8.3- Improper Integrals: Infinite Limits of Integration

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In this section, we consider evaluating definite integrals on intervals of infinite length, for example,

$$\int_a^\infty f(x)\ dx.$$

We might naively try to use the FTC and say that this integral should be equal to $F(\infty) - F(a)$, where F is an antiderivative of f. The problem is that the term $F(\infty)$ does not make sense, and so we need to modify our reasoning. While we can't evaluate a function at the 'point' ∞, we can take the limit of the function at ∞. It follows that we can make sense of integrating a function over an interval of infinite length if we define the integral in terms of a limit.

Definition. *We use the following limits to define the three types of* ***improper integrals*** *on intervals of infinite length: For any value a,*

- $\displaystyle\int_a^\infty f(x)\ dx = \lim_{b\to\infty}\int_a^b f(x)\ dx$
- $\displaystyle\int_{-\infty}^a f(x)\ dx = \lim_{b\to-\infty}\int_b^a f(x)\ dx$
- $\displaystyle\int_{-\infty}^\infty f(x)\ dx = \lim_{b\to-\infty}\int_b^0 f(x)\ dx + \lim_{b\to\infty}\int_0^b f(x)\ dx$

If the limits on the right hand side exist and have finite values, then the improper integral is said to ***converge****. Otherwise, the improper integral* ***diverges***

Example 1. *Compute the following improper integrals or show that they diverge.*

(a) $\displaystyle\int_0^\infty e^{-x}\ dx$

(b) $\displaystyle\int_2^\infty \frac{1}{x\ln x}\ dx$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{\sqrt{x^2+4}}\, dx$

(d) $\displaystyle\int_{-\infty}^{2} \frac{1}{4+x^2}\, dx$

Example 2. *For what values of $p > 0$ does the integral*

$$\int_1^{\infty} \frac{1}{x^p}\, dx$$

converge? ***Note:*** *This computation will be important for material in Chapter 9.*

Solution.