ECON550: Problem Set 5

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Fall 2020

Problem 1

(a) We have

$$P(|(\hat{\theta}-\theta)| > 2/\sqrt{n}) = P(|\sqrt{n}(\hat{\theta}-\theta)| > 2)$$

Now, since $\sqrt{n}(\hat{\theta}-\theta) \to_d N(0,\theta^2)$, by symmetry of the normal distribution we can approximate this then as

$$P(|\sqrt{n}(\hat{\theta}-\theta)| > 2) \approx 2P(\sqrt{n}(\hat{\theta}-\theta) > 2)$$

$$= 2(1-\Phi(2/\sqrt{\theta^2})) = 2(1-\Phi(2/|\theta|))$$

(b) Since $\sqrt{n}(\hat{\theta}-\theta) \to_d N(0,\theta^2)$ and $\hat{\theta} \to_p \theta$, we have $\sqrt{n}(\hat{\theta}-\theta)/|\hat{\theta}| \to_d N(0,\theta^2/|\theta|^2) = N(0,1)$. Hence, we get

$$P(|(\hat{\theta}-\theta)| > 2/\sqrt{n}) = P(|\sqrt{n}(\hat{\theta}-\theta)| > 2)$$

$$= P(|(\sqrt{n}(\hat{\theta}-\theta))/\hat{\theta}| > 2/|\hat{\theta}|)$$

$$\approx 2(1-\Phi(2/|\hat{\theta}|))$$

Problem 2

As in PS5, we define

$$\hat{\lambda}_n = \frac{1}{\bar{X}_n}$$

The variance of X_i is given by

$$\int_0^\infty \left(x-\frac{1}{\lambda}\right)^2 \lambda e^{-\lambda x}\, dx = \frac{1}{\lambda^2}$$

Then by the CLT $\sqrt{n}(\bar{X}_n - 1/\lambda) \to_d N(0,1/\lambda^2)$

Problem 3

The objective can be rewritten as

$$(Y-X\hat{\beta})'(Y-X\hat{\beta}) = Y'Y - Y'X\hat{\beta} - X'Y\hat{\beta} + X'X\hat{\beta}^2 = Y'Y - 2X'Y\hat{\beta} + X'X\hat{\beta}^2$$

The FOC:

$$-2X'Y + 2X'X\hat{\beta} = 0$$

$$X'Y = X'X\hat{\beta}$$

Since $X'X$ is nonsingular,

$$\hat{\beta} = (X'X)^{-1}(X'Y)$$

Problem 4

We can use the delta method. By Slutsky's theorem,

$$g(\hat{\theta}) \to g(\theta) = \begin{bmatrix} \theta_1 - \theta_2 \\ \theta_1\theta_3 \end{bmatrix}$$

Then $G(\theta)$ is

$$\begin{bmatrix} 1 & -1 & 0 \\ \theta_3 & 0 & \theta_1 \end{bmatrix}$$

So by the delta method,

$$\sqrt{n}(g(\hat{\theta}) - g(\theta)) \to_d N(0, G(\theta)\Sigma G(\theta)')$$

Problem 5

Define $\overline{X_nY_n} = n^{-1}\sum_{i=1}^n X_iY_i$. Then we use, as our estimator,

$$\hat{\rho} = \frac{\overline{X_nY_n} - \bar{X}_n\bar{Y}_n}{\sqrt{\hat{S}_{Xn}\hat{S}_{Yn}}}$$

If we define $\overline{X_n^2} = n^{-1}\sum_{i=1}^n X_i^2$ and $\overline{Y_n^2} = n^{-1}\sum_{i=1}^n Y_i^2$, then we get $\hat{S}_{Xn} = n^{-1}(\overline{X_n^2} - \bar{X}_n^2)$ and $\hat{S}_{Yn} = n^{-1}(\overline{Y_n^2} - \bar{Y}_n^2)$. Note that by the WLLN,

$$\hat{Z} = \begin{bmatrix} \overline{X_nY_n} \\ \overline{X_n^2} \\ \overline{Y_n^2} \\ \bar{X}_n \\ \bar{Y}_n \end{bmatrix} \to_p \begin{bmatrix} Cov(X,Y) + \mu_X\mu_Y \\ \sigma_X^2 + \mu_X^2 \\ \sigma_Y^2 + \mu_Y^2 \\ \mu_X \\ \mu_Y \end{bmatrix} = Z$$

Define the variance matrix:

$$\Sigma = \begin{bmatrix} Var(XY) & Cov(XY,X^2) & Cov(XY,Y^2) & Cov(XY,X) & Cov(XY,Y) \\ Cov(X^2,XY) & Var(X^2) & Cov(X^2,Y^2) & Cov(X^2,X) & Cov(X^2,Y) \\ Cov(Y^2,XY) & Cov(Y^2,X^2) & Var(Y^2) & Cov(Y^2,X) & Cov(Y^2,Y) \\ Cov(X,XY) & Cov(X,X^2) & Cov(X,Y^2) & Var(X) & Cov(X,Y) \\ Cov(Y,XY) & Cov(Y,X^2) & Cov(Y,Y^2) & Cov(Y,X) & Var(Y) \end{bmatrix}$$

By the CLT,

$$\sqrt{n}(\hat{Z} - Z) \to_d N(0, \Sigma)$$

Then, if we define:

$$g(Z) = \frac{\overline{X_n Y_n} - \bar{X}_n \bar{Y}_n}{\sqrt{(\overline{X_n^2} - \bar{X}_n^2)(\overline{Y_n^2} - \bar{Y}_n^2)}}$$

we can apply the multivariate delta method. Define

$$G(Z)' = \begin{bmatrix} \frac{1}{\sqrt{(Z_2 - Z_4^2)(Z_3 - Z_5^2)}} \\ -\frac{(Z_1 - Z_4 Z_5)}{2\sqrt{(Z_2 - Z_4^2)^3 (Z_3 - Z_5^2)}} \\ -\frac{(Z_1 - Z_4 Z_5)}{2\sqrt{(Z_2 - Z_4^2)(Z_3 - Z_5^2)^3}} \\ \frac{(Z_1 - Z_4 Z_5)}{\sqrt{(Z_2 - Z_4^2)^3 (Z_3 - Z_5^2)}} Z_4 - \frac{Z_5}{\sqrt{(Z_2 - Z_4^2)(Z_3 - Z_5^2)}} \\ \frac{(Z_1 - Z_4 Z_5)}{\sqrt{(Z_2 - Z_4^2)(Z_3 - Z_5^2)^3}} Z_5 - \frac{Z_4}{\sqrt{(Z_2 - Z_4^2)(Z_3 - Z_5^2)}} \end{bmatrix}$$

Then by the multivariate delta method,

$$\sqrt{n}(\hat{\rho} - \rho) = \sqrt{n}(g(\hat{Z}) - g(Z)) \to_d N(0, G(Z)\Sigma G(Z)')$$

Problem 6

(a) We have

$$\hat{\beta}_n = \left(\sum_{i=1}^{n} X_i X_i' \right)^{-1} \sum_{i=1}^{n} X_i Y_i$$

$$= \left(n^{-1} \sum_{i=1}^{n} X_i X_i' \right)^{-1} \left(n^{-1} \sum_{i=1}^{n} X_i Y_i \right)$$

Note by WLLN

$$\left(n^{-1} \sum_{i=1}^{n} X_i X_i' \right)^{-1} \to_p E[X_i X_i']$$

$$\left(n^{-1} \sum_{i=1}^{n} X_i Y_i \right) \to_p E[X_i Y_i]$$

Hence, using the rules of convergence in probability and Slutsky's theorem (and the fact that Σ_X is positive definite)

$$\hat{\beta}_n = \left(n^{-1} \sum_{i=1}^{n} X_i X_i' \right)^{-1} \left(n^{-1} \sum_{i=1}^{n} X_i Y_i \right)$$

$$\to_p (E[X_i X_i'])^{-1} E[X_i Y_i]$$

$$= (\Sigma_X - \mu_X^2 I) E[X_i X_i' \beta_0 + X_i U_i]$$

$$= (\Sigma_X - \mu_X^2 I)^{-1} ((\Sigma_X - \mu_X^2 I)\beta + E[X_i U_i])$$

$$= (\Sigma_X - \mu_X^2 I)^{-1}((\Sigma_X - \mu_X^2 I)\beta + Cov[X_i U_i])$$

$$= (\Sigma_X - \mu_X^2 I)^{-1}(\Sigma_X - \mu_X^2 I)\beta$$

$$= \beta_0$$

Hence $\hat{\beta}_n$ is consistent

(b) If $Cov(X_i, U_i) \neq 0$, then we get

$$\hat{\beta}_n \to_p E[X_i X_i']^{-1}(E[X_i X_i']\beta + Cov[X_i U_i])$$

$$= \beta_0 + E[X_i X_i']^{-1} Cov(X_i, U_i)$$

Problem 7

(a) The minimand is

$$(\hat{\pi}_n - \hat{A}_n\gamma)'(\hat{\pi}_n - \hat{A}_n\gamma)$$

$$= \hat{\pi}_n'\hat{\pi}_n - (\hat{A}_n\gamma)'\hat{\pi}_n - \hat{\pi}_n'\hat{A}_n\gamma + (\hat{A}_n\gamma)'(\hat{A}_n\gamma)$$

$$= \hat{\pi}_n'\hat{\pi}_n - 2\hat{\pi}_n'\hat{A}_n\gamma + (\hat{A}_n\gamma)'(\hat{A}_n\gamma)$$

Taking the FOC, we get

$$2\hat{\pi}_n'\hat{A}_n = 2\gamma'\hat{A}_n'\hat{A}_n$$

$$\hat{\pi}_n'\hat{A}_n = \gamma'\hat{A}_n'\hat{A}_n$$

Since $\hat{A}_n$ has full column rank, $\hat{A}_n'\hat{A}_n$ is invertible, and hence

$$\hat{\gamma}' = (\hat{\pi}_n'\hat{A}_n)(\hat{A}_n'\hat{A}_n)^{-1}$$

$$\hat{\gamma} = (\hat{A}_n'\hat{A}_n)^{-1}(\hat{A}_n'\hat{\pi}_n)$$

(b) Define $g(x, y) = (x'x)^{-1}(x'y)$. Then $\hat{\gamma} = g(\hat{A}_n, \hat{\pi_n})$. Since A is full column rank, we have by Slutsky's theorem that

$$\hat{\gamma} = g(\hat{A}_n, \hat{\pi_n}) \to_p g(A, \pi_0) = (A'A)^{-1}(A'\pi_0)$$

Problem 8

(a) The conditional expectation is:

$$E[\hat{\beta}] = E[(X'X)^{-1}(X'Y)|X] = E[(X'X)^{-1}(X'(\beta_0 X + U))|X]$$

$$= E[\beta_0 + (X'X)^{-1}(X'U)|X]$$

$$= \beta_0 + (X'X)^{-1}X'E[U|X]$$

$$= \beta_0$$

since $E[U|X] = 0$. Hence $\hat{\beta}$ is unbiased.

(b)

$$V[\hat{\beta}] = V[(X'X)^{-1}(X'Y)|X] = V[(X'X)^{-1}(X'(\beta_0 X + U))|X]$$
$$= V[\beta_0 + (X'X)^{-1}(X'U)|X]$$
$$= V[(X'X)^{-1}(X'U)|X]$$
$$= (X'X)^{-1}X'V[U|X]X(X'X)^{-1}$$
$$= \sigma^2(X'X)^{-1}X'X(X'X)^{-1}$$
$$= \sigma^2(X'X)^{-1}$$

(c) We know that by WLLN, $X'X/n = n^{-1}\sum X_i X_i' \to_p \Sigma_X$. By Slutsky's theorem, we get $(X'X/n)^{-1} \to_p \Sigma_X^{-1}$ which exists because Σ_X is positive definite. Now, we note that

$$E[X_i U_i] = E[E[X_i U_i|X_i]] = 0$$

$$V[X_i U_i] = E[X_i X_i' U_i^2] = E[X_i X_i' E[U_i^2|X_i]] = \sigma^2 \Sigma_X$$

so by the CLT

$$\sqrt{n}(X'U/n) = \sqrt{n}(X'U/n - E[X'U/n]) \to_d N(0, \sigma^2\Sigma_X)$$

Using these two, we have

$$\sqrt{n}(\hat{\beta} - \beta_0) = \sqrt{n}((X'X)^{-1}(X'Y) - \beta_0)$$
$$= \sqrt{n}((X'X)^{-1}X'(\beta_0 X + U) - \beta_0)$$
$$= \sqrt{n}(X'X)^{-1}X'U$$
$$= (X'X/n)^{-1} \cdot \sqrt{n}(X'U/n)$$
$$\to_d \Sigma_X^{-1}N(0, \sigma^2\Sigma_X)$$
$$= N(0, \sigma^2\Sigma_X^{-1})$$

(d) We need to estimate $\sigma^2\Sigma_X^{-1}$. Consider:

$$\hat{\sigma}^2 = \frac{1}{n}(Y - X\hat{\beta})'(Y - X\hat{\beta})(X'X/n)^{-1}$$

$$= \frac{1}{n}(Y - X(X'X)^{-1}(X'Y))'(Y - X(X'X)^{-1}(X'Y))(X'X/n)^{-1}$$
$$= \frac{1}{n}((I - X(X'X)^{-1}X')Y)'((I - X(X'X)^{-1}X')Y)(X'X/n)^{-1}$$
$$= \frac{1}{n}Y'(I - X(X'X)^{-1}X')'((I - X(X'X)^{-1}X')Y)(X'X/n)^{-1}$$

$$= \frac{1}{n} Y'(I - 2X(X'X)^{-1}X' + X(X'X)^{-1}X'X(X'X)^{-1}X')Y(X'X/n)^{-1}$$

$$= \frac{1}{n} Y'(I - 2X(X'X)^{-1}X' + X(X'X)^{-1}X')Y(X'X/n)^{-1}$$

$$= \frac{1}{n} Y'(I - X(X'X)^{-1}X')Y(X'X/n)^{-1}$$

Now, we plug in for Y

$$= \frac{1}{n}(X\beta_0 + U)'(I - X(X'X)^{-1}X')(X\beta_0 + U)(X'X/n)^{-1}$$

$$= \frac{1}{n}(X\beta_0 + U)'(X\beta_0 + U - X(X'X)^{-1}X'X\beta_0 - X(X'X)^{-1}X'XU)(X'X/n)^{-1}$$

$$= \frac{1}{n}(X\beta_0 + U)'(X\beta_0 + U - X\beta_0 - X(X'X)^{-1}X'U)(X'X/n)^{-1}$$

$$= \frac{1}{n}(X\beta_0 + U)'(U - X(X'X)^{-1}X'U)(X'X/n)^{-1}$$

$$= \frac{1}{n}(X\beta_0 + U)'(I - X(X'X)^{-1}X')U(X'X/n)^{-1}$$

Applying the same process to $(X\beta_0 + U)'(I - X(X'X)^{-1}X')$ we get

$$= \frac{1}{n} U'(I - X(X'X)^{-1}X')U(X'X/n)^{-1}$$

$$= \frac{1}{n}(U'U - (U'X)(X'X)^{-1}(X'U))(X'X/n)^{-1}$$

$$= (U'U/n - (U'X/n)(X'X/n)^{-1}(X'U/n))(X'X/n)^{-1}$$

Now, by the WLLN, we know $(X'U/n) \to_p E[X_iU_i] = E[E[X_iU_i|X_i]] = 0$. Also by the WLLN, $U'U/n \to_p E[U_i^2] = \sigma^2$. Finally from the previous part, we know $(X'X/n)^{-1} \to_p \Sigma_X^{-1}$. Combining,

$$(U'U/n - (U'X/n)(X'X/n)^{-1}(X'U/n))(X'X/n)^{-1} \to_p (\sigma^2 - 0 \cdot \Sigma_X^{-1} \cdot 0)\Sigma_X^{-1} = \sigma^2\Sigma_X^{-1}$$

and hence $\hat{\sigma}^2$ is our consistent estimator.