Problem Set 6

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Problem 1

(a) From the moment equations,

$$E(1 - w\theta) = 0$$

$$E(w)\theta = 1$$

$$\theta = 1$$

and

$$E(\theta^3 - x) = 0$$

$$E(x) = 1$$

Hence, the optimal weighting matrix is

$$E(g'g)^{-1} = E\begin{bmatrix} (\theta^3 - x)^2 & (\theta^3 - x)(1 - w\theta) \\ (\theta^3 - x)(1 - w\theta) & (1 - w\theta)^2 \end{bmatrix}^{-1}$$

$$= E\begin{bmatrix} (1 - x)^2 & (1 - x)(1 - w) \\ (1 - x)(1 - w) & (1 - w)^2 \end{bmatrix}^{-1}$$

$$= E\begin{bmatrix} x^2 - 2x + 1 & 1 - x - w + wx \\ 1 - x - w + wx & w^2 - 2w + 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 3 - 2 + 1 & 1 - 1 - 1 + 1/2 \\ 1 - 1 - 1 + 1/2 & 2 - 2 + 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & -1/2 \\ -1/2 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 4/7 & 2/7 \\ 2/7 & 8/7 \end{bmatrix}$$

(b) (i) From lecture, we know

$$\sqrt{n}(\hat{\theta} - \theta_0) \to_d N(0, ((G'WG)^{-1})'G'W\Omega W'G(G'WG)^{-1})$$

When $W = \Omega^{-1}$, we get

$$\begin{aligned}
&((G'\Omega^{-1}G)^{-1})'G'(\Omega^{-1})'G(G'\Omega^{-1}G)^{-1} \\
&= (G'\Omega^{-1}G)^{-1} \\
&= \left(\left(\frac{\partial}{\partial\theta}Eg\right)'\Omega^{-1}\left(\frac{\partial}{\partial\theta}Eg\right)\right)^{-1} \\
&= \left(\begin{bmatrix} 3 & -1 \end{bmatrix}\begin{bmatrix} 4/7 & 2/7 \\ 2/7 & 8/7 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)^{-1} \\
&= \left(\frac{4}{7}9 + \frac{2}{7}(-3) + \frac{2}{7}(-3) + \frac{8}{7}\right)^{-1} \\
&= \frac{7}{32}
\end{aligned}$$

So

$$\sqrt{n}(\hat{\theta} - \theta_0) \to_d N(0, 7/32)$$

(ii) If $W = I$, then

$$\begin{aligned}
((G'WG)^{-1})'G'W\Omega W'G(G'WG)^{-1} &= ((G'G)^{-1})'G'\Omega G(G'G)^{-1} \\
&= \frac{1}{100}\begin{bmatrix} 3 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1/2 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} \\
&= \frac{1}{100}\left(2(9) - \frac{1}{2}(-3) - \frac{1}{2}(-3) + 1\right) \\
&= \frac{11}{50}
\end{aligned}$$

So

$$\sqrt{n}(\hat{\theta} - \theta_0) \to_d N(0, 11/50)$$

Problem 2

(a) The Poisson pdf of $y|x$ is

$$e^{-e^{x'\theta}}\frac{(e^{x'\theta})^y}{y!}$$

So the likelihood is

$$\prod_{i=1}^{n} e^{-e^{x_i'\theta}}\frac{(e^{x_i'\theta})^{y_i}}{y_i!}$$

The log-likelihood is then

$$\sum_{i=1}^{n} -e^{x_i'\theta} + y_i x_i'\theta - \log(y_i!)$$

(b) (i) We have $\hat{\theta}_{ML}$ satisfies

$$0 = \sum_{i=1}^{n} -x_{i,j} e^{x_i'\hat{\theta}_{ML}} + y_i x_{i,j}$$

$$0 = \sum_{i=1}^{n} x_i (y_i - e^{x_i'\hat{\theta}_{ML}})$$

Define $g(y, x, \theta) = x(y - e^{x'\theta})$. Then $\hat{\theta}_{ML}$ is the GMM estimator associated with this $Eg = 0$ moment condition. We just need to show that θ_0 satisfies the moment condition and the SOC holds. First, clearly

$$Eg(y, x, \theta_0) = E(x(y - e^{x'\theta_0})) = E(E(x(y - e^{x'\theta_0})|x))$$

$$= E(xE(y|x) - xe^{x'\theta_0}) = E(xe^{x'\theta_0} - xe^{x'\theta_0}) = 0$$

So θ_0 satisfies the moment condition, and we know the GMM estimator is consistent so $\hat{\theta}_{ML}$ is consistent as long as the SOC holds:

$$G = \frac{\partial}{\partial \theta} Eg = -E(xx'e^{x'\theta_0})$$

Then

$$-v'E(xx'e^{x'\theta})v = -E(vxx'v'e^{x'\theta}) = -E(||vx||^2 e^{x'\theta}) < 0$$

as long as $\exists x$ such that $P(vx) > 0$ for all $v \neq 0$.

(ii) Note that G is symmetric from the previous part. The GMM variance under identity weight matrix is

$$((G'G)^{-1})'G'\Omega G(G'G)^{-1} = G^{-1}\Omega G^{-1}$$

Assuming this is nondegenerate (G invertible), then since $H(\hat{\theta})^{-1}$ is a consistent estimator of G^{-1}, we only need G invertible for $H(\hat{\theta})^{-1}\Omega(\hat{\theta})H(\hat{\theta})^{-1}$ to be a consistent estimator of the asymptotic variance. However, for $-H(\hat{\theta})^{-1}$ to be a consistent estimator of the asymptotic variance, we need

$$\Omega = -G^{-1}$$

which happens iff

$$E(x(y - e^{x'\theta})(x(y - e^{x'\theta}))'|x) = E((y - e^{x'\theta})^2 xx'|x) = E(xx'e^{x'\theta}|x)$$

$$E((y - e^{x'\theta})^2|x) = e^{x'\theta} \implies V(y|x) = e^{x'\theta}$$

Hence we also need $V(y|x) = e^{x'\theta}$ for $-H(\hat{\theta})^{-1}$ to be a consistent estimator of the variance.