8.4- Improper Integrals: Infinite Integrands

Name/ Uid:____ Date:____________

Recall that the Fundamental Theorem of Calculus says that if f is **continuous** on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

If f is not continuous on $[a, b]$, then this theorem is simply not true. If, for example, our function f has a vertical asymptote at $x = a$, then the area under the graph might very well be infinite.

Definition. *We use the following limits to define three improper integrals involving discontinuous integrands:*

- *If f is continuous on $[a, b)$ and discontinuous at b, then*

$$\int_a^b f(x)\ dx = \lim_{t \to b^-} \int_a^t f(x)\ dx.$$

- *If f is continuous on $(a, b]$ and discontinuous at a, then*

$$\int_a^b f(x)\ dx = \lim_{t \to a^+} \int_t^b f(x)\ dx.$$

- *If f is continuous on $[a, b]$ except at a point $a < c < b$, then*

$$\int_a^b f(x)\ dx = \lim_{t \to c^-} \int_a^t f(x)\ dx + \lim_{t \to c^+} \int_t^b f(x)\ dx.$$

Again, if the limits on the right exist and are finite, then we say that the improper integral converges. Otherwise, it diverges.

Example 1. *Compute the following improper integrals or show that they diverge.*

(a) $\displaystyle\int_0^1 \frac{\ln x}{x}\ dx$

(b) $\displaystyle\int_{-1}^2 \frac{x}{\sqrt{4 - x^2}}\ dx$

(c) $\displaystyle\int_0^{\pi/4} \sqrt{\sin x}\cot x \; dx$

(d) $\displaystyle\int_{\pi/4}^{3\pi/4} \sec^2 x \; dx$

Example 2. *For what values of $p > 0$ is the integrand*

$$\int_0^1 \frac{1}{x^p} \; dx$$

convergent?

Solution.