$\boxed{7.3}$ Trigonometric Integrals

Recall your trig identities on the top of your handout

There are 5 types of trig questions that we worry about:

1. $\int \sin^n x\,dx$ & $\int \cos^n x\,dx$
2. $\int \sin^m x \cos^n x\,dx$
3. $\int \sin mx \cos nx\,dx$, $\int \sin mx \sin nx\,dx$, $\int \cos mx \cos nx\,dx$
4. $\int \tan^n x\,dx$, $\int \cot^n x\,dx$
5. $\int \tan^m x \sec^n x\,dx$, $\int \cot^m x \csc^n x\,dx$

Type 1: $\int \sin^n x\,dx$ or $\int \cos^n x\,dx$

- n odd → use $\sin^2 x + \cos^2 x = 1$, want to factor out a sin or cos

for example: $\int \sin^5 x\,dx = \int \sin x(\sin^4 x)\,dx = \int \sin x(\sin^2 x)^2\,dx$

$$= \int \sin x(1-\cos^2 x)^2\,dx$$
$$= \int \sin x(1-2\cos^2 x+\cos^4 x)\,dx$$
$$= \int (1-2\cos^2 x+\cos^4 x)\sin x\,dx \qquad u=\cos x,\ du=-\sin x\,dx$$
$$= -\int (1-2u^2+u^4)\,du$$
$$= -\left[u-\tfrac{2}{3}u^3+\tfrac{1}{5}u^5\right]+C = -\cos x+\tfrac{2}{3}\cos^3 x-\tfrac{1}{5}\cos^5 x+C$$

- n even → most likely half-angle identities

$$\int \sin^2 x\,dx = \int \tfrac{1}{2}-\tfrac{1}{2}\cos 2x\,dx$$
$$= \tfrac{1}{2}x-\tfrac{1}{4}\sin 2x+C$$

OR

$$\int \cos^4 x\,dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$$
$$= \tfrac{1}{4}\int 1+2\cos 2x+\cos^2 2x\,dx$$
$$= \tfrac{1}{4}\int dx+\tfrac{2}{4}\int \cos 2x\,dx+\tfrac{1}{4}\int \cos^2 2x\,dx$$
$$= \tfrac{1}{4}x+\tfrac{1}{4}\sin(2x)+\tfrac{1}{4}\int \tfrac{1}{2}+\tfrac{1}{2}\cos 4x\,dx$$
$$= \tfrac{1}{4}x+\tfrac{1}{4}\sin(2x)+\tfrac{1}{8}x+\tfrac{1}{32}\sin(4x)+C$$
$$= \tfrac{3}{8}x+\tfrac{1}{4}\sin(2x)+\tfrac{1}{32}\sin(4x)+C$$

Type 2: $\int \sin^m x \cos^n x\,dx$

- If m or n is odd positive integer and the other exponent is any other number, we factor out sin x or cos x and use $\sin^2 x+\cos^2 x=1$

for example: $\int \sin^3 x\cos^2 x\,dx = \int \sin x(\sin^2 x)\cos^2 x\,dx$

$$= \int \sin x(1-\cos^2 x)\cos^2 x\,dx \qquad u=\cos x,\ du=-\sin x\,dx$$
$$= -\int (1-u^2)u^2\,du$$
$$= -\int u^2-u^4\,du = -\left[\tfrac{1}{3}u^3-\tfrac{1}{5}u^5\right]+C = -\tfrac{1}{3}\cos^3 x+\tfrac{1}{5}\cos^5 x+C$$

• similarly as in type 1, if m and n are even, we use the half angle formulas

for example: $\int \sin^2 x \cos^2 x \, dx$

$= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)\left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right) dx$

$= \int \frac{1}{4} - \frac{1}{4}\cos^2(2x) \, dx$

$= \int \frac{1}{4} - \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right) dx$

$= \int \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos(4x) \, dx = \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$

Type 3: $\int \sin mx \cos nx \, dx$, $\int \sin mx \sin nx \, dx$, $\int \cos mx \cos nx \, dx$

• use the following identities

1. $\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x]$
2. $\sin mx \sin nx = -\frac{1}{2}[\cos(m+n)x - \cos(m-n)x]$
3. $\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x]$

for example [look @ #5]: $\int \sin(5x)\cos(2x) \, dx$

$= \int \frac{1}{2}[\sin(5+2)x + \sin(5-2)x] \, dx$

$= \int \frac{1}{2}\sin(7x) + \frac{1}{2}\sin(3x) \, dx$

$= -\frac{1}{2}\cos(7x) \cdot \frac{1}{7} - \frac{1}{2} \cdot \frac{1}{3}\cos(3x) + C$

Type 4: $\int \tan^n x \, dx$, $\int \cot^n x \, dx$

• when its tan → factor out $\tan^2 x = \sec^2 x - 1$

• when its cot → factor out $\cot^2 x = \csc^2 x - 1$

for example: $\int \cot^4 x \, dx = \int \cot^2 x (\csc^2 x - 1) \, dx$

$= \int \cot^2 x \csc^2 x - \cot^2 x \, dx$

$= \int \cot^2 x \csc^2 x \, dx - \int \csc^2 x - 1 \, dx$

$u = \cot x$
$du = -\csc^2 x \, dx$

$= -\int u^2 \, du - \int \csc^2 x - 1 \, dx$

$= -\frac{1}{3}\cot^3 x + \cot x + x + C$

for example: $\int \tan^5 x \, dx = \int \tan^3 x \tan^2 x \, dx$

$= \int \tan^3 x (\sec^2 x - 1) \, dx$

$= \int \tan^3 x \sec^2 x - \tan^3 x \, dx$

$= \int \tan^3 x \sec^2 x - \tan x (\sec^2 x - 1) \, dx$

$= \int \tan^3 x \sec^2 x - \tan x \sec^2 x + \tan x \, dx$

$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C$

Type 5: $\int \tan^m x \sec^n x \, dx$ (*), $\int \cot^m x \csc^n x \, dx$

• if n is even in (*), pull out a factor of $\sec^2 x$ and then use above identity to write a polynomial in $\tan x$. then $u = \tan x$

for example: $\int \sec^4 x \tan x \, dx$

$= \int \sec^2 x \cdot \sec^2 x \cdot \tan x \, dx$

$= \int \sec^2 x \cdot (\tan^2 x + 1) \tan x \, dx$

$= \int (\sec^2 x \tan^2 x + \sec^2 x) \tan x \, dx$

$= \int \sec^2 x \tan^3 x + \sec^2 x \tan x \, dx$

$u = \tan x$
$du = \sec^2 x \, dx$

$= \int u^3 + u \, du = \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x + C$

- If m is odd, pull a factor of $\sec x \tan x$ and then use identity. $u = \sec x$

<u>for example:</u> $\int \sec^3 x \tan^3 x \, dx = \int \sec x \tan x \,(\sec^2 x \tan^2 x)\, dx$

$$= \int \sec x \tan x \,(\sec^2 x\,(\sec^2 x - 1))\, dx$$

$$= \int \sec x \tan x \,(\sec^4 x - \sec^2 x)\, dx$$

$u = \sec x$
$du = \sec x \tan x$

$$= \int u^4 - u^2 \, du$$

$$= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$$