ECON550: Problem Set 1

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HMC Exercises (7th edition)

1.2.1(c)

$$C_1 \cup C_2 = \{(x, y) \ : \ 1 < x < 3, 1 < y < 3 \mid 0 < x < 1, 1 < x < 2\}$$

$$C_1 \cap C_2 = \{(x, y) \ : \ 1 < x < 2, 1 < y < 2\}$$

1.2.2(a)

$$C^c = \{x \ : \ 0 < x \leq 5/8\}$$

1.2.5(a)

$$C_1 \cap (C_2 \cup C_3) = (C_1 \cap C_2) \cup (C_1 \cap C_3)$$

For $x \in C_1 \cap (C_2 \cup C_3)$, $x \in C_1$ and either $x \in C_2$ or $x \in C_3$. Hence, $x \in C_1 \cap C_2$ or $x \in C_1 \cap C_3$, so $x \in (C_1 \cap C_2) \cup (C_1 \cap C_3)$. The statements follow in reverse, so these two sets are equal. We can also see this through the Venn diagram, as recommended.

1.2.8(a)

$$\lim_{k \to \infty} C_k = \{x \ : \ 0 < x < 3\}$$

1.2.9(b)

$$\lim_{k \to \infty} C_k = \emptyset$$

For any $2 + \epsilon$, $\exists k$ such that $1/k < \epsilon$. Hence $2 + \epsilon \notin C_k$, so the limit just the empty set.

1.2.10

$$Q(C_1) = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} = \frac{80}{81}$$

$$Q(C_2) = 1$$

1.2.12

$$Q(C_1) = \int_{-1}^{1} \int_{-1}^{1} x^2 + y^2 \ dx \ dy = \frac{8}{3}$$

$$Q(C_2) = \int_{-1}^{1} 2t^2 \ dt = \frac{4}{3}$$

1.3.2

$$P(C_1) = 1/4$$

$$P(C_2) = 1/13$$

$$P(C_1 \cap C_2) = 1/52$$

$$P(C_1 \cup C_2) = 4/13$$

1.3.6 This is not a probability set function because

$$\int_C e^{-|x|} dx = \int_{-\infty}^{\infty} e^{-|x|} dx = 2 \neq 1$$

With a constant normalization factor of 1/2, this will be a probability set function.

1.3.7 Note that since $C_1 \setminus (C_1 \cap C_2)$ is disjoint from $C_1 \cap C_2$, by additivity, we must have

$$P(C_1 \setminus (C_1 \cap C_2)) + P(C_1 \cap C_2) = P(C_1)$$

By nonnegativity,

$$P(C_1 \setminus (C_1 \cap C_2)) \geq 0$$

Hence

$$P(C_1) - P(C_1 \cap C_2) \geq 0$$

$$P(C_1 \geq P(C_1 \cap C_2))$$

Further, $C_2 \setminus (C_1 \cap C_2)$ is disjoint from C_1, so by additivity

$$P(C_2 \setminus (C_1 \cap C_2)) + P(C_1) = P(C_2 \cup C_1)$$

By nonnegativity, $P(C_2 \setminus (C_1 \cap C_2)) \geq 0$, so once again, we find

$$P(C_2 \cup C_1) \geq P(C_1)$$

Finally, by finite additivity

$$P(C_1) + P(C_2) = P(C_1 \setminus (C_1 \cap C_2)) + P(C_1 \cap C_2) + P(C_2 \setminus (C_1 \cap C_2)) + P(C_1 \cap C_2)$$

$$= P(C_1 \cup C_2) + P(C_1 \cap C_2)$$

Since $P(C_1 \cap C_2) \geq 0$ by nonnegativity, we get

$$P(C_1) + P(C_2) \geq P(C_1 \cup C_2)$$

and we are done.

1.3.20 Take

$$C_k = (a - 1/k, a + 1/k)$$

for k large enough that this is a subinterval of $(0, 1)$. This is possible because $(0, 1)$ is open, and hence we can always find some neighborhood of a contained in $(0, 1)$. Now, we have that

$$\lim_{k\to\infty} C_k = \{a\}$$

Using expression (1.3.8), we get

$$P(\{a\}) = \lim_{k\to\infty} P(C_k) = \lim_{k\to\infty} \frac{1}{2k} = 0$$

1.3.22 Consider any $A \in \mathcal{B}$. Then by definition $A \in \mathcal{E} \supset \mathcal{D}$ for all σ-fields $\mathcal{E}$. But since $\mathcal{E}$ is a σ-field, $A^c \in \mathcal{E}$ for all $\mathcal{E}$. Then, by definition of $\mathcal{B}$, $A^c \in \mathcal{B}$. Hence $\mathcal{B}$ is closed under complements.

Now, we just have to show closure under countable union. Suppose $A_1, A_2, ... \in \mathcal{B}$. Then $A_i \in \mathcal{E} \supset \mathcal{D}$ for all σ-fields $\mathcal{E}$. Since $\mathcal{E}$ are σ-fields, they are all closed under countable union, and hence $\cup A_i \in \mathcal{E}$ for all σ-fields $\mathcal{E} \supset \mathcal{D}$. Thus, by definition of $\mathcal{B}$, $\cup A_i \in \mathcal{B}$. Hence $\mathcal{B}$ is closed under countable union, and hence $\mathcal{B}$ is a σ-field.

1.3.23 Note $(b, \infty) \in \mathcal{B}_0$, so by closure under complements, $(-\infty, b]$. Hence, for $b > a$, by closure under union, since $(-\infty, b] \in \mathcal{B}_0$ and $[a, \infty) \in \mathcal{B}_0$, we must have $[a, b] \in \mathcal{B}_0$. Hence, $\mathcal{B}_0$ contains all the closed intervals as well.

Since $\mathcal{B}_0$ contains all closed intervals, $[a, a] = \{a\} \in \mathcal{B}_0$. Hence, if $b < a$, since $(b, a) \in \mathcal{B}_0$, we have $(b, a) \cup [a, a] = (b, a] \in \mathcal{B}_0$. Similarly, if $a < b$, then since $(a, b) \in \mathcal{B}_0$, we have $(a, b) \cup [a, a] = [a, b) \in \mathcal{B}_0$. Thus, $\mathcal{B}_0$ contains all the half-open intervals as well.

Problem 1

A σ-field must be closed under complements and countable unions. Hence, we can take the following procedure to enumerate the sets in the σ-field generated by $F = \{A, B, C\}$:

1. Add all complements of sets in F to F.
2. Add unions of every subset of sets in F to F.
3. Repeat 1 and 2 until no more new sets are added to F.

It is very clear that the resulting F is closed under complements; since step 1 in the procedure ensures that for any $S \in F$, $S^c \in F$.

We claim that F is finite. To show this, consider the function $f : \Omega \to \{0, 1\}^3$ defined by

$$f(\omega) = (\mathbb{1}_{\omega\in A}, \mathbb{1}_{\omega\in B}, \mathbb{1}_{\omega\in C})$$

where $\mathbb{1}_{\omega\in S}$ is 1 if $\omega \in S$ and 0 otherwise. We first argue that every set in F is the union of preimages of f, or $\cup_i f^{-1}(v_i)$ where $v_i \in \{0, 1\}^3$.

We first note that if

$$S_1 = \bigcup_{v \in T_1} f^{-1}(v)$$

and

$$S_2 = \bigcup_{v \in T_2} f^{-1}(v)$$

then

$$S_1 \cup S_2 = \bigcup_{v \in T_1 \cup T_2} f^{-1}(v)$$

Further, we note that if

$$S = \bigcup_{v \in T} f^{-1}(v)$$

then

$$S^c = \bigcup_{v \in T^c} f^{-1}(v)$$

Hence, we have shown that unions of preimages of f are closed under unions and complements. We also know that

$$A = \bigcup_{(i,j)\in\{0,1\}^2} f^{-1}(\{1, i, j\})$$

$$B = \bigcup_{(i,j)\in\{0,1\}^2} f^{-1}(\{i, 1, j\})$$

$$C = \bigcup_{(i,j)\in\{0,1\}^2} f^{-1}(\{i, j, 1\})$$

Hence, by our generation of F, we know that every element of F must be a union of preimages of f. Since there are $2^3 = 8$ preimages, F can have at most $2^8 = 256$ different elements. Hence, since the generation can produce at most a finite number of sets, we only have to show closure of F under finite union. But closure under finite union follows immediately from our generation process; by step 2, we ensure that every finite union of sets in F is in F. Hence, we prove that our method will terminate and produces the σ-field generated by A, B, C.

Problem 2

(a) Define

$$A_i = \left(\frac{1}{2^i}, \frac{1}{2^{i-1}}\right]$$

Note that all of the A_i are disjoint with each other, and their union $\cup_{i=1}^{\infty} A_i$ is $(0, 1]$.

(b) To show this is a field, we show this is closed under complements and finite unions.

We first show closure under complements. Consider $A = \cup_{j=1}^{J}(a_j, b_j]$, where $0 \leq a_1 \leq b_1 \leq a_2 ... \leq b_J \leq 1$.

If we define $b_0 = 0$, $a_{j+1} = 1$. Let us define $b_0 = 0$ and $a_{J+1} = 1$. Then we have that

$$A^c = \cup_{j=1}^{J+1}(b_{j-1}, a_j]$$

Note that we may have some $b_{i-1} = a_i$ for some i, in which case we say the interval is simply the null set (which doesn't affect the union). But we can clearly see that A^c is also a union of disjoint half-open half-closed subintervals, and hence $A^c \in \mathcal{F}_0$

Now, we show closure under finite union. By definition $\mathcal{F}_0$ contains all finite unions of disjoint half-open half-closed subintervals of $(0, 1]$. We first show that a union of two non-disjoint half-closed subintervals is a half-open half-closed subinterval. WLOG, let the intervals be $(a_1, b_1]$ and $(a_2, b_2]$, and WLOG suppose $a_1 < b_1$, $a_2 < b_2$. Since these are nondisjoint, $a_2 \leq b_1$. But the union of these two is then given by $(a_1, b_2]$. Now, for any finite union of sets $A_1, A_2 ... A_n \in \mathcal{F}_0$, we note that the union $\cup A_i$ is a finite number of potentially non-disjoint half-open half-closed intervals. However, by the statement we just showed, we can repeatedly union pairs of non-disjoint intervals together into single half-open half-closed intervals until we are left with a set of disjoint half-open half-closed intervals. Since there are only a finite number of potentially non-disjoint half-open half-closed intervals we need to union, we will be left with a finite number of disjoint half-open half-closed intervals. Hence the union $\cup A_i \in \mathcal{F}_0$, so we have shown $\mathcal{F}_0$ is a field.

(c) We show that the measure μ satisfies the three axioms.

First, we show that the measure is always nonnegative. Consider $\mu(A)$, where $A = \cup_{j=1}^{J}(a_j, b_j]$. By definition, $b_i > a_i \ \forall i$. Thus,

$$\mu(A) = \sum_{j=1}^{J}(b_j - a_j) \geq 0$$

Second, we show that $\mu(\Omega) = 1$. But this follows by definition:

$$\mu(\Omega) = \mu((0, 1]) = 1 - 0 = 1$$

Last, we show the countable additivity property of μ on $\mathcal{F}_0$. Let $A_1, A_2, ...$ be a countable sequence of disjoint sets in $\mathcal{F}_0$, such that $\cup_{i=1}^{\infty} A_i \in \mathcal{F}_0$. Each A_i is a finite union of disjoint half-open half-closed intervals $\cup_{j=1}^{J}(a_{ij}, b_{ij}]$, so since all the A_i are disjoint by assumption, every $(a_{ij}, b_{ij}]$ is disjoint from $(a_{i'j'}, b_{i'j'}]$, where at least one of $i \neq i'$ and $j \neq j'$ is true. We have

$$\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} \cup_{j=1}^{J_i} (a_{ij}, b_{ij}]$$

Since a countable union of finite unions is also countable, and we argued that every pair of intervals is disjoint, we can invoke the suggested assumption of countable additivity of μ on intervals to obtain:

$$\mu\left(\cup_{i=1}^{\infty} A_i\right) = \mu\left(\cup_{i=1}^{\infty} \cup_{j=1}^{J_i} (a_{ij}, b_{ij}]\right)$$

$$= \cup_{i=1}^{\infty} \cup_{j=1}^{J_i} \mu((a_{ij}, b_{ij}])$$

$$= \cup_{i=1}^{\infty} \cup_{j=1}^{J_i} (b_{ij} - a_{ij})$$

$$= \cup_{i=1}^{\infty} \mu(A_i)$$

as desired. Hence we have shown μ satisfies the three axioms of probability.

Problem 3

To show this is a σ-field, we show that it is closed under complements and is closed under countable union. These together imply that $\Omega \in F$, since for any $A \in F$, $A^c \in F$ and hence $A \cup A^c = \Omega \in F$.

To show closure under complement, we consider some arbitrary subset C of $\Omega \in F$. By definition, this is generated by some $B \in \mathcal{B}(\Omega_X)$, such that

$$C = \{\omega \in \Omega \ : \ X(\omega) \in B\}$$

Since $\mathcal{B}(\Omega_X)$ is a σ-field, $B^c \in \mathcal{B}(\Omega_X)$. We claim

$$C' = \{\omega \in \Omega \ : \ X(\omega) \in B^c\}$$

is the complement of C. For any arbitrary $\omega \notin C$, $X(\omega) \notin B$, implying $X(\omega) \in B^c$ and hence $\omega \in C'$. Therefore $C^c \subseteq C'$. Further, for any $\omega \in C'$, $X(\omega) \in B^c$ and hence $X(\omega) \notin B$, so $\omega \notin C$. So $C' \subseteq C^c$, and hence $C' = C^c \in F$. So F is closed under complements.

Now, we show closure under countable unions. Suppose sets $C_1, C_2, ... \in F$, and let their respective sets B_i be such that

$$C_i = \{\omega \in \Omega \ : \ X(\omega) \in B_i\}$$

Then let $B^* = \cup B_i$, and define

$$C^* = \{\omega \in \Omega \ : \ X(\omega) \in B^*\}$$

We claim $C^* = \cup C_i$. First, consider $\omega \in C^*$. Then $X(\omega) \in B^*$, by definition of C^*. So $X(\omega) \in B_k$ for some k, since $B^* = \cup B_i$. Hence $\omega \in C_k$, so $\omega \in \cup C_i$. Therefore, $C^* \subseteq \cup C_i$.

Now, suppose $\omega \in \cup C_i$. Then $\omega \in C_k$ for some k. This implies $X(\omega) \in B_k$, so $X(\omega) \in B^*$. Hence $\omega \in C^*$. Therefore $\cup C_i \subseteq C^*$.

Together, these two imply $C^* = \cup C_i \in F$. Therefore, F is closed under countable union and complements, so by our earlier argument, $\Omega \in F$, and so this is a σ-field.