7.4 Rationalizing Substitutions

we first focus on integrands involving $\sqrt[n]{ax+b}$:

- take $u = (ax+b)^{1/n}$
- then $u^n = ax+b \Rightarrow \frac{u^n - b}{a} = x \Rightarrow \frac{n u^{n-1}}{a} du = dx$

for example:

$$\int x^3 \sqrt[3]{x+1}\, dx \qquad u = (x+1)^{\frac{1}{3}} \Rightarrow u^3 = x+1$$

$$u^3 - 1 = x$$

$$3u^2\, du = dx$$

$$= \int (u^3-1)\cdot u \cdot 3u^2\, du$$

$$= \int (u^4 - u) 3u^2\, du = \int 3u^6 - 3u^3\, du$$

$$= \frac{3}{7}(x+1)^{7/3} - \frac{3}{4}(x+1)^{4/3} + C$$

Integrands involving: $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, & $\sqrt{x^2-a^2}$

- we consider the following substitutions

1. $\sqrt{a^2-x^2} \Rightarrow x = a\sin t \quad t \in [-\pi/2, \pi/2]$
2. $\sqrt{a^2+x^2} \Rightarrow x = a\tan t \quad t \in (-\pi/2, \pi/2)$
3. $\sqrt{x^2-a^2} \Rightarrow x = a\sec t \quad t \in [0, \pi], \neq \pi/2$

applying the subs:

1. $\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2 t} = a\sqrt{1-\sin^2 t} = a\sqrt{\cos^2 t} = a\cos t$
2. $\sqrt{a^2+x^2} = \sqrt{a^2 + a^2\tan^2 t} = a\sqrt{1+\tan^2 t} = a\sqrt{\sec^2 t} = a\sec t$
3. $\sqrt{x^2-a^2} = \sqrt{a^2 - a^2\sec^2 t} = a\sqrt{\tan^2 t} = \pm a\tan t$

for example:

$$\int \frac{1}{x^2\sqrt{x^2-4}}\, dx$$

this is #3 so our $a = 2$ and $x = 2\sec t$

$dx = 2\sec t \tan t\, dt$

$$= \int \frac{1}{4\sec^2 t \cdot \sqrt{4\sec^2 t - 4}} \cdot 2\sec t \tan t\, dt$$

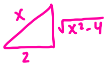
$$= \int \frac{2\sec t \tan t\, dt}{4\sec^2 t \cdot 2\sqrt{\tan^2 t}} = \int \frac{\sec t \tan t\, dt}{4\sec^2 t \cdot \tan t}$$

$$= \frac{1}{4}\int \frac{dt}{\sec t} = \frac{1}{4}\int \cos t\, dt$$

$$= \frac{1}{4}\sin t + C = \frac{1}{4}\left(\frac{\sqrt{x^2-4}}{x}\right) + C$$

but... $t = ?$ from $x = 2\sec t$

$\frac{x}{2} = \sec t \Rightarrow t = \sec^{-1}\left(\frac{x}{2}\right)$



for example:

$$\int_0^{4/3} \sqrt{16-9x^2}\, dx$$

$$= \int_0^{4/3} \sqrt{4^2 - (3x)^2}\, dx = \int_0^{4/3} \sqrt{9\left(\left(\tfrac{4}{3}\right)^2 - x^2\right)}\, dx$$

$$= \int_0^{4/3} 3\cdot\sqrt{(4/3)^2 - x^2}\, dx \qquad x = \frac{4}{3}\sin t$$

$$dx = \frac{4}{3}\cos t\, dt$$

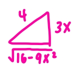
$$= \int 3\cdot\sqrt{(4/3)^2 - (4/3)^2\sin^2 t} \cdot \frac{4}{3}\cos t\, dt$$

$$= \int 3\cdot\frac{4}{3}\cdot\frac{4}{3}\sqrt{1-\sin^2 t}\cdot\cos t\, dt$$

$$= \int 3\cdot\frac{16}{9}\cos^2 t\, dt = \frac{16\cdot 3}{9}\int \cos^2 t\, dt$$

$$= \frac{16\cdot 3}{9}\int \frac{1}{2} + \frac{1}{2}\cos(2t)\, dt$$

$$= \frac{16\cdot 3}{9}\left(\frac{1}{2}t + \frac{1}{4}\sin(2t)\right) \qquad t = \sin^{-1}\left(\frac{3x}{4}\right)$$



$$= \frac{16\cdot 3}{9}\left(\frac{1}{2}\sin^{-1}\left(\frac{3x}{4}\right) + \frac{1}{4}\sin\left(2\sin^{-1}\left(\frac{3x}{4}\right)\right)\right)\Bigg|_0^{4/3}$$

$$= \frac{16\cdot 3}{9}\left[\frac{1}{2}\cdot\frac{\pi}{2} + 0 - 0\right] = \frac{\pi}{4}\cdot\frac{16\cdot 3}{9} = \frac{4\pi}{3}$$

one last example:

$$\int \frac{x^3}{\sqrt{x^2+100}}\,dx \qquad x = 10\tan t \qquad dx = 10\sec^2 t$$

$$= \int \frac{1000\tan^3 t}{\sqrt{100(1+\tan^2 t)}} \cdot 10\sec^2 t\,dt$$

$$= \int \frac{1000\tan^3 t}{10 \cdot \sec t} \cdot 10\sec^2 t\,dt$$

$$= \int 1000\tan^3 t\sec t\,dt = 1000\int \tan t \cdot \tan^2 t \cdot \sec t\,dt$$

$$= 1000\int \tan t\,(-1+\sec^2 t)\sec t\,dt \qquad u = \sec t \quad du = \sec t\tan t\,dt$$

$$x = 10\sec t \qquad t = \sec^{-1}\left(\frac{x}{10}\right)$$

$$= 1000\int -1+u^2\,du = 1000\left[-u+\frac{1}{3}u^3\right]+c$$

$$= 1000\left[-\sec t+\frac{1}{3}\sec^3 t\right]+c$$

$$= 1000\left[-\frac{x}{10}+\frac{1}{3}\left(\frac{x}{10}\right)^3\right]+c$$

$$= \boxed{-100x+\frac{x^3}{3}+c}$$