ECON550: Problem Set 5

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Problem 1

By the definition of convergence in distribution, we must have that

$$\lim_{n\to\infty} \Pr(Y_n \leq y) = \phi(y)$$

Then

$$\begin{aligned}\lim_{n\to\infty} \Pr(-Y_n \leq y) &= \lim_{n\to\infty} \Pr(Y_n \geq -y)\\ &= \lim_{n\to\infty} (1 - \Pr(Y_n \leq -y))\\ &= 1 - \lim_{n\to\infty} \Pr(Y_n \leq -y)\\ &= 1 - \phi(-y)\\ &= \phi(y)\end{aligned}$$

Hence $-Y_n \to_d Z$.

Problem 2

Consider an arbitrary element $\hat{\Sigma}_n[i,j]$.

$$\hat{\Sigma}_n[i,j] = \frac{1}{n}\left(\sum_{t=1}^{n}(X_t[i] - \overline{X}_n[i])(X_t[j] - \overline{X}_n[j])\right)$$

$$= \frac{1}{n}\left(\sum_{t=1}^{n}(X_t[i]X_t[j] - \overline{X}_n[i]X_t[j] - \overline{X}_n[j]X_t[i] + \overline{X}_n[j]\overline{X}_n[i])\right)$$

$$= \frac{1}{n}\left(\sum_{t=1}^{n} X_t[i]X_t[j] - \overline{X}_n[i]\sum_{t=1}^{n} X_t[j] - \overline{X}_n[j]\sum_{t=1}^{n} X_t[i] + n\overline{X}_n[j]\overline{X}_n[i]\right)$$

$$= \frac{1}{n}\left(\sum_{t=1}^{n} X_t[i]X_t[j] - n\overline{X}_n[i]\overline{X}_n[j] - n\overline{X}_n[j]\overline{X}_n[i] + n\overline{X}_n[j]\overline{X}_n[i]\right)$$

$$= \frac{1}{n}\left(\sum_{t=1}^{n} X_t[i]X_t[j] - n\overline{X}_n[i]\overline{X}_n[j]\right)$$
$$= \frac{1}{n}\left(\sum_{t=1}^{n} X_t[i]X_t[j]\right) - \overline{X}_n[i]\overline{X}_n[j]$$

Now, by the WLLN, the first term converges in probability to $E[X[i]X[j]]$, $\overline{X}_n[i] \to_p \mu[i]$, $\overline{X}_n[j] \to_p \mu[j]$. Then from the rules for convergence in probability shown in lecture, we get

$$\hat{\Sigma}_n[i,j] \to_p E[X[i]X[j]] - \mu[i]\mu[j] = \Sigma[i,j] + \mu[i]\mu[j] - \mu[i]\mu[j] = \Sigma[i,j]$$

Hence

$$\hat{\Sigma}_n \to_p \Sigma$$

Problem 3

Suppose $\sqrt{n}(\hat{\theta_n} - \theta) \to_d Z = N(0, \Sigma)$. Once again by the mean value theorem, $\exists$ a vector θ_n^* which elementwise is between $\theta, \hat{\theta_n}$ such that

$$\sqrt{n}\left(g(\hat{\theta}_n) - g(\theta)\right) = G(\theta_n^*)\sqrt{n}(\hat{\theta_n} - \theta)$$

Again,

$$G(\theta_n^*) \to_p G(\theta)$$

Then

$$\sqrt{n}\left(g(\hat{\theta}_n) - g(\theta)\right) = G(\theta_n^*)\sqrt{n}(\hat{\theta_n} - \theta)$$
$$\to_d G(\theta)N(0, \Sigma) = N(0, G\Sigma G')$$

Problem 4

We know that

$$H_{n,1} \to_p \mu$$

and by the WLLN:

$$H_{n,2} \to_p E[X^2] = \sigma^2 + \mu^2$$

Let $H = (\mu,\ \sigma^2 + \mu^2)'$. Further we note that if we take

$$Y_i = \begin{bmatrix} X_i \\ X_i^2 \end{bmatrix}$$

$$Y = \begin{bmatrix} X \\ X^2 \end{bmatrix}$$

We compute the variance matrix of Y, Σ. Then Y has the variance matrix given by:

$$\Sigma = \begin{bmatrix} \sigma^2 & E[X^3] - \mu(\sigma^2 + \mu^2) \\ E[X^3] - \mu(\sigma^2 + \mu^2) & E[X^4] - (\sigma^2 + \mu^2)^2 \end{bmatrix}$$

Then by the multivariate CLT, we have

$$\sqrt{n}(H_n - H) \to_d N(0, \Sigma)$$

Problem 5

Suppose $Y_n \to_d c$. Then by definition,

$$\lim_{n\to\infty} \Pr(Y_n \le y) = \mathbb{1}_{c \le y}$$

$$\lim_{n\to\infty} \Pr(Y_n \ge y) = \lim_{n\to\infty} 1 - \Pr(Y_n \le y) = 1 - \mathbb{1}_{c\le y} = \mathbb{1}_{c \ge y}$$

Since $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr(Y_n > c + \epsilon) = 0$$

$$\lim_{n\to\infty} \Pr(Y_n < c - \epsilon) = 0$$

Then

$$\begin{aligned} \lim_{n\to\infty} \Pr(|Y_n - c| > \epsilon) &= \lim_{n\to\infty} \Pr(Y_n > c + \epsilon) + \lim_{n\to\infty} \Pr(Y_n < c - \epsilon) \\ &= 0 \end{aligned}$$

And therefore $Y_n \to_p c$.

Now suppose $Y_n \to_p c$. Then for any $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr(|Y_n - c| > \epsilon) = 0$$

$$\lim_{n\to\infty} \Pr(Y_n < c - \epsilon) \le \lim_{n\to\infty} \Pr(|Y_n - c| > \epsilon) = 0$$

$$\lim_{n\to\infty} \Pr(Y_n < c - \epsilon) = 0$$

This implies that if $y < c$

$$\lim_{n\to\infty} \Pr(Y_n < y) = 0$$

Similarly,

$$\lim_{n\to\infty} \Pr(Y_n > c + \epsilon) \le \lim_{n\to\infty} \Pr(|Y_n - c| > \epsilon) = 0$$

so

$$\lim_{n\to\infty} \Pr(Y_n > c + \epsilon) = 0$$

This implies that if $y \ge c$,

$$\lim_{n\to\infty} \Pr(Y_n < y) = \lim_{n\to\infty} 1 - \Pr(Y_n > y) = 1$$

And hence together with the prevvious observation

$$\lim_{n\to\infty} \Pr(Y_n < y) = \mathbb{1}_{c\le y}$$

This implies that $Y_n \to_d c$. So $Y_n \to_p c \iff Y_n \to_d c$.

Problem 6

(a) By the delta method,

$$\sqrt{n}(\hat{\theta}_n - \theta)'c = \sqrt{n}(\hat{\theta}_n' c - \theta' c) \to_d W'c = N(0, c'\Sigma c)$$

(b) We can just take $c = (1, 0, 0, 0, ...)$ from the previous part to

$$\sqrt{n}(\hat{\theta}_{n,1} - \theta_1) \to_d N(0, \Sigma_{11})$$

(c) We can apply the CMT using $h(z) = z'z$. Then

$$n(\hat{\theta}_n - \theta)'(\hat{\theta}_n - \theta) = (\sqrt{n}(\hat{\theta}_n - \theta))'(\sqrt{n}(\hat{\theta}_n - \theta))$$

$$\to W'W$$

(d) We can take $c = (1, -1, 0, 0, ...)$ from part (a) to get

$$\sqrt{n}((\hat{\theta}_{n,1} - \hat{\theta}_{n,2}) - (\theta_1 - \theta_2)) \to_d N(0, \Sigma_{11} - 2\Sigma_{12} + \Sigma_{22})$$

Problem 7

$$V((X_i - \mu)^2) = E[(X_i - \mu)^4] - E[(X_i - \mu)^2]^2$$

$$= E[X_i^4 - 4X_i^3\mu + 6X_i^2\mu^2 - 4X_i\mu^3 + \mu^4] - (E[X_i^2] - \mu^2)^2$$

$$= E[X_i^4] - 4\mu E[X_i^3] + 6E[X_i^2]\mu^2 - 4\mu^4 + \mu^4 - (E[X_i^2]^2 - 2\mu^2 E[X_i^2] + \mu^4)$$

$$= E[X_i^4] - 4\mu E[X_i^3] + 8E[X_i^2]\mu^2 - 4\mu^4 - E[X_i^2]^2$$

Define

$$Y_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k$$

By the WLLN,

$$Y_k \to_p E[X^k]$$

As a note, $Y_1 \to_p \mu$. Then we know by applications of the convergence rules and Slutsky's theorem,

$$Y_4 - 4Y_1Y_3 + 8Y_2Y_1^2 - 4Y_1^4 - Y_2^2 \to_p E[X_i^4] - 4\mu E[X_i^3] + 8E[X_i^2]\mu^2 - 4\mu^4 - E[X_i^2]^2 = V(X_i - \mu^2)$$

And hence that is our estimator.

Problem 8

(i) We already know the variances, so consider

$$\frac{\sum_{i=1}^n (X_i - \overline{X}_n)(Y_i - \overline{Y}_n)}{(n-1)\sqrt{\sigma_X^2 \sigma_Y^2}}$$

The expectation is

$$E\left[\frac{\sum_{i=1}^n (X_i - \overline{X}_n)(Y_i - \overline{Y}_n)}{(n-1)\sqrt{\sigma_X^2 \sigma_Y^2}}\right]$$

$$= \frac{\sum_{i=1}^n E[(X_i - \overline{X}_n)(Y_i - \overline{Y}_n)]}{(n-1)\sqrt{\sigma_X^2 \sigma_Y^2}}$$

$$= \frac{\sum_{i=1}^n E[X_i Y_i] - nE[\overline{X}_n \overline{Y}_n]}{(n-1)\sqrt{\sigma_X^2 \sigma_Y^2}}$$

$$= \frac{\sum_{i=1}^n (Cov(X,Y) + E[X]E[Y]) - nCov(\overline{X}_n, \overline{Y}_n) - nE[\overline{X}_n]E[\overline{Y}_n]}{(n-1)\sqrt{\sigma_X^2 \sigma_Y^2}}$$

$$= \frac{nCov(X,Y) + nE[X]E[Y] - n^{-1}Cov(\sum_i X_i, \sum_i Y_I) - nE[\overline{X}_n]E[\overline{Y}_n]}{(n-1)\sqrt{\sigma_X^2 \sigma_Y^2}}$$

$$= \frac{nCov(X,Y) + nE[X]E[Y] - Cov(X,Y) - nE[X]E[Y]}{(n-1)\sqrt{\sigma_X^2 \sigma_Y^2}}$$

$$= \frac{(n-1)Cov(X,Y)}{(n-1)\sqrt{\sigma_X^2 \sigma_Y^2}}$$

$$= \rho$$

so this is unbiased.

(ii) For consistency, we rewrite this as

$$\frac{\sum_{i=1}^n (X_i - \overline{X}_n)(Y_i - \overline{Y}_n)}{(n-1)\sqrt{\sigma_X^2 \sigma_Y^2}}$$

$$= \frac{n}{n-1}\left(\frac{\frac{1}{n}\sum_{i=1}^n X_i Y_i - \overline{X}_n \overline{Y}_n)}{\sqrt{\sigma_X^2 \sigma_Y^2}}\right)$$

By WLLN,

$$\frac{1}{n}\sum_{i=1}^n X_i Y_i \to_p E[XY]$$

$$\overline{X}_n \to_p E[X]$$

$$\overline{Y}_n \to_p E[Y]$$

and

$$n/(n-1) \to_p 1$$

then applying the properties shown in lecture,

$$\frac{n}{n-1}\left(\frac{\frac{1}{n}\sum_{i=1}^n X_iY_i - \overline{X}_n\overline{Y}_n)}{(n-1)\sqrt{\sigma_X^2\sigma_Y^2}}\right) \to_p \frac{E[XY]-E[X]E[Y]}{\sqrt{\sigma_X^2\sigma_Y^2}} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \rho$$

So this estimator is consistent.

(iii) Lastly, we want to characterize

$$\begin{aligned}
&\sqrt{n}(\hat{\rho}-\rho) \\
&= \sqrt{n}\left(\frac{\sum_{i=1}^n (X_i-\overline{X}_n)(Y_i-\overline{Y}_n)}{(n-1)\sigma_X\sigma_Y} - \rho\right) \\
&= \sqrt{n}\left(\frac{\sum_{i=1}^n X_iY_i - n\overline{X}_n\overline{Y}_n}{(n-1)\sigma_X\sigma_Y} - \rho\right) \\
&= \frac{1}{\sigma_X\sigma_Y}\left(\frac{\sqrt{n}}{n-1}\sum_{i=1}^n X_iY_i - \frac{n\sqrt{n}}{(n-1)}\overline{X}_n\overline{Y}_n - \sqrt{n}\sigma_X\sigma_Y\rho\right) \\
&= \frac{1}{\sigma_X\sigma_Y}\left(\frac{\sqrt{n}}{n-1}\sum_{i=1}^n X_iY_i - \frac{n\sqrt{n}}{(n-1)}\overline{X}_n\overline{Y}_n - \sqrt{n}\sigma_{XY}\right) \\
&= \frac{1}{\sigma_X\sigma_Y}\left(\frac{\sqrt{n}}{n-1}\sum_{i=1}^n (X_i-\mu_X)(Y_i-\mu_Y) - \frac{n\sqrt{n}}{n-1}(\overline{X}_n\overline{Y}_n - \mu_X\overline{Y}_n - \mu_Y\overline{X}_n + \mu_X\mu_Y) - \sqrt{n}\sigma_{XY}\right) \\
&= \frac{1}{\sigma_X\sigma_Y}\left(\frac{n\sqrt{n}}{n-1}\left(\frac{1}{n}\sum_{i=1}^n (X_i-\mu_X)(Y_i-\mu_Y) - \sigma_{XY}\right) - \frac{n}{n-1}\sqrt{n}(\overline{X}_n-\mu_X)(\overline{Y}_n-\mu_Y) - \sqrt{n}\sigma_{XY} + \sigma_{XY}\frac{n\sqrt{n}}{n-1}\right) \\
&= \frac{1}{\sigma_X\sigma_Y}\left(\frac{n\sqrt{n}}{n-1}\left(\frac{1}{n}\sum_{i=1}^n (X_i-\mu_X)(Y_i-\mu_Y) - \sigma_{XY}\right) - \frac{n}{n-1}\sqrt{n}(\overline{X}_n-\mu_X)(\overline{Y}_n-\mu_Y) + \sigma_{XY}\frac{\sqrt{n}}{n-1}\right)
\end{aligned}$$

Now, from the CLT, we have

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n (X_i-\mu_X)(Y_i-\mu_Y) - \sigma_{XY}\right) \to_d N(0, V((X-\mu_X)(Y-\mu_Y)))$$

$$\sqrt{n}(\overline{X}_n-\mu_X) \to_d N(0,\sigma_X^2)$$

by the WLLN, we have

$$(\overline{Y}_n-\mu_Y) \to_d 0$$

Additionally,

$$\frac{n}{n-1} \to_p 1$$

$$\frac{\sqrt{n}}{n-1} \to_p 0$$

Combining, we get

$$\frac{n\sqrt{n}}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_X)(Y_i-\mu_Y)-\sigma_{XY}\right)\to_d N(0,V((X-\mu_X)(Y-\mu_Y)))$$

$$\frac{n}{n-1}\sqrt{n}(\overline{X}_n-\mu_X)(\overline{Y}_n-\mu_Y)\to_d 0\cdot N(0,\sigma_X^2)=0$$

$$\sigma_{XY}\frac{\sqrt{n}}{n-1}\to_d 0$$

So together, we get

$$\sqrt{n}(\hat{\rho}-\rho)=\frac{1}{\sigma_X\sigma_Y}\left(\frac{n\sqrt{n}}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_X)(Y_i-\mu_Y)-\sigma_{XY}\right)-\frac{n}{n-1}\sqrt{n}(\overline{X}_n-\mu_X)(\overline{Y}_n-\mu_Y)+\sigma_{XY}\frac{\sqrt{n}}{n-1}\right)$$

$$\to_d \frac{1}{\sigma_X\sigma_Y}N(0,V((X-\mu_X)(Y-\mu_Y)))=N\left(0,\frac{V((X-\mu_X)(Y-\mu_Y)))}{\sigma_X^2\sigma_Y^2}\right)$$