$\boxed{7.5}$ Integration of Rational Functions using Partial Fractions

- I think this section is best displayed through examples

(1) $\int \frac{1}{x^2-2x-8}\,dx$ distinct linear factors

$= \int \frac{1}{(x-4)(x+2)}\,dx$

$\frac{A}{x-4} + \frac{B}{x+2} = \frac{1}{(x-4)(x+2)}$

$A(x+2) + B(x-4) = 1$

$Ax + 2A + Bx - 4B = 1 + 0x$

$x: A + B = 0$ $\quad$ $c: 2A - 4B = 1$

$A = -B$ $\quad$ $2(-B) - 4B = 1$

$A = 1/6$ $\quad$ $-6B = 1$

$B = -1/6$

$= \int \frac{1/6}{x-4} + \frac{-1/6}{x+2}\,dx$

$= \frac{1}{6}\ln|x-4| - \frac{1}{6}\ln|x+2| + C$

(2) $\int \frac{1}{x^3+x}\,dx$ distinct linear factors

$= \int \frac{1}{x(x^2+1)}$

$\frac{A}{x} + \frac{Bx+C}{x^2+1} =$ $A(x^2+1) + (Bx+C)x = 1$

$Ax^2 + A + Bx^2 + Cx = 1 + 0x + 0x^2$

- $A + B = 0$ $\quad$ - $A = 1$ $\quad$ - $C = 0$

$B = -1$

$= \int \frac{1}{x} - \frac{x}{x^2+1}\,dx$

$= \ln|x| - \frac{1}{2}\ln|x^2+1| + C$

(3) $\int \frac{x+3}{(x-1)^2}\,dx$ Repeated Linear Factors

$= \int \frac{x+3}{(x-1)(x-1)}\,dx$

$\frac{A}{x-1} + \frac{B}{(x-1)^2} =$ $A(x-1) + B = x+3$

$Ax - A + B = x + 3$

$A = 1$ $\quad$ $-1 + B = 3$

$B = 4$

$= \int \frac{1}{x-1} + \frac{4}{(x-1)^2}\,dx$ $\quad$ $\frac{4}{u^2} = 4u^{-2}$

$= \ln|x-1| - \frac{4}{x-1} + C$

(4) $\int \frac{-x^2-10}{x^3+2x^2+10x}\,dx = \int \frac{-(x^2+10)}{x(x^2+2x+10)}\,dx$

$\frac{A}{x} + \frac{Bx+C}{x^2+2x+10} =$ $A(x^2+2x+10) + (Bx+C)x = -(x^2+10)$

- $Ax^2 + Bx^2 = -x^2 \Rightarrow -1 + B = -1 \Rightarrow B = 0$
- $2Ax + Cx = 0 \Rightarrow -2 + C = 0 \Rightarrow C = 2$
- $10A = -10 \Rightarrow A = -1$

$= \int -\frac{1}{x} + \frac{2}{x^2+2x+10}\,dx$

$= \int -\frac{1}{x} + \int \frac{2}{(x^2+2x+4)+6}$

$= -\ln|x| + \int \frac{2}{(x+2)^2+6}\,dx$

$= -\ln|x| + 2\int \frac{1}{(x+2)^2+6}\,dx$

$= \quad \cdots \quad + \frac{2}{6}\int \frac{1}{\left(\frac{x+2}{\sqrt{6}}\right)^2+1}\,dx$ $\quad$ $u = \frac{x+2}{\sqrt{6}}$ $\quad$ $\sqrt{6}\,du = dx$

$= \cdots \quad + \frac{1}{3}\cdot\sqrt{6}\int \frac{1}{u^2+1}\,du = -\ln|x| + \frac{\sqrt{6}}{3}\tan^{-1}\left(\frac{x+2}{\sqrt{6}}\right) + C$