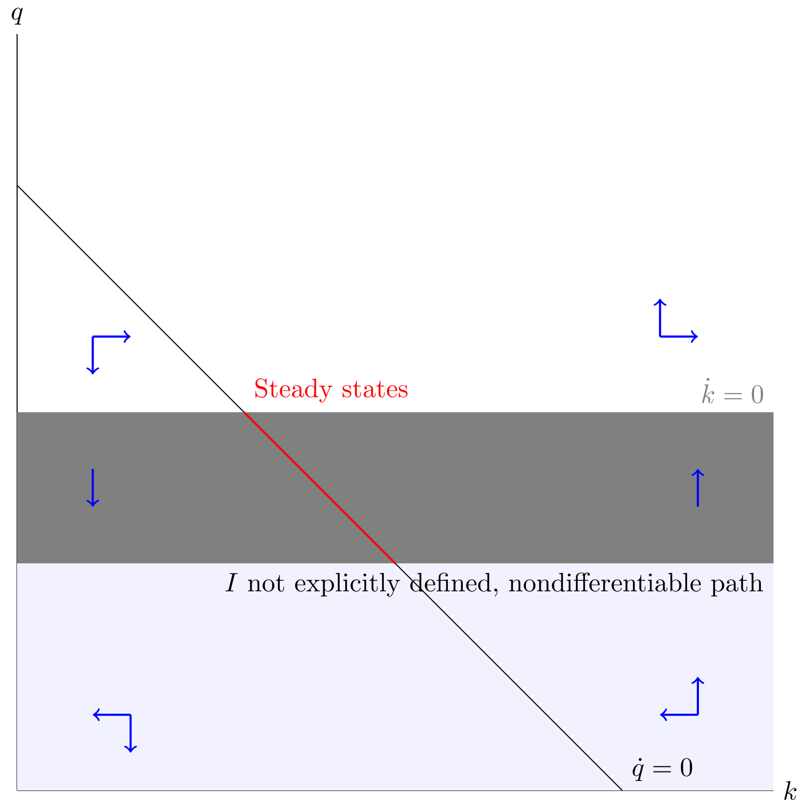
$$\dot{q} = 0$$

implies, with the previous part's FOCs,

$$\frac{a - k}{p^K} = rq$$

$$q = \frac{a-k}{rp^K}$$

Phase diagram on the next page.



(5) If $k = a - rp^k(1-\omega)$, then $\frac{a-k}{rp^k} = 1-\omega = q$. Increasing a shifts the $\dot{q}=0$ line up. If the new a' is such that

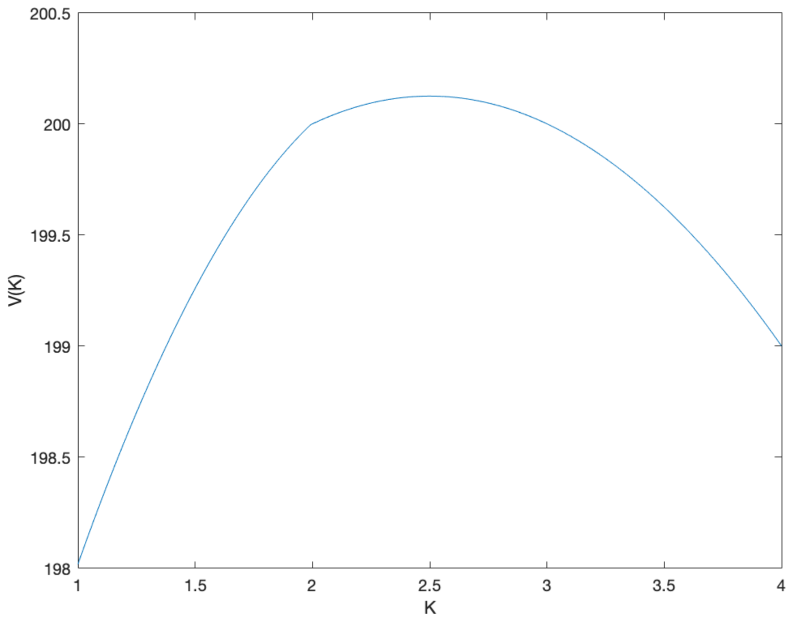
$$\frac{a' - (a - rp^k(1-\omega))}{rp^k} > 1$$

$$a' - a > rp^k\omega$$

Then we jump to the regime above the inaction band and under the $\dot{q}=0$ line, and capital increases to a new steady state. However, if $0 < a' - a < rp^k\omega$, then only the level of q jumps up to a new steady state, and the firm makes no changes in capital. The adjustment costs add this friction to investment; due to the inaction region, the firm does not respond to this positive shock in productivity.

Problem 3

See code. The plot of $V(K)$ is below



The plot of $I(V')$ is below (V' = marginal q)