

## Problem Set 4

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### 7.2

Using the WLLN, we have

$$\frac{1}{n} \sum_i X_i X_i' \rightarrow_p E[XX']$$

Also, we have

$$\frac{1}{n} \sum_i X_i Y_i \rightarrow_p E[XY] = E[\beta X X' + X e] = \beta E[XX']$$

Further, since  $\lambda$  is constant, we have

$$\frac{\lambda}{n} I_k \rightarrow_p 0$$

Therefore,

$$\begin{aligned} \hat{\beta} &= \left( \frac{1}{n} \sum_i X_i X_i' + \frac{\lambda}{n} I_k \right)^{-1} \left( \frac{1}{n} \sum_i X_i Y_i \right) \\ &\rightarrow_p (E[XX'])^{-1} (\beta E[XX']) \\ &= \beta \end{aligned}$$

Hence this estimator is consistent.

### 7.3

See the previous problem. In this case, we have

$$\begin{aligned} \hat{\beta} &= \left( \frac{1}{n} \sum_i X_i X_i' + \frac{\lambda}{n} I_k \right)^{-1} \left( \frac{1}{n} \sum_i X_i Y_i \right) \\ &= \left( \frac{1}{n} \sum_i X_i X_i' + c I_k \right)^{-1} \left( \frac{1}{n} \sum_i X_i Y_i \right) \\ &\rightarrow_p \beta (E[XX'] + c I_k)^{-1} E[XX'] \end{aligned}$$

Since  $(E[XX'] + c I_k)^{-1} E[XX'] \neq 1$  generically, this is not consistent.

## 7.8

$$\begin{aligned}
\sqrt{n}(\hat{\sigma}^2 - \sigma^2) &= \sqrt{n} \left( \frac{1}{n} \sum_i (Y_i - X_i' \hat{\beta})^2 - E[(Y - X' \beta)^2] \right) \\
&= \sqrt{n} \left( \frac{1}{n} \sum_i (Y_i - X_i' \beta + X_i' \beta - X_i' \hat{\beta})^2 - E[(Y - X' \beta)^2] \right) \\
&= \sqrt{n} \left( \frac{1}{n} \sum_i (Y_i - X_i' \beta)^2 + 2 \frac{1}{n} \sum_i (X_i' \beta - X_i' \hat{\beta})' (Y_i - X_i' \beta) + \frac{1}{n} \sum_i (X_i' \beta - X_i' \hat{\beta})^2 - E[(Y - X' \beta)^2] \right) \\
&= \sqrt{n} \left( \frac{1}{n} \sum_i (Y_i - X_i' \beta)^2 + 2 \frac{1}{n} \sum_i (\beta - \hat{\beta})' (X_i Y_i - X_i X_i' \beta) + \frac{1}{n} \sum_i (\beta - \hat{\beta}) X_i X_i' (\beta - \hat{\beta})' - E[(Y - X' \beta)^2] \right) \\
&= \sqrt{n} \left( \frac{1}{n} \sum_i (Y_i - X_i' \beta)^2 + 2 \frac{1}{n} \sum_i (\beta - \hat{\beta})' (X_i e_i) + (\beta - \hat{\beta})' \left( \frac{1}{n} \sum_i X_i X_i' \right) (\beta - \hat{\beta}) - E[(Y - X' \beta)^2] \right) \\
&= \sqrt{n}(\beta - \hat{\beta})' \left( 2 \frac{1}{n} \sum_i X_i e_i \right) + \sqrt{n}(\beta - \hat{\beta})' \left( \left( \frac{1}{n} \sum_i X_i X_i' \right) (\beta - \hat{\beta}) \right) + \sqrt{n} \left( \frac{1}{n} \sum_i e_i^2 - E[e^2] \right)
\end{aligned}$$

Now, we know that  $\sqrt{n}(\beta - \hat{\beta}) \rightarrow_d N(0, V_\beta)$ . By the WLLN, we also have

$$\frac{1}{n} \sum_i X_i e_i \rightarrow_p E[Xe] = 0$$

$$\frac{1}{n} \sum_i X_i X_i' \rightarrow_p E[XX']$$

Finally, from consistency of  $\hat{\beta}$ , we get:

$$\beta - \hat{\beta} \rightarrow_p \beta - \beta = 0$$

So putting these together, we get

$$\begin{aligned}
\sqrt{n}(\beta - \hat{\beta})' \left( 2 \frac{1}{n} \sum_i X_i e_i \right) &\rightarrow_d 0 \\
\sqrt{n}(\beta - \hat{\beta})' \left( \left( \frac{1}{n} \sum_i X_i X_i' \right) (\beta - \hat{\beta}) \right) &\rightarrow_d 0
\end{aligned}$$

And by the CLT,

$$\sqrt{n} \left( \frac{1}{n} \sum_i e_i^2 - E[e^2] \right) \rightarrow_d N(0, V(e))$$

Hence,

$$\sqrt{n}(\hat{\sigma}^2 - \sigma^2) \rightarrow_d N(0, V(e))$$

## 7.10

(a) The point estimator is just  $x'\hat{\beta}$ .

(b) To estimate variance, we have

$$V(x'\beta) = x'V(\beta)x$$

So

$$\widehat{V(x'\beta)} = x'\widehat{V(\beta)}x = x'\hat{\sigma}^2(x'x)^{-1}x = \hat{\sigma}^2x'(x'x)^{-1}x$$

## 7.28

(a) See code.

$$\hat{\beta}_1, \hat{s}_1 \approx 0.0904, 0.003$$

$$\hat{\beta}_2, \hat{s}_2 \approx 0.0354, 0.003$$

$$\hat{\beta}_3, \hat{s}_3 \approx -0.0465, 0.005$$

$$\hat{\beta}_4, \hat{s}_4 \approx 1.1852, 0.046$$

(b) The marginal value of experience is given by

$$\beta_2 + 2\beta_3(\text{experience})/100 = \beta_2 + \beta_3/5$$

where we have evaluated at  $\text{experience} = 10$  as the problem asks.

$$\theta = \frac{\beta_1}{\beta_2 + \beta_3/5}$$

$$\hat{\theta} \approx 3.4683$$

(c) Let  $g(\beta) = \beta_1/(\beta_2 + \beta_3/5)$ . Then

$$G(\beta) = \frac{\partial g}{\partial \beta} = \begin{bmatrix} \frac{1}{\beta_2 + \beta_3/5} & -\frac{\beta_1}{(\beta_2 + \beta_3/5)^2} & -\frac{\beta_1/5}{(\beta_2 + \beta_3/5)^2} & 0 \end{bmatrix}$$

Then by the delta method,

$$\sqrt{n}(\hat{\theta} - \theta) \rightarrow_d N(0, G(\beta)V_\beta G(\beta)')$$

The estimated standard deviation is  $\hat{s}_\theta = \sqrt{\widehat{G(\beta)}\widehat{V_\beta}G(\beta)'} \approx 0.2267$ .

(d) The confidence interval is

$$\begin{aligned} & \left[ \hat{\theta} - \hat{s}_\theta z_{.95}, \hat{\theta} + \hat{s}_\theta z_{.95} \right] \\ & \approx [3.0954, 3.8413] \end{aligned}$$

(e) The desired estimate is  $h(\beta) = 12\beta_1 + 20\beta_2 + 4\beta_3 + \beta_4$ . Define

$$H(\beta) = \begin{bmatrix} 12 & 20 & 4 & 1 \end{bmatrix}$$

$$\sqrt{n}(\widehat{h(\beta)} - h(\beta)) \rightarrow_d N(0, H(\beta)V_\beta H(\beta)')$$

The standard deviation estimator is then  $\hat{s}_h = \widehat{H(\beta)}\widehat{V_\beta}\widehat{H(\beta)}' \approx 0.0117$  Our confidence interval is then

$$\left[ \hat{h} - \hat{s}_h z_{.975}, \hat{h} + \hat{s}_h z_{.975} \right]$$

$$\approx [2.7693, 2.8150]$$