1. Find the integrals:

a) $\int \frac{\ln(2x)}{x} dx$

Answer. Make the substitution $u = \ln(2x)$, $du = dx/x$ (yes, that is correct!), leading to

$$\int \frac{\ln(2x)}{x} dx = \int u\,du = \frac{(\ln(2x))^2}{2} + C .$$

b) $\int \frac{e^{x+1}}{e^x} dx$

Answer. Since $e^{x+1} = e^x \cdot e^1$, we have

$$\int \frac{e^{x+1}}{e^x} dx = e \int dx = ex + C .$$

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2. Integrate $\int \frac{u^2 + 1}{u^2(u-1)} du$

Answer. We integrate by parts; that is, we find A, B, C such that

$$\frac{u^2+1}{u^2(u-1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-1} .$$

Putting the right side over a common denominator and equating numerators gives $u^2 + 1 = Au(u-1) + B(u-1) + Cu^2$. Now evaluate at the roots:

$$\text{At } u = 0 : \qquad 1 = B(-1) \quad \text{so that} \quad B = -1 ,$$

$$\text{At } u = 1 : \qquad 1^2 + 1 = C \quad \text{so that} \quad C = 2 .$$

Now we equate the coefficients of u^2 : $1 = A + C$, so that $A = -C = -2$. This gives us

$$\int \frac{u^2+1}{u^2(u-1)} du = \int \frac{-2}{u} du + \int \frac{-1}{u^2} du + \int \frac{2}{u-1} du = -2\ln|u| + u^{-1} + 2\ln|u-1| + C .$$

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3. $\int x \arctan x\, dx$

Answer. We integrate by parts, letting

$$u = \arctan x, \ dv = x\,dx, \quad du = \frac{1}{1+x^2} dx, \ v = \frac{x^2}{2} .$$

Then

$$\int x \arctan x\, dx = \frac{x^2}{2}(\arctan x) - \frac{1}{2}\int \frac{x^2}{1+x^2} dx = \frac{x^2 \arctan x}{2} - \frac{1}{2}(x - \arctan x) + C .$$

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4. The deer population in Sad Valley, Idaho was 1200 in the year 2000, and in 2002 is 1450. Assuming continuous growth at the same rate, in what year will the population reach 2400?

Answer. The growth equation is $P = P_0 e^{rt}$, where t is the number of years after 2000. We are given $P_0 = 1200$, $P(2) = 1450$ and are asked; for what t do we have $P(t) = 2400$? First, find r: From $1450 = 1200 e^{2r}$, we get

$$r = \frac{1}{2}\ln(\frac{1450}{1200}) = .0946 .$$

Now, solve $1200 e^{.0946t} = 2400$, or $t = \ln 2/.0946 = 7.326$ years.

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5. Find the limit. Show your work.

a) **Answer**. $\lim\limits_{x\to 4} \dfrac{\sin(\pi x)}{x^2 - 16} =^{l'H} \lim\limits_{x\to 4} \dfrac{\pi\cos(\pi x)}{2x} = \dfrac{\pi}{8}$.

b) **Answer**. $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{2x^2} =^{l'H} \lim\limits_{x\to 0} \dfrac{e^x - 1}{4x} =^{l'H} \lim\limits_{x\to 0} \dfrac{e^x}{4} = \dfrac{1}{4}$.

c) **Answer**. $\lim_{x\to\infty} \frac{x^3}{e^x} =^{l'H} \lim_{x\to\infty} \frac{3x^2}{e^x} =^{l'H} \lim_{x\to\infty} \frac{6x}{e^x} =^{l'H} \lim_{x\to\infty} \frac{6}{e^x} = 0$.

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6. Do the integrals converge? If so, evaluate:

a) **Answer**. $\displaystyle\int_1^\infty e^{-2\theta}\,d\theta = \lim\limits_{A\to\infty}\int_1^A e^{-2\theta}\,d\theta = \lim\limits_{A\to\infty} \dfrac{e^{-2\theta}}{-2}\Big|_1^A = \dfrac{1}{2}$.

b) **Answer**. $\displaystyle\int_0^\infty \dfrac{dt}{1+t}$ diverges since $\int_0^A dt/(1+t) = \ln(1+A) \to \infty$ as $A \to \infty$.

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7. Here are some series and sequences. Write the letter C if there is convergence, and the letter D if not.

a) $\lim\limits_{n\to\infty} \dfrac{n!}{(2n)!}$ b) $\lim\limits_{n\to\infty} \dfrac{2^n}{n^2}$ c) $\lim\limits_{n\to\infty} \dfrac{3n^2 - 2n + 1}{4n^3 + 1}$ d) $\Sigma_{n=1}^\infty \dfrac{n!}{(2n)!}$ e) $\Sigma_{n=1}^\infty \dfrac{n}{n^2 + 1}$

f) $\Sigma_{n=1}^\infty n(n-1)(n-3)2^n$ g) $\Sigma_{n=1}^\infty \dfrac{n(\ln(n))^2}{n^3 + 1}$ h) $\Sigma_{n=1}^\infty \dfrac{1}{(2n+1)(2n+2)}$

Answer.
   a) C since the denominator is greater than $n+1$ times the numerator,
   b) D by l'Hôpital's rule,
   c) C since the degree of the denominator is one more than the degree of the numerator,
   d) C by the ratio test,
   e) D since the degree of the denominator is only one more than the degree of the numerator,
   f) D since the general term does not converge to zero.
   g) C since the degree of the denominator is more than one more than the degree of the numerator,
   h) C by comparison with the series $\Sigma(1/n^2)$.

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8. Find the focus of the parabola given by the equation $y^2 - 8x + 2y + 17 = 0$.

Answer. Complete the square;

$$y^2 + 2y + 1 = 8x - 16 \quad\text{or}\quad (y+1)^2 = 8(x-2) .$$

The vertex is at $(2, -1)$ and the parabola opens to the right. Since $4p = 8$, $p = 2$, and the focus is 2 units right of the vertex, so is at $(4, -1)$.

9. Find the area of the region enclosed by the curve given in polar coordinates by $r = 2\cos\theta$.

Answer. This is the circle of radius 1 centered at the point (1,0) so has area π. If you did not recognize the curve, you integrated, but only from 0 to π, since that is all you need to traverse the whole boundary. Thus

$$Area = \int_0^\pi \frac{1}{2}r^2 d\theta = \frac{1}{2}\int_0^\pi (2\cos\theta)^2 d\theta = \int_0^\pi (1 + \cos(2\theta))d\theta = \pi .$$

10. a) Find the general solution of the differential equation $y'' + 4y' + 5y = 0$.

Answer. The auxiliary equation, $r^2 + 4r + 5 = 0$ has the roots $r = 2 \pm i$. Thus the general solution is

$$y_h = e^{2x}(A\cos x + B\sin x) .$$

b) Solve the initial value problem:

$$y'' + 4y' + 5y = 5 , \qquad y(0) = 0, y'(0) = 0 .$$

Answer. A particular solution is the constant function $y_p = 1$. Thus the general solution is $y = y_p + y_h = 1 + e^{2x}(A\cos x + B\sin x)$. We solve for A and B from the initial conditions:

$$0 = 1 + A , \quad 0 = 2A + B , \quad \text{so} \quad A = -1, B = 2$$

and the solution is $y = 1 + e^{2x}(-\cos x + 2\sin x)$.