Section 6.2- Inverse Functions and Their Derivatives

Name/ Uid:______________________________________ Date:___________

Definition. *A function f is* ***one-to-one*** *if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. In other words, f is one-to-one if no two distinct inputs give the same output.*

The graph of a one-to-one function passes the **horizontal line test**: every horizontal line intersects the graph at most once.

Theorem 1. *If f is strictly monotonic on its domain (either strictly decreasing or strictly increasing), then f is one-to-one.*

Example 1. *Determine whether or not the following functions are one-to-one:*

(a) $f(x) = x^2$

(b) $f(x) = x^3 + x - 1$.

(c) $f(x) = \sqrt{\frac{1}{x-3}}, \quad x > 3$.

Definition. *If f is one-to-one with domain D and range R, the* ***inverse*** *of f (or f-inverse) is the function with domain R and range D, denoted f^{-1}, which satisfies*

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(y)) = y$$

whenever $x \in D$ and $y \in R$.

Example 2. *Use the definition above to show that the function $g(x) = \frac{1}{3}x - 6$ is the inverse to $f(x) = 3x + 18$.*

Finding A Formula for the Inverse

It is not always possible to find a formula for the inverse of a given one-to-one function f, but in simple cases you can use the following procedure:

1. Write $y = f(x)$.
2. Switch x and y to make the equation $x = f(y)$.

3. Solve for y. The resulting function is f-inverse, i.e. $y = f^{-1}(x)$.

Example 3. *Use the procedure above to find $f^{-1}(x)$ for the given $f(x)$.*

(a) $f(x) = 5x - 3$.

(b) $f(x) = x^2 - 2x + 9, \qquad x > 1$ ***Hint:*** *Complete the square!*

(c) $f(x) = \sqrt{x}$ *(Be careful!)*

The Derivative of the Inverse

Theorem 2 (Inverse Function Theorem)**.** *Suppose f is differentiable and strictly monotonic on the interval I. Suppose that x is a point in the range of f such that $f'(f^{-1}(x)) \neq 0$. Then f^{-1} is differentiable at x and*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Note: This formula is particularly useful when you can't explicitly solve for $f^{-1}(x)$.

Example 4. *Use the Inverse Function Theorem to find $(f^{-1})'(1)$ for*

(a) $f(x) = x^3 + x - 1$.

(b) $f(x) = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$.