Problem Set 3

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7.2

Using the WLLN, we have

$$\frac{1}{n}\sum_i X_i X_i' \to_p E[XX']$$

Also, we have

$$\frac{1}{n}\sum_i X_i Y_i \to_p E[XY] = E[\beta XX' + Xe] = \beta E[XX']$$

Further, since λ is constant, we have

$$\frac{\lambda}{n} I_k \to_p 0$$

Therefore,

$$\begin{aligned}
\hat{\beta} &= \left(\frac{1}{n}\sum_i X_i X_i' + \frac{\lambda}{n} I_k\right)^{-1}\left(\frac{1}{n}\sum_i X_i Y_i\right) \\
&\to_p (E[XX'])^{-1}(\beta E[XX']) \\
&= \beta
\end{aligned}$$

Hence this estimator is consistent.

7.3

See the previous problem. In this case, we have

$$\begin{aligned}
\hat{\beta} &= \left(\frac{1}{n}\sum_i X_i X_i' + \frac{\lambda}{n} I_k\right)^{-1}\left(\frac{1}{n}\sum_i X_i Y_i\right) \\
&= \left(\frac{1}{n}\sum_i X_i X_i' + cI_k\right)^{-1}\left(\frac{1}{n}\sum_i X_i Y_i\right) \\
&\to_p \beta(E[XX'] + cI_k)^{-1} E[XX']
\end{aligned}$$

Since $(E[XX'] + cI_k)^{-1}E[XX'] \neq 1$ generically, this is not consistent.

7.8

$$\sqrt{n}\left(\hat{\sigma}^2-\sigma^2\right)=\sqrt{n}\left(\frac{1}{n}\sum_i(Y_i-X_i'\hat{\beta})^2-E[(Y-X'\beta)^2]\right)$$

$$=\sqrt{n}\left(\frac{1}{n}\sum_i(Y_i-X_i'\beta+X_i'\beta-X_i'\hat{\beta})^2-E[(Y-X'\beta)^2]\right)$$

$$=\sqrt{n}\left(\frac{1}{n}\sum_i(Y_i-X_i'\beta)^2+2\frac{1}{n}\sum_i(X_i'\beta-X_i'\hat{\beta})'(Y_i-X_i'\beta)+\frac{1}{n}\sum_i(X_i'\beta-X_i'\hat{\beta})^2-E[(Y-X'\beta)^2]\right)$$

$$=\sqrt{n}\left(\frac{1}{n}\sum_i(Y_i-X_i'\beta)^2+2\frac{1}{n}\sum_i(\beta-\hat{\beta})'(X_iY_i-X_iX_i'\beta)+\frac{1}{n}\sum_i(\beta-\hat{\beta})X_iX_i'(\beta-\hat{\beta})'-E[(Y-X'\beta)^2]\right)$$

$$=\sqrt{n}\left(\frac{1}{n}\sum_i(Y_i-X_i'\beta)^2+2\frac{1}{n}\sum_i(\beta-\hat{\beta})'(X_ie_i)+(\beta-\hat{\beta})\left(\frac{1}{n}\sum_i X_iX_i'\right)(\beta-\hat{\beta})'-E[(Y-X'\beta)^2]\right)$$

$$=\sqrt{n}(\beta-\hat{\beta})'\left(2\frac{1}{n}\sum_i X_ie_i\right)+\sqrt{n}(\beta-\hat{\beta})'\left(\left(\frac{1}{n}\sum_i X_iX_i'\right)(\beta-\hat{\beta})\right)+\sqrt{n}\left(\frac{1}{n}\sum_i e_i^2-E[e^2]\right)$$

Now, we know that $\sqrt{n}(\beta-\hat{\beta})\to_d N(0,V_\beta)$. By the WLLN, we also have

$$\frac{1}{n}\sum_i X_ie_i\to_p E[Xe]=0$$

$$\frac{1}{n}\sum_i X_iX_i'\to_p E[XX']$$

Finally, from consistency of $\hat{\beta}$, we get:

$$\beta-\hat{\beta}\to_p\beta-\beta=0$$

So putting these together, we get

$$\sqrt{n}(\beta-\hat{\beta})'\left(2\frac{1}{n}\sum_i X_ie_i\right)\to_d 0$$

$$\sqrt{n}(\beta-\hat{\beta})'\left(\left(\frac{1}{n}\sum_i X_iX_i'\right)(\beta-\hat{\beta})\right)\to_d 0$$

And by the CLT,

$$\sqrt{n}\left(\frac{1}{n}\sum_i e_i^2-E[e^2]\right)\to_d N(0,V(e))$$

Hence,

$$\sqrt{n}\left(\hat{\sigma}^2-\sigma^2\right)\to_d N(0,V(e))$$

7.10

(a) The point estimator is just $x'\hat{\beta}$.

(b) To estimate variance, we have

$$V(x'\beta) = x'V(\beta)x$$

So

$$\widehat{V(x'\beta)} = x'\widehat{V(\beta)}x = x'\widehat{\sigma}^2(x'x)^{-1}x = \widehat{\sigma}^2 x'(x'x)^{-1}x$$

7.28

(a) See code.

$$\hat{\beta}_1, \hat{s}_1 \approx 0.0904, 0.003$$

$$\hat{\beta}_2, \hat{s}_2 \approx 0.0354, 0.003$$

$$\hat{\beta}_3, \hat{s}_3 \approx -0.0465, 0.005$$

$$\hat{\beta}_4, \hat{s}_4 \approx 1.1852, 0.046$$

(b) The marginal value of experience is given by

$$\beta_2 + 2\beta_3(\mathit{experience})/100 = \beta_2 + \beta_3/5$$

where we have evaluated at $\mathit{experience} = 10$ as the problem asks.

$$\theta = \frac{\beta_1}{\beta_2 + \beta_3/5}$$

$$\hat{\theta} \approx 3.4683$$

(c) Let $g(\beta) = \beta_1/(\beta_2 + \beta_3/5)$. Then

$$G(\beta) = \frac{\partial g}{\partial \beta} = \begin{bmatrix} \frac{1}{\beta_2+\beta_3/5} & -\frac{\beta_1}{(\beta_2+\beta_3/5)^2} & -\frac{\beta_1/5}{(\beta_2+\beta_3/5)^2} & 0 \end{bmatrix}$$

Then by the delta method,

$$\sqrt{n}(\hat{\theta} - \theta) \to_d N(0, G(\beta)V_\beta G(\beta)')$$

The estimated standard deviation is $\hat{s}_\theta = \sqrt{\widehat{G(\beta)}\widehat{V_\beta}\widehat{G(\beta)}'} \approx 0.2267$.

(d) The confidence interval is

$$\left[\hat{\theta} - \frac{1}{\sqrt{n}}\hat{s}_\theta z_{.95}, \hat{\theta} + \frac{1}{\sqrt{n}}\hat{s}_\theta z_{.95}\right]$$

$$\approx [3.4626, 3.4741]$$

(e) The desired estimate is $h(\beta) = 12\beta_1 + 20\beta_2 + 4\beta_3 + \beta_4$. Define

$$H(\beta) = \begin{bmatrix} 12 & 20 & 4 & 1 \end{bmatrix}$$

$$\sqrt{n}(\widehat{h(\beta)} - h(\beta)) \to_d N(0, H(\beta)V_\beta H(\beta)')$$

The standard deviation estimator is then $\hat{s}_h = \widehat{H(\beta)}\widehat{V_\beta}\widehat{H(\beta)}' \approx 0.0117$ Our confidence interval is then

$$\left[\hat{h} - \frac{1}{\sqrt{n}}\hat{s}_h z_{.975}, \hat{h} + \frac{1}{\sqrt{n}}\hat{s}_h z_{.975}\right]$$

$$\approx [2.7918, 2.7925]$$