7.6- Strategies for Integration

Name/ Uid:____ Date:___________

At this point, we have a two fundamental techniques that we can use to evaluate integrals: substitution and integration by parts. In certain cases, there are additional tricks we can use, including:

- Techniques for integrating trigonometric integrals (Section 7.3)
- Rationalizing substitution (Section 7.4)
- Decomposition of a rational function into partial fractions (Section 7.5)

Given an integral, it is not always easy to tell which technique should be used to evaluate it. However, developing the ability to determine which technique or combination of techniques to use is important. Truly, the only way to get good at this is to do many examples. There are, however, a couple additional rules of thumb that can be helpful:

- **Simplify the integrand, if possible-** Sometimes it is helpful to use algebra or trig identities to simplify the integrand. This is, after all, the premise behind integrating rational functions using partial fraction decomposition.

Example 1. *Integrate* $\int \frac{\sin^2 x}{\cos^4 x}\,dx$ ***Hint:*** *Write in terms of tangent and secant.*

Solution.

$$= \int \frac{\sin^2(x)}{\cos^2(x)} \cdot \frac{1}{\cos^2(x)}\,dx$$

$$= \int \tan^2 x \cdot \sec^2 x\,dx \qquad \begin{aligned} u &= \tan x \\ du &= \sec^2 x\,dx \end{aligned}$$

$$= \int u^2\,du$$

$$= \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C$$

Example 2. *Integrate* $\int \frac{1}{\sqrt{x+1}-\sqrt{x}}\,dx$ ***Hint:*** *Multiply by the conjugate.*

Solution.

$$= \int \frac{1}{\sqrt{x+1}-\sqrt{x}} \cdot \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}\,dx$$

$$= \int \frac{\sqrt{x+1}+\sqrt{x}}{(x+1)-x}\,dx$$

$$= \int \sqrt{x+1}+\sqrt{x}\,dx$$

$$= \int (x+1)^{\frac{1}{2}} + x^{\frac{1}{2}}\,dx$$

$$= \frac{2}{3}(x+1)^{\frac{3}{2}} + \frac{2}{3}x^{\frac{3}{2}} + C$$

- **Look for Substitution-** Sometimes substitution $u = g(x)$ can be helpful even in instances where a du is not obvious in the integrand. If some complicated composition of functions appears in your integrand, you might try making a substitution and then trying to find a clever way of writing the remaining terms.

Example 3. *Use the substitution* $u = \sqrt{x}$ *and integration by parts to evaluate* $\int e^{\sqrt{x}}\, dx$.

Solution.

$u = \sqrt{x}$

$du = \frac{1}{2}\frac{1}{\sqrt{x}}\,dx$

$= \int e^u \cdot 2u\, du$

$= fg - \int g\, df$

$f = 2u \quad g = e^u$

$df = 2du \quad dg = e^u\, du$

$= 2ue^u - \int 2e^u\, du$

$= 2ue^u - 2e^u + C$

$= 2\sqrt{x}\, e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$

Example 4. *Compute the integral* $\int x^5\sqrt{x^2-9}\, dx$ *in two different ways: 1) use the rationalizing substitution* $x = 3\sec t$ *and 2) use the regular substitution* $u = x^2 - 9$.

Solution.

1. $x = 3\sec t$

$dx = 3\sec t \tan t\, dt$

$\int 3^5 \sec^5 t \cdot \sqrt{9\sec^2 t - 9}\, dt$

$= \int 3^5 \sec^5 t \cdot 3\tan t\, dt$

$= \int 3^6 \sec^2 t \cdot \sec^2 t \cdot \sec t \cdot \tan t\, dt$

$= 3^6 \int u^2 \cdot u^2\, du$

$u = \sec t$

$du = \sec t \tan t\, dt$

$= 3^6 \int u^4\, du$

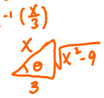
$= 3^6 \cdot \frac{1}{5} u^5 + C$

$= 3^6 \cdot \frac{1}{5}\sec^5 t + C$

$x = 3\sec t$

$t = \sec^{-1}\left(\frac{x}{3}\right)$

$= 3^6 \cdot \frac{1}{5}\sec^5\left(\sec^{-1}\left(\frac{x}{3}\right)\right) + C$



$= 3^6 \cdot \frac{1}{5}\left[\frac{x}{3}\right]^5 + C$

$= \frac{3x^5}{5} + C$

2. $\int x^5 \sqrt{x^2-9}\, dx$

$u = (x^2-9)^{\frac{1}{2}}$

$u^2 = x^2 - 9$

$u^2 + 9 = x^2$

$2u\,du = 2x\,dx$

$u\,du = x\,dx$

$= \int x^2 \cdot x^2 \cdot x\sqrt{x^2-9}\, dx$

$= \int (u^2+9)^2 \cdot u^2\, du$

$= \int (u^4 + 18u^2 + 81)u^2\, du$

$= \int u^6 + 18u^4 + 81u^2\, du$

$= \frac{1}{7}u^7 + \frac{18}{5}u^5 + \frac{81}{3}u^3 + C$

$= \frac{1}{7}(x^2-9)^{7/2} + \frac{18}{5}(x^2-9)^{5/2} + \frac{81}{3}(x^2-9)^{3/2} + C$

Remark 1. *It is important to realize that, even with all of these techniques at your disposal, there are many integrands which you* ***cannot*** *integrate. It can be shown that there are some antiderivatives that cannot be expressed in terms of elementary functions (polynomials, trig functions, exponentials, inverses, etc). For example, the integral*

$$\int e^{-x^2}\, dx$$

is of fundamental importance in statistics and probability, but cannot be expressed in terms of elementary functions. To evaluate forms of this integral and others, you need to use numerical approximation, a computer, or a chart.