7.5- Integration of Rational Functions Using Partial Fractions

Name/ Uid:____ Date:___________

The Fundamental Theorem of Algebra implies that every polynomial (with real coefficients) can be factored into a product of linear factors (factors of the form $(ax+b)$) and irreducible quadratic factors (factors of the form ax^2+bx+c, with $b^2-4ac<0$). In this section, we use this factorization to decompose a rational function into **partial fractions**, a sum of simpler rational functions.

We will focus our attention on rational functions $r(x)=\frac{p(x)}{q(x)}$ where the degree of p is **less** then the degree of q. If it happens that the degree of p is greater than or equal to the degree of q, we first do polynomial long division and then use the techniques described below on the remainder.

To decompose a rational function into partial fractions you need to know the general form you expect your answer to take. That way, you can write out this form using arbitrary constants, then solve for the constants using algebra. Whether or not you can solve for the constants depends on whether you guessed the 'correct' form.

Principles of Decomposing $\frac{p(x)}{q(x)}$ into Partial Fractions

- Factor the denominator q into a product of linear factors and irreducible quadratic factors.
- Write $\frac{p(x)}{q(x)}$ as a sum of rational functions, where the denominators of each term in the sum are given by the factors of $q(x)$.
- The numerator of each term should be a generic polynomial of degree one less than the degree of the factor in the denominator.
- For each factor of q with multiplicity greater than one, include terms with denominators which are the factor raised to all powers from 1 up to the original multiplicity.
- Solve for the constants in your numerators by comparing coefficients of powers of x in the numerator of your decomposition with the coefficients of $p(x)$.

Here are these principles applied to some examples:

rational function	**partial fraction form**
$\frac{x}{(x+1)(x-5)(x+3)}$	$\frac{A_1}{x+1}+\frac{A_2}{x-5}+\frac{A_3}{x+3}$
$\frac{2}{x^2(x-1)(x^2+1)}$	$\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{x-1}+\frac{A_4x+A_5}{x^2+1}$
$\frac{x^3-9}{(x-1)^3(x^2+x+4)^2}$	$\frac{A_1}{x-1}+\frac{A_2}{(x-1)^2}+\frac{A_3}{(x-1)^3}+\frac{A_4x+A_5}{(x^2+x+4)}+\frac{A_6x+A_7}{(x^2+x+4)^2}$

Example 1. *Use partial fractions to evaluate* $\int \frac{1}{x^2-2x-8}\,dx$.

Solution.

Example 2. *Use partial fractions to evaluate* $\int \frac{1}{x^3 + x}\, dx$.

Solution.

Example 3. *Use partial fractions to evaluate* $\int \frac{x+3}{(x-1)^2}\, dx$.

Solution.

Example 4. *Use partial fractions to evaluate* $\int \frac{-x^2 - 10}{x^3 + 2x^2 + 10x}\, dx$.

Solution.