Problem Set 3

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Problem 1

We know

$$\hat{\beta} = \begin{bmatrix} 1 & n^{-1}\sum \tilde{x}_i \\ n^{-1}\sum \tilde{x}_i & n^{-1}\sum \tilde{x}_i^2 \end{bmatrix}^{-1} \begin{bmatrix} n^{-1}\sum y_i \\ n^{-1}\sum \tilde{x}_i y_i \end{bmatrix}$$

$$= \frac{1}{(n^{-1}\sum \tilde{x}_i^2) - \bar{\tilde{x}}^2} \begin{bmatrix} n^{-1}\sum \tilde{x}_i^2 & -n^{-1}\sum \tilde{x}_i \\ -n^{-1}\sum \tilde{x}_i & 1 \end{bmatrix} \begin{bmatrix} n^{-1}\sum y_i \\ n^{-1}\sum \tilde{x}_i y_i \end{bmatrix}$$

$$\hat{\beta}_2 = \frac{1}{(n^{-1}\sum \tilde{x}_i^2) - \bar{\tilde{x}}^2} \left(-\bar{\tilde{x}} n^{-1} \sum y_i + n^{-1} \sum \tilde{x}_i y_i\right)$$

$$= \frac{1}{(n^{-1}\sum \tilde{x}_i^2) - \bar{\tilde{x}}^2} \left(n^{-1} \sum (\tilde{x}_i - \bar{\tilde{x}}) y_i\right)$$

$$= \sum \frac{n^{-1}(\tilde{x}_i - \bar{\tilde{x}})}{(n^{-1}\sum \tilde{x}_i^2) - \bar{\tilde{x}}^2} y_i$$

$$= \sum \frac{\tilde{x}_i - \bar{\tilde{x}}}{\sum \tilde{x}_i^2 - n\bar{\tilde{x}}^2} y_i$$

$$= \sum \frac{\tilde{x}_i - \bar{\tilde{x}}}{\sum \tilde{x}_i^2 - 2n\bar{\tilde{x}}^2 + n\bar{\tilde{x}}^2} y_i$$

$$= \sum \frac{\tilde{x}_i - \bar{\tilde{x}}}{\sum \tilde{x}_i^2 - 2\bar{\tilde{x}}\sum \tilde{x}_i + n\bar{\tilde{x}}^2} y_i$$

$$= \sum \frac{\tilde{x}_i - \bar{\tilde{x}}}{\sum \tilde{x}_i^2 - \sum 2\bar{\tilde{x}}\tilde{x}_i + \sum \bar{\tilde{x}}^2} y_i$$

$$= \sum \frac{\tilde{x}_i - \bar{\tilde{x}}}{\sum (\tilde{x}_j - \bar{\tilde{x}})^2} y_i$$

$$= \sum \omega_i y_i$$

Problem 2

(a) We know

$$\gamma = E[xx']^{-1}E[xy]$$

$$= E\begin{bmatrix} 1 & w_1 & w_2 \\ w_1 & w_1^2 & w_1 w_2 \\ w_2 & w_1 w_2 & w_2^2 \end{bmatrix}^{-1} E\begin{bmatrix} 1(w_2^2+e) \\ w_1(w_2^2+e) \\ w_2(w_2^2+e) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & Ew_1^2 & 0 \\ 0 & 0 & Ew_2^2 \end{bmatrix}^{-1} \begin{bmatrix} E(w_2^2+e) \\ E(w_1(w_2^2+e)) \\ E(w_2(w_2^2+e)) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/Ew_1^2 & 0 \\ 0 & 0 & 1/Ew_2^2 \end{bmatrix} \begin{bmatrix} E(w_2^2) \\ E(w_1 w_2^2) \\ E(w_2^3) \end{bmatrix}$$

$$= \begin{bmatrix} E(w^2) \\ E(w_1 w_2^2)/E(w_1^2) \\ E(w_2^3)/Ew_2^2 \end{bmatrix}$$

(b) The average causal effect should be 0, since $m(x)$ has no dependence on w_1. However, $\gamma_1 = E(w_1 w_2^2)/E(w_1^2)$ which might not be 0. So γ_1 doesn't necessarily estimate the average causal effect of w_1.