ECON550: Problem Set 2

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HMC Exercises (7th edition)

1.4.1

$$P(C_2 \cup C_3 \cup ...|C_1) = \frac{P(C_1 \cap (C_2 \cup C_3 \cup ...))}{P(C_1)}$$

$$= \frac{P((C_1 \cap C_2) \cup (C_1 \cap C_3) \cup ...))}{P(C_1)}$$

Since all the $C_2, C_3, ...$ are disjoint,

$$= \frac{P(C_1 \cap C_2) + P(C_1 \cap C_3) + P(C_1 \cap C_4) + ...}{P(C_1)}$$

$$= P(C_2|C_1) + P(C_3|C_1) + P(C_4|C_1) + ...$$

1.4.2

$$P(C_1 \cap C_2 \cap C_3 \cap C_4) = \frac{P(C_1)}{P(C_1)} \frac{P(C_1 \cap C_2)}{P(C_1 \cap C_2)} \frac{P(C_1 \cap C_2 \cap C_3)}{P(C_1 \cap C_2 \cap C_3)} P(C_1 \cap C_2 \cap C_3 \cap C_4)$$

$$= P(C_1) \frac{P(C_1 \cap C_2)}{P(C_1)} \frac{P(C_1 \cap C_2 \cap C_3)}{P(C_1 \cap C_2)} \frac{P(C_1 \cap C_2 \cap C_3 \cap C_4)}{P(C_1 \cap C_2 \cap C_3)}$$

$$= P(C_1)P(C_2|C_1)P(C_3|C_1 \cap C_2)P(C_4|C_1 \cap C_2 \cap C_3)$$

Only using Bayes' rule.

1.4.9 Let $P(a)$ denote the probability of drawing exactly a blue chips from bowl 1. The probability of drawing a blue chip from the second bowl is then given by

$$P(0)\frac{0}{5} + P(1)\frac{1}{5} + P(2)\frac{2}{5} + P(3)\frac{3}{5} + P(4)\frac{4}{5}$$

$$\frac{6}{252}\frac{0}{5} + \frac{60}{252}\frac{1}{5} + \frac{120}{252}\frac{2}{5} + \frac{60}{252}\frac{3}{5} + \frac{6}{252}\frac{4}{5}$$

The probability of drawing exactly 3 blue chips from bowl I and then drawing a blue chip from bowl II is

$$\frac{60}{252}\frac{3}{5}$$

Thus, the conditional probability desired is

$$\frac{180}{60+240+180+24} = \frac{5}{14}$$

1.4.12(c) C_1 and C_2^c are independent, so

$$P(C_1 \cap C_2^c) = 0.42$$

Hence

$$P(C_1 \cup C_2^c) = P(C_1) + P(C_2^c) - P(C_1 \cap C_2^c) = 0.88$$

1.4.23

$$P(C_1 \cup C_2 \cup C_3) = P(C_1) + P(C_2) + P(C_3) - P(C_1 \cap C_2) - P(C_2 \cap C_3) - P(C_1 \cap C_3) + P(C_1 \cap C_2 \cap C_3)$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{6} - \frac{1}{12} - \frac{1}{8} + \frac{1}{24} = \frac{3}{4}$$

1.5.1 The induced probability is:

$$P_X(0) = \frac{9}{13}$$

$$P_X(1) = \frac{1}{13}$$

$$P_X(2) = \frac{1}{13}$$

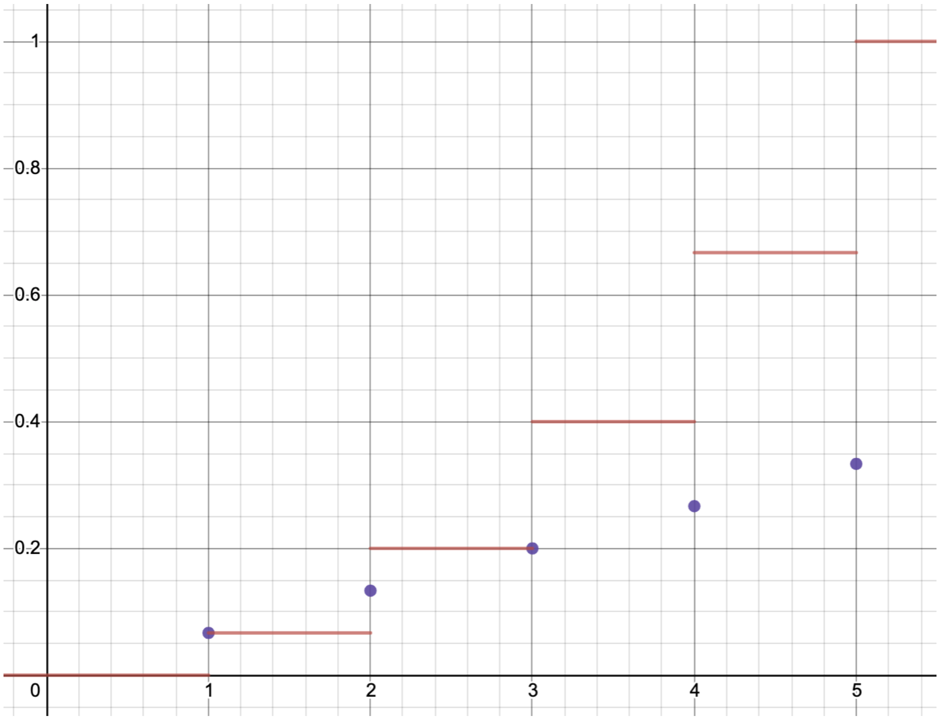
$$P_X(3) = \frac{1}{13}$$

$$P_X(4) = \frac{1}{13}$$

1.5.4(c) The cdf is given by:

$$\begin{cases} 0 & x < 1 \\ 1/15 & 1 \le x < 2 \\ 3/15 & 2 \le x < 3 \\ 6/15 & 3 \le x < 4 \\ 10/15 & 4 \le x < 5 \\ 1 & 5 \le x \end{cases}$$

We have sketched the CDF in red and the PDF in purple in the next page.



1.5.6

$$P_X(D_1) = \int_1^2 \frac{2x}{9} dx = 1/9$$

1.5.8(d) This probability is 0; there is only a nonzero probability for set with a nonempty intersection with $[-1, 1]$.

1.6.3(a) The pmf is

$$p(x) = \frac{1}{6}\left(\frac{5}{6}\right)^{x-1}$$

1.6.3(b)

$$\sum p(x) = \sum \frac{1}{6}\left(\frac{5}{6}\right)^{x-1}$$

$$= \frac{1/6}{1-(5/6)} = 1$$

1.6.3(d)

$$F(x) = \sum_{k=1}^{x} \frac{1}{6}\left(\frac{5}{6}\right)^{x-1} = 1 - \left(\frac{5}{6}\right)^{x}$$

1.7.1 CDF:

$$F_X(x) = \frac{\sqrt{x}}{10}$$

PMF:

$$f_X(x) = \frac{1}{5\sqrt{x}}$$

1.7.6(a)

$$P(|X| < 1) = \int_{-1}^{1} x^2/18 \; dx = 1/3$$

$$P(X^2 < 9) = 1$$

1.7.7

$$P_X(C_1 \cup C_2) = 1/2 + (1/20) = 11/20$$

$$P_X(C_1 \cap C_2) = 0$$

1.7.9(b) Median is

$$x^3 = \frac{1}{2} \implies x = \frac{1}{\sqrt[3]{2}}$$

1.7.15 Negating the expression, we get

$$-P(X > z) \leq -P(Y > z)$$

$$1 - P(X > z) \leq 1 - P(Y > z)$$

$$P(X \leq z) \leq P(Y \leq z)$$

$$F_X(z) \leq F_Y(z)$$

Strict inequality holding for at least one z.

Problem 4

We know that $\lim_{x\to-\infty} F_X(x) = \lim_{x\to-\infty} Pr(X \leq x)$

$$= \lim_{x\to-\infty} 1 - Pr(X > x)$$

$$= 1 - \lim_{x\to-\infty} Pr(X > x)$$

But as $x \to -\infty$ the interval $(x, \infty) \to \mathbb{R}$, and hence $\lim_{x\to-\infty} Pr(X > x) = Pr(X \in \mathbb{R}) = 1$. Therefore

$$\lim_{x\to-\infty} F_X(x) = 1 - \lim_{x\to-\infty} Pr(X > x)$$

$$= 1 - 1 = 0$$

as desired.

Problem 5

To show this is a σ-field, we show this is closed under complements and finite unions. To show closure under complements, consider some $S \in \mathcal{B}(\Omega_X)$. Then we know $\exists B \in \mathcal{B}$ such that

$$S = B \cap \Omega_X$$

$$\Omega_X \setminus S = \Omega_X \setminus (B \cap \Omega_X) =$$

$$= \Omega_X \setminus B = B^c \cap \Omega_X$$

Since $\mathcal{B}$ is a σ-field, $B^c \in \mathcal{B}$, and hence $S = B^c \cap \Omega_X$ is in $\mathcal{B}(\Omega_X)$.

Now, we show closure under countable union. Consider $S_1, S_2, ... \in \mathcal{B}(\Omega_X)$. We know $\exists B_1, B_2, ... \in \mathcal{B}$ such that $S_i = B_i \cap \Omega_X$. Let

$$S = \cup_i S_i$$

$$= \cup_i (B_i \cap \Omega_X)$$

$$= (\cup_i B_i) \cap (\cup_i \Omega_X)$$

$$= (\cup_i B_i) \cap \Omega_X$$

But $\cup B_i \in \mathcal{B}$ since $\mathcal{B}$ is a σ-field, and so we have that $S \in \mathcal{B}(\Omega_X)$, and hence we have the closure properties. So $\mathcal{B}(\Omega_X)$ is a σ-field.

Problem 6

Let s_1, s_2 be simple functions. Then there exists some disjoint sets A_j such that

$$s_1(x) = \sum_{j=1}^{M} c_j 1_{A_j}$$

$$s_2(x) = \sum_{j=1}^{M} d_j 1_{A_j}$$

Then

$$\int (s_1 + s_2) d\mu = \sum_{j=1}^{M} (c_j + d_j) \mu(A_j)$$

$$= \sum_{j=1}^{M} c_j \mu(A_j) + d_j \mu(A_j)$$

$$= \sum_{j=1}^{M} c_j \mu(A_j) + \sum_{j=1}^{M} d_j \mu(A_j)$$

$$= \int s_1 d\mu + \int s_2 d\mu$$

Further, we have that

$$\int a s_1 d\mu = \sum_{j=1}^{M} a c_j \mu(A_j)$$

$$= a \sum_{j=1}^{M} c_j \mu(A_j)$$

$$= a \int s_1 d\mu$$

So we have linearity of the Lesbesgue integral on simple functions.

Problem 7

We first show linearity for nonnegative, measureable functions. Suppose we have two such functions f, g. Then, we note that we can construct two increasing sequences of simple functions $\{s_n\} \to f, \{s'_n\} \to g$, such that by the monotone convergence

$$\int f d\mu = \lim_{n\to\infty} \int s_n d\mu$$

$$\int g d\mu = \lim_{n\to\infty} \int s'_n d\mu$$

So

$$\int f d\mu + \int g d\mu = \lim_{n\to\infty} \left(\int s_n d\mu + \int s'_n d\mu \right)$$

By linearity of simple functions as we showed in the previous problem,

$$\int f d\mu + \int g d\mu = \lim_{n\to\infty} \left(\int s_n d\mu + \int s'_n d\mu \right)$$

$$= \lim_{n\to\infty} \left(\int (s_n + s'_n) d\mu \right)$$

We note then that the sequence $s^*_n = s_n + s'_n$ must also be increasing, since both s_n and s'_n are increaasing. Further, $\{s^*_n\} \to f + g$. Hence

$$\int f d\mu + \int g d\mu = \lim_{n\to\infty} \left(\int (s_n + s'_n) d\mu \right)$$

$$= \int (f + g) d\mu$$

We also know that for scalar a, $\{a s_n\} \to af$, and hence

$$\int a f d\mu = \lim_{n\to\infty} \int a s_n d\mu$$

$$= \lim_{n\to\infty} a \int s_n d\mu$$

$$= a \lim_{n\to\infty} \int s_n d\mu$$

$$= a \int f d\mu$$

So Lebesgue integration is linear for nonnegative, measureable functions.

Now more generally, suppose f, g are integrable. Then we have for $f = f^+ - f^-$, $g = g^+ - g^-$. f^+, g^+, f^-, g^- all nonnegative. Then

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

$$\int g d\mu = \int g^+ d\mu - \int g^- d\mu$$

$$\int f d\mu + \int g d\mu = \int f^+ d\mu - \int f^- d\mu + \int g^+ d\mu - \int g^- d\mu$$

$$= \int (f^+ + g^+) d\mu - \int (f^- + g^-) d\mu$$

$$= \int ((f^+ + g^+) - (f^- + g^-)) d\mu$$

$$= \int (f + g) d\mu$$

using the linearity of nonnegative functions we proved earlier. Further,

$$\int a f d\mu = \int a f^+ d\mu - \int a f^- d\mu$$

$$= a \left(\int f^+ d\mu - \int f^- d\mu \right)$$

$$= a \int f d\mu$$

So we have shown linearity of the Lebesgue integral on integrable functions.