ECON550: Problem Set 9

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Problem 1

Taking the canonical projection into the first dimension, we get

$$\sqrt{n}\left(\hat{\beta}_{n,1} - \beta_{0,1}\right) \to_d N(0, \Sigma_{11})$$

$$\frac{\sqrt{n}}{\sqrt{\hat{\Sigma_{11}}}}\left(\hat{\beta}_{n,1} - \beta_{0,1}\right) \to_d N(0, 1)$$

The confidence interval is then

$$\left[\hat{\beta}_{n,1} - \frac{\sqrt{\hat{\Sigma_{11}}}}{\sqrt{n}} z_{1-\alpha/2}, \hat{\beta}_{n,1} + \frac{\sqrt{\hat{\Sigma_{11}}}}{\sqrt{n}} z_{1-\alpha/2}\right]$$

$$= [3.2 - \frac{\sqrt{1.7}}{\sqrt{n}} 2.576, 3.2 + \frac{\sqrt{1.7}}{\sqrt{n}} 2.576]$$

Problem 2

(a) Taking the canonical projection mapping into 1d, we can apply the CMT to get that $\sqrt{n}(\hat{\theta}_{n,1} - \theta_{0,1}) \to_d N(0, \Sigma_{11})$ Since $\hat{\Sigma_{11}} \to \Sigma_{11}$, we get

$$\frac{\sqrt{n}}{\sqrt{\hat{\Sigma}_{11}}}(\hat{\theta}_{n,1} - \theta_{0,1}) \to_d N(0, 1)$$

So the interval is

$$\left[\hat{\theta}_{n,1} - \frac{\sqrt{\hat{\Sigma}_{11}}}{\sqrt{n}} z_{1-\alpha/2}, \hat{\theta}_{n,1} + \frac{\sqrt{\hat{\Sigma}_{11}}}{\sqrt{n}} z_{1-\alpha/2}\right]$$

$$= \left[1.2 - \frac{1}{\sqrt{n}} 1.96, 1.2 + \frac{1}{\sqrt{n}} 1.96\right]$$

(b) We use the delta method to characterize the asymptotic distribution of $\sqrt{n}\left(\hat{\theta}_{n,1}\hat{\theta}_{n,2} - \theta_{0,1}\theta_{0,2}\right)$. Take $g(x, y) = xy$. Then $G(\theta_0)$ is

$$G(\theta_0) = \begin{bmatrix} \theta_{0,2} & \theta_{0,1} \end{bmatrix}$$

We have that $G(\hat{\theta}_n) \to_p G(\theta_0)$, $\hat{\Sigma} \to_p \Sigma$, so we have that

$$\frac{\sqrt{n}}{\sqrt{G(\hat{\theta}_n)\hat{\Sigma}G(\hat{\theta}_n)'}}\left(\hat{\theta}_{n,1}\hat{\theta}_{n,2} - \theta_{0,1}\theta_{0,2}\right) \to_d N(0,1)$$

So the interval is

$$\left[\hat{\theta}_{n,1}\hat{\theta}_{n,2} - \frac{\sqrt{G(\hat{\theta}_n)\hat{\Sigma}G(\hat{\theta}_n)'}}{\sqrt{n}} z_{1-\alpha/2}, \hat{\theta}_{n,1}\hat{\theta}_{n,2} + \frac{\sqrt{G(\hat{\theta}_n)\hat{\Sigma}G(\hat{\theta}_n)'}}{\sqrt{n}} z_{1-\alpha/2}\right]$$

$$= [2.76 - \frac{\sqrt{8.41}}{\sqrt{n}} 1.645, 2.76 + \frac{\sqrt{8.41}}{\sqrt{n}} 1.645]$$

Problem 3

(a) The interval is given by

$$\left[\hat{\beta}_{LS} - \sqrt{\frac{\hat{\sigma}^2}{ns_X^2}} t_{n-2,1-\alpha/2}, \hat{\beta}_{LS} + \sqrt{\frac{\hat{\sigma}^2}{ns_X^2}} t_{n-2,1-\alpha/2}\right]$$

$$= \left[1.4 - \frac{2}{15} t_{23,0.975}, 1.4 + \frac{2}{15} t_{23,0.975}\right]$$

$$= [1.12, 1.68]$$

(b) Exact, since the t statistic follows the t distribution.

(c) The one sided interval is

$$\left[\hat{\beta}_{LS} - \sqrt{\frac{\hat{\sigma}^2}{ns_X^2}} t_{n-2,1-\alpha}, \infty\right)$$

$$= [1.07, \infty)$$

Problem 4

(a) The log-likelihood is

$$k \log \theta + (n-k) \log(1-\theta)$$

where k is the number of i such that $X_i = 1$. The maximization FOC is

$$\frac{k}{\hat{\theta}} = \frac{n-k}{1-\hat{\theta}}$$

$$\hat{\theta}(n-k) = k(1-\hat{\theta})$$

$$\hat{\theta}n = k$$

$$\hat{\theta} = k/n = n^{-1} \sum_i X_i = \bar{X}_n$$

So this is our MLE.

(b) The log-likelihood is (dropping constant terms)

$$-\frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum_i (X_i - \mu)^2$$

Solving the FOC for μ, we get $\mu = \bar{X}_n$. Solving the FOC for σ^2,

$$\frac{n}{2} = \frac{1}{2\sigma^2} \sum_i (X_i - \bar{X}_n)^2$$

$$\hat{\sigma}^2 = n^{-1} \sum_i (X_i - \bar{X}_n)^2$$

Problem 5

(a)

$$EX_1 = \int_0^\infty \lambda x e^{-\lambda x} = -xe^{-\lambda x}\Big|_0^\infty + \int e^{-\lambda x}\Big|_0^\infty$$

$$= 0 - \frac{-1}{\lambda} = 1/\lambda$$

(b) The log likelihood is

$$n \log \lambda - \lambda \sum X_i$$

The MLE is

$$\hat{\lambda} = \frac{n}{\sum X_i}$$

(c) By the WLLN $\bar{X}_n \to_p EX_1 = 1/\lambda$ so by Slutsky's theorem,

$$\hat{\lambda} \to_p 1/(1/\lambda) = \lambda$$

And hence $\hat{\lambda}$ is consistent.

(d) We note that

$$E[X_1^2] = \int_0^\infty \lambda x^2 e^{-\lambda x} = \frac{2}{\lambda^2}$$

$$VX_1 = E[X_1^2] - (EX_1)^2 = \frac{1}{\lambda^2}$$

By the CLT, $\sqrt{n}(\bar{X}_n - 1/\lambda) \to_d N(0, 1/\lambda^2)$. By the delta method, taking $g(x) = 1/x$, $g'(1/\lambda) = -\lambda^2$ so we have

$$\sqrt{n}(\hat{\lambda} - \lambda) \to_d N(0, \lambda^2)$$

Problem 6

Let $g(a,b) = \sqrt{b}/a$. Then the desired distribution is

$$\sqrt{n}\left(g\left(\begin{bmatrix}\bar{X}_n \\ S_{X_n}^2\end{bmatrix}\right) - g\left(\begin{bmatrix}\mu \\ \sigma^2\end{bmatrix}\right)\right)$$

By the delta method, we have

$$G(\theta_0) = \begin{bmatrix}\frac{-\sigma}{\mu^2} & \frac{1}{2\mu\sigma}\end{bmatrix}$$

So

$$\sqrt{n}\left(g\left(\begin{bmatrix}\bar{X}_n \\ S_{X_n}^2\end{bmatrix}\right) - g\left(\begin{bmatrix}\mu \\ \sigma^2\end{bmatrix}\right)\right) \to_d N(0, G(\theta_0)\Sigma G(\theta_0)')$$

By Slutsky's theorem,

$$\sqrt{S_{X_n}^2} \to_p \sigma$$

and we know

$$\bar{X_n} \to_p \mu$$

Hence by the rules of convergence in probability, the estimator

$$\hat{G} = \begin{bmatrix}\frac{-\sqrt{S_{X_n}^2}}{\bar{X_n}^2} & \frac{1}{2\bar{X_n}\sqrt{S_{X_n}^2}}\end{bmatrix} \to_p \begin{bmatrix}\frac{-\sigma}{\mu^2} & \frac{1}{2\mu\sigma}\end{bmatrix} = G(\theta_0)$$

We know

$$(\hat{G}\hat{\Sigma}\hat{G}')^{-1/2}\sqrt{n}\left(\frac{S_{X_n}}{\bar{X_n}} - \frac{\sigma}{\mu}\right) \to_p N(0,1)$$

So the desired confidence interval is

$$\left[\frac{S_{X_n}}{\bar{X_n}} - n^{-1/2}(\hat{G}\hat{\Sigma}\hat{G}')^{1/2} z_{0.975}, \frac{S_{X_n}}{\bar{X_n}} + n^{-1/2}(\hat{G}\hat{\Sigma}\hat{G}')^{1/2} z_{0.975}\right]$$

To show that this does indeed asymptotically contain the true value σ/μ with probability 0.95, we note that in PS8 we already showed

$$(\hat{G}\hat{\Sigma}\hat{G}')^{-1/2}\sqrt{n}\left(\frac{S_{X_n}}{\bar{X_n}} - \frac{\sigma}{\mu}\right) \to_p N(0,1)$$

Hence

$$\begin{aligned}
P\left(\sigma/\mu \in \left[\frac{S_{X_n}}{\bar{X_n}} - n^{-1/2}(\hat{G}\hat{\Sigma}\hat{G}')^{1/2} z_{0.975}, \frac{S_{X_n}}{\bar{X_n}} + n^{-1/2}(\hat{G}\hat{\Sigma}\hat{G}')^{1/2} z_{0.975}\right]\right) \\
= P\left(\left|\frac{S_{X_n}}{\bar{X_n}} - \sigma/\mu\right| \leq n^{-1/2}(\hat{G}\hat{\Sigma}\hat{G}')^{1/2} z_{0.975}\right) \\
= P\left(\hat{G}\hat{\Sigma}\hat{G}')^{-1/2}\sqrt{n}\left|\frac{S_{X_n}}{\bar{X_n}} - \frac{\sigma}{\mu}\right| \leq z_{0.975}\right) \\
= 0.95
\end{aligned}$$

Problem 7

Fix $\epsilon > 0$. We wish to show that for c_n, as $n \to \infty$, $P(|\bar{X}_n| \geq \epsilon) \to 0$. Rewriting $\bar{X}_n$ as suggested, we get

$$P(|\bar{X}_n| \geq \epsilon) = P\left(\left|n^{-1}\sum X_i\{|X_i| \leq c_n\}\right| + \left|n^{-1}\sum X_i\{|X_i| > c_n\}\right| \geq \epsilon\right)$$

$$\leq P\left(\left|n^{-1}\sum X_i\{|X_i| \leq c_n\}\right| \geq \epsilon\right) + P\left(\left|n^{-1}\sum X_i\{|X_i| > c_n\}\right| \geq \epsilon\right)$$

$$= P\left(\left|\sum X_i\{|X_i| \leq c_n\}\right| \geq n\epsilon\right) + P\left(\left|n^{-1}\sum X_i\{|X_i| > c_n\}\right| \geq \epsilon\right)$$

$$\leq \frac{c_n^2}{(n\epsilon)^2} + P\left(\left|n^{-1}\sum X_i\{|X_i| > c_n\}\right| \geq \epsilon\right)$$

where we applied Chebyshev to the left term, and noticed that $X_i\{X_i \leq c_n\}$ is at most c_n. We can apply Markov to the right term to get

$$\leq \frac{c_n^2}{(n\epsilon)^2} + \frac{E\left[|\sum X_i\{|X_i| > c_n\}|\right]}{n\epsilon}$$

$$=\leq \frac{c_n^2}{(n\epsilon)^2} + \frac{E\left[|X_i\{|X_i| > c_n\}|\right]}{\epsilon}$$

Note that this is dominated by $|x|$, and so by the DCT, $E\left[|X_i\{|X_i| > c_n\}|\right] \leq E|X|$, and as $c_n \to \infty$, $\{|X_i| > c_n\} \to 0$, so we have that as $n \to \infty$, the right term approaches 0. Take $c_n = n^{1/3}$. Then the left term also disappears as $n \to \infty$, so as $n \to \infty, P(|\bar{X}_n| \geq \epsilon) \to 0$ for fixed ϵ, and hence we are done.

Problem 8

(a) By the CLT, $\sqrt{n}(\bar{X}_n - \mu) \to_d N(0, \sigma^2)$. Then by the delta method, taking $g(x) = 1/x$, we have $g'(\mu) = -\frac{1}{\mu^2}$, so

$$\sqrt{n}(1/\bar{X}_n - 1/\mu) \to_d N(0, \sigma^2/\mu^4)$$

(b) By the CLT, $\sqrt{n}\bar{X}_n \to_d N(0, \sigma^2)$. Applying the CMT,

$$n^{-1/2}(1/\bar{X}_n) \to_d 1/N(0, \sigma^2)$$

Note in this case b_n is 0.

Problem 9

(a) We have

$$\frac{\partial \log f}{\partial \theta} = \frac{x}{\theta} - \frac{1-x}{1-\theta}$$

$$\frac{\partial^2 \log f}{\partial \theta^2} = -\frac{x}{\theta^2} - \frac{1-x}{(1-\theta)^2}$$

$$E\left[\frac{\partial^2 \log f}{\partial \theta^2}\right] = -\frac{\theta}{\theta^2} - \frac{1-\theta}{(1-\theta)^2} = -\frac{1}{\theta} - \frac{1}{1-\theta}$$

$$I(\theta) = \frac{1}{\theta} + \frac{1}{1-\theta}$$

$$I(\theta)^{-1} = \frac{1}{\frac{1}{\theta} + \frac{1}{1-\theta}} = \theta(1-\theta)$$

By Slutsky's theorem, since $\bar{X}_n \to_p \theta$ from WLLN, $\bar{X}_n(1-\bar{X}_n) \to_p \theta(1-\theta)$. By Slutsky, $\sqrt{\bar{X}_n(1-\bar{X}_n)} \to_p \sqrt{\theta(1-\theta)}$.

(b) We have

$$\frac{\partial \log f}{\partial \mu} = \frac{x-\mu}{\sigma^2}$$

$$\frac{\partial \log f}{\partial \sigma^2} = -\frac{1}{2\sigma^2} + \frac{(x-\mu)^2}{2\sigma^4}$$

$$\frac{\partial^2 \log f}{\partial \mu^2} = -\frac{1}{\sigma^2}$$

$$\frac{\partial^2 \log f}{\partial \mu \partial \sigma^2} = -\frac{x-\mu}{\sigma^4}$$

$$\frac{\partial^2 \log f}{(\partial \sigma^2)^2} = \frac{1}{2\sigma^4} - \frac{(x-\mu)^2}{\sigma^6}$$

$$E\left[\frac{\partial^2 \log f}{\partial \mu^2}\right] = -\frac{1}{\sigma^2}$$

$$E\left[\frac{\partial^2 \log f}{\partial \mu \partial \sigma^2}\right] = 0$$

$$E\left[\frac{\partial^2 \log f}{(\partial \sigma^2)^2}\right] = \frac{1}{2\sigma^4} - \frac{\sigma^2}{\sigma^6} = -\frac{1}{2\sigma^4}$$

The information matrix is then

$$I(\theta) = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{bmatrix}$$

I is nicely diagonal, so its inverse is

$$I(\theta)^{-1} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix}$$

Now, $\hat{S}_X^2$ is a consistent estimator of σ^2 as we have shown in class, so

$$\begin{bmatrix} \hat{S}_X^2 & 0 \\ 0 & 2\hat{S}_X^4 \end{bmatrix} \to_p I(\theta)^{-1}$$

by Slutsky's theorem. Finally, $\sqrt{\hat{S}_X^2}$ is a consistent estimator of the error of $\sqrt{n}(\bar{X}_n - \mu)$ and $\sqrt{2}\hat{S}_X^2$ is a consistent estimator of the error of $\sqrt{n}(\hat{S}_X^2 - \sigma^2)$.

Problem 10

(a) f is a probability density, so

$$1 = \int f(y,\theta)d\mu$$

Then by assumption iii, we have

$$0 = \frac{\partial}{\partial\theta}\int f(y,\theta)d\mu = \int \frac{\partial}{\partial\theta} f(y,\theta)d\mu$$

Then we have

$$\begin{aligned} E_\theta \frac{\partial}{\partial\theta}\log f &= \int \left(\frac{\partial}{\partial\theta}\log f(y,\theta)\right) f(y,\theta)d\mu \\ &= \int \left(\frac{\partial}{\partial\theta} f(y,\theta)\right) d\mu \\ &= 0 \end{aligned}$$

(b) From the previous part, we have f is a probability density, so

$$1 = \int f(y,\theta)d\mu$$

Then by assumption iv, we have

$$0 = \frac{\partial^2}{\partial\theta\partial\theta'}\int f(y,\theta)d\mu = \int \frac{\partial^2}{\partial\theta\partial\theta'} f(y,\theta)d\mu$$

Since we have

$$\left(\frac{\partial}{\partial\theta}\log f(y,\theta)\right) f(y,\theta) = \frac{\partial}{\partial\theta} f(y,\theta)$$

we can differentiate both sides to get

$$\left(\frac{\partial^2}{\partial\theta\partial\theta'}\log f(y,\theta)\right) f(y,\theta) + \left(\frac{\partial}{\partial\theta}\log f(y,\theta)\right)\frac{\partial}{\partial\theta'} f(y,\theta) = \frac{\partial^2}{\partial\theta\partial\theta'} f(y,\theta)$$

Integrating, we get

$$\begin{aligned} \int \left(\frac{\partial^2}{\partial\theta\partial\theta'}\log f(y,\theta)\right) f(y,\theta) + \left(\frac{\partial}{\partial\theta}\log f(y,\theta)\right)\frac{\partial}{\partial\theta'} f(y,\theta) &= \int \frac{\partial^2}{\partial\theta\partial\theta'} f(y,\theta) = 0 \\ \int \left(\frac{\partial^2}{\partial\theta\partial\theta'}\log f(y,\theta)\right) f(y,\theta) + \int \left(\frac{\partial}{\partial\theta}\log f(y,\theta)\right)\frac{\partial}{\partial\theta'} f(y,\theta) &= 0 \\ E_\theta\left(\frac{\partial^2}{\partial\theta\partial\theta'}\log f(y,\theta)\right) + \int \left(\frac{\partial}{\partial\theta}\log f(y,\theta)\right)\left(\frac{\partial}{\partial\theta'}\log f(y,\theta)\right) f(y,\theta) &= 0 \\ E_\theta\left(\frac{\partial^2}{\partial\theta\partial\theta'}\log f(y,\theta)\right) + E_\theta\left[\left(\frac{\partial}{\partial\theta}\log f(y,\theta)\right)\left(\frac{\partial}{\partial\theta'}\log f(y,\theta)\right)\right] &= 0 \\ -E_\theta\left(\frac{\partial^2}{\partial\theta\partial\theta'}\log f(y,\theta)\right) &= E_\theta\left[\left(\frac{\partial}{\partial\theta}\log f(y,\theta)\right)\left(\frac{\partial}{\partial\theta'}\log f(y,\theta)\right)\right] \end{aligned}$$

Problem 11

(a) Taking the log-likelihood, we get

$$\mathcal{L} = n \log 2 + \sum_i \log X_i - 2 \log \theta - \infty(\theta < \max X_i)$$

Since X_i cannot be more than θ, we have to constrain the log-likelihood to $\theta \geq \max X_i$. Since the remainder of this expression is strictly decreasing in θ, the MLE is then the smallest allowable θ, so

$$\hat{\theta} = \max X_i$$

(b) The CDF of $\hat{\theta}$ is given by $F_{\hat{\theta}}(t) = F_{X_i}(t)^n = (t/\theta)^{2n}$. So $f_{\hat{\theta}}(t) = \frac{2n}{\theta}(t/\theta)^{2n-1}$. Then $E[\hat{\theta}]$ is

$$\int_0^\theta 2n(t/\theta)^{2n} = \frac{2n\theta}{2n+1}$$

So the desired constant is

$$\frac{2n+1}{2n}$$