Section 6.5- Exponential Growth and Decay

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Many quantities change at a rate proportional to their size. If $y(t)$ represents a quantity at time t, then this statement implies that $y(t)$ solves the differential equation

$$\boxed{\frac{dy}{dt} = ky} \tag{1}$$

for some k, which can be positive or negative depending on whether the quantity is growing or decaying. This quantity k is called the **relative growth or decay rate**. The function

$$\boxed{y(t) = Ce^{kt}} \tag{2}$$

for any constant C, satisfies Eq. (1). Since $y(0) = C$, C can be interpreted as an initial quantity. It turns out that functions of this form are the **only** solutions to Eq. (1).

Population Growth

As the population increases, so does the number of offspring born in a given year (this is the rate of change of the population). Therefore, it is reasonable to believe that a population $y(t)$ at time t satisfies Eq. (1) for some positive k. It follows that the population as a function of time, $y(t)$, satisfies Eq. (2) where C is the population at time $t = 0$.

Example 1. *The population of the United States was 250 million in 1990 and 282 million in 2000. Assuming the population grows exponentially, find the population in 2014.* ***Note:*** *the actual population in 2014 was 319 million.*

Solution.

Radioactive Decay

Certain elements decay spontaneously into other elements by emitting radiation. It is impossible to tell when a given atom will decay, instead one can give a probability that a given atom will decay in a certain time frame. It follows that the more atoms, the more decays in a given time frame. If $y(t)$ denotes the mass of the substance at time t, then $y(t)$ satisfies Eq. (1) and has the form of Eq. (2). In this application, the constant k is negative and is related to the **half-life** of the element: the amount of time it takes for half of the mass to decay. If T denote the half-life, then the relationship is $k = -\frac{\ln 2}{T}$.

Example 2. *The half-life of sodium-22 is* 2.6 *years. A sample of sodium-22 had an initial mass of* 10 *grams. Now* 3 *grams remain. How much time has elapsed?*

Solution.

Newton's Law of Cooling

Newton's Law of Cooling states that a warm object cools at a rate which is proportional to the difference between the temperature and its surroundings (the law also applies to warming). If $T(t)$ denote the temperature of the object at time t and A denotes the temperature of the surroundings, then this implies that T satisfies the differential equation

$$\frac{dT}{dt} = k(T - A).$$

This is a slightly different equation than Eq. (1). However, if we make the change of variables $y(t) = T(t) - A$, then $\frac{dy}{dt} = \frac{dT}{dt}$, and so y satisfies Eq. (1). It follows that since $y(t) = Ce^{kt}$, then $T(t) = Ce^{kt} + A$ with $T(0) = C + A$.

Example 3. *A turkey is taken out of a* $400°$ *degree oven and left to cool in a* $70°$ *room.* 15 *minutes later, its temperature is* $370°$*. How long will it take before the turkey is* $250°$*?*

Solution.

Continuously Compound Interest & The Number e as a Limit

The number e, the base of the natural logarithm, can be obtained by taking the following limit:

$$\boxed{e = \lim_{n\to\infty} (1 + \frac{1}{n})^n}$$

Suppose a principal of A_0 dollars is invested in an account which pays an annual interest rate of r (a percentage written as a decimal) compound n times a year. After t years the account will contain

$$A(t) = A_0(1 + \frac{r}{n})^{nt}$$

dollars. If we let $n \to \infty$ in this expression, then we are compounding the interest continually and by our limit definition of e we get that

$$A(t) = \lim_{n\to\infty} A_0(1 + \frac{r}{n})^{nt} = A_0 \left(\lim_{n\to\infty} (1 + \frac{r}{n})^{\frac{n}{r}} \right)^{rt} = A_0 \left(\lim_{m\to 0} (1 + \frac{1}{m})^m \right)^{rt} = A_0 e^{rt}$$

where we have made the substitution $m = \frac{n}{r}$.

Example 4. $\$1000$ *is invested in a bank account paying* 3% *interest and no further investments are made. How much money is in the account after* 10 *years if*

(a) the interest is compound annually?

(b) the interest is compound monthly?

(c) the interest is compound continuously?