Section 6.1- The Natural Logarithm Function

Name/ Uid:____ Date:___________

Definition (The Natural Logarithm Function)**.** *For $x > 0$, we define the* ***natural logarithm*** *of x by*

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

From the First Fundamental Theorem of Calculus, it follows immediately that for $x > 0$,

$$D_x(\ln x) = \frac{1}{x}.$$

By considering $x > 0$ and $x < 0$ separately, we obtain the more general

$$D_x(\ln|x|) = \frac{1}{x} \Longleftrightarrow \int \frac{1}{x}\, dx = \ln|x| + C \tag{1}$$

Example 1. *Use (1) to evaluate the following:*

(a) $D_x(x \ln x)$

(b) $D_x(\ln(x^2 + 9))$

(c) $\displaystyle\int \frac{1}{2x+3}\, dx$

(d) $\displaystyle\int \tan x\, dx$ ***Hint:*** $\tan x = \frac{\sin x}{\cos x}$.

Theorem 1 (Properties of the Natural Logarithm)**.** *For any positive numbers a and b and rational number r,*

(i) $\ln 1 = 0$

(ii) $\ln(ab) = \ln a + \ln b$

(iii) $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

(iv) $\ln(a^r) = r \ln a$

Example 2. *Suppose that* $\ln a = 2$, $\ln b = 5$, *and* $\ln c = -1$. *Evaluate*

(a) $\ln(ab)$

(b) $\ln(a^3 b^{-2})$

(c) $\ln\left(\frac{ac^2}{b^3}\right)$

Logarithmic Differentiation

The natural logarithm of a function containing powers, products, and quotients can sometimes be more easily differentiated than the function itself. For example, suppose that $y = \dfrac{(x-1)^3}{\sqrt{x}(x+5)^2}$. Taking the natural logarithm of both sides and using the properties in Theorem 1 gives

$$\ln y = \ln\left(\frac{(x-1)^3}{\sqrt{x}(x+5)^2}\right) = 3\ln(x-1) - \frac{1}{2}\ln x - 2\ln(x+5)$$

Now we differentiate both sides (with respect to x), remembering to use the chain rule on $\ln y$ and get

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x-1} - \frac{1}{2x} - \frac{2}{x+5}$$

and finally we solve for $\frac{dy}{dx}$ to get

$$\frac{dy}{dx} = y\left(\frac{3}{x-1} - \frac{1}{2x} - \frac{2}{x+5}\right) = \frac{(x-1)^3}{\sqrt{x}(x+5)^2}\left(\frac{3}{x-1} - \frac{1}{2x} - \frac{2}{x+5}\right).$$

Example 3. *Use logarithmic differentiation to find* $\frac{dy}{dx}$ *when*

(a) $y = x^{2/5}(x+3)^4$

(b) $y = \dfrac{1}{x^2 + 2x - 15}$

(c) $y = \dfrac{\sqrt[3]{2x+5}}{x^2(3x^2+1)}$