8.1 Indeterminate Forms of Type 0/0

Motivation:

what happens when we try to take the limit of:

(a) $\lim_{x\to 3} \frac{x^2-9}{x^2-x-6} = \lim_{x\to 3} \frac{9-9}{9-9} = \lim_{x\to 3} \frac{0}{0}$ How do we deal with this?

- factoring: $\lim_{x\to 3} \frac{(x+3)(x-3)}{(x-3)(x+2)} = \lim_{x\to 3} \frac{x+3}{x+2} = \frac{6}{5}$

(b) $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{f(a)-f(a)}{a-a} = \frac{0}{0}$ what now?

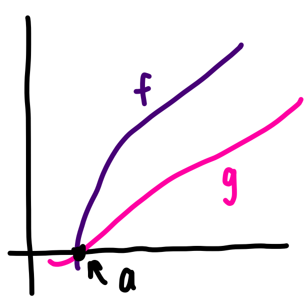
lets introduce a standard procedure for dealing w/ limits of 0/0 form

L'Hôpital's Rule (1696):

suppose $\lim_{x\to u} f(x) = \lim_{x\to u} g(x) = 0$ ^[$g'(x)\neq 0$ in a nhbd of limit point]. If $\lim_{x\to u} \frac{f'(x)}{g'(x)}$ exists in either finite or infinite sense then $\lim_{x\to u} \frac{f(x)}{g(x)} = \lim_{x\to u} \frac{f'(x)}{g'(x)}$

Rmk: holds for one sided limits & limits at infinity

why is this true? (sketch of proof)



- when we zoom in near $x=a$, f and g look like linear functions in a neighborhood of a.
- recall that linearization is given by $L(x) = f(a) + f'(a)(x-a) = f'(a)(x-a)$
  - similar for g
- it follows:
  $\lim_{x\to a} \frac{f(x)}{g(x)} \approx \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$
- relies heavily on continuity

EX: $\lim_{x\to 0} \frac{\sin x}{x}$

1. check if in 0/0 form — $\frac{\sin(0)}{0} = \frac{0}{0}$ ✓
2. $\lim_{x\to 0} \frac{\sin x}{x} \overset{LH}{=} \lim_{x\to 0} \frac{\cos(x)}{1} = \frac{\cos(0)}{1} = 1$

EX: $\lim_{x\to\pi} \frac{\tan(2x)}{x-\pi} \overset{L'H}{=} \lim_{x\to\pi} \frac{2\sec^2(2x)}{1} = \frac{2\sec^2(2\pi)}{1} = 2$

EX: $\lim_{x\to 1^+} \frac{\ln x}{(x-1)^2} \overset{L'H}{=} \lim_{x\to 1^+} \frac{1/x}{2(x-1)} = \frac{1}{0} = +\infty$

EX: $\lim_{x\to\pi} \frac{1+\cos x}{(x-\pi)^2} = \lim_{x\to\pi} \frac{-\sin x}{2(x-\pi)} \overset{L'H}{=} \lim_{x\to\pi} \frac{-\cos x}{2} = \frac{1}{2}$

EX: $\lim_{x\to 0} \frac{e^x-1-x}{x^2\cos x} \overset{L'H}{=} \lim_{x\to 0} \frac{e^x-1}{2x\cos x - x^2\sin x} \overset{L'H}{=} \lim_{x\to 0} \frac{e^x}{2\cos x - 2x\sin x - (2x\sin x + ..)}$

$= \frac{1}{2}$

EX: $\lim_{x\to 0} \frac{\sin(x^2)\tan x}{x^2}$

$\overset{L'H}{=} \lim_{x\to 0} \frac{\cos(x^2)\cdot 2x\cdot\tan x + \sec^2 x\sin(x^2)}{2x}$

$\overset{L'H}{=} \lim_{x\to 0}$ EWW...

Trap: $\lim_{x\to 0} \frac{\sin x}{x^2} = \lim_{x\to 0} \frac{\cos x}{2x} = \infty$