Section 6.4- General Exponential and Logarithmic Functions

Name/ Uid:____ Date:___________

Definition. *The **exponential function with base** $a > 0$ is defined for any x as*

$$a^x = (e^{\ln a})^x = e^{x \ln a}.$$

Theorem 1 (Properties of the Exponential:)**.** *For $a, b > 0$ and any x, y,*

(a) $a^{x+y} = a^x a^y$.

(b) $a^{x-y} = \frac{a^x}{a^y}$.

(c) $(a^x)^y = a^{xy}$.

(d) $(ab)^x = a^x b^x$

(e) $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

The derivative is easy to calculate using the Chain Rule:

$$\boxed{D_x(a^x) = D_x(e^{x \ln a}) = e^{x \ln a} \ln a = a^x \ln a}$$

It follows that a^x is increasing if $a > 1$ and a^x is decreasing if $0 < a < 1$. Assuming $a \neq 1$, we have

$$\boxed{\int a^x \; dx = \frac{a^x}{\ln a} + C}$$

Example 1. *Compute the following derivatives or indefinite integrals:*

(a) $D_x(4^{x^2-x}) =$

(b) $\int \frac{2^{\sqrt{x}}}{\sqrt{x}} \; dx =$

General Logarithms

Definition. *The **logarithmic function with base** $a > 0$, written $\log_a x$, is defined as the inverse to the function a^x. In particular,*

$$\log_a x = y \Leftrightarrow a^y = x.$$

Remark 1. *The inverse relationship between a^x and $\log_a x$ implies that*

$$a^{\log_a x} = x \qquad \log_a (a^y) = y$$

for all y and $x > 0$.

Theorem 2 (Properties of the Logarithm)**.** *For positive x, y and rational r*

(a) $\log_a(xy) = \log_a x + \log_a y$.

(b) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.

(c) $\log_a(x^r) = r\log_a x$.

Since $x = a^{\log_a x}$, differentiating both sides of this equation with the Chain Rule gives

$$1 = a^{\log_a x}(\ln a)D_x(\log_a x) = x(\ln a)D_x(\log_a x)$$

which implies that

$$\boxed{D_x(\log_a x) = \frac{1}{x\ln a}}$$

This formula also gives a relationship between $\ln x$ and $\log_a x$. Since

$$D_x(\log_a x) = D_x(\frac{\ln x}{\ln a}),$$

it follows that

$$\log_a x = \frac{\ln x}{\ln a} + C$$

for some C. Setting $x = 1$ yields $C = 0$, or

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

Example 2. *Compute the following derivatives:*

(a) $D_x(\log_2(\sqrt[3]{x})) =$

In the next two problems, you will be asked to compute the derivative of a function of the form $y = a(x)^{b(x)}$. There are two (equivalent) methods for doing this:

1. *Logarithmic differentiation: write $\ln y = \ln(a(x)^{b(x)}) = b(x)\ln a(x)$, then differentiate.*
2. *Write $a(x) = e^{\ln a(x)}$, so $y = \left(e^{\ln a(x)}\right)^{b(x)} = e^{b(x)\ln a(x)}$.*

(b) *Use method 1 described above to find $D_x((\sin x)^x)$.*

(c) *Use method 2 described above to find $D_x(x^{\sin x})$.*