ECON550: Problem Set 4

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Fall 2020

HMC Exercises (7th edition)

2.1.1

$$P(0 < X_1 < 1/2,\ 1/4 < X_2 < 1) = \int_0^{1/2} \int_{1/4}^{1} 4x_1 x_2 = \frac{1}{4}\left(1 - \frac{1}{16}\right) = \frac{15}{64}$$

$$P(X_1 = X_2) = 0$$

$$P(X_1 < X_2) = \int_0^1 \int_0^{x_2} 4x_1 x_2 \ dx_1 \ dx_2 = \int_0^1 2x_2(x_2^2) = \frac{2}{4} = 1/2$$

$$P(X_1 \le X_2) = 1/2$$

2.1.6

$$P(Z \le 0) = 0$$

$$P(Z \le 6) = \int_0^6 \int_0^{6-x} e^{-x-y} \ dy \ dx = \int_0^6 \left(e^{-x} - e^{-6}\right) dy = 1 - e^{-6} - 6e^{-6} \approx 0.9826$$

$$F_Z(t) = \int_0^t \int_0^{t-x} e^{-x-y} \ dy \ dx = \int_0^t e^{-x} - e^{-t} \ dx = 1 - e^{-t} - te^{-t}$$

$$f_Z(t) = te^{-t}$$

where $t \ge 0$.

2.1.10 Marginal distribution:

$$f_{X_1}(x_1) = \int_{x_1}^1 15x_1^2 x_2 \ dx_2 = \frac{15}{2} x_1^2(1 - x_1^2)$$

$$f_{X_2}(x_2) = \int_0^{x_2} 15x_1^2 x_2 \ dx_1 = 5x_2^4$$

$$P(X_1 + X_2 \le 1) = \int_0^{1/2} \int_{x_1}^{1-x_1} 15x_1^2 x_2 \ dx_2 \ dx_1 = \int_0^{1/2} (15/2)x_1^2(1 - 2x_1) \ dx_1 = (5/16) - (15/64) = 5/64$$

2.1.16

$$P(2X+3Y<1)=\int_0^{1/2}\int_0^{(1-2x)/3}6(1-x-y)\ dy\ dx=\int_0^{1/2}6\left(\frac{(1-x)(1-2x)}{3}-\frac{(1-2x)^2}{18}\right)$$

$$=\frac{1}{3}\int_0^{1/2}(5-14x+8x^2)=\frac{1}{3}((5/2)-(7/4)+(1/3))=13/36$$

2.3.1 Conditional mean:

$$E[X_2|X_1=x_1]=\int_0^1 x_2\frac{x_1+x_2}{\frac{1}{2}+x_1}\ dx_2=\frac{x_1/2+1/3}{\frac{1}{2}+x_1}=\frac{3x_1+2}{6x_1+3}$$

Conditional variance:

$$E[X_2^2|X_1=x_1]=\int_0^1 x_2^2\frac{x_1+x_2}{\frac{1}{2}+x_1}=\frac{x_1/3+1/4}{x_1+1/2}=\frac{4x_1+3}{12x_1+6}$$

$$E[X_2^2|X_1=x_1]-(E[X_2|X_1=x_1])^2=\frac{4x_1+3}{12x_1+6}-\left(\frac{3x_1+2}{6x_1+3}\right)^2$$

$$=\frac{(4x_1+3)(2x_1+1)}{6(2x_1+1)^2}-\frac{(3x_1+2)(3x_1+2)}{9(2x_1+1)^2}=\frac{6x_1^2+6x_1+1}{18(2x_1+1)^2}$$

2.3.2a, b To determine c_1 and c_2, we need normalization to 1. For c_1, we get

$$\int_0^{x_2}c_1x_1/x_2^2\ dx_1=1$$

So we need $c_1=2$. Likewise, we get that $c_2=5$. Then the joint pdf is

$$f(x_1,x_2)=f_{1|2}(x_1|x_2)f_2(x_2)=10x_1x_2^2$$

$$0<x_1<x_2<1$$

2.3.10 The marginal probability function of x_1:

$$p_1(0)=4/18$$

$$p_1(1)=7/18$$

$$p_1(2)=7/18$$

The marginal probability function of x_2:

$$p_2(0)=11/18$$

$$p_2(1)=7/18$$

The conditional means are:

$$E[X_1|X_2=0]=16/11$$

$$E[X_1|X_2=1]=5/7$$

$$E[X_2|X_1=0]=3/4$$

$$E[X_2|X_1=1]=3/7$$

$$E[X_2|X_1=2]=1/7$$

2.4.1a

$$\frac{Cov(X,Y)}{\sqrt{V(X)V(Y)}}=\frac{(2/3)}{(2/3)}=1$$

2.4.6

$$E[Y|x]=\int_{-x}^{x}\frac{y}{2x}=\frac{(1/2)x^2-(1/2)(-x)^2}{2x}=0$$

and hence this is a straight line.

2.4.11 Consider $Y=X_1+X_2$. Then by linearity of expectation, $E[Y]=\mu_1+\mu_2$. Also

$$E[Y^2]=E[X_1^2+2X_1X_2+X_2^2]=E[X_1]^2+E[X_2]^2+2E[X_1X_2]$$

$$\begin{aligned}V[Y]=E[Y^2]-E[Y]^2&=E[X_1]^2+E[X_2]^2+2E[X_1X_2]-(\mu_1+\mu_2)^2\\&=E[X_1]^2+E[X_2]^2+2E[X_1X_2]-\mu_1^2-\mu_2^2-2\mu_1\mu_2\\&=\sigma^2+\sigma^2+2Cov(X_1,X_2)\\&=2\sigma^2+2Cov(X_1,X_2)\\&=2\sigma^2(1+\rho)\end{aligned}$$

Then by Chebyshev, we have

$$P(|Y-\mu_1-\mu_2|\geq k\sigma)\leq\frac{2\sigma^2(1+\rho)}{(k\sigma)^2}=\frac{2(1+\rho)}{k^2}$$

2.5.2 Can't be independent because the support is non-rectangular, as we argued in class. Since $x_1<x_2$, we have dependence.

2.5.5 The desired probability is 5

$$\frac{2}{3}+\frac{5}{8}-\frac{5}{12}=\frac{7}{8}$$

2.5.8 These are not independent, the support isn't a rectangle.

$$E(X|y)=\int_{y}^{1}\frac{3x^2}{(3/2)(1-y^2)}\frac{2}{3}$$

2.5.11 Let X_1 be the first midpoint, X_2 the second. Then

$$P(X_1 > X_2 + 1) = P(X_1 - X_2 > 1) = \int_7^1 4 \int_6^{x-1} \frac{1}{196}\ dy\ dx = \frac{1}{8}$$

$$P(X_2 > X_1 + 1) = P(X_2 - X_1 > 1) = \int_6^2 0 \int_0^{y-1} \frac{1}{196}\ dy\ dx = \frac{311}{392}$$

$$1 - (P(X_1 > X_2 + 1) + P(X_2 > X_1 + 1)) = \frac{4}{49}$$

2.8.1

$$\begin{aligned}
(n-1)^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 &= (n-1)^{-1} \sum_{i=1}^{n} X_i^2 - 2\bar{X}X_i + \bar{X}^2 \\
&= (n-1)^{-1} \left(\sum_{i=1}^{n} X_i^2 - \sum_{i=1}^{n} 2\bar{X}X_i + \sum_{i=1}^{n} \bar{X}^2 \right) \\
&= (n-1)^{-1} \left(\sum_{i=1}^{n} X_i^2 - 2\bar{X} \sum_{i=1}^{n} X_i + n\bar{X}^2 \right) \\
&= (n-1)^{-1} \left(\sum_{i=1}^{n} X_i^2 - 2n\bar{X}^2 + n\bar{X}^2 \right) \\
&= (n-1)^{-1} \left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2 \right)
\end{aligned}$$

2.8.4

$$E[X_1 X_2] = \mu_1\mu_2 + Cov(X_1, X_2) = \mu_1\mu_2$$

$$\begin{aligned}
V(X_1X_2) = E[X_1^2 X_2^2] - \mu_1^2\mu_2^2 &= E[X_1]^2 E[X_2]^2 - \mu_1^2\mu_2^2 \\
&= (\sigma_1^2 + \mu_1^2)(\sigma_2^2 + \mu_2^2) - \mu_1^2\mu_2^2 \\
&= \sigma_1^2\sigma_2^2 + \mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2
\end{aligned}$$

2.8.10

$$V(X + 2Y) = Cov(X + 2Y, X + 2Y) = Var(X) + 4Var(Y) + 4Cov(X, Y) = 15$$

$$Cov(X, Y) = 3/4$$

$$\rho = Cov(X, Y)/\sqrt(V(X)V(Y)) = \frac{3}{8\sqrt{2}}$$

2.8.16

$$V\left(\sum X_i\right) = \sum V(X_i) + \sum_i \sum_{j \neq i} Cov(X_i, X_j) = 10 \cdot (5) + 90 \cdot (0.5) = 50 + 45 = 95$$

2.8.18 Consider $f(x) = \sqrt{x}$. By concavity and Jensen's inequality,

$$E[f(S^2)] < f(E[S^2])$$

$$E[S] < f(\sigma^2) = \sigma$$

as desired.

2.6.1a

$$f_X(x) = \frac{2x+2}{3}$$

$$f_Y(y) = \frac{2y+2}{3}$$

$$f_Z(z) = \frac{2z+2}{3}$$

2.6.1b

$$P(0 < X, Y, Z < 1/2) = \int_0^{1/2} \int_0^{1/2} \int_0^{1/2} \frac{2x+2y+2z}{3} = \int_0^{1/2} \int_0^{1/2} \frac{1}{12} + \frac{y+z}{3} = \int_0^{1/2} \frac{1}{24} + \frac{1}{24} + \frac{z}{6} = \frac{1}{24} + \frac{1}{48} = \frac{1}{16}$$

$$P(0 < X < 1/2) = P(0 < Y < 1/2) = P(0 < Z < 1/2) = \int_0^{1/2} \frac{2x+2}{3} = \frac{1}{12} + \frac{1}{3} = \frac{5}{12}$$

2.6.1c No. We can verify using part b that $P(A \cap B \cap C) \neq P(A)P(B)P(C)$

2.6.1e The joint cdf is

$$\int \int \int f(x, y, z) = \frac{x^2 + y^2 + z^2}{3}$$

The marginal cdfs are given by

$$F_X(x) = \frac{x^2 + 2x}{3}$$

$$F_Y(y) = \frac{y^2 + 2y}{3}$$

$$F_Z(z) = \frac{z^2 + 2z}{3}$$

2.6.1f Conditional distribution:

$$f_{X,Y|Z} = \frac{f(x, y, z)}{f_Z(z)} = \frac{2x+2y+2z}{2z+2} = \frac{x+y+z}{z+1}$$

$$E[X + Y|Z = z] = \int \int (x+y) \frac{x+y+z}{z+1}$$

$$= \frac{1}{z+1} \int \int x^2 + x(y+z) + xy + y^2 + yz = \frac{1}{z+1} \int \frac{1}{3} + y + (z/2) + y^2 + yz$$

$$= \frac{1}{z+1}\left(\frac{1}{3}+\frac{1}{2}+z+\frac{1}{3}\right)$$
$$= \frac{6z+7}{6z+6}$$

3.3.2 The PDF is given by:

$$f(x) = \frac{x^{3/2}e^{-x/2}}{3\sqrt{2\pi}}$$

By numerically bashing or consulting a table, we get that $c = 0.831$, $d = 12.833$

3.4.2

$$P(X < 60) = P((X-75)/10 < 1.5) = 0.0668$$

$$P(70 < X < 100) = P(-0.5 < (X-75)/10 < 2.5) = 0.6853$$

3.4.3 We can take $b \approx 1.645$.

3.4.18 Skewness:

$$\frac{E[(X-\mu)^3]}{\sigma^3} = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \int \frac{x^3e^{-x^2/2}}{\sqrt{2\pi}} = 0$$

Kurtosis

$$\frac{E[(X-\mu)^4]}{\sigma^4} = \int \frac{x^4e^{-x^2/2}}{\sqrt{2\pi}} = 3$$

3.5.8 We rewrite

$$f(x,y) = \frac{1}{2\pi}\left(\exp\left(-\frac{1}{2}(x^2+y^2)\right) + xy\exp\left(-(x^2+y^2-1)\right)\right)$$

Note that when we integrate through the domain of only x or only y, the second term disappears due to symmetry about 0, and we will be left with

$$f_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)$$

Hence the marginal distributions are normal. However, the pdf is still joint, as the distribution depends on both x, y and characterizes the realization on X, Y.

3.5.10 By HMC Theorem 3.5.1, we have that

$$\mu_Z = a\mu_X + b\mu_Y = 0$$

$$\sigma_Z^2 = a^2\sigma_X^2 + 2ab\rho\sigma_X\sigma_Y + b^2\sigma_Y^2 = a^2 + b^2 + 2ab\rho$$

Then Z is distributed as $N(\mu_Z, \sigma_Z^2)$.

3.5.15 Take $a = [1/n, 1/n, 1/n, ...]$. Then by Theorem 3.5.1, we have the distribution of $\bar{X}$ is given by $N(a\mu, a\Sigma a')$. If all of its components have the same mean μ, then the distribution is then $N(\mu, a\Sigma a')$.

Problem 1

We know that

$$V(AX+b) = E[(AX+b) - E[AX+b])(AX+b) - E[AX+b])']$$
$$= E[(AX - AE[X])(AX - AE[X])']$$
$$= E[A(X - E[X])(X - E[X])'A']$$
$$= AE[(X - E[X])(X - E[X])']A'$$
$$= AV(X)A'$$

Problem 2

$$Cov(AX+BY, CZ+DW) = E[(AX+BY - E[AX+BY])(CZ+DW - E[CZ+DW])']$$
$$= E[(AX+BY - AE[X] - BE[Y])(CZ+DW - CE[Z] - DE[W])']$$
$$= E[(A(X - E[X]) + B(Y - E[Y]))(C(Z - E[Z]) + D(W - E[W]))']$$
$$= E[A(X-E[X])(Z-E[Z])'C' + B(Y-E[Y])(Z-E[Z])'C' + A(X-E[X])(W-E[W]))'D' + B(Y-E[Y])(W-E[W]))'D']$$
$$= AE[(X - E[X])(Z - E[Z])']C' + BE[(Y - E[Y])(Z - E[Z])']C'$$
$$+AE[(X - E[X])(W - E[W]))']D' + BE[(Y - E[Y])(W - E[W]))']D'$$
$$= ACov(X, Z)C' + BCov(Y, Z)C' + ACov(X, W)D' + BCov(Y, W)D'$$