Problem Set 10

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Problem 1

(a) To pin down β, we have

$$\frac{1}{T}\sum_{t=1}^{T} y_{it} = u_i + \left(\frac{1}{T}\sum_{t=1}^{T} x_{it}'\right)\beta + \frac{1}{T}\sum_{t=1}^{T} e_{it}$$

$$y_{it} - \frac{1}{T}\sum_{t=1}^{T} y_{it} = \left(x_{it}' - \frac{1}{T}\sum_{t=1}^{T} x_{it}'\right)\beta + e_{it} - \frac{1}{T}\sum_{t=1}^{T} e_{it}$$

$$\left(y_{it} - \frac{1}{T}\sum_{t=1}^{T} y_{it}\right) - \left(x_{it}' - \frac{1}{T}\sum_{t=1}^{T} x_{it}'\right)\beta = e_{it} - \frac{1}{T}\sum_{t=1}^{T} e_{it}$$

Since $E[e_{it}|x_{i1}, x_{i2}, ...] = 0$, we get $E[x_{is}e_{it}] = 0$, and hence

$$E\left[\left(x_{it} - \frac{1}{T}\sum_{t=1}^{T} x_{it}\right)\left(\left(y_{it} - \frac{1}{T}\sum_{t=1}^{T} y_{it}\right) - \left(x_{it}' - \frac{1}{T}\sum_{t=1}^{T} x_{it}'\right)\beta\right)\right]$$

$$= E\left[\left(x_{it} - \frac{1}{T}\sum_{t=1}^{T} x_{it}\right)\left(e_{it} - \frac{1}{T}\sum_{t=1}^{T} e_{it}\right)\right] = 0$$

For μ_u, we have

$$E\left[\frac{1}{T}\sum_{t=1}^{T}(y_{it} - x_{it}'\beta) - \mu_u\right] = E\left[\frac{1}{T}\sum_{t=1}^{T}(u_i + e_{it}) - \mu_u\right]$$

$$= E\left[u_i + \frac{1}{T}\sum_{t=1}^{T} e_{it} - \mu_u\right] = \mu_u + 0 - \mu_u = 0$$

For μ_{ux}, we get

$$E\left[\frac{1}{T}\sum_{t=1}^{T} x_{it}(y_{it} - x_{it}'\beta) - \mu_{ux}\right] = E\left[\frac{1}{T}\sum_{t=1}^{T} x_{it}(u_i + e_{it}) - \mu_{ux}\right]$$

$$= E\left[\frac{1}{T}\sum_{t=1}^{T} x_{it}u_i + \frac{1}{T}\sum_{t=1}^{T} x_{it}e_{it} - \mu_{ux}\right] = \mu_{ux} + 0 - \mu_{ux} = 0$$

Lastly, for σ_u^2, we note that

$$E\left[\frac{1}{\binom{T}{2}}\sum_{s\neq t\in\{1,2,...T\}}(y_{is}-x_{is}'\beta)(y_{it}-x_{it}'\beta)-\mu_u^2-\sigma_u^2\right]=E\left[\frac{1}{\binom{T}{2}}\sum_{s\neq t\in\{1,2,...T\}}(u_i+e_{is})(u_i+e_{it})-\mu_u^2-\sigma_u^2\right]$$

$$=E\left[\frac{1}{\binom{T}{2}}\sum_{s\neq t\in\{1,2,...T\}}(u_i^2+e_{is}u_i+e_{it}u_i+e_{is}e_{it})-\mu_u^2-\sigma_u^2\right]$$

$$=E\left[u_i^2+\frac{1}{\binom{T}{2}}\sum_{s\neq t\in\{1,2,...T\}}(e_{is}u_i+e_{it}u_i+e_{is}e_{it})-\mu_u^2-\sigma_u^2\right]=E[u_i^2]-\mu_u^2-\sigma_u^2=\sigma_u^2-\sigma_u^2=0$$

So altogether:

$$g(x_i,y_i,\beta,\mu_u,\mu_{ux},\sigma_u^2)=\begin{bmatrix}\left(x_{it}-\frac{1}{T}\sum_{t=1}^T x_{it}\right)\left(\left(y_{it}-\frac{1}{T}\sum_{t=1}^T y_{it}\right)-\left(x_{it}'-\frac{1}{T}\sum_{t=1}^T x_{it}'\right)\beta\right)\\ \frac{1}{T}\sum_{t=1}^T(y_{it}-x_{it}'\beta)-\mu_u\\ \frac{1}{T}\sum_{t=1}^T x_{it}(y_{it}-x_{it}'\beta)-\mu_{ux}\\ \frac{1}{\binom{T}{2}}\sum_{s\neq t\in\{1,2,...T\}}(y_{is}-x_{is}'\beta)(y_{it}-x_{it}'\beta)-\mu_u^2-\sigma_u^2\end{bmatrix}$$

And we will have a unique solution as long as

$$\left(x_{it}-\frac{1}{T}\sum_{t=1}^T x_{it}\right)\left(x_{it}'-\frac{1}{T}\sum_{t=1}^T x_{it}'\right)$$

is invertible.

(b) We have that

$$E\left[\prod_{t=1}^l y_{it}\right]=E\left[\prod_{t=1}^l(u_i+e_{it})\right]$$

$$=E\left[\prod_{t=1}^l u_i\right]$$

since $E[e_{it}|u_i,...e_{is},s\neq t]=0$. Hence a consistent estimator is

$$\frac{1}{n}\sum_{i=1}^n\left(\prod_{t=1}^l y_{it}\right)$$

We need $T\geq l$.

Problem 2

We have

$$f(y_{it}|u,\sigma^2)=\frac{1}{\sigma}\phi\left(\frac{y_{it}-u_i}{\sigma}\right)$$

So the log-likelihood is

$$l(\mu, \sigma^2) = \sum_{i=1}^{n} \sum_{t=1}^{2} \left(\log \phi \left(\frac{y_{it} - u_i}{\sigma} \right) - \log \sigma \right)$$

$$= \sum_{i=1}^{n} \sum_{t=1}^{2} \left(-\frac{1}{2} \left(\frac{y_{it} - u_i}{\sigma} \right)^2 - \frac{1}{2} \log(2\pi) - \frac{1}{2} \log \sigma^2 \right)$$

Taking the FOCs,

$$\frac{\partial l}{\partial u_i} = -\sum_{t=1}^{2} \left(\frac{y_{it} - u_i}{\sigma} \right) = 0$$

$$\sum_{t=1}^{2} y_{it} - 2u_i = 0$$

$$\hat{u_i} = \frac{y_{i1} + y_{i2}}{2}$$

and

$$\frac{\partial l}{\partial \sigma^2} = \sum_{i=1}^{n} \sum_{t=1}^{2} \left(\frac{1}{2} \left(\frac{(y_{it} - u_i)^2}{(\sigma^2)^2} \right) - \frac{1}{2} \frac{1}{\sigma^2} \right) = 0$$

$$\sum_{i=1}^{n} \sum_{t=1}^{2} \left((y_{it} - u_i)^2 \right) - 2n\sigma^2 = 0$$

$$\hat{\sigma}^2 = \frac{1}{2n} \sum_{i=1}^{n} \sum_{t=1}^{2} \left((y_{it} - u_i)^2 \right)$$

$$= \frac{1}{2n} \sum_{i=1}^{n} \sum_{t=1}^{2} \left(y_{it} - \frac{y_{i1} + y_{i2}}{2} \right)^2$$

$$= \frac{1}{2n} \sum_{i=1}^{n} \left(\frac{y_{i1} - y_{i2}}{2} \right)^2 + \left(\frac{y_{i1} - y_{i2}}{2} \right)^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left(\frac{y_{i1} - y_{i2}}{2} \right)^2$$

$$= \frac{1}{4n} \sum_{i=1}^{n} (y_{i1} - y_{i2})^2$$

This probability limit is

$$\hat{\sigma}^2 = \frac{1}{4} \left(\frac{1}{n} (y_{i1} - yi2)^2 \right) \rightarrow_p \frac{1}{4} E(y_{i1} - y_{i2})^2$$

$$= \frac{1}{4} E(e_{i1} - e_{i2})^2 = \frac{1}{4}(2\sigma^2) = \frac{1}{2}\sigma^2$$