ECON550: Problem Set 3

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HMC Exercises (7th edition)

1.8.2 We have

$$\mathbb{E}[X] = \int_{-2}^{4} \frac{x(x+2)}{18}\ dx = 2$$

$$\mathbb{E}[(X+2)^3] = \int_{-2}^{4} \frac{(x+2)^4}{18}\ dx = \frac{432}{5}$$

1.8.3

$$\mathbb{E}[X] = 3$$

$$\mathbb{E}[X^2] = 11$$

$$\mathbb{E}[(X+2)^2] = \mathbb{E}[X^2] + 4\mathbb{E}[X] + 4 = 11 + 12 + 4 = 27$$

1.8.6

$$\mathbb{E}[X(1-X)] = \mathbb{E}[X] - \mathbb{E}[X^2] = \frac{3}{4} - \frac{3}{5} = \frac{3}{20}$$

1.8.10 X is symmetrically distributed about 0 because the distribution function only uses x^2, which is symmetric about 0. However, $E(X) \neq 0$ because the integral

$$\int \frac{1}{\pi} \frac{x}{x^2+1}$$

doesn't converge.

1.9.3a From a symmetry observation,

$$\mu = \frac{1}{2}$$

The variance is

$$\sigma^2 = \int_0^1 (x-0.5)^2 6x(1-x)\ dx = 0.05$$

$$\sigma = \frac{\sqrt{5}}{10}$$

So the desired probability is

$$\int_{\frac{1}{2}-\frac{\sqrt{5}}{5}}^{\frac{1}{2}+\frac{\sqrt{5}}{5}} 6x(1-x)\ dx = \frac{11}{5\sqrt{5}} \approx 0.98$$

1.9.6 We use linearity of expectation:

$$E[(X-\mu)/\sigma] = \frac{E(X)-\mu}{\sigma} = 0$$

since $\mu = E[X]$. Also,

$$E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = \frac{E[X^2-2X\mu+\mu^2]}{\sigma^2} = \frac{E[X^2]-E[X]^2}{\sigma^2} = 1$$

Finally,

$$E\left[\exp\left(t\frac{X-\mu}{\sigma}\right)\right] = e^{-t\mu/\sigma}E\left[\exp\left(t\frac{X}{\sigma}\right)\right] = e^{-t\mu/\sigma}M\left(\frac{t}{\sigma}\right)$$

1.9.20

$$E[X] = \int_0^b xf(x)\ dx = xF(x)|_0^b - \int_0^b F(x)dx = b - \int_0^b F(x) = \int_0^b (1-F(x))\ dx$$

1.9.24 The expected value is

$$\mu = \int x \sum_i c_i f_i = \int \sum_i c_i(xf_i) = \sum_i \int c_i(xf_i) = \sum_i c_i \int xf_i = \sum_i c_i\mu_i$$

The variance is

$$\sigma^2 = \int (x-\mu)^2 \sum_i c_i f_i = \sum_i c_i \int (x-\mu)^2 f_i = \sum_i c_i \int (x^2 - 2\mu x + \mu^2 - 2\mu_i + 2\mu_i + \mu_i^2 - \mu_i^2) f_i$$

$$= \sum_i c_i \int (x^2 - 2\mu_i + \mu_i^2 + 2\mu_i - 2\mu x + \mu^2 - \mu_i^2) f_i$$

$$= \sum_i c_i \int (x^2 - 2\mu_i x + \mu_i^2) f_i + (2\mu_i x - 2\mu x + \mu^2 - \mu_i^2) f_i$$

$$= \sum_i c_i \int (x-\mu_i)^2 f_i + (2\mu_i - 2\mu) x f_i + (\mu^2 - \mu_i^2) f_i$$

$$= \sum_i c_i \left(\sigma_i^2 + (2\mu_i - 2\mu)\mu_i + (\mu^2 - \mu_i^2)\right)$$

$$= \sum_i c_i \left(\sigma_i^2 + 2\mu_i^2 - 2\mu\mu_i + \mu^2 - \mu_i^2\right)$$

$$= \sum_i c_i \left(\sigma_i^2 + \mu_i^2 - 2\mu\mu_i + \mu^2\right)$$

$$= \sum_i c_i \left(\sigma_i^2 + (\mu_i - \mu)^2\right)$$

1.10.3 The variance is

$$\sigma^2 = E[X^2] - E[X]^2 = 4$$

$$\sigma = 2$$

So

$$P(-2 < X < 8) = 1 - P(|X-3| \geq 5) = 1 - P(|X-\mu| \geq \frac{5}{2}\sigma) \geq 1 - \frac{4}{25} = \frac{21}{25}$$

1.10.4 By Markov's inequality,

$$P(X \geq a) = P(e^{tX} \geq e^{ta}) \leq \frac{E[e^{tX}]}{e^t a} = e^{-at}M(t)$$

Also, by Markov's inequality,

$$P(X \leq a) = P(e^{-tX} \leq e^{-ta}) = P(e^{tX} \geq e^{ta}) \leq \frac{E[e^{tX}]}{e^{at}} = e^{-at}M(t)$$

1.10.6a The function $\phi(x) = 1/x$ is convex (second derivative is $2/x^3$, positive for positive x), so by Jensen's inequality,

$$E[1/X] = E[\phi(X)] \geq \phi(E[X]) = 1/E[X]$$

1.10.6b The function $\phi(x) = -\log x$ is convex, since it has second derivative $1/x^2$, which is positive for positive x. So by Jensen's,

$$E[-\log X] = E[\phi(X)] \geq \phi(E[X]) = -\log E[X]$$

3.1.3 Using linearity and the fact that a binomial RV has expectation np and variance $np(1-p)$, we get

$$E\left[\frac{X}{n}\right] = \frac{np}{n} = p$$

$$E\left[\left(\frac{X}{n} - p\right)^2\right] = E\left[\frac{(X-pn)^2}{n^2}\right] = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$$

3.1.6 The probability of no successes is given by

$$\left(\frac{3}{4}\right)^n$$

We need

$$(3/4)^n \leq 0.3$$

$$n \geq \log_{3/4} 0.3 \approx 4.185$$

Hence, we need $n \geq 5$ to ensure that the probability of at least one success is at least 0.7.

3.1.14 The probability $X < 3$ is given by

$$(1/3) + (2/9) + (4/27) = \frac{19}{27}$$

So the probability $X \geq 3$ is

$$\frac{8}{27}$$

So the conditional PMF is given by

$$p(x|x \geq 3) = \frac{9}{8}\left(\frac{2}{3}\right)^x = \frac{1}{3}\left(\frac{2}{3}\right)^{x-3}$$

3.1.23 The desired probabilities are

$$P(X \geq k+j | X \geq k) = \frac{P(X \geq k+j)}{P(X \geq k)}$$

$$= \frac{(1-p)^{k+j}}{(1-p)^k} = (1-p)^j = P(X \geq j)$$

Problem 2

We will assume g is measureable, as in the question prompt. We note that ν satisfies countable disjoint additivity due to additivity of the Lebesgue integral:

$$\nu(\cup A_i) = \int \sum_i 1_{A_i} g \, d\mu = \sum_i \int_{A_i} g d\mu = \sum_i \nu(A_i)$$

(in the second step, we skip the result that the textbook showed using monotone convergence). Hence, we just need nonnegativity. That is, we will require

$$\int_A g \, d\mu \geq 0$$

So g is nonnegative.

Problem 3

(a) Suppose $\rho(A) = 0$. Then since $\mu << \rho$, $\mu(A) = 0$. Since $\nu << \mu$, $\nu(A) = 0$. Hence $\nu << \rho$.

(b) By Radon-Nikodym, let

$$\nu(A) = \int_A f d\rho$$

$$\nu(A) = \int_A g d\mu$$

$$\mu(A) = \int_A h d\rho$$

Note we have

$$\nu(A) = \int 1_A g d\mu = \int 1_A g h d\rho = \int_A g h d\rho$$

But since this holds for all A, we must have

$$f = gh$$

$$\frac{d\nu}{d\rho} = (d\nu/d\mu)(d\mu/d\rho)$$

Problem 4

(a) Clearly, for any A,

$$\mu(A) = \int_A d\mu = \int_A \frac{d\mu}{d\mu} d\mu$$

Hence $d\mu/d\mu = 1$.

(b) We have:

$$\mu(A) = \int_A d\mu = \int_A (d\mu/d\nu) d\nu = \int_A (d\mu/d\nu)(d\nu/d\mu) d\mu$$

Hence

$$(d\mu/d\nu)(d\nu/d\mu) = d\mu/d\mu = 1$$

So we get the desired result.