ECON 511 Problem Set 9

Nicholas Wu

Spring 2021

Problem 1

1 The firm problem is

$$\max P(s^t)A_t(s^t)N_t(s^t) - W_t(s^t)N_t(s^t)$$

so the FOC is

$$P(s^t)A_t(s^t) = W_t(s^t)$$

2 The consumer problem is

$$\max \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) \left(\log C_t(s^t) + \log \left(\frac{M_t(s^t)}{P_t(s^t)} \right) - \frac{N_t(s^t)^{1+\varphi}}{1+\varphi} \right)$$

subject to

$$P_t(s^t)C_t(s^t) + M_t(s^t) + Q_t(s^t)B_t(s^t) \leq M_{t-1}(s^{t-1}) + B_{t-1}(s^{t-1}) + W_t(s^t)N_t(s^t) - T_t(s^t)$$

so the Lagrangian is

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) \left(\log C_t(s^t) + \log \left(\frac{M_t(s^t)}{P_t(s^t)} \right) - \frac{N_t(s^t)^{1+\varphi}}{1+\varphi} \right)$$

$$-\lambda_t(s^t)(P_t(s^t)C_t(s^t) + M_t(s^t) + Q_t(s^t)B_t(s^t) - M_{t-1}(s^{t-1}) - B_{t-1}(s^{t-1}) - W_t(s^t)N_t(s^t) + T_t(s^t))$$

The FOCs are then

$$\beta^t \frac{\pi(s^t)}{C_t(s^t)} = \lambda_t(s^t)P_t(s^t)$$

$$\beta^t \pi(s^t) N_t(s^t)^{\varphi} = \lambda_t(s^t)W_t(s^t)$$

$$\lambda_t(s^t)Q_t(s^t) = \sum_{s^{t+1}} \lambda_{t+1}(s^{t+1})$$

$$\beta^t \frac{\pi(s^t)}{M_t(s^t)} = \lambda_t(s^t) - \sum_{s^{t+1}} \lambda_{t+1}(s^{t+1})$$

3 Eliminating $\lambda_t(s^t)$, we get

$$\beta^t \pi(s^t) N_t(s^t)^{\varphi} = \beta^t \frac{\pi(s^t)}{C_t(s^t)P_t(s^t)} W_t(s^t)$$

$$N_t(s^t)^{\varphi} = \frac{W_t(s^t)}{C_t(s^t)P_t(s^t)}$$

For Q, we get

$$\beta^t \frac{\pi(s^t)}{C_t(s^t)P_t(s^t)} Q_t(s^t) = \sum_{s^{t+1}} \beta^{t+1} \frac{\pi(s^{t+1})}{C_{t+1}(s^{t+1})P_{t+1}(s^{t+1})}$$

$$\frac{\pi(s^t)Q_t(s^t)}{C_t(s^t)P_t(s^t)} = \sum_{s^{t+1}} \frac{\beta\pi(s^{t+1})}{C_{t+1}(s^{t+1})P_{t+1}(s^{t+1})}$$

$$Q_t(s^t) = \sum_{s^{t+1}} \frac{\beta\pi(s^{t+1}|s_t)C_t(s^t)P_t(s^t)}{C_{t+1}(s^{t+1})P_{t+1}(s^{t+1})}$$

$$Q_t(s^t) = E_t \frac{\beta C_t(s^t)P_t(s^t)}{C_{t+1}(s^{t+1})P_{t+1}(s^{t+1})}$$

and for M, we have

$$\beta^t \frac{\pi(s^t)}{M_t(s^t)} = \beta^t \frac{\pi(s^t)}{C_t(s^t)P_t(s^t)} - \sum_{s^{t+1}} \frac{\beta^{t+1}\pi(s^{t+1})}{C_{t+1}(s^{t+1})P_{t+1}(s^{t+1})}$$

$$\frac{1}{M_t(s^t)} = \frac{1}{C_t(s^t)P_t(s^t)} - \sum_{s^{t+1}} \frac{\beta\pi(s^{t+1}|s^t)}{C_{t+1}(s^{t+1})P_{t+1}(s^{t+1})}$$

$$\frac{1}{M_t(s^t)} = \frac{1}{C_t(s^t)P_t(s^t)} - E_t \frac{\beta}{C_{t+1}(s^{t+1})P_{t+1}(s^{t+1})}$$

$$\frac{1}{M_t(s^t)} = \frac{1 - Q_t(s^t)}{C_t(s^t)P_t(s^t)}$$

4 From market clearing, we have

$$A_t(s^t)N_t(s^t) = C_t(s^t)$$

From the firm FOC

$$P(s^t)A_t(s^t) = W_t(s^t)$$

From the previous part

$$N_t(s^t)^{\varphi} = \frac{W_t(s^t)}{C_t(s^t)P_t(s^t)}$$

so

$$N_t(s^t)^{\varphi} = \frac{P(s^t)A_t(s^t)}{C_t(s^t)P_t(s^t)} = \frac{A_t(s^t)}{C_t(s^t)} = \frac{1}{N_t(s^t)}$$

$$N_t(s^t)^{\varphi+1} = 1$$

$$(1+\varphi)\log N_t = 0$$

$$n_t = 0$$

Since

$$A_t(s^t)N_t(s^t) = C_t(s^t)$$

$$\log A_t(s^t) + \log N_t(s^t) = \log C_t(s^t)$$

$$a_t + n_t = c_t$$

$$a_t = c_t$$

And since $Y_t = C_t$, by market clearing, $y_t = c_t = a_t$.

5 Log linearizing, we get

$$\varphi n_t = w_t - c_t - p_t$$

$$Q_t(s^t) = E_t \frac{\beta C_t(s^t)P_t(s^t)}{C_{t+1}(s^{t+1})P_{t+1}(s^{t+1})}$$

$$-i_t = -\rho - E_t(p_{t+1} - p_t) - E_t(c_{t+1} - c_t)$$

$$c_t = \rho - i_t + E_t\pi_{t+1} + E_t c_{t+1}$$

$$m_t = c_t + p_t - \log(1 - Q_t)$$

6 From a constant velocity of money, we get

$$p_t + c_t - m_t = p_{t-1} + c_{t-1} - m_{t-1}$$

$$\pi_t + c_t - c_{t-1} + m_{t-1} - m_t = 0$$

$$\pi_t + a_t - a_{t-1} + m_{t-1} - m_t = 0$$

$$\pi_t + \ln(1 + \gamma_a) - \ln(1 + \gamma_m) + \epsilon_a^t - \epsilon_m^t$$

$$\pi_t = \ln(1 + \gamma_m) - \ln(1 + \gamma_a) + \epsilon_m^t - \epsilon_a^t$$

Plugging into Euler eq

$$c_t = \rho + i_t - E_t\pi_{t+1} - E_t c_{t+1}$$

$$= \rho - i_t + E_t \left(\ln(1 + \gamma_m) - \ln(1 + \gamma_a) + \epsilon_m^t - \epsilon_a^t\right) + E_t c_{t+1}$$

$$= \rho - i_t + \ln(1 + \gamma_m) - \ln(1 + \gamma_a) + E_t c_{t+1}$$

$$= \rho - i_t + \ln(1 + \gamma_m) - \ln(1 + \gamma_a) + E_t a_{t+1}$$

$$= \rho - i_t + \ln(1 + \gamma_m) - \ln(1 + \gamma_a) + a_t + \ln(1 + \gamma_a)$$

$$= \rho - i_t + \ln(1 + \gamma_m) + a_t$$

Since $c_t = a_t$,

$$\rho - i_t + \ln(1 + \gamma_m) = 0$$

$$i_t = \rho + \ln(1 + \gamma_m)$$

Now, since

$$\frac{1}{M_t(s^t)} = \frac{1 - Q_t(s^t)}{C_t(s^t)P_t(s^t)}$$

$$V = \frac{PY}{M} = \frac{PC}{M} = 1 - Q \approx i_t = \rho + \ln(1 + \gamma_m)$$

as desired.

7 By constant velocity of money,

$$p_t + c_t - m_t = \log V \approx \log(\rho + \ln(1 + \gamma_m))$$

Since money only factors into the utility via $m_t - p_t = \log(M_t/P_t)$ we have that

$$m_t - p_t = a_t - \log V \approx \log(\rho + \ln(1 + \gamma_m))$$

So only γ_m affects the utility.

8 The social planner problem is

$$\max \ln C_t + \ln \frac{M_t}{P_t} - \frac{N_t^{1+\varphi}}{1+\varphi}$$

subject to $C_t = A_t N_t$. The FOC in M gives

$$\frac{1}{M} = 0$$

From before we hhave

$$0 = \frac{1}{M} = \frac{1 - Q}{PC}$$

$$0 = \frac{PC}{M} = 1 - Q = i$$

Hence $i_t = 0$.

Problem 2

1 Plugging into the EE:

$$\hat{Y}_t = E_t\hat{Y}_{t+1} - \sigma(\hat{i}_t - E_t\hat{\pi}_{t+1} - \hat{r}_t^n)$$

$$\hat{Y}_t = E_t\hat{Y}_{t+1} - \sigma(\phi_y\hat{Y}_t + \phi_r\hat{r}_t^n - E_t\hat{\pi}_{t+1} - \hat{r}_t^n)$$

$$\hat{Y}_t = E_t\hat{Y}_{t+1} - \sigma\phi_y\hat{Y}_t - \sigma\phi_r\hat{r}_t^n + \sigma E_t\hat{\pi}_{t+1} + \sigma\hat{r}_t^n$$

$$(1 + \sigma\phi_y)\hat{Y}_t = E_t\hat{Y}_{t+1} + \sigma E_t\hat{\pi}_{t+1} + \sigma(1 - \phi_r)\hat{r}_t^n$$

$$(1 + \sigma\phi_y)\hat{Y}_t - E_t\hat{Y}_{t+1} - \sigma(1 - \phi_r)\hat{r}_t^n = \sigma E_t\hat{\pi}_{t+1}$$

$$E_t\hat{\pi}_{t+1} = \frac{1 + \sigma\phi_y}{\sigma}\hat{Y}_t - \frac{1}{\sigma}E_t\hat{Y}_{t+1} - (1 - \phi_r)\hat{r}_t^n$$

Then we can take any arbitrary random δ_t such that $E_t\delta_{t+1} = 0$, and if

$$\hat{\pi}_{t+1} = \frac{1+\sigma\phi_y}{\sigma}\hat{Y}_t - \frac{1}{\sigma}E_t\hat{Y}_{t+1} - (1-\phi_r)\hat{r}_t^n + \delta_{t+1}$$

So there cannot be a unique solution.

2 From the hint, we take

$$\hat{i}_t = r_t + E_t\hat{\pi}_{t+1}$$

$$\hat{i}_{t+1} = \phi_i\hat{i}_t + \phi_\pi\hat{\pi}_{t+1}$$

$$E_t\hat{i}_{t+1} = \phi_i\hat{i}_t + \phi_\pi E_t\hat{\pi}_{t+1}$$

$$E_t\hat{i}_{t+1} - \phi_i\hat{i}_t = \phi_\pi E_t\hat{\pi}_{t+1}$$

$$\hat{i}_t = r_t + \frac{1}{\phi_\pi}E_t\hat{i}_{t+1} - \frac{\phi_i}{\phi_\pi}\hat{i}_t$$

$$\frac{\phi_\pi+\phi_i}{\phi_\pi}\hat{i}_t = r_t + \frac{1}{\phi_\pi}E_t\hat{i}_{t+1}$$

$$\hat{i}_t = \frac{\phi_\pi}{\phi_\pi+\phi_i}r_t + \frac{1}{\phi_\pi+\phi_i}E_t\hat{i}_{t+1}$$

$$\hat{i}_t = \left(\frac{1}{\phi_\pi+\phi_i}\right)^T E_t\hat{i}_{t+T} + \sum_{\tau=t}^{T}\frac{\phi_\pi}{(\phi_\pi+\phi_i)^{\tau-t+1}}r_\tau$$

For a unique solution, we need $|\phi_\pi + \phi_i| > 1$.

3 From the EE, we have

$$\hat{i}_t = r_t + E_t\hat{\pi}_{t+1}$$

$$\phi_\pi E_t\hat{\pi}_{t+1} = r_t + E_t\hat{\pi}_{t+1}$$

$$(\phi_\pi - 1)E_t\hat{\pi}_{t+1} = r_t$$

$$E_t\hat{\pi}_{t+1} = \frac{1}{\phi_\pi - 1}r_t$$

Then as part 1, we can take some δ_t such that $E_t\delta_{t+1} = 0$,

$$\hat{\pi}_{t+1} = \frac{1}{\phi_\pi - 1}r_t + \delta_{t+1}$$

If $\phi_\pi = 1$, then $r = 0$, which is not possible. Hence all allowable values of ϕ_π, there is no unique bounded solution.