Problem Set 8

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Problem 1

$$\gamma(k) = cov(y_t, y_{t-k})$$
$$= cov\left(\mu + \epsilon_t + \sum_{j=1}^{q} \theta_j \epsilon_{t-j}, \mu + \epsilon_{t-k} + \sum_{j=1}^{q} \theta_j \epsilon_{t-k-j}\right)$$
$$= cov\left(\epsilon_t + \sum_{j=1}^{q} \theta_j \epsilon_{t-j}, \epsilon_{t-k} + \sum_{j=1}^{q} \theta_j \epsilon_{t-k-j}\right)$$
$$= E\left[\theta_k \epsilon_{t-k}^2 + \sum_{j=k+1}^{q} \theta_j \theta_{j-k} \epsilon_{t-j}^2\right]$$
$$= \sigma^2 \left(\theta_k + \sum_{j=1}^{q-k} \theta_j \theta_{j+k}\right)$$

if $k \leq q$. If $k > q$, then the covariance is just 0.

Problem 2

From the theorem in the notes, we know that y_t is strictly stationary and ergodic if

$$\left|\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}\right| < 1$$

If $\phi_1^2 + 4\phi_2 \geq 0$, we need to show

$$\frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} < 1$$
$$\frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} > -1$$

With some algebra, we get

$$\phi_1 + \sqrt{\phi_1^2 + 4\phi_2} < 2$$

$$\phi_1 - \sqrt{\phi_1^2 + 4\phi_2} > -2$$

$$\sqrt{\phi_1^2 + 4\phi_2} < 2 - \phi_1$$

$$\sqrt{\phi_1^2 + 4\phi_2} < 2 + \phi_1$$

$$\phi_1^2 + 4\phi_2 < 4 - 4\phi_1 + \phi_1^2$$

$$\phi_1^2 + 4\phi_2 < 4 + 4\phi_1 + \phi_1^2$$

$$\phi_2 + \phi_1 < 1$$

$$\phi_2 - \phi_1 < 1$$

with the additional constraint that $-2 \leq \phi_1 \leq 2$. So for this case, we have:

$$\phi_1^2 + 4\phi_2 \geq 0$$

$$-2 \leq \phi_1 \leq 2$$

$$\phi_2 + \phi_1 < 1$$

$$\phi_2 - \phi_1 < 1$$

Alternatively, if $\phi_1^2 + 4\phi_2 < 0$, then the roots are imaginary, and so we get

$$\frac{\sqrt{\phi_1^2 - (\phi_1^2 + 4\phi_2)}}{2} < 1$$

$$-\phi_2 < 1$$

$$\phi_2 > -1$$

Hence in this case, the constraints are

$$\phi_1^2 + 4\phi_2 < 0$$

$$\phi_2 > -1$$

Problem 3

(a) We have

$$var(\sqrt{T}\bar{y}_T) = \frac{1}{T} cov\left(\sum_{j=1}^{T} y_j, \sum_{j=1}^{T} y_j\right)$$

$$= \frac{1}{T} \sum_{j=1}^{T} \sum_{k=1}^{T} cov(y_j, y_k)$$

$$= \frac{1}{T}\sum_{j=1}^{T}\sum_{k=1}^{T}\gamma(|j-k|)$$

$$= \frac{1}{T}\left(\sum_{j=1}^{T}\sum_{k=j}^{j}\gamma(0) + \sum_{j=1}^{T}\sum_{k=j+1}^{T}\gamma(k-j) + \sum_{j=1}^{T}\sum_{k=1}^{j-1}\gamma(j-k)\right)$$

$$= \frac{1}{T}\left(T\gamma(0) + 2\sum_{j=1}^{T}\sum_{k=j+1}^{T}\gamma(k-j)\right)$$

$$= \frac{1}{T}\left(T\gamma(0) + 2\sum_{j=1}^{T-1}\sum_{i=1}^{T-j}\gamma(i)\right)$$

$$= \frac{1}{T}\left(T\gamma(0) + 2\sum_{i=1}^{T-1}\sum_{j=1}^{T-i}\gamma(i)\right)$$

$$= \frac{1}{T}\left(T\gamma(0) + 2\sum_{i=1}^{T-1}(T-i)\gamma(i)\right)$$

$$= \gamma_0 + \frac{2}{T}\sum_{i=1}^{T-1}(T-i)\gamma(i)$$

(b) From the previous part, we have

$$var(\sqrt{T}\bar{y}_T) = \gamma_0 + \frac{2}{T}\sum_{i=1}^{T-1}(T-i)\gamma(i)$$

$$= \gamma_0 + 2\sum_{i=1}^{T-1}\frac{T-i}{T}\gamma(i)$$

$$= \gamma_0 + 2\sum_{i=1}^{\infty}\frac{T-i}{T}\gamma(i)1\{i<T\}$$

Now, we have that since $0 \le \frac{T-i}{T} < 1$ and $0 \le 1\{i<T\} < 1$,

$$\left|\frac{T-i}{T}\gamma(i)1\{i<T\}\right| = \frac{T-i}{T}1\{i<T\}|\gamma(i)| < |\gamma(i)|$$

Also,

$$\lim_{T\to\infty}\frac{T-i}{T}\gamma(i)1\{i<T\} = \gamma(i)$$

and by assumption,

$$\sum_{i=1}^{\infty}|\gamma(i)| < \infty$$

So by the dominated convergence theorem,

$$\lim_{T\to\infty} var(\sqrt{T}\bar{y}_T) = \lim_{T\to\infty} \gamma_0 + 2\sum_{i=1}^{\infty} \frac{T-i}{T}\gamma(i)1\{i<T\}$$

$$= \gamma_0 + 2\lim_{T\to\infty} \sum_{i=1}^{\infty} \frac{T-i}{T}\gamma(i)1\{i<T\}$$

$$= \gamma_0 + 2\sum_{i=1}^{\infty} \lim_{T\to\infty} \frac{T-i}{T}\gamma(i)1\{i<T\}$$

$$= \gamma_0 + 2\sum_{i=1}^{\infty} \gamma(i)$$

$$= \gamma_0 + \sum_{i=1}^{\infty} \gamma(i) + \sum_{i=1}^{\infty} \gamma(i)$$

$$= \gamma_0 + \sum_{i=1}^{\infty} \gamma(-i) + \sum_{i=1}^{\infty} \gamma(i)$$

$$= \gamma_0 + \sum_{i=-1}^{-\infty} \gamma(i) + \sum_{i=1}^{\infty} \gamma(i)$$

$$= \sum_{i=-\infty}^{\infty} \gamma(i)$$

as desired.

Problem 4

(a) We have

$$\gamma(k) = cov(y_t, y_{t-k})$$

$$= cov(\alpha y_{t-1} + \epsilon_t, y_{t-k})$$

$$= \alpha cov(y_{t-1}, y_{t-k}) + cov(\epsilon_t, y_{t-k})$$

$$= \alpha\gamma(k-1)$$

$$= \alpha^k\gamma(0)$$

And we have

$$\gamma(0) = var(y_t) = \alpha^2 var(y_{t-1}) + \sigma^2 = \alpha^2\gamma(0) + \sigma^2$$

$$\gamma(0)(1-\alpha^2) = \sigma^2$$

$$\gamma(0) = \frac{\sigma^2}{1-\alpha^2}$$

and hence

$$\gamma(k) = \alpha^k \frac{\sigma^2}{1-\alpha^2}$$

(b) From the previous question, we have

$$var(\sqrt{T}\bar{y}_T) \to \sum_{k=-\infty}^{\infty} \gamma(k)$$

$$= \gamma(0) + 2\sum_{k=1}^{\infty} \gamma(k)$$

$$= \frac{\sigma^2}{1-\alpha^2} + 2\sum_{k=1}^{\infty} \alpha^k \frac{\sigma^2}{1-\alpha^2}$$

$$= \frac{\sigma^2}{1-\alpha^2} + \frac{2\alpha}{1-\alpha}\frac{\sigma^2}{1-\alpha^2}$$

$$= \frac{\sigma^2}{1-\alpha^2}\left(1 + \frac{2\alpha}{1-\alpha}\right)$$

$$= \frac{\sigma^2}{1-\alpha^2}\left(\frac{1+\alpha}{1-\alpha}\right)$$

$$= \frac{\sigma^2}{(1-\alpha)^2}$$

$$= \left(\frac{\sigma}{1-\alpha}\right)^2$$