Section 6.6- First-Order Linear Differential Equations

Name/ Uid:____ Date:___________

In this section we introduce the standard technique for solving **first-order linear differential equations**. First-order linear equations take the form

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$$

for some functions P, Q. The method we will use to find solutions to such equations is the method of the **integrating factor**. Roughly speaking, an integrating factor is a function that we multiply both sides of the equation by so that the left hand side of the equation becomes the derivative of some function times y. We can then obtain the solution to the equation with an integral.
For example, suppose my equation was

$$\frac{dy}{dx} + \frac{1}{x}y = x$$

Notice that if I multiply both sides by x, then the equation becomes

$$x\frac{dy}{dx} + y = x^2 \Longleftrightarrow D_x(xy) = x^2$$

Integrating both sides now yields

$$xy = \int x^2\ dx = \frac{x^3}{3} + C \Rightarrow y = \frac{1}{x}\left(\frac{x^3}{3} + C\right) = \frac{x^2}{3} + \frac{C}{x}.$$

where C can be any constant. This family of solutions is called the **general solution**. We can solve for C explicitly if we are given an **initial condition**, a point on the graph $y(a) = b$. Once we find a specific value for C, we have a **particular solution**.
The factor x that I multiplied the equation by was the integrating factor in this example. In general, we compute the integrating factor

$$\mu(x) = e^{\int P(x)\ dx}.$$

This function has the property that $\mu'(x) = \mu(x)P(x)$. With this as our $\mu(x)$, then we have that

$$D_x(\mu(x)y) = \mu(x)Q(x) \Rightarrow \mu(x)y = \int \mu(x)Q(x)\ dx \Rightarrow \boxed{y = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)\ dx\right)}$$

Example 1. *Use an integrating factor to find the general solution to*

$$\frac{dy}{dx} = 3y + 2.$$

Solution.

Example 2. *Solve the IVP*

$$2\frac{dy}{dx} - y = e^{x/3}, \qquad y(0) = 1.$$

Solution.

Example 3. *Solve the IVP*

$$\frac{dy}{dx} - \frac{2}{x}y = x^2 \cos x, \qquad y(\pi) = 1.$$

Solution.

Example 4. *Suppose a 50L tank is initially full of brine which has a concentration .2 kg/L. Brine with a concentration of .4 kg/L flows into the take at a rate of 3 L/min. The liquid in the tank is continually stirred, and is drained from the tank at a rate of 3 L/min. How long will it be before there are 15 kg of salt dissolved in the tank? Let $y(t)$ denote the number of kilograms of salt in the tank after t minutes. Find a differential equation that y solves along with an initial condition.*

Solution.