ECON 511 Problem Set 7

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Problem 1

1 The free entry condition implies $V = 0$ and hence from HJB:

$$0 = -c + q(\theta)J$$

$$\frac{c}{q(\theta)} = J$$

Using the steady state values, we have

$$w = (1-\beta)z + \beta p + \beta\theta$$

$$\begin{aligned}
J &= \frac{p-w}{r+\lambda} \\
&= \frac{p-(1-\beta)z-\beta p+\beta c\theta}{r+\lambda} = \frac{c}{q(\theta)} \\
p-(1-\beta)z-\beta p-\beta c\theta &= \frac{c}{q(\theta)}(r+\lambda) \\
\beta p+\beta c\theta &= p-(1-\beta)z-\frac{c}{q(\theta)}(r+\lambda) \\
\beta c\theta &= (1-\beta)p-(1-\beta)z-\frac{c}{q(\theta)}(r+\lambda) \\
\beta\theta &= (1-\beta)\frac{p-z}{c}-\frac{r+\lambda}{q(\theta)}
\end{aligned}$$

2 Taking the partial derivative wrt p, we get

$$\beta\frac{\partial\theta}{\partial p} = (1-\beta)\frac{1}{c} + \frac{r+\lambda}{q(\theta)^2}\frac{\partial q}{\partial\theta}\frac{\partial\theta}{\partial p}$$

$$\begin{aligned}
\beta\frac{\partial\theta}{\partial p} &= (1-\beta)\frac{1}{c} + \frac{r+\lambda}{\theta q(\theta)}\eta(\theta)\frac{\partial\theta}{\partial p} \\
\left(\beta-\frac{r+\lambda}{\theta q(\theta)}\eta(\theta)\right)\frac{\partial\theta}{\partial p} &= (1-\beta)\frac{1}{c}
\end{aligned}$$

$$\frac{p}{\theta}\frac{\partial \theta}{\partial p} = \frac{1-\beta}{\beta - \frac{r+\lambda}{\theta q(\theta)}\eta(\theta)}\frac{p}{c\theta}$$

$$\epsilon_{\theta,p} = \frac{1-\beta}{\beta - \frac{r+\lambda}{\theta q(\theta)}\eta(\theta)}\frac{p}{c\theta}$$

3 The $\frac{r+\lambda}{\theta q(\theta)}\eta(\theta)$ term is negligible, and β 1/2, so

$$\epsilon_{\theta,p} \sim \frac{p}{c\theta}$$

Since $\frac{r+\lambda}{q(\theta)}$ is also negligible, from part 1,

$$\beta\theta \sim (1-\beta)\frac{p-z}{c}$$

$$\theta \sim \frac{p-z}{c}$$

since $\beta \sim 1/2$. Hence

$$\epsilon_{\theta,p} \sim \frac{p}{c\theta} \sim \frac{p}{p-z}$$

The RHS is a measure of unproductivity; the larger p is relative to z, the smaller the RHS gets, and vice versa.

4 From the previous part, we have

$$\frac{p}{p-z} \approx 20$$

$$\frac{p-z}{p} \approx 0.05$$

$$\frac{z}{p} \approx 0.95$$

$$\frac{p}{z} \approx 1.05$$

This says that the value of a filled job isn't that much better than the unemployment benefit, which implies that job loss/unemployment is not that bad from the social planner perspective.

5 The HJB equations are

$$rU(p) = z + \sqrt{\theta(p)}(W(p) - U(p)) + \alpha \mathbb{E}_{p'}[U(p') - U(p) \mid p]$$

$$rJ(p) = p - w(p) - \lambda(J(p) - V(p)) + \alpha \mathbb{E}_{p'}[J(p') - J(p) \mid p]$$

$$rV(p) = -c + \frac{1}{\sqrt{\theta(p)}}(J(p) - V(p)) + \alpha \mathbb{E}_{p'}[V(p') - V(p) \mid p]$$

6 Since $\beta = 0.5$, the Nash bargaining solution gives

$$S(p) = W(p) + J(p) - U(p) - V(p) = 2(W(p) - U(p)) = 2(J(p) - V(p))$$

Using the HJB equations, we get

$$r(J(p)-V(p)) = p-w(p)-\lambda(J(p)-V(p))+\alpha\mathbb{E}_{p'}[J(p')-J(p) \mid p]+c-\frac{1}{\sqrt{\theta(p)}}(J(p)-V(p))-\alpha\mathbb{E}_{p'}[V(p')-V(p) \mid p]$$

$$(r+\lambda)(J(p)-V(p)) = p-w(p)+\alpha\mathbb{E}_{p'}[(J(p')-V(p'))-(J(p)-V(p)) \mid p]+c-\frac{1}{\sqrt{\theta(p)}}(J(p)-V(p))$$

$$2\left(r+\lambda+\frac{1}{\sqrt{\theta(p)}}\right)(J(p)-V(p)) = 2p-2w(p)+2c+\alpha\mathbb{E}_{p'}[S(p')-S(p) \mid p]$$

$$\left(r+\lambda+\frac{1}{\sqrt{\theta(p)}}\right)S(p) = 2p-2w(p)+2c+\alpha\mathbb{E}_{p'}[S(p')-S(p) \mid p]$$

Similarly,

$$r(W(p)-U(p)) = w(p)-\lambda(W(p)-U(p))+\alpha\mathbb{E}_{p'}[W(p')-W(p) \mid p]-z-\sqrt{\theta(p)}(W(p)-U(p))-\alpha\mathbb{E}_{p'}[U(p')-U(p) \mid p]$$

$$\left(r+\lambda+\sqrt{\theta(p)}\right)(W(p)-U(p)) = w(p)-z+\alpha\mathbb{E}_{p'}[(W(p')-U(p'))-(W(p)-U(p)) \mid p]$$

$$\left(r+\lambda+\sqrt{\theta(p)}\right)S(p) = 2w(p)-2z+\alpha\mathbb{E}_{p'}[S(p')-S(p) \mid p]$$

$$\left(r+\lambda+\sqrt{\theta(p)}\right)S(p) = 2w(p)-2z+\alpha\mathbb{E}_{p'}[S(p')-S(p) \mid p]$$

Adding the last equation to the last equation from the other part, we get

$$\left(2r+2\lambda+\sqrt{\theta(p)}+\frac{1}{\sqrt{\theta(p)}}\right)S(p) = 2p+2c-2z+2\alpha\mathbb{E}_{p'}[S(p')-S(p) \mid p]$$

$$\left(r+\lambda+\frac{1}{2}\sqrt{\theta(p)}+\frac{1}{2\sqrt{\theta(p)}}\right)S(p) = p+c-z+\alpha\mathbb{E}_{p'}[S(p')-S(p) \mid p]$$

7 From free entry, we get

$$J(p) = c\sqrt{\theta(p)}$$

$$S(p) = 2(J(p)-V(p)) = 2c\sqrt{\theta(p)}$$

Plugging in, we get

$$\left(r+\lambda+\frac{1}{2}\sqrt{\theta(p)}+\frac{1}{2\sqrt{\theta(p)}}\right)2c\sqrt{\theta(p)} = p+c-z+\alpha\mathbb{E}_{p'}[2c\sqrt{\theta(p')}-2c\sqrt{\theta(p)} \mid p]$$

$$\left(2(r+\lambda)\sqrt{\theta(p)}+\theta(p)+1\right)c = p+c-z+2c\alpha\mathbb{E}_{p'}[\sqrt{\theta(p')}-\sqrt{\theta(p)} \mid p]$$

$$\left(2(r+\lambda)\sqrt{\theta(p)}+\theta(p)\right)c = p-z+2c\alpha\mathbb{E}_{p'}[\sqrt{\theta(p')}-\sqrt{\theta(p)} \mid p]$$

8 See code.

9 The ratio is approximately 3.5. This is much much lower than the empirical observations, implying the model under these parameters is not a good approximation of reality.

Problem 2

1 The assumption in the class version was that $d_t \geq \underline{d} > 0$, but in this case $d_t \to 0$, so no such $\underline{d}$ exists.

2 From the lecture slides:

$$\phi_t^s U'(d_t) = \lim_{n\to\infty} \sum_{j=1}^{n-1} \beta^j U'(d_{t+j}) d_{t+j} + \lim_{n\to\infty} \beta^n \phi_{t+n}^s U'(d_{t+n})$$

At time 0,

$$\phi_0^s U'(c_0) = \beta U'(d_1)(\phi_1^s + d_1)$$

Using the fact that $c_0 = d_0 - M\phi_0^m$,

$$\phi_0^s = \beta \frac{d_0 - M\phi_0^m}{d_1}(\phi_1^s + d_1)$$

Now

$$\frac{\phi_t^s}{d_t} = \lim_{n\to\infty} \sum_{j=1}^{n-1} \beta^j + \lim_{n\to\infty} \beta^n \phi_{t+n}^s \frac{1}{d_{t+n}}$$

$$\frac{\phi_t^s}{\beta^t \delta d_0} = \lim_{n\to\infty} \sum_{j=1}^{n-1} \beta^j + \lim_{n\to\infty} \phi_{t+n}^s \frac{1}{\beta^t \delta d_0}$$

Assuming no bubbles, $\lim_{n\to\infty} \phi_{t+n}^s = 0$ so

$$\phi_t^s = \frac{\beta^{t+1}\delta d_0}{1-\beta} + \lim_{n\to\infty} \phi_{t+n}^s = \frac{\beta^{t+1}\delta d_0}{1-\beta}$$

Plugging back into ϕ_1^s we get

$$\phi_0^s = \beta \frac{d_0 - M\phi_0^m}{\beta\delta d_0}\left(\frac{\beta^2 \delta d_0}{1-\beta} + \beta\delta d_0\right) = \beta(d_0 - M\phi_0^m)\left(\frac{\beta}{1-\beta} + 1\right)$$

$$= \frac{\beta(d_0 - M\phi_0^m)}{1-\beta}$$

3 Plugging in,

$$R_{t+1} = \frac{\phi_{t+1}^s + d_{t+1}}{\phi_t^s} = \frac{\frac{\beta^{t+2}\delta d_0}{1-\beta} + \beta^{t+1}\delta d_0}{\frac{\beta^{t+1}\delta d_0}{1-\beta}} = \frac{\frac{\beta}{1-\beta} + 1}{\frac{1}{1-\beta}} = 1$$

4 From the FOCs,

$$\phi_t^m \lambda_t = \mu_t + \beta \phi_{t+1}^m \lambda_{t+1}$$

The initial $t = 0$ gives

$$\phi_0^m(d_0 - M\phi_0^m) = \beta\phi_1^m(d_1)$$
$$\frac{\phi_0^m}{d_0 - M\phi_0^m} = \frac{\beta\phi_1^m}{d_1}$$
$$\frac{\phi_1^m}{\delta d_0} = \frac{1}{d_0 - M\phi_0^m}\phi_0^m$$
$$\phi_1^m = \frac{\delta d_0}{d_0 - M\phi_0^m}\phi_0^m$$

Iterating,

$$\phi_t^m = \frac{\delta d_0}{d_0 - M\phi_0^m}\phi_0^m$$

Note there is no t-dependence.

5 Since ϕ_t^m is constant over time for $t \geq 1$, the returns to money and equity are both identically 0. (the only difference is between time 0 and 1).

6 Suppose, for sake of contradiction, $\phi_m^0 > 0$. The PDV of all fruit from time t on is:

$$\sum_{j=0}^{\infty} \beta^j \frac{U'(d_{t+j})}{U'(d_t)} d_{t+j} = \sum_{j=0}^{\infty} \beta^j d_t = \frac{d_t}{1-\beta}$$

which goes to 0 since $d_t \to 0$. But this implies that at some t, $\phi_t^m M$ will be larger than the value of the entire economy, implying that at that time it is not optimal for the consumer to set $m_{t+1} = M$ since the consumer can spend $d_t/\phi_t^m < M$ to buy the entire economy and have money left over. Hence we have a contradiction, and so we must have $\phi_t^m = \phi_0^m = 0$.

Problem 3

1 From the FOCs, we get

$$U'(c_t) = \lambda_t$$
$$\phi_t^m \lambda_t = \mu_t + \beta\phi_{t+1}^m \lambda_{t+1}$$

Combining, we get

$$\phi_t^m U'(c_t) = \mu_t + \beta\phi_{t+1}^m U'(c_{t+1})$$

Assuming $m_t > 0$, we get

$$\frac{U'(c_t)}{\beta U'(c_{t+1})}\phi_t^m = \phi_{t+1}^m$$

Repeating, we get

$$\frac{U'(c_0)}{\beta^t U'(c_t)}\phi_0^m = \phi_t^m$$

Adding market clearing, we get

$$\frac{U'(d_0 - \phi_0^m M)}{\beta^t U'(d_t)} \phi_0^m = \phi_t^m$$

as desired.

2 Since $d_t \to \underline{d} > 0$, taking the limit

$$\lim_{t\to\infty} \phi_t^m = \lim_{t\to\infty} \frac{U'(d_0 - \phi_0^m M)}{\beta^t U'(d_t)} \phi_0^m = \phi_0^m \frac{U'(d_0 - \phi_0^m M)}{U'(d_t)} \lim_{t\to\infty} \frac{1}{\beta^t} \to \infty$$

hence the consumer wealth

$$\phi_t^s + d_t + \phi_t^m M \to \infty$$

which implies that the consumer at some point can buy the economy and have money left over, violating market clearing.

3 From the FOCs, we get

$$U'(c_t) = \lambda_t$$

$$\phi_t^s \lambda_t = \beta \lambda_{t+1} (\phi_{t+1}^s + d_{t+1})$$

Plugging in, we get

$$\phi_t^s U'(c_t) = \beta U'(c_{t+1}) \phi_{t+1}^s + \beta U'(c_{t+1}) d_{t+1}$$

$$U'(c_t) \phi_t^s = \frac{1}{\beta} U'(c_{t-1}) \phi_{t-1}^s - U'(c_t) d_t$$

$$= \frac{1}{\beta} \left(\frac{1}{\beta} U'(c_{t-2}) \phi_{t-2}^s - U'(c_{t-1}) d_{t-1} \right) - U'(c_t) d_t$$

$$= \frac{1}{\beta^2} U'(c_{t-2}) \phi_{t-2}^s - \frac{1}{\beta} U'(c_{t-1}) d_{t-1} - U'(c_t) d_t$$

$$= \frac{1}{\beta^t} U'(c_0) \phi_0^s - \sum_{j=0}^{t} \frac{1}{\beta^{t-j}} U'(c_j) d_j$$

So we get

$$\phi_t^s = \frac{U'(c_0)}{\beta^t U'(c_t)} \phi_0^s - \sum_{j=0}^{t} \frac{U'(c_j)}{\beta^{t-j} U'(c_t)} d_j$$

$$= \frac{U'(c_0)}{\beta^t U'(c_t)} \phi_0^s - \frac{1}{\beta^t} \sum_{j=0}^{t} \beta^j \frac{U'(c_j)}{U'(c_t)} d_j$$

$$= \frac{1}{\beta^t} \left(\frac{U'(c_0)}{U'(c_t)} \phi_0^s - \sum_{j=0}^{t} \beta^j \frac{U'(c_j)}{U'(c_t)} d_j \right)$$

4 By examining the previous expression, as long as the parenthesized stuff goes to 0 at a rate on the order of β^t, the limit $\lim_{t\to\infty} \phi_t^s < \infty$; hence it is possible for $\phi_0^s \neq 0$ and $\lim_{t\to\infty} \phi_t^s < \infty$.