Section 6.9- The Hyperbolic Functions and Their Inverses

Name/ Uid:____ Date:___________

Hyperbolic functions arise naturally in many applications. Hyperbolic sine and hyperbolic cosine are defined as

$$\boxed{\begin{aligned} \sinh x &= \tfrac{e^x - e^{-x}}{2} \\ \cosh x &= \tfrac{e^x + e^{-x}}{2} \end{aligned}}$$

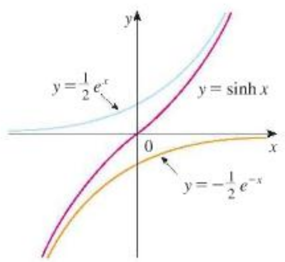
Notice that

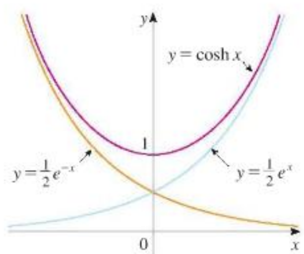
$$\cosh^2 x - \sinh^2 x = \frac{1}{4}\left(e^x + e^{-x}\right)^2 - \frac{1}{4}\left(e^x - e^{-x}\right)^2 = 1$$

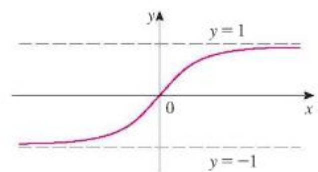
which indicates that the point $(\cosh t, \sinh t)$ lies on the hyperbola $x^2 - y^2 = 1$. Contrast this to the typical trigonometric functions: the point $(\cos t, \sin t)$ lies on the unit circle $x^2 + y^2 = 1$.
The remaining hyperbolic functions are defined in analogy to the trigonometric functions:

$$\boxed{\begin{array}{ll} \tanh x = \frac{\sinh x}{\cosh x} & \coth x = \frac{\cosh x}{\sinh x} \\ \operatorname{csch} x = \frac{1}{\sinh x} & \operatorname{sech} x = \frac{1}{\cosh x} \end{array}}$$

The graphs of $\sinh x$, $\cosh x$, and $\tanh x$ appear below:







$y = \sinh x$ $\quad$ $y = \cosh x$ $\quad$ $y = \tanh x$

There is an analogy between identities of hyperbolic functions and those of trigonometric functions. Some of the more useful are:

$\sinh(-x) = -\sinh x$	$\cosh(-x) = \cosh x$
$\cosh^2 x - \sinh^2 x = 1$	$1 - \tanh^2 x = \operatorname{sech}^2 x$
$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$	$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Derivatives of Hyperbolic Functions

Since $\sinh x$ and $\cosh x$ are defined in terms of exponentials, they are easy to differentiate. For example,

$$D_x \sinh x = D_x\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x.$$

It follows by a similar computation that $D_x \cosh x = \sinh x$, and now the quotient rule gives the remaining derivatives:

$D_x \sinh x = \cosh x$	$D_x \operatorname{csch} x = -\operatorname{csch} x \coth x$
$D_x \cosh x = \sinh x$	$D_x \operatorname{sech} x = -\operatorname{sech} x \tanh x$
$D_x \tanh x = \operatorname{sech}^2 x$	$D_x \coth x = -\operatorname{csch}^2 x$

Example 1. *Compute the following derivatives or indefinite integrals.*

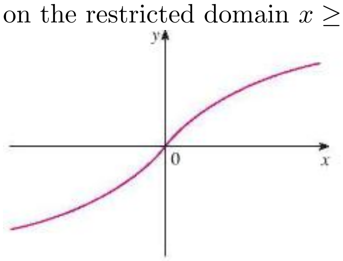
(a) $D_x(\sinh^2(x^3)) =$

(b) $\displaystyle\int \coth x \; dx =$

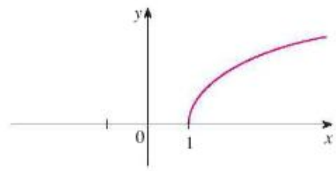
(c) $\displaystyle\int \cosh^5 x \sinh x \; dx =$

Inverse Hyperbolic Functions

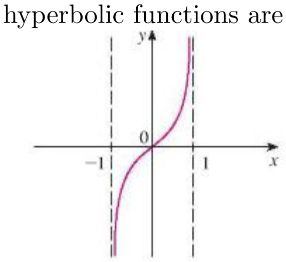
We define the inverse hyperbolic functions in the same way we defined the inverse trigonometric functions in section 6.8. Note that $\sinh x$ and $\tanh x$ are one-to-one on their domains, but $\cosh x$ is only one-to-one on the restricted domain $x \geq 0$. The graphs of the three main inverse hyperbolic functions are below:



$y = \sinh^{-1} x$



$y = \cosh^{-1} x$



$y = \tanh^{-1} x$

Since the hyperbolic functions are defined in terms of exponentials, it is reasonable to expect that the inverse hyperbolic functions can be defined in terms of the natural log. This is in fact the case.

$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$	$x \in \mathbb{R}$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$	$x \geq 1$
$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	$-1 < x < 1$

Derivatives of Inverse Hyperbolic Functions

These identities make differentiating inverse hyperbolic functions easy.

$D_x \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$	$D_x \operatorname{csch}^{-1} x = -\frac{1}{\lvert x\rvert\sqrt{x^2+1}}$
$D_x \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$	$D_x \operatorname{sech}^{-1} x = -\frac{1}{x\sqrt{1-x^2}}$
$D_x \tanh^{-1} x = \frac{1}{1-x^2}$	$D_x \coth^{-1} x = \frac{1}{1-x^2}$

Example 2. *Prove the identity*

$$\tan^{-1}(\sinh t) = \sin^{-1}(\tanh t)$$

Hint: *Differentiate both sides of the equation and observe that the values agree at $t = 0$.*