ECON 511 Problem Set 4

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Problem 1

The farmer's law of motion on land holdings is given by:

$$K_t = \frac{1}{u_t}((a + q_t)K_{t-1} - RB_{t-1})$$

$$B_t = \frac{1}{R}q_{t+1}K_t$$

Immediately in the period of the shock, we have

$$u_t K_t = (a + \Delta a + q_t)K^* - RB^* = (a + \Delta a + q_t - q^*)K^*$$

Taking the first order approximation around steady state, we get

$$u(K^*)K^* + u'(K^*)K^*(K_t - K^*) + u(K^*)(K_t - K^*) = (a + \Delta a + q_t - q^*)K^*$$

Dividing out K^*, we get

$$u(K^*) + u'(K^*)K^*\hat{K}_t + u(K^*)\hat{K}_t = a + \Delta a + q_t - q^*$$

Using the log-linear approximation on q, we get

$$u(K^*) + u'(K^*)K^*\hat{K}_t + u(K^*)\hat{K}_t = a + \Delta a + \hat{q}_t q^*$$

We know

$$u(K^*) = a = \frac{R-1}{R}q^*$$

Dividing out $u(K^*)$ we get

$$1 + \frac{u'(K^*)}{u(K^*)}K^*\hat{K}_t + \hat{K}_t = 1 + \Delta + \frac{R}{R-1}\hat{q}_t$$

$$\frac{u'(K^*)}{u(K^*)}K^*\hat{K}_t + \hat{K}_t = \Delta + \frac{R}{R-1}\hat{q}_t$$

After the shock,

$$u(K_t)K_t = (a + q_t - q_t)K_{t-1}$$

since no other shocks are experienced and the agents know this, the q terms cancel and hence

$$u(K_t)K_t = aK_{t-1}$$

Again taking the first order approximation and dividing out K^*,

$$u(K^*) + u'(K^*)K^*\hat{K}_t + u(K^*)\hat{K}_t = a\hat{K}_{t-1} + a$$

Using again that $u(K^*) = a$, we get

$$\frac{u'(K^*)}{u(K^*)}K^*\hat{K}_t + \hat{K}_t = \hat{K}_{t-1}$$

$$\left(\frac{u'(K^*)}{u(K^*)}K^* + 1\right)\hat{K}_t = \hat{K}_{t-1}$$

$$\hat{K}_t = \hat{K}_{t-1}\left(\frac{u(K^*)}{u'(K^*)K^* + u(K^*)}\right)$$

Since $u'(K^*)K^*$ is positive, the parenthesized expression is less than 1, and hence $\hat{K}_t \to 0$, so $K_t \to K^*$. So the economy returns to steady state at a constant rate.

Now, by definition of u,

$$q_t = u_t + \frac{q_{t+1}}{R} = u_t + \frac{u_{t+1}}{R} + \frac{q_{t+2}}{R^2}$$

$$= \sum_{s=0}^{\infty}\frac{u_s}{R^s}$$

Taking the Taylor expansions around steady state, we get

$$\hat{q}_t q^* = \sum_{s=0}^{\infty}\frac{\hat{u}_s u^*}{R^s}$$

Using $u^* = q^*(R-1)/R$,

$$\hat{q}_t = \frac{R-1}{R}\sum_{s=0}^{\infty}\frac{\hat{u}_s}{R^s}$$

Now, we have

$$u(K_t)K_t = aK_{t-1}$$

$$u(K_t) = a\frac{K_{t-1}}{K_t}$$

Using then we have

$$u^*\hat{u} + u^* = a\frac{K_{t-1}}{K_t}$$

$$\hat{u} + 1 = \frac{K_{t-1}}{K_t}$$

$$\hat{u} = \frac{K_{t-1} - K_t}{K_t}$$

Taking the approximation around K^*, we get

$$\hat{u} = \hat{K}_{t-1} - \hat{K}_t$$

Using the derived law of motion for $\hat{K}$, we get

$$\hat{u} = \left(\frac{u'(K^*)}{u(K^*)}K^* + 1\right)\hat{K}_t - \hat{K}_t$$

$$\hat{u} = \left(\frac{u'(K^*)}{u(K^*)}K^*\right)\hat{K}_t$$

Pluggin in our expression for $\hat{q}$, we get

$$\hat{q}_t = \frac{R-1}{R}\sum_{s=0}^{\infty}\frac{\hat{u}_s}{R^s}$$

$$= \frac{R-1}{R}\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\frac{u'(K^*)}{u(K^*)}K^*\right)\hat{K}_{t+s}$$

$$= \frac{R-1}{R}\left(\frac{u'(K^*)}{u(K^*)}K^*\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{K}_{t+s}$$

Again using our derived law of motion for $\hat{K}$, we get

$$\hat{q}_t = \frac{R-1}{R}\left(\frac{u'(K^*)}{u(K^*)}K^*\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{K}_t\left(\frac{u(K^*)}{u'(K^*)K^* + u(K^*)}\right)^s$$

$$\hat{q}_t = \frac{R-1}{R}\left(\frac{u'(K^*)}{u(K^*)}K^*\right)\hat{K}_t\sum_{s=0}^{\infty}\left(\frac{u(K^*)}{R(u'(K^*)K^* + u(K^*))}\right)^s$$

$$= \frac{R-1}{R}\left(\frac{u'(K^*)}{u(K^*)}K^*\right)\hat{K}_t\frac{1}{1-\left(\frac{u(K^*)}{R(u'(K^*)K^*+u(K^*))}\right)}$$

$$= \frac{R-1}{R}\left(\frac{u'(K^*)}{u(K^*)}K^*\right)\hat{K}_t\frac{R(u'(K^*)K^* + u(K^*))}{R(u'(K^*)K^* + u(K^*)) - u(K^*)}$$

$$\frac{R}{R-1}\hat{q}_t = \left(\frac{u'(K^*)}{u(K^*)}K^*\right)\hat{K}_t\frac{R(u'(K^*)K^* + u(K^*))}{R(u'(K^*)K^* + u(K^*)) - u(K^*)}$$

Now, we have the earlier condition:

$$\frac{u'(K^*)}{u(K^*)}K^*\hat{K}_t + \hat{K}_t = \Delta + \frac{R}{R-1}\hat{q}_t$$

Plugging in, we get

$$\frac{u'(K^*)}{u(K^*)}K^*\hat{K}_t + \hat{K}_t = \Delta + \left(\frac{u'(K^*)}{u(K^*)}K^*\right)\hat{K}_t\frac{R(u'(K^*)K^* + u(K^*))}{R(u'(K^*)K^* + u(K^*)) - u(K^*)}$$

$$\frac{u'(K^*)}{u(K^*)}K^*\hat{K}_t + \hat{K}_t - \left(\frac{u'(K^*)}{u(K^*)}K^*\right)\hat{K}_t\frac{R(u'(K^*)K^* + u(K^*))}{R(u'(K^*)K^* + u(K^*)) - u(K^*)} = \Delta$$

Substituting for $\eta = u(K^*)/(u'(K^*)K^*)$, we get

$$\frac{1}{\eta}\hat{K}_t + \hat{K}_t - \left(\frac{1}{\eta}\right)\hat{K}_t\frac{R(1+\eta)}{R(1+\eta)-\eta} = \Delta$$

$$\left(\frac{1}{\eta} + 1 - \left(\frac{1}{\eta}\right)\frac{R(1+\eta)}{R(1+\eta)-\eta}\right)\hat{K}_t = \Delta$$

$$\frac{\eta+1}{\eta}\left(1 - \frac{R}{R(1+\eta)-\eta}\right)\hat{K}_t = \Delta$$

$$\hat{K}_t = \frac{\eta}{1+\eta}\left(\frac{R(1+\eta)-\eta}{R(1+\eta)-\eta-R}\right)\Delta$$

$$= \frac{\eta}{1+\eta}\left(\frac{R+R\eta-\eta}{R\eta-\eta}\right)\Delta$$

$$= \frac{1}{1+\frac{1}{\eta}}\left(1 + \frac{R}{R\eta-\eta}\right)\Delta$$

$$= \frac{1}{1+\frac{1}{\eta}}\left(1 + \frac{R}{R-1}\frac{1}{\eta}\right)\Delta$$

which matches the notes. We can also solve for $\hat{q}_t$:

$$\hat{q}_t = \frac{R-1}{R}\left(\frac{u'(K^*)}{u(K^*)}K^*\right)\hat{K}_t\frac{R(u'(K^*)K^* + u(K^*))}{R(u'(K^*)K^* + u(K^*)) - u(K^*)}$$

$$\hat{q}_t = \frac{R-1}{R}\left(\frac{1}{\eta}\right)\frac{1}{1+\frac{1}{\eta}}\left(1 + \frac{R}{R-1}\frac{1}{\eta}\right)\Delta\frac{R(1+\eta)}{R(1+\eta)-\eta}$$

$$\hat{q}_t = \frac{R-1}{R}\left(\frac{1}{\eta}\right)\frac{\eta}{1+\eta}\left(\frac{R+R\eta-\eta}{R\eta-\eta}\right)\Delta\frac{R+R\eta}{R+R\eta-\eta}$$

$$\hat{q}_t = \frac{R-1}{R}\frac{1}{1+\eta}\left(\frac{R+R\eta}{R\eta-\eta}\right)\Delta$$

$$\hat{q}_t = \frac{1}{1+\eta}\left(\frac{1+\eta}{\eta}\right)\Delta$$

$$\hat{q}_t = \frac{1}{\eta}\Delta$$

as noted before. So we already have the law of motion for $\hat{K}$, and the initial conditions for $\hat{K}_t$. The law of motion on $\hat{q}_t$ is given by what we found earlier

$$\hat{q}_\tau = \frac{R-1}{R}\left(\frac{u'(K^*)}{u(K^*)}K^*\right)\hat{K}_\tau\frac{R(u'(K^*)K^* + u(K^*))}{R(u'(K^*)K^* + u(K^*)) - u(K^*)}$$

and the initial condition is $\hat{q}_t = \frac{1}{\eta}\Delta$.

Problem 2

(1) The consumer problem:

$$\max E \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\theta}}{1-\theta}$$

subject to

$$c_t + b_{t+1} = e_t w_t + (1+r_t) b_t$$

$$\lim_{t\to\infty} b_t \prod_{i=1}^{t} \frac{1}{1+r_i} = 0$$

(2) The budget constraint

$$c_t + b_{t+1} = e_t w_t + (1+r_t) b_t$$

$$c_t = e_t w_t + (1+r_t) b_t - b_{t+1}$$

so we get

$$\max E \sum_{t=0}^{\infty} \beta^t \frac{(e_t w_t + (1+r_t) b_t - b_{t+1})^{1-\theta}}{1-\theta}$$

subject to

$$\lim_{t\to\infty} b_t \prod_{i=1}^{t} \frac{1}{1+r_i} = 0$$

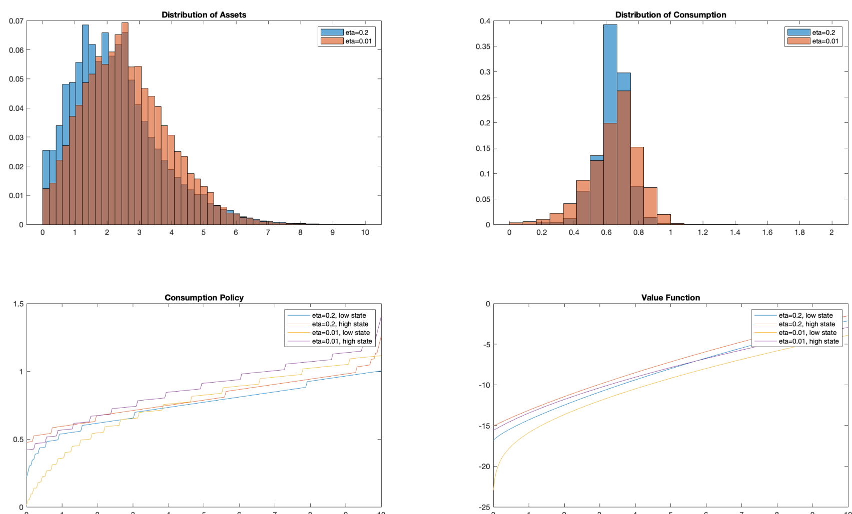
(3) If $\underline{b} = 0$, then bonds and capital will be perfect substitutes, so since $b_t \geq 0$ always, we can just have investment in capital instead of bonds, so $b_t = k_t$. If $\underline{b} < 0$, then it is possible to borrow using bonds, which is impossible with capital.

(4), (5) The Bellman eq is given by

$$V(k,e) = \max_{k'} \left(\frac{(ew(k) + (1+r(k))k - k')^{1-\theta}}{1-\theta} + \beta E_e V(k', e) \right)$$

See figures for graphs.

η	W	K	L	w	r
0.2	-0.4677	0.2315	0.5000	1.0966	0.0382
0.01	-0.4813	0.2577	0.5000	1.1550	0.0260


Distribution of Assets
eta=0.2
eta=0.01
Distribution of Consumption
eta=0.2
eta=0.01
Consumption Policy
eta=0.2, low state
eta=0.2, high state
eta=0.01, low state
eta=0.01, high state
Value Function
eta=0.2, low state
eta=0.2, high state
eta=0.01, low state
eta=0.01, high state