ECON550: Problem Set 8

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Midterm Problem 6

(a) Define the statistic $\hat{\theta}$ as

$$\hat{\theta} = n^{-1}\sum_{i=1}^{n} \log\left(\frac{X_i}{Y_i}\right) - \log\left(\frac{n^{-1}\sum_{i=1}^{n} X_i}{n^{-1}\sum_{i=1}^{n} Y_i}\right)$$

By the WLLN,

$$n^{-1}\sum_{i=1}^{n} \log\left(\frac{X_i}{Y_i}\right) \to_p E\log\left(\frac{X_i}{Y_i}\right)$$

$$n^{-1}\sum_{i=1}^{n} X_i \to_p EX_i$$

$$n^{-1}\sum_{i=1}^{n} Y_i \to_p EY_i$$

We require $E\log\left(\frac{X_i}{Y_i}\right) < \infty$, $EX_i < \infty$, $EY_i < \infty$. Now, by Slutsky's theorem and the rules of convergence in probability, we get

$$\log\left(\frac{n^{-1}\sum_{i=1}^{n} X_i}{n^{-1}\sum_{i=1}^{n} Y_i}\right) \to_p \log\left(\frac{EX_i}{EY_i}\right)$$

Applying the rules of convergence in probability, we get

$$\hat{\theta} = n^{-1}\sum_{i=1}^{n} \log\left(\frac{X_i}{Y_i}\right) - \log\left(\frac{n^{-1}\sum_{i=1}^{n} X_i}{n^{-1}\sum_{i=1}^{n} Y_i}\right)$$

$$\to_p E\log\left(\frac{X_i}{Y_i}\right) + \log\left(\frac{EX_i}{EY_i}\right)$$

(b) Define g as

$$g(A, B, C) = A - \log(B/C)$$

Define

$$Z_i = \left(\begin{bmatrix} \log(X_i/Y_i) \\ X_i \\ Y_i \end{bmatrix}\right)$$

Let

$$\bar{Z}_i = n^{-1} \sum_{i=1}^{n} Z_i$$

Then

$$\hat{\theta} = g(\bar{Z}_i)$$

Note that

$$\theta = g(EZ_i) = E \log\left(\frac{X_i}{Y_i}\right) + \log\left(\frac{EX_i}{EY_i}\right)$$

Define the variance matrix Σ as the variance matrix $Var(Z_i)$. By the CLT,

$$\sqrt{n}(\bar{Z}_i - EZ_i) \to_d N(0, \Sigma)$$

By the delta method,

$$G(\theta) = \begin{bmatrix} 1 & -1/EX_i & 1/EX_y \end{bmatrix}$$

so

$$\sqrt{n}(\hat{\theta} - \theta) = \sqrt{n}(g(\bar{Z}_i) - EZ_i) \to_d N(0, G(\theta)\Sigma G(\theta)')$$

(c) We can estimate $G(\theta)$ with

$$\hat{G} = \begin{bmatrix} 1 & -1/\bar{X}_i & 1/\bar{Y}_i \end{bmatrix}$$

By Slutsky's theorem, since $\bar{X}_i \to_p EX_i$ and $\bar{Y}_i \to_p EY_i$,

$$\hat{G} = \begin{bmatrix} 1 & -1/\bar{X}_i & 1/\bar{Y}_i \end{bmatrix} \to_p \begin{bmatrix} 1 & -1/EX_i & 1/EX_y \end{bmatrix} = G(\theta)$$

Now, we just need to estimate Σ. We can take

$$\hat{\Sigma} = n^{-1} \sum_{i=1}^{N} Z_i Z_i' - \bar{Z}_i \bar{Z}_i{}'$$

Assuming $||EZ_i Z_i'||, |EZ_i| < \infty$, by the WLLN,

$$n^{-1} \sum_{i=1}^{N} Z_i Z_i' \to EZ_i Z_i'$$

Therefore, by Slutsky's theorem

$$\hat{\Sigma} \to_p \Sigma$$

Hence, applying Slutsky's theorem again,

$$\hat{G}\hat{\Sigma}\hat{G}'$$

converges in probability to the desired variance.

Midterm Problem 7

(a) We have

$$\frac{\bar{Y}_i}{\bar{X}_i} = \frac{\sqrt{n}\bar{Y}_i}{\sqrt{n}\bar{X}_i}$$

Define

$$Z_i = \begin{bmatrix} Y_i \\ X_i \end{bmatrix}$$

Note

$$EZ_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $\Sigma = Var(Z_i)$. We have by the CLT

$$\sqrt{n}\bar{Z}_i \to_d \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \sim N(0, \Sigma)$$

Let $g(A, B) = A/B$. Note

$$\frac{\bar{Y}_i}{\bar{X}_i} = g(\bar{Z}_i)$$

Then by the CMT,

$$\frac{\bar{Y}_i}{\bar{X}_i} = \frac{\sqrt{n}\bar{Y}_i}{\sqrt{n}\bar{X}_i} \to_d g(N(0, \Sigma)) = \frac{N_1}{N_2}$$

(b) We consider

$$\frac{\bar{Y}_i}{\sqrt{n}\bar{X}_i}$$

We know $\bar{Y}_i \to_p \mu_Y$. Applying the rules for convergence in distribution, we get, via the CMT again,

$$\frac{\bar{Y}_i}{\sqrt{n}\bar{X}_i} \to_d \frac{\mu_Y}{N_2}$$

where N_2 is from the previous part.

Problem 1

$$\hat{\beta}_n = \left(\sum_{i=1}^{n} X_i X_i'\right)^{-1} \sum_{i=1}^{n} X_i Y_i$$

$$= \left(n^{-1} \sum_{i=1}^{n} X_i X_i'\right)^{-1} \left(n^{-1} \sum_{i=1}^{n} X_i Y_i\right)$$

Note by WLLN

$$\left(n^{-1} \sum_{i=1}^{n} X_i X_i'\right)^{-1} \to_p E[X_i X_i']$$

$$\left(n^{-1}\sum_{i=1}^{n} X_i Y_i\right) \to_p E[X_i Y_i]$$

Applying Slutsky and the rules of convergence in probability, and using the fact that Σ is positive definite,

$$\hat{\beta}_n = \left(n^{-1}\sum_{i=1}^{n} X_i X_i'\right)^{-1}\left(n^{-1}\sum_{i=1}^{n} X_i Y_i\right)$$

$$\to_p (E[X_i X_i'])^{-1} E[X_i Y_i]$$

$$= (E[X_i X_i'])^{-1} E[X_i X_i'\beta + X_i U_i]$$

$$= \beta + (E[X_i X_i'])^{-1} E[X_i U_i]$$

Note that the other term will generally not be zero if $E[U_i|X_i]$ is not zero.

Problem 2

(a) We have

$$\sqrt{n}(be\hat{t}a_n - \beta_0) = \sqrt{n}\left((X'X)^{-1}(X'Y) - \beta_0\right)$$

$$= \sqrt{n}\left((X'X)^{-1}(X'(X\beta_0 + U)) - \beta_0\right)$$

$$= \sqrt{n}\left((X'X)^{-1}(X'X\beta_0 + X'U)) - \beta_0\right)$$

$$= \sqrt{n}\left((X'X)^{-1}(X'U)\right)$$

$$= \sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_i U_i\right)\right)$$

Now

$$\left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right) \to_p \Sigma_X$$

since Σ_X is positive definite, by Slutsky's theorem,

$$\left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \to_p \Sigma_X^{-1}$$

Additionally, since $E[X_i U_i] = 0$, by the CLT

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i U_i\right) \to_d N(0, Var(X_i U_i))$$

So all together

$$\sqrt{n}(be\hat{t}a_n - \beta_0) = \sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_i U_i\right)\right) \to_d N(0, \Sigma_X^{-1} Var(X_i U_i)(\Sigma_X^{-1})')$$

$$= N(0, \Sigma_X^{-1} E(U_i X_i X_i' U_i)(\Sigma_X^{-1})')$$

(b) The variance is $\Sigma_X^{-1} E(U_i X_i X_i' U_i)(\Sigma_X^{-1})')$. To get a consistent estimator of this expression, we just need a consistent estimator for Σ_X, and a consistent estimator for $E(U_i X_i X_i' U_i)$, and we can then apply the rules of convergence in probability and Slutsky's theorem to get the overall variance estimator (since Σ_X is positive definite).

By the WLLN, we already know

$$n^{-1} \sum X_i X_i' \to_p \Sigma_X$$

Hence we just have to construct an estimator for $E(U_i X_i X_i' U_i)$. Consider estimating the (a, b) element as

$$\hat{\Sigma}_{ab} = n^{-1} \sum_{i=1}^{n} (Y_i - X_i'\hat{\beta}) X_{ai} X_{bi} (Y_i - X_i'\hat{\beta})$$

$$= n^{-1} \sum_{i=1}^{n} (X_i'(\beta_0 - \hat{\beta}) + U_i) X_{ai} X_{bi} (X_i'(\beta_0 - \hat{\beta}) + U_i)$$

$$= n^{-1} \sum_{i=1}^{n} X_i'(\beta_0 - \hat{\beta}) X_{ai} X_{bi} X_i'(\beta_0 - \hat{\beta}) + U_i X_{ai} X_{bi} X_i'(\beta_0 - \hat{\beta}) + X_i'(\beta_0 - \hat{\beta}) X_{ai} X_{bi} U_i + U_i X_{ai} X_{bi} U_i$$

We consider this one term at a time. The first term:

$$n^{-1} \sum_{i=1}^{n} (\beta_0 - \hat{\beta})' X_i X_{ai} X_{bi} X_i'(\beta_0 - \hat{\beta}) = (\beta_0 - \hat{\beta})' \left(n^{-1} \sum_{i=1}^{n} X_i X_{ai} X_{bi} X_i' \right) (\beta_0 - \hat{\beta})$$

As long as $E[||X_i||^4] < \infty$, by the WLLN, the sum

$$\left(n^{-1} \sum_{i=1}^{n} X_i X_{ai} X_{bi} X_i' \right)$$

converges to something finite. Then because $(\beta_0 - \hat{\beta}) \to_p 0$, this term converges in probability to 0. Similarly, the second/third terms are

$$n^{-1} \sum_{i=1}^{n} U_i X_{ai} X_{bi} X_i'(\beta_0 - \hat{\beta}) = \left(n^{-1} \sum_{i=1}^{n} U_i X_{ai} X_{bi} X_i' \right) (\beta_0 - \hat{\beta})$$

By Cauchy Schwarz, $E||U_i X_{ai} X_{bi} X_i'|| \leq \sqrt{E||(U_i X_{ai})^2|| E||(X_{bi} X_i')^2||} < \infty$ due to finite second moment on X_i. Hence, since $(\beta_0 - \hat{\beta}) \to_p 0$, the second and third terms also converge in probability to 0.

Finally, we consider the last term. By the WLLN

$$n^{-1} \sum_{i=1}^{n} U_i X_{ai} X_{bi} U_i \to_p E(U_i X_i X_i' U_i)_{ab}$$

as desired. Hence $\hat{\Sigma}$ consistently estimates $E(U_i X_i X_i' U_i)$. Thus, all together, the desired estimator is

$$\left(n^{-1}\sum X_i X_i'\right)^{-1}\hat{\Sigma}\left(\left(n^{-1}\sum X_i X_i'\right)^{-1}\right)'$$

Problem 3

$$\bar{X_n} \sim N(\mu, \frac{2}{n})$$

So the confidence interval is

$$\left[2 - \frac{\sqrt{2}}{\sqrt{n}} z_{1-\alpha/2}, 2 + \frac{\sqrt{2}}{\sqrt{n}} z_{1-\alpha/2}\right]$$

For $\alpha = 0.05$:

$$\left[2 - \frac{\sqrt{2}}{\sqrt{n}}(1.96), 2 + \frac{\sqrt{2}}{\sqrt{n}}(1.96)\right]$$

For $\alpha = 0.1$:

$$\left[2 - \frac{\sqrt{2}}{\sqrt{n}}(1.645), 2 + \frac{\sqrt{2}}{\sqrt{n}}(1.645)\right]$$

Problem 4

(a) The interval formula is

$$\left[\bar{X_n} - \frac{\sqrt{S^2_{X_n}}}{\sqrt{n}} t_{n-1,1-\alpha/2}, \bar{X_n} + \frac{\sqrt{S^2_{X_n}}}{\sqrt{n}} t_{n-1,1-\alpha/2}\right]$$

We just plug the values in then. For 90%:

$$[79.2, 83.2]$$

For 95%:

$$[78.8, 83.6]$$

For 99%:

$$[77.9, 84.5]$$

(b) We use the same formula from the previous problem part. Plugging in numbers and computing, we get:

$$[3.68, 5.72]$$

(c) The confidence interval is

$$[\bar{X_n} - z_{1-\alpha/2}\sqrt{\sigma^2/n}, \bar{X_n} + z_{1-\alpha/2}\sqrt{\sigma^2/n}]$$

Setting it equal to the other interval, we get

$$\sigma/4 = z_{1-\alpha/2}\sqrt{\sigma^2/n}$$

$$\sqrt{n} = 4z_{1-\alpha/2}$$

$$n \geq 62$$

(d) (a)

$$P(a < (n-1)S^2/\sigma^2 < b) = 0.95$$

$$P\left(\frac{a}{(n-1)S^2} < 1/\sigma^2 < \frac{b}{(n-1)S^2}\right) = 0.95$$

$$P\left(\frac{(n-1)S^2}{a} > \sigma^2 > \frac{(n-1)S^2}{b}\right) = 0.95$$

(b) We want to choose b such that

$$P\left(\frac{(n-1)S^2}{\sigma^2} < b\right) = 0.975$$

Since $X \sim \chi^2_{n-1}$, $b \approx 17.535$. Also, we choose a similarly

$$P\left(\frac{(n-1)S^2}{\sigma^2} < a\right) = 0.025$$

so $a \approx 2.180$. Then the confidence interval is

$$\left[\frac{(n-1)S^2}{b}, \frac{(n-1)S^2}{a}\right] \approx [3.62, 29.10]$$

(c) If μ is known, we don't use the sample variance, and instead use

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$$

Then the statistic nS^2/σ^2 is distributed as χ^2_n (one fewer degree of freedom). We then find a, b such that

$$P\left(\frac{nS^2}{\sigma^2} < a\right) = 0.025$$

$$P\left(\frac{nS^2}{\sigma^2} < b\right) = 0.975$$

The confidence interval is then

$$\left[\frac{nS^2}{b}, \frac{nS^2}{a}\right]$$

Problem 5

(a) We know that $\bar{X}_n \to_p \mu, S^2_{X_n} \to_p \sigma^2$ from results in lectures/problem sets. So we seek to characterize

$$\sqrt{n}\left(\begin{bmatrix}\bar{X}_n \\ S^2_{X_n}\end{bmatrix} - \begin{bmatrix}\mu \\ \sigma^2\end{bmatrix}\right)$$

We first consider

$$\sqrt{n}\left(\begin{bmatrix}\bar{X}_n \\ \hat{S}^2_{X_n}\end{bmatrix} - \begin{bmatrix}\mu \\ \sigma^2\end{bmatrix}\right) = \sqrt{n}\left(n^{-1}\sum_{i=1}^{n}\begin{bmatrix}X_i \\ (X_i - \bar{X}_n)^2\end{bmatrix} - \begin{bmatrix}\mu \\ \sigma^2\end{bmatrix}\right)$$
$$= \sqrt{n}\left(n^{-1}\sum_{i=1}^{n}\begin{bmatrix}X_i \\ (X_i - \mu)^2\end{bmatrix} - \begin{bmatrix}\mu \\ \sigma^2\end{bmatrix}\right) + \sqrt{n}\begin{bmatrix}0 \\ (\bar{X}_n - \mu)^2\end{bmatrix}$$

Now, by the CLT $\sqrt{n}(\bar{X}_n - \mu) \to_d N(0, \sigma^2)$, so $\sqrt{n}(\bar{X}_n - \mu)^2 \to_p 0$ since $\bar{X}_n - \mu \to_p 0$. And the first term, by the CLT, converges to

$$\sqrt{n}\left(n^{-1}\sum_{i=1}^{n}\begin{bmatrix}X_i \\ (X_i - \mu)^2\end{bmatrix} - \begin{bmatrix}\mu \\ \sigma^2\end{bmatrix}\right) \to_d N(0, \Sigma)$$

where

$$\Sigma = \begin{bmatrix}\sigma^2 & E[(X_i - \mu)^3] \\ E[(X_i - \mu)^3] & E[(X_i - \mu)^4] - \sigma^4\end{bmatrix}$$

So we have

$$\sqrt{n}\left(\begin{bmatrix}\bar{X}_n \\ \hat{S}^2_{X_n}\end{bmatrix} - \begin{bmatrix}\mu \\ \sigma^2\end{bmatrix}\right) \to_d N(0, \Sigma)$$

Now, since

$$\begin{bmatrix}\bar{X}_n \\ S^2_{X_n}\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & n/(n-1)\end{bmatrix}\begin{bmatrix}\bar{X}_n \\ \hat{S}^2_{X_n}\end{bmatrix}$$

And we know

$$\begin{bmatrix}1 & 0 \\ 0 & n/(n-1)\end{bmatrix} \to I$$

We get that

$$\sqrt{n}\left(\begin{bmatrix}\bar{X}_n \\ S^2_{X_n}\end{bmatrix} - \begin{bmatrix}\mu \\ \sigma^2\end{bmatrix}\right)$$
$$= \sqrt{n}\left(\begin{bmatrix}1 & 0 \\ 0 & n/(n-1)\end{bmatrix}\begin{bmatrix}\bar{X}_n \\ \hat{S}^2_{X_n}\end{bmatrix} - \begin{bmatrix}\mu \\ \sigma^2\end{bmatrix}\right)$$
$$= \sqrt{n}\begin{bmatrix}1 & 0 \\ 0 & n/(n-1)\end{bmatrix}\left(\begin{bmatrix}\bar{X}_n \\ \hat{S}^2_{X_n}\end{bmatrix} - \begin{bmatrix}\mu \\ \sigma^2\end{bmatrix}\right) + \sqrt{n}\begin{bmatrix}0 \\ \sigma^2/(n-1)\end{bmatrix}$$
$$\to_d N(0, \Sigma) + 0 = N(0, \Sigma)$$

(b) Note that

$$S_{X_n}^2 \to_p \sigma^2$$

Further, by Slutsky's theorem, $S_{X_n}^4 \to_p \sigma^4$. We just need to find estimators for $E[(X_i - \mu)^3]$ and $E[(X_i - \mu)^4]$. To estimate $E[(X_i - \mu)^3]$ we consider

$$n^{-1} \sum (X_i - \bar{X}_n)^3 = n^{-1} \left(\sum X_i^3 + \sum 3X_i^2 \bar{X}_n + \sum 3X_i \bar{X}_n^2 - \sum \bar{X}_n^3 \right)$$

$$\to_p EX_i^3 - 3EX_i^2 \mu + 3EX_i \mu^2 - \mu^3 = E[(X_i - \mu)^3]$$

Where we just applied the WLLN and Slutsky's theorem. Similarly

$$n^{-1} \sum (X_i - \bar{X}_n)^4 = n^{-1} \left(\sum X_i^3 - \sum 4X_i^3 \bar{X}_n + \sum 6X_i^2 \bar{X}_n^2 - \sum 4X_i \bar{X}_n^3 + \sum \bar{X}_n^4 \right)$$

$$\to_p EX_i^3 - 4EX_i^3 \mu + 6EX_i^2 \mu^2 - 4EX_i \mu^3 + \mu^4 = E[(X_i - \mu)^4]$$

(c) Let $g(a, b) = \sqrt{b}/a$. Then the desired distribution is

$$\sqrt{n} \left(g\left(\begin{bmatrix} \bar{X}_n \\ S_{X_n}^2 \end{bmatrix} \right) - g\left(\begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix} \right) \right)$$

By the delta method, we have

$$G(\theta_0) = \begin{bmatrix} \frac{-\sigma}{\mu^2} & \frac{1}{2\mu\sigma} \end{bmatrix}$$

So

$$\sqrt{n} \left(g\left(\begin{bmatrix} \bar{X}_n \\ S_{X_n}^2 \end{bmatrix} \right) - g\left(\begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix} \right) \right) \to_d N(0, G(\theta_0)\Sigma G(\theta_0)')$$

(d) By Slutsky's theorem,

$$\sqrt{S_{X_n}^2} \to_p \sigma$$

and we know

$$\bar{X}_n \to_p \mu$$

Hence by the rules of convergence in probability, the estimator

$$\hat{G} = \begin{bmatrix} \frac{-\sqrt{S_{X_n}^2}}{\bar{X}_n{}^2} & \frac{1}{2\bar{X}_n \sqrt{S_{X_n}^2}} \end{bmatrix} \to_p \begin{bmatrix} \frac{-\sigma}{\mu^2} & \frac{1}{2\mu\sigma} \end{bmatrix} = G(\theta_0)$$

We know

$$(\hat{G}\hat{\Sigma}\hat{G}')^{-1/2} \sqrt{n} \left(\frac{S_{X_n}}{\bar{X}_n} - \frac{\sigma}{\mu} \right) \to_p N(0, 1)$$

So the desired confidence interval is

$$\left[\frac{S_{X_n}}{\bar{X}_n} - n^{-1/2} (\hat{G}\hat{\Sigma}\hat{G}')^{1/2} z_{0.975}, \frac{S_{X_n}}{\bar{X}_n} + n^{-1/2} (\hat{G}\hat{\Sigma}\hat{G}')^{1/2} z_{0.975} \right]$$

Problem 6

Pick an orthonormal matrix $M \in \mathbb{R}^{n^2}$, where the first of the column vectors is along the direction $\vec{1}$. Consider MX, where X is the vector $(X_1, X_2, ...X_n)$. Then MX is a linear operation on normal random variables, and hence is jointly normal. The variance matrix is given by $M(\sigma^2 I)M' = \sigma^2 MM' = \sigma^2 I$ by orthonormality of M. Since covariance 0 suffices for independence in jointly normal random variables, this implies that $MX_1, MX_2, ...$ are all pairwise, since the variance matrix of MX_1 only has nonzero diagonal entries.

We know, by construction of M, that $MX_1 = \frac{1}{\sqrt{n}}(\sum X_i) = \sqrt{n}\bar{X}_n$. Further,

$$\sum_{i=2}^{n}(MX_i)^2 = M'X'XM - n\bar{X}_n^2 = \sum_{i=1}^{n}(X_i - \bar{X}_n)^2 = (n-1)S_{X_n}^2$$

Since $MX_2^2, MX_3^2...$ are independent of MX_1, and $(n-1)S_{X_n}^2$ is a linear combination of these, it follows that $(n-1)S_{X_n}^2$ is independent of $\sqrt{n}\bar{X}_n$, which implies $\bar{X}_n$ is independent of $S_{X_n}^2$.

Problem 7

(a) Expanding, we get

$$\left(n^{-1}\sum(Z_i - \bar{Z}_n)X_i\right)^{-1} n^{-1}\sum(Z_i - \bar{Z}_n)Y_i$$

$$= \left(n^{-1}\sum(Z_i - \bar{Z}_n)X_i\right)^{-1} n^{-1}\sum(Z_i - \bar{Z}_n)(\alpha_0 + \beta_0 X_i + U_i)$$

$$= \left(n^{-1}\sum(Z_i - \bar{Z}_n)X_i\right)^{-1} n^{-1}\sum(Z_i - \bar{Z}_n)\alpha_0 + \beta_0 + \left(n^{-1}\sum(Z_i - \bar{Z}_n)X_i\right)^{-1} n^{-1}\sum(Z_i - \bar{Z}_n)U_i$$

Note that $\sum(Z_i - \bar{Z}_n) = 0$ so

$$= \beta_0 + \left(n^{-1}\sum(Z_i - \bar{Z}_n)X_i\right)^{-1} n^{-1}\sum(Z_i - \bar{Z}_n)U_i$$

Now, the probability limit

$$n^{-1}\sum(Z_i - \bar{Z}_n)X_i = n^{-1}\sum(Z_i - \mu_Z)X_i - (\bar{Z}_n - \mu_Z)\bar{X}_n$$

$$= n^{-1}\sum(Z_i - \mu_Z)(X_i - \mu_X) - (\bar{Z}_n - \mu_Z)\bar{X}_n$$

Using WLLN and the fact that $(\bar{Z}_n - \mu_Z) \to_p 0$ and $\bar{X}_n \to_p \mu_X$, so

$$n^{-1}\sum(Z_i - \bar{Z}_n)X_i = n^{-1}\sum(Z_i - \mu_Z)(X_i) - (\bar{Z}_n - \mu_Z)\bar{X}_n \to_p E[(Z_i - \mu_Z)X_i] = E[X_i Z_i] - \mu_Z\mu_X = Cov(X_i, Z_i)$$

Similarly

$$n^{-1}\sum(Z_i - \bar{Z}_n)U_i = n^{-1}\sum(Z_i - \mu_Z)(U_i) - (\bar{Z}_n - \mu_Z)\bar{U_n} \to_p E[(Z_i - \mu_Z)U_i] = E[U_i Z_i] - \mu_Z\mu_U = Cov(U_i, Z_i)$$

So the probability limit:

$$\hat{\beta}_{IV} = \beta_0 + \left(n^{-1}\sum(Z_i - \bar{Z}_n)X_i\right)^{-1} n^{-1}\sum(Z_i - \bar{Z}_n)U_i$$

$$\to_p \beta_0 + \frac{Cov(Z_i, U_i)}{Cov(Z_i, X_i)}$$

For this limit to exist, we need the denominator to not be zero, so $Cov(Z_i, X_i) \neq 0$

(b) For consistency, we need the term

$$\frac{Cov(Z_i, U_i)}{Cov(Z_i, X_i)}$$

to be zero. This happens iff $Cov(Z_i, U_i) = 0$.

(c) From the rewriting of $\hat{\beta}_{IV}$ from part a, we have

$$\sqrt{n}(\hat{\beta}_{IV} - \beta_0) = \sqrt{n}\left(n^{-1}\sum(Z_i - \bar{Z}_n)X_i\right)^{-1} n^{-1}\sum(Z_i - \bar{Z}_n)U_i$$

$$= \sqrt{n}\left(n^{-1}\sum(Z_i - \bar{Z}_n)X_i\right)^{-1}\left(n^{-1}\sum(Z_i - \mu_Z)(U_i) - (\bar{Z}_n - \mu_Z)\bar{U_n}\right)$$

$$= \left(n^{-1}\sum(Z_i - \bar{Z}_n)X_i\right)^{-1}\left(\sqrt{n}n^{-1}\sum(Z_i - \mu_Z)(U_i) - \sqrt{n}(\bar{Z}_n - \mu_Z)\bar{U_n}\right)$$

Note that $(\bar{Z}_n - \mu_Z) \to_p 0$, $\sqrt{n}\bar{U}_n \to_d N(0, \sigma_U^2)$, so the term $\sqrt{n}(\bar{Z}_n - \mu_Z)\bar{U}_n \to_d 0$. We know the term $\left(n^{-1}\sum(Z_i - \bar{Z}_n)X_i\right)^{-1} \to_p Cov(X_i, Z_i)^{-1}$ by Slutsky and the result from the previous parts, and

$$\sqrt{n}n^{-1}\sum(Z_i - \mu_Z)(U_i) \to_p N(0, Var(Z_i - \mu_Z)U_i)$$

by the CLT. Hence

$$\sqrt{n}(\hat{\beta}_{IV} - \beta_0) = \left(n^{-1}\sum(Z_i - \bar{Z}_n)X_i\right)^{-1}\left(\sqrt{n}n^{-1}\sum(Z_i - \mu_Z)(U_i) - \sqrt{n}(\bar{Z}_n - \mu_Z)\bar{U_n}\right)$$

$$\to_d Cov(X_i, Z_i)^{-1}N(0, Var((Z_i - \mu_Z)U_i)) = N\left(0, \frac{Var((Z_i - \mu_Z)U_i)}{Cov(X_i, Z_i)^2}\right)$$