Section 6.7- Approximations for Differential Equations

Name/ Uid:____ Date:___________

A general first order differential equation has the form

$$y' = f(x, y) \tag{1}$$

There are many, even relatively simple, choices for f where the solution to this DE cannot be written in terms of elementary functions (polynomials, trig functions, exponentials, etc.). So it is in our interest to think about ways in which we might approximate solutions instead of trying to write down exact expressions for them.

Slope fields (also sometimes called **direction fields**) provide a nice way of visualizing solutions to such an equation. The key fact here is that $y' = \frac{dy}{dx}$ represents the slope of the tangent line to the function y. So you have a solution to Eq. (1) that passes through the point (x, y) in the xy-plane, then it must have slope $f(x, y)$ there. If we plot a small line of slope $f(x, y)$ at the point (x, y)...and do this for many values of (x, y) we obtain a slope field, and solutions to Eq. (1) can be visually obtained by drawing curves in this field for which the little slope lines are tangent at every point.

Example 1. *The slope field for the equation*

$$y' = x + y$$

is given below in Figure A. Sketch a few solutions. In particular sketch the solutions that pass through the points $(0, 1)$, $(2, -1)$, $(0, -3)$ *and* $(0, -1)$.

Example 2. *Let* $v(t)$ *denote the speed of an object falling through the air in meters per second. It is reasonable to assume that the force of air resistance is proportional to the velocity of the object...this constant of proportionality depends on the shape and size of the object, density of air, etc. Assume the velocity a falling object satisfies the differential equation*

$$v' = 9.8 - .1v$$

Observe the slope field for this equation below in Figure B. What do you notice about the solutions as $t \to \infty$*? What does this correspond to physically?*

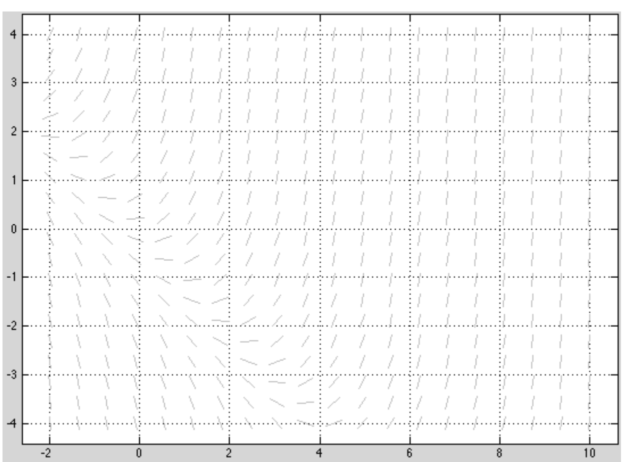

Figure A

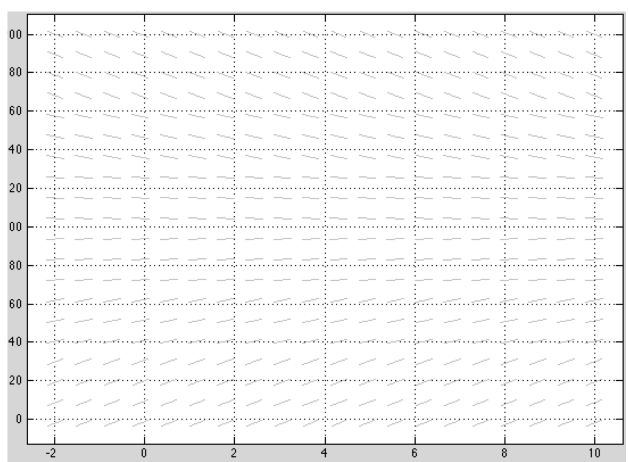

Figure B

Euler's Method

Let's assume that we are give a first-order initial value problem of the form

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

If can't find the solution analytically (by taking integrals, manipulation, etc) then perhaps we could try to approximate the solution using numerical methods.
We know the value of the solution at time x_0, since $y(x_0) = y_0$. Let's try to approximate the solution at time $x_0 + h$, where we think of h as a 'small step'. In particular, we'd like to estimate $y(x_0 + h)$. If we approximate y by its tangent line at x_0 and remember that the slope of this tangent line is $f(x_0, y_0)$, we obtain

$$y(x_0 + h) \approx y(x_0) + hf(x_0, y_0).$$

We now have an approximation to the value of $y(x_0 + h)$, the solution at time $x_0 + h$. We could repeat this argument to then get an estimate of the value of y another step h over, $y(x_0 + 2h)$. This iterative procedure is called **Euler's Method**.

Algorithm 1 (Euler's Method)**.** *To approximate solutions to the initial value problem*

$$y' = f(x, y), \qquad y(x_0) = y_0$$

we define $x_n = x_0 + hn$ *and*

$$y_{n+1} = y_n + hf(x_n, y_n)$$

for $n \geq 0$*.* y_n *is then regarded as an approximation to* $y(x_n)$*, the value of the true solution at* $x = x_n$*.*

Example 3. *Use Euler's method to approximate the value of* $y(1)$*, where* y *solves*

$$y' = y + 1 \qquad y(0) = 0.$$

Use 5 steps ($h = .2$)*.*

Solution.

Remark 1. *Note that we actually know the solution to that equation (it is both separable and linear), namely* $y(x) = e^x - 1$*. And* $y(1) \approx 1.718$*. So we are close, but not that close.*

Example 4. *Use Euler's Method with* $h = .5$ *to estimate* $y(4)$*, where* y *solves*

$$y' = y^2 - x \qquad y(2) = 1.$$

Solution.