ECON550: Problem Set 5

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HMC Exercises (7th edition)

5.1.1 Suppose $a_n \to_p a$. Then we have

$$Pr(|a_n - a| \geq \epsilon) \to 0$$

as $n \to \infty$. But since a_n is deterministic,

$$Pr(|a_n - a| \geq \epsilon) = 1(|a_n - a| \geq \epsilon)$$

Since this goes to 0, there exists some N such that for all $n > N$, $|a_n - a| < \epsilon$, and hence $a_n \to a$.

Now, for the reverse implication, suppose $a_n \to a$. Then for any ϵ, $\exists N$ such that $\forall n > N$, $|a_n - a| < \epsilon$. That implies that $Pr(|a_n - a| \geq \epsilon) = 0$ for all $n > N$, and hence $a_n \to_p a$.

5.1.2

(a) Note that since we know $Y_n = \sum_{i=1}^{N} X_i$ for X_i independent Bernoulli random variables. Then we have by the WLLN,

$$Y_n/n = \bar{X}_n \to_p \mu_X = p$$

(b) This follows immediately from Rules 1 and 2. Since $1 \to_p 1$, $-Y_n/n \to_p -p$, we have $1 - Y_n/n \to_p 1 - p$.

(c) From parts a and b, we know

$$1 - Y_n/n \to_p 1 - p$$

$$Y_n/n \to_p p$$

Therefore, by Rule 3,

$$(Y_n/n)(1 - Y_n/n) \to_p p(1 - p)$$

Problem 2

Since Σ is positive definite, it is diagonalizable as $B\Gamma B'$, where $BB' = I$. Then using multiplicativity of the determinant :

$$(\det(\Sigma))^{-1/2} = (\det(B\Gamma B'))^{-1/2}$$

$$= (\det(B)\det(\Gamma)\det(B'))^{-1/2}$$

$$= (\det(B)\det(\Gamma)\det(B'))^{-1/2}$$

Note that since $BB' = I$, $\det(B)\det(B') = \det(I) = 1$. Hence:

$$(\det(\Sigma))^{-1/2} = (\det(B)\det(\Gamma)\det(B'))^{-1/2}$$

$$= (\det(\Gamma))^{-1/2}$$

$$= \det(B)(\det(\Gamma))^{-1/2}\det(B')$$

Now, since Γ is diagonal, $\det(\Gamma)^{-1/2} = \det(\Gamma^{-1/2})$. So

$$(\det(\Sigma))^{-1/2} = \det(B)(\det(\Gamma))^{-1/2}\det(B')$$

$$= \det(B)\det(\Gamma^{-1/2})\det(B')$$

$$= \det(B\Gamma^{-1/2}B') = \det(\Sigma^{-1/2})$$

Problem 3

Let

$$\bar{\mu}_n = n^{-1}\sum_{i=1}^{n} E(X_i)$$

Then we have, from Markov,

$$P(|\bar{X}_n - \bar{\mu}_n| > \epsilon) = P((\bar{X}_n - \bar{\mu}_n)^2 > \epsilon^2)$$

$$\leq \frac{E(\bar{X}_n - \bar{\mu}_n)^2}{\epsilon^2}$$

$$= \frac{\sum_i \sum_j E((X_i - \mu_i)(X_j - \mu_j))}{n^2\epsilon^2}$$

$$= \frac{\sum_{i=1}^n \sum_{j=1}^n Cov(X_i, X_j)}{n^2\epsilon^2}$$

$$= \frac{(1/n)\sum_{i=1}^n \sum_{j=1}^n Cov(X_i, X_j)}{n\epsilon^2}$$

$$= \frac{(1/n)\left(n\tau_0 + 2\sum_{i=1}^n \sum_{j=1}^{i-1} Cov(X_i, X_j)\right)}{n\epsilon^2}$$

$$= \frac{\tau_0 + \frac{2}{n}\sum_{i=1}^n \sum_{j=1}^{i-1} \tau_{i-j}}{n\epsilon^2}$$

$$= \frac{\tau_0 + 2\sum_{i=1}^{n-1} \frac{(n-i)}{n}\tau_i}{n\epsilon^2}$$

$$\leq \frac{\tau_0 + 2\sum_{i=1}^{n-1} \tau_i}{n\epsilon^2}$$

Now, by the assumption in the problem, the sum in the numerator is bounded, so the entire numerator is finite. Hence as $n \to \infty$, the relevant probability decreases on the order of $1/(n\epsilon^2)$, so we have that the expression $\bar{X}_n - \bar{\mu}_n \to 0$.

Problem 4

- It suffices to show this for X_i's; it holds for the Y_i's symmetrically. Consider

$$S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}(X_i^2 - 2X_i\bar{X}_n + \bar{X}_n^2)$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}X_i^2 - \frac{2}{n-1}\bar{X}_n\sum_{i=1}^{n}X_i + \frac{1}{n-1}\sum_{i=1}^{n}\bar{X}_n^2$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}X_i^2 - \frac{2n}{n-1}\bar{X}_n^2 + \frac{n}{n-1}\bar{X}_n^2$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}X_i^2 - \frac{n}{n-1}\bar{X}_n^2$$

$$= \frac{n}{n-1}\left(\frac{1}{n}\left(\sum_{i=1}^{n}X_i^2\right) - \bar{X}_n^2\right)$$

  Now, by the weak law of large numbers:

$$\frac{1}{n}\left(\sum_{i=1}^{n}X_i^2\right) \to_p E[X_i^2] = \sigma_X^2 + \mu_X^2$$

  Further $\bar{X}_n \to_p \mu_X$, so by Rule 3, $\bar{X}_n^2 \to_p \mu_X^2$. Lastly, we note that as $n \to \infty$, $n/(n-1) \to_p 1$, so we get

$$S_X^2 \to_p 1(\sigma_X^2 + \mu_X^2 - \mu_X^2) = \sigma_X^2$$

- Now, we examine

$$S_{XY} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)(Y_i - \bar{Y}_n)$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n}(X_iY_i - \bar{X}_nY_i - \bar{Y}_nX_i + \bar{X}_n\bar{Y}_n)\right)$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n}(X_iY_i) - n\bar{X}_n\bar{Y}_n - n\bar{Y}_n\bar{X}_n + n\bar{X}_n\bar{Y}_n)\right)$$

$$= \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n}(X_iY_i) - \bar{X}_n\bar{Y}_n)\right)$$

Now, once again, we know $n/(n-1) \to_p 1$, by the WLLN $\frac{1}{n}\sum_{i=1}^{n}(X_iY_i) \to_p E[XY]$, $\bar{X}_n \to_p \mu_X$, and $\bar{Y}_n \to_p \mu_Y$. Hence we have

$$S_{XY} \to_p E[XY] - \mu_X\mu_Y = Cov(X,Y) = \sigma_{XY}$$

- We take our statistic to be

$$\hat{\rho} = \frac{S_{XY}}{\sqrt{S_X^2 S_Y^2}}$$

By Rule 3 and part a, $S_X^2 S_Y^2 \to_p \sigma_X^2\sigma_Y^2$. By part b, $S_{XY} \to_p \sigma_{XY}$. Finally, by Slutsky and rule 3, we get that

$$\hat{\rho} \to_p \frac{\sigma_{XY}}{\sqrt{\sigma_X^2\sigma_Y^2}} = \rho_{XY}$$

Problem 5

- We compute, using integration by parts

$$\int_0^{\infty} \lambda x e^{-\lambda x} = -xe^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x}$$

$$= \frac{1}{\lambda}(1 - e^{-\lambda\infty}) = \frac{1}{\lambda}$$

- We take the estimator: $\hat{\lambda} = n/\sum X_i$. We note that

$$\hat{\lambda} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} X_i}$$

We know from the WLLN that $\frac{1}{n}\sum_{i=1}^{n} X_i \to_p E[X] = \frac{1}{\lambda}$. Hence we have from Slutsky's theorem that

$$\hat{\lambda} \to_p \frac{1}{(1/\lambda)} = \lambda$$