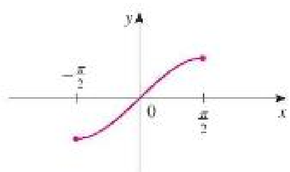
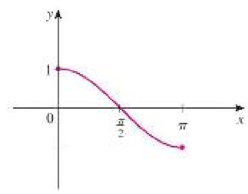
Section 6.8- The Inverse Trigonometric Functions and Their Derivatives

Name/ Uid:____ Date:___________

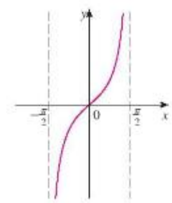
We would like to define inverse functions to our familiar trigonometric functions like sine, cosine, tangent, etc. However, these functions are not one-to-one on their natural domains. For example, $\sin x$ has natural domain $\mathbb{R}$, but it does not pass the horizontal line test. However, $\sin x$ is one-to-one for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Similarly, $\cos x$ is one-to-one for $0 \leq x \leq \pi$ and $\tan x$ is one-to-one for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. The graphs of $\sin x$, $\cos x$, and $\tan x$ respectively on these restricted domains are shown below:



$y = \sin x$

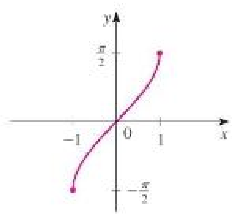


$y = \cos x$

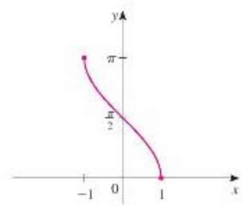


$y = \tan x$

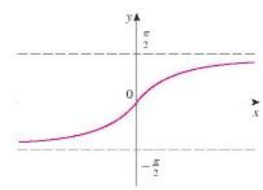
We then define the functions $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$ to be the inverses of these one-to-one functions. The graphs of $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$ respectively are shown below:



$y = \sin^{-1} x$



$y = \cos^{-1} x$



$y = \tan^{-1} x$

$$\boxed{\begin{array}{lcll} x = \sin^{-1} y & \iff & y = \sin x & \text{and} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ x = \cos^{-1} y & \iff & y = \cos x & \text{and} \quad 0 \leq x \leq \pi \\ x = \tan^{-1} y & \iff & y = \tan x & \text{and} \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \end{array}}$$

Notation 1. *The inverse trigonometric functions are sometimes denoted by the prefix 'arc', i.e.* $\sin^{-1} x = \arcsin x$.

Example 1. *Evaluate the following:*

(a) $\sin^{-1}(1) =$

(b) $\cos^{-1}(\frac{1}{2}) =$

(c) $\sin(\sin^{-1}(\frac{1}{2})) =$

(d) $\sin^{-1}(\sin(\frac{3\pi}{2})) =$

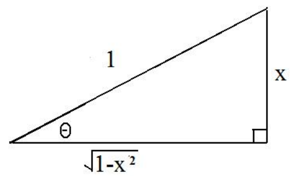
(e) $\cos^{-1}(\sin(-\frac{\pi}{3})) =$

Derivatives of Inverse Trigonometric Functions

Derivatives of the inverse trig functions can be found by using the general relationship between the derivative of a function and that of its inverse. For example, if $f(x) = \sin x$, then $f'(x) = \cos x$ and so for any $-1 < x < 1$,

$$D_x \sin^{-1} x = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(\sin^{-1} x)}.$$

To find $\cos(\sin^{-1} x)$, we set $\theta = \sin^{-1} x$ and observe that we can draw the following right triangle:



It follows that $\cos\theta = \sqrt{1-x^2}$, and so $D_x \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$. The derivatives of the other inverse trig functions can be found similarly. We have the following derivatives:

$D_x \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$D_x \csc^{-1} x = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$D_x \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$	$D_x \sec^{-1} x = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$D_x \tan^{-1} x = \frac{1}{1+x^2}$	$D_x \cot^{-1} x = -\frac{1}{1+x^2}$

These derivatives give rise to antiderivatives of course. Because of the sign relationships exhibited above, we only need to remember three antiderivatives

$$\boxed{\begin{aligned} \int \frac{1}{\sqrt{1-x^2}}dx &= \sin^{-1} x + C \\ \int \frac{1}{1+x^2}dx &= \tan^{-1} x + C \\ \int \frac{1}{x\sqrt{x^2-1}}dx &= \sec^{-1} |x| + C \end{aligned}}$$

Example 2. *Find the following derivatives and indefinite integrals*

(a) $D_x(\tan^{-1}(\sqrt{x}))$

(b) $D_x(\sin(\cos^{-1}(x)))$

(c) $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$

(d) $\displaystyle\int \frac{e^{2x}}{1+e^{4x}}\,dx$

(e) $\displaystyle\int \frac{1}{x^2+2x+5}\,dx$ ***Hint:*** $x^2+2x+5 = (x+1)^2+4$