Problem Set 2

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Problems

(3.2) The OLS coefficient of the regression of Y on Z is

$$(Z'Z)^{-1}(Z'Y) = (C'X'XC)^{-1}(C'X'Y) = C^{-1}(X'X)^{-1}(C')^{-1}C'X'Y = C^{-1}(X'X)^{-1}X'Y = C^{-1}\beta$$

where β is the OLS coefficient from the regression of Y on X.

The residual of OLS of Y on Z is

$$Y - ZC^{-1}\beta = Y - X\beta$$

which is exactly the residual of OLS of Y on X.

(3.4) We have

$$e = Y - X\beta$$

$$X_2'e = X_2'Y - X_2'X(X'X)^{-1}X'Y$$

$$= X_2'Y - X_2'\begin{bmatrix} X_1 & X_2 \end{bmatrix}\left(\begin{bmatrix} X_1' \\ X_2' \end{bmatrix}\begin{bmatrix} X_1 & X_2 \end{bmatrix}\right)^{-1}\begin{bmatrix} X_1' \\ X_2' \end{bmatrix} Y$$

$$= X_2'Y - \begin{bmatrix} X_2'X_1 & X_2'X_2 \end{bmatrix}\left(\begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix}\right)^{-1}\begin{bmatrix} X_1'Y \\ X_2'Y \end{bmatrix}$$

Since $\begin{bmatrix} X_2'X_1 & X_2'X_2 \end{bmatrix}\left(\begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix}\right)^{-1}$ gives the bottom k_2 rows of the identity matrix, we have

$$= X_2'Y - X_2'Y = 0$$

So $X_2'e = 0$.

(3.5) The regression coefficient is

$$(X'X)^{-1}(X'e) = (X'X)^{-1}(X'(Y - X\beta))$$

$$= (X'X)^{-1}(X'Y - X'X(X'X)^{-1}(X'Y))$$

$$= (X'X)^{-1}(X'Y - X'Y)$$
$$= 0$$

(3.6) The OLS coefficient is given by

$$(X'X)^{-1}(X'\hat{Y}) = (X'X)^{-1}(X'(X(X'X)^{-1}X'Y)) = (X'X)^{-1}(X'X)(X'X)^{-1}X'Y = (X'X)^{-1}X'Y$$

(3.10)

$$P = X'(X'X)^{-1}X = \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} \left(\begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} X_1 & X_2 \end{bmatrix}$$
$$= \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} \left(\begin{bmatrix} X_1'X_1 & 0 \\ 0 & X_2'X_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} X_1 & X_2 \end{bmatrix}$$
$$= \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} \begin{bmatrix} (X_1'X_1)^{-1} & 0 \\ 0 & (X_2'X_2)^{-1} \end{bmatrix} \begin{bmatrix} X_1 & X_2 \end{bmatrix}$$
$$= X_1'(X_1'X_1)^{-1}X_1 + X_2'(X_2'X_2)^{-1}X_2$$
$$= P_1 + P_2$$

(3.11) Denote $1_k = \begin{bmatrix} 1 & 1 & ... & 1 \end{bmatrix}$. Then

$$n^{-1}1_n\hat{Y} = n^{-1}1_nX(X'X)^{-1}X'Y$$

Since X contains a constant,

$$X = \begin{bmatrix} c1_n' & X_2 \end{bmatrix}$$
$$n^{-1}1_n\hat{Y} = n^{-1}1_n \begin{bmatrix} c1_n' & X_2 \end{bmatrix} \left(\begin{bmatrix} nc^2 & c1_nX_2 \\ cX_2'1_n & X_2'X_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} c1_n \\ X_2' \end{bmatrix} Y$$
$$n^{-1}1_n\hat{Y} = n^{-1} \begin{bmatrix} nc & 1_nX_2 \end{bmatrix} \left(\begin{bmatrix} nc^2 & c1_nX_2 \\ cX_2'1_n & X_2'X_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} c1_n \\ X_2' \end{bmatrix} Y$$
$$n^{-1}1_n\hat{Y} = n^{-1}c^{-1} \begin{bmatrix} nc^2 & c1_nX_2 \end{bmatrix} \left(\begin{bmatrix} nc^2 & c1_nX_2 \\ cX_2'1_n & X_2'X_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} c1_n \\ X_2' \end{bmatrix} Y$$

Since

$$\begin{bmatrix} nc^2 & c1_nX_2 \\ cX_2'1_n & X_2'X_2 \end{bmatrix} \left(\begin{bmatrix} nc^2 & c1_nX_2 \\ cX_2'1_n & X_2'X_2 \end{bmatrix} \right)^{-1} = I$$
$$\begin{bmatrix} \begin{bmatrix} nc^2 & c1_nX_2 \end{bmatrix} \\ \begin{bmatrix} cX_2'1_n & X_2'X_2 \end{bmatrix} \end{bmatrix} \left(\begin{bmatrix} nc^2 & c1_nX_2 \\ cX_2'1_n & X_2'X_2 \end{bmatrix} \right)^{-1} = I$$

$$\begin{bmatrix} \begin{bmatrix} nc^2 & c1_nX_2 \end{bmatrix} \left(\begin{bmatrix} nc^2 & c1_nX_2 \\ cX_2'1_n & X_2'X_2 \end{bmatrix} \right)^{-1} \\ \begin{bmatrix} cX_2'1_n & X_2'X_2 \end{bmatrix} \left(\begin{bmatrix} nc^2 & c1_nX_2 \\ cX_2'1_n & X_2'X_2 \end{bmatrix} \right)^{-1} \end{bmatrix} = I$$

Hence

$$\begin{bmatrix} nc^2 & c1_nX_2 \end{bmatrix} \left(\begin{bmatrix} nc^2 & c1_nX_2 \\ cX_2'1_n & X_2'X_2 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \end{bmatrix}$$

So,

$$\begin{aligned} n^{-1}1_n\hat{Y} &= n^{-1}c^{-1} \begin{bmatrix} nc^2 & c1_nX_2 \end{bmatrix} \left(\begin{bmatrix} nc^2 & c1_nX_2 \\ cX_2'1_n & X_2'X_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} c1_n \\ X_2' \end{bmatrix} Y \\ &= n^{-1}c^{-1} \begin{bmatrix} 1 & 0 & 0 & ... & 0 \end{bmatrix} \begin{bmatrix} c1_n \\ X_2' \end{bmatrix} Y \\ &= n^{-1}c^{-1} \begin{bmatrix} c1_nY \end{bmatrix} \\ &= n^{-1}1_nY \\ &= \bar{Y} \end{aligned}$$

as desired.

(3.13)

(a) Let $D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$. Then

$$D'D = \begin{bmatrix} D_1'D_1 & D_1'D_2 \\ D_2'D_1 & D_2'D_2 \end{bmatrix} = \begin{bmatrix} N_M & 0 \\ 0 & N_W \end{bmatrix}$$

where N_M, N_W denote the number of men and women respectively. Hence

$$(D'D)^{-1} = \begin{bmatrix} N_M & 0 \\ 0 & N_W \end{bmatrix}^{-1} = \begin{bmatrix} 1/N_M & 0 \\ 0 & 1/N_W \end{bmatrix}$$

So

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = (D'D)^{-1}D'y = \begin{bmatrix} (1/N_M)D_1'Y \\ (1/N_W)D_2'Y \end{bmatrix} = \begin{bmatrix} \bar{Y_1} \\ \bar{Y_2} \end{bmatrix}$$

(b) The Y transformation normalizes Y by subtracting out the subgroup mean, i.e. subtracting the average of the men's Y for men and the average of the women's Y for women. X is the same; subtracts the average X vector of men for men, and the average X vector of women for women.

(c) Note that by part a, Y^* is the regression residual of Y on $D = [D_1\ D_2]$. Similarly, by part a, X^* is the regression residual of X on $D = [D_1\ D_2]$. By Frisch-Waugh-Lovell, then, $\tilde{\beta} = \hat{\beta}$, since we are regressing the residuals of Y on D on the residuals of X on D.