ECON 511 Problem Set 4

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Problem 1

(1) The maximization problem is given by

$$V = \max \mathbb{E}\left[\int_{t=0}^{\infty} e^{-\rho t}\left(z_t k_t - i_t - \frac{i_t^2}{2}\right)\ dt\right]$$

subject to

$$\dot{k}_t = i_t - \delta k_t$$

$$dz_t = \mu\ dt + \sigma\ dW_t$$

(2) The HJB equation is

$$\rho V(k,z) = \max_i \left[zk - i - \frac{i^2}{2} + V_k(k,z)(i - \delta k) + V_z(k,z)\mu + \frac{1}{2}V_{zz}(k,z)\sigma^2\right]$$

(3) The FOC on i is

$$-1 - i + V_k(k,z) = 0$$

$$i = V_k(k,z) - 1$$

Plugging in, we have

$$\rho V(k,z) = zk - V_k(k,z) + 1 - \frac{(V_k(k,z)-1)^2}{2} + V_k(k,z)(V_k(k,z) - 1 - \delta k) + V_z(k,z)\mu + \frac{1}{2}V_{zz}(k,z)\sigma^2$$

$$= zk - V_k(k,z) + 1 - \frac{(V_k(k,z))^2 - 2V_k(k,z) + 1}{2} + V_k(k,z)(V_k(k,z) - 1 - \delta k) + V_z(k,z)\mu + \frac{1}{2}V_{zz}(k,z)\sigma^2$$

$$= zk - V_k(k,z) + \frac{1}{2} + \frac{1}{2}(V_k(k,z))^2 - \delta k V_k(k,z) + V_z(k,z)\mu + \frac{1}{2}V_{zz}(k,z)\sigma^2$$

We assume the quadratic functional form:

$$V(k,z) = a_0 + a_1 k + a_2 z + a_3 z^2 + a_4 zk$$

Then

$$V_k(k,z) = a_1 + a_4 z$$

$$V_z(k,z) = a_2 + 2a_3 z + a_4 k$$

$$V_{zz}(k,z) = 2a_3$$

Starting with the zk coefficient as in the hint, we have:

$$\rho a_4 = 1 - \delta a_4$$

$$a_4 = \frac{1}{\rho + \delta}$$

We then solve for the z^2 coefficients on both sides:

$$\rho a_3 = \frac{1}{2}(a_4^2)$$

$$a_3 = \frac{1}{2\rho(\rho+\delta)^2}$$

We can now solve for a_1 by examining coefficient on k:

$$\rho a_1 = -\delta a_1 + a_4 \mu$$

$$a_1(\rho + \delta) = a_4 \mu$$

$$a_1 = \frac{\mu}{(\rho+\delta)^2}$$

And now for a_2 by examining the coefficient on z:

$$\rho a_2 = -a_4 + a_1 a_4 + 2a_3 \mu$$

$$\rho a_2 = -\frac{1}{\rho+\delta} + \frac{\mu}{(\rho+\delta)^3} + \frac{\mu}{\rho(\rho+\delta)^2}$$

$$\rho a_2 = -\frac{\rho(\rho+\delta)^2}{\rho(\rho+\delta)^3} + \frac{\rho\mu}{\rho(\rho+\delta)^3} + \frac{\mu(\rho+\delta)}{\rho(\rho+\delta)^3}$$

$$a_2 = \frac{\rho\mu + \mu(\rho+\delta) - \rho(\rho+\delta)^2}{\rho^2(\rho+\delta)^3}$$

Finally, the constant term a_0:

$$\rho a_0 = \frac{1}{2} - a_1 + \frac{1}{2}a_1^2 + a_2\mu + a_3\sigma^2$$

$$a_0 = \frac{1}{2\rho} - \frac{a_1}{\rho} + \frac{a_1^2}{2\rho} + \frac{\mu a_2}{\rho} + \frac{a_3\sigma^2}{\rho}$$

$$a_0 = \frac{1}{2\rho} - \frac{\mu}{\rho(\rho+\delta)^2} + \frac{\mu^2}{2\rho(\rho+\delta)^4} + \frac{\rho\mu^2 + \mu^2(\rho+\delta) - \mu\rho(\rho+\delta)^2}{\rho^3(\rho+\delta)^3} + \frac{\sigma^2}{2\rho^2(\rho+\delta)^2}$$

For the sake of saving ugliness, we will not write V out explicitly, since $V = a_0 + a_1 k + a_2 z + a_3 z^2 + a_4 zk$ and we have defined all the coefficients already.

(4) The only coefficient featuring σ^2 is a_0. So

$$\frac{\partial V}{\partial \sigma^2} = \frac{\partial a_0}{\partial \sigma^2} = \frac{a_3}{\rho} = \frac{1}{2\rho^2(\rho+\delta)^2}$$

This is positive, hence the economic value of uncertainty is positive. This is because the firm's net present value of profits are convex in z. Increasing both ρ and δ decreases the value of uncertainty. The higher the discount rate, the worse the future profits and hence the less value of future fluctuations of z. The higher δ, the less convex the profits.

(5) $i = V_k(k,z) - 1$ so

$$i^* = a_1 + a_4 z - 1 = \frac{\mu}{(\rho+\delta)^2} + \frac{1}{\rho+\delta}z - 1$$

This is decreasing in ρ and δ and increasing in μ and z. This is similar to the q-model in that it is independent of k and increasing in productivity, and implies firms invest for $q > 1$ and disinvest for $q < 1$. Investment increases in μ, since that results in expected higher future profitability, and decreases in ρ, δ for the same reasons as the previous part, that they decrease the value of the future profits.

Problem 2

(1) K^* satisfies:

$$K^* = \arg\max zk^\theta - rk$$

The FOC gives

$$\theta z (K^*)^{\theta-1} = r$$

$$K^* = \left(\frac{r}{\theta z}\right)^{\frac{1}{\theta-1}}$$

(2) We have $e^x = K/K^*$, so $K = K^* e^x$. So the flow revenue

$$zK^\theta - rK = z(K^* e^x)^\theta - rK^* e^x$$

$$= z(K^*)^\theta e^{\theta x} - rK^* e^x$$

Plugging in K^*, we get

$$= z\left(\frac{r}{\theta z}\right)^{\frac{\theta}{\theta-1}} e^{\theta x} - r\left(\frac{r}{\theta z}\right)^{\frac{1}{\theta-1}} e^x$$

$$= \left(\frac{r^\theta}{\theta z}\right)^{\frac{1}{\theta-1}} \left(\frac{1}{\theta} e^{\theta x} - e^x\right)$$

Approximating the expression around 0 using a Taylor expansion, we have

$$= \left(\frac{r^\theta}{\theta z}\right)^{\frac{1}{\theta-1}} \left(\frac{1}{\theta}\left(1 + \theta x + \frac{1}{2}\theta^2 x^2\right) - \left(1 + x + \frac{1}{2}x^2\right)\right)$$

$$= \left(\frac{r^\theta}{\theta z}\right)^{\frac{1}{\theta-1}} \left(\left(\frac{1}{\theta} + \frac{1}{2}\theta x^2\right) - \left(1 + \frac{1}{2}x^2\right)\right)$$

$$= \left(\frac{r^\theta}{\theta z}\right)^{\frac{1}{\theta-1}} \left(\left(\frac{1}{\theta} - 1\right) + \frac{1}{2}x^2 (\theta - 1)\right)$$

$$= r\left(\frac{r}{\theta z}\right)^{\frac{1}{\theta-1}} \left(\left(\frac{1}{\theta} - 1\right) + \frac{1}{2}x^2 (\theta - 1)\right)$$

$$= rK^* \left(\left(\frac{1}{\theta} - 1\right) + \frac{1}{2}x^2 (\theta - 1)\right)$$

(3) The nonrecursive problem is given by (using the approximation from part 2)

$$\max_x \mathbb{E}\left[\int_0^\infty e^{-rt} \left(rK_t^* \left(\left(\frac{1}{\theta} - 1\right) + \frac{1}{2}x_t^2 (\theta - 1)\right) - K_t^* F' 1_{x_t \neq 0}\right) dt\right]$$

subject to

$$K_t^* = \left(\frac{r}{\theta z_t}\right)^{\frac{1}{\theta-1}}$$

$$x_t = \log(K_t/K_t^*)$$

$$\log z_t = \sigma W_t$$

Rewriting the objective by converting to K_t^*/K_0^*, we have

$$\max_x \mathbb{E}\left[K_0^* \int_0^\infty e^{-rt} \left(r\left(\frac{K_t^*}{K_0^*}\right) \left(\left(\frac{1}{\theta} - 1\right) + \frac{1}{2}x_t^2 (\theta - 1)\right) - \frac{K_t^*}{K_0^*} F' 1_{x_t \neq 0}\right) dt\right]$$

subject to

$$\frac{K_t^*}{K_0^*} = \left(\frac{z_0}{z_t}\right)^{\frac{1}{\theta-1}}$$

$$x_t = \log(K_t/(K_t^*/K_0^*)K_0^*) = \log\left(\frac{K_t}{K_0}\frac{K_0}{K_0^*}\frac{K_0^*}{K_t^*}\right) = \log(K_t/K_0) + x_0 + \frac{1}{\theta - 1}\log(z_t/z_0)$$

$$\log z_t = \sigma W_t$$

Adding a dummy variable $y_t = K_t^*/K_0^*$, we get

$$\max_x \mathbb{E}\left[K_0^* \int_0^\infty e^{-rt} \left(ry_t \left(\left(\frac{1}{\theta} - 1\right) + \frac{1}{2}x_t^2 (\theta - 1)\right) - y_t F' 1_{x_t \neq 0}\right) dt\right]$$

subject to

$$y_t = \left(\frac{z_0}{z_t}\right)^{\frac{1}{\theta-1}}$$

$$x_t = \log(K_t/K_0) + x_0 + \frac{1}{\theta - 1}\log(z_t/z_0)$$

$$\log z_t = \sigma W_t$$

Note now that the only way K_0^* factors into the maximization problem is just as a constant term. Hence the value of the firm is linear in K_0^*, so the value of the firm is homogeneous of degree 1 in K^*. The law of motion of x is then

$$x = \log(K_t/K_t^*)$$

$$dx = -d\log K^*$$

Note that $\log K^* = c - \frac{1}{\theta-1}\log z$, where the constant term c has no time dependence. Hence in the inaction region close to $x = 0$, the law of motion of x follows:

$$dx = -d\log K^* = d\left(\frac{1}{\theta-1}\log z\right) = \frac{\sigma}{\theta-1}\ dW$$

(4) Note that by definition of K^*, we have

$$K_t^* = \left(\frac{r}{\theta z_t}\right)^{\frac{1}{\theta-1}}$$

$$d\log K_t^* = -\frac{1}{\theta-1}d\log z_t = \frac{\sigma}{1-\theta}dW$$

Using Ito's Lemma, we get

$$dK_t^* = K_t^* d\log K_t^* + \frac{1}{2}K_t^*(d\log K_t^*)^2$$

$$dK_t^* = K_t^*\frac{\sigma}{1-\theta}dW + \frac{1}{2}K_t^*\left(\frac{\sigma}{1-\theta}\right)^2 dt$$

Let

$$\pi(x) = r\left(\left(\frac{1}{\theta}-1\right) + \frac{1}{2}x^2(\theta-1)\right)$$

We have the following two laws of motion:

$$dK_t^* = K_t^*\frac{\sigma}{1-\theta}dW + \frac{1}{2}K_t^*\left(\frac{\sigma}{1-\theta}\right)^2 dt$$

$$dx = \frac{\sigma}{\theta-1}\ dW$$

where the second comes from the previous problem part. The 2-variable HJB is then

$$rV = K^*\pi(x) + V_k\left(\frac{1}{2}K^*\left(\frac{\sigma}{1-\theta}\right)^2\right) + (V_x)(0) + \frac{1}{2}V_{kk}\left(K^*\frac{\sigma}{1-\theta}\right)^2 + \frac{1}{2}V_{xx}\left(\frac{\sigma}{\theta-1}\right)^2 + V_{kx}K^*\frac{\sigma}{1-\theta}\frac{\sigma}{\theta-1}$$

$$rV = K^*\pi(x) + \frac{1}{2}\left(\frac{\sigma}{1-\theta}\right)^2\left(K^*V_k + V_{kk}(K_t^*)^2 + V_{xx} - 2V_{kx}K^*\right)$$

Using $V = K^*W$, we get that $V_k = W$, $V_{kk} = 0$, $V_{kx} = W'$, $V_{xx} = K^*W''$, so

$$rK^*W = K^*\pi(x) + \frac{1}{2}\left(\frac{\sigma}{1-\theta}\right)^2(K^*W + K^*W'' - 2W')$$

$$rW = \pi(x) + \frac{1}{2}\left(\frac{\sigma}{1-\theta}\right)^2 (W + W'' - 2W')$$

(5) The upper band bound is $\overline{x}$, and let the optimal adjustment at that bound be $\overline{x}^*$. Similarly, let the lower band bound be $\underline{x}$ and the optimal adjustment be $\underline{x}^*$. We have

$$W(\overline{x}) = W(\overline{x}^*) - F'$$

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Optimality of $\overline{x}^*, \underline{x}^*$ imply

$$W'(\overline{x}^*) = 0$$

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Finally, together with the previous conditions, since F' is a constant, we get

$$W'(\overline{x}) = 0$$

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(6) We first can plug the form into the HJB equation and match coefficients of terms: this gives 5 equations on 7 unknown variables (a, b, c, d, f, g, h). We also have the conditions from the previous part; this gives us 4 more unknowns: $\overline{x}$, $\overline{x}^*$, $\underline{x}$, $\underline{x}^*$, and 6 more equations. All together, we have 11 equations on 11 unknowns, so we can solve this system of equations to find the desired parameters.