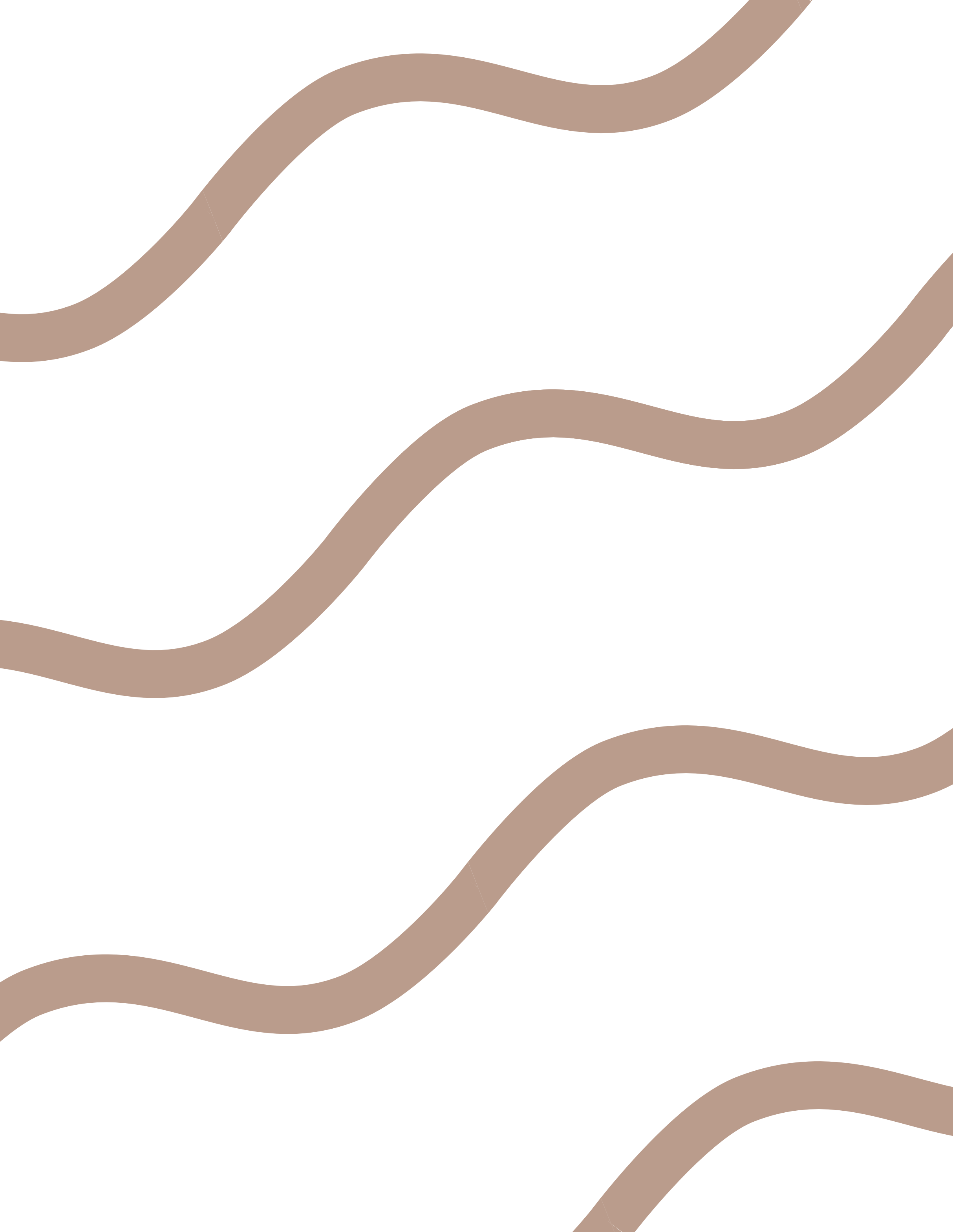

Chapter 3.

CES

$$U_j = \Big(\sum_{\omega} a_{ij}(\omega)^{\frac{1}{\sigma}} \; q_{ij}(\omega)^{\frac{\sigma-1}{\sigma}} \Big)^{\frac{\sigma}{\sigma-1}}$$

$$\max U_j \quad \text{s.t.} \quad \sum p_{ij} q_{ij} \le X_j \qquad // \lambda$$

$$\frac{\partial \mathcal{L}}{\partial q_{ij}} = \Big(\sum_{\omega} a_{ij}(\omega)^{\frac{1}{\sigma}} \, q_{ij}(\omega)^{\frac{\sigma-1}{\sigma}} \Big)^{\frac{1}{\sigma-1}} \cdot a_{ij}(\omega)^{\frac{1}{\sigma}} \cdot q_{ij}(\omega)^{-\frac{1}{\sigma}}$$

$$= \lambda \, p_{ij}(\omega)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \iff X_j = \sum p_{ij} \cdot q_{ij}$$

$$\frac{a_{ij}(\omega)^{\frac{1}{\sigma}} \cdot q_{ij}(\omega)^{-\frac{1}{\sigma}}}{a_{ij}(\omega')^{\frac{1}{\sigma}} \cdot q_{ij}(\omega')^{-\frac{1}{\sigma}}} = \frac{p_{ij}(\omega)}{p_{ij}(\omega')}$$

$$\Rightarrow X_j = \frac{1}{a_{ij}(\omega)} \, q_{ij}(\omega) \, p_{ij}(\omega)^{\sigma} \, P_j^{1-\sigma}$$

$$P_j = \Big(\sum a_{ij}(\omega) \, p_{ij}(\omega)^{1-\sigma} \Big)^{\frac{1}{1-\sigma}}$$

$$U_j = \Big(\sum a_{ij}(\omega)^{\frac{1}{\sigma}} \cdot \big(a_{ij}(\omega) \cdot X_j \cdot P_{ij}^{-\sigma} \cdot P_j^{\sigma-1} \big)^{\frac{\sigma-1}{\sigma}} \Big)^{\frac{\sigma}{\sigma-1}}$$

$$= \left(\sum a_{ij}(\omega) \cdot X_j^{\frac{\sigma-1}{\sigma}} \cdot p_{ij}^{1-\sigma} \cdot P_j^{\frac{(\sigma-1)^2}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

$$= P_j^{\sigma-1} \cdot X_j \left(\sum a_{ij}(\omega) \cdot p_{ij}^{1-\sigma} \right)^{\frac{\sigma}{\sigma-1}}$$

$$= P_j^{\sigma-1} \cdot X_j \cdot P_j^{-\sigma} = \frac{X_j}{P_j}$$

CES demand.

$$\boxed{q_{ij}(\omega) = a_{ij}(\omega)\, p_{ij}(\omega)^{-\sigma} X_j P_j^{\sigma-1}}$$

this comes purely from demand side assumption, no market structure assumption

$$= a_{ij}(\omega) \cdot U_j \left(\frac{p_{ij}(\omega)}{P_j} \right)^{-\sigma}$$

$$X_{ij}(\omega) = p_{ij}(\omega)\, q_{ij}(\omega) = a_{ij}(\omega)\, p_{ij}(\omega)^{1-\sigma} \cdot X_j \cdot P_j^{\sigma-1}$$

Armington

$$p_{ij} = \tau_{ij} \frac{w_i}{A_i}$$

$$\Rightarrow X_{ij}(\omega) = a_{ij}\, \tau_{ij}^{1-\sigma} \left(\frac{w_i}{A_i}\right)^{1-\sigma} \cdot X_j P_j^{\sigma-1}$$

$$Y_i = \sum_j X_{ij} = \sum_j a_{ij}\, \tau_{ij}^{1-\sigma} \left(\frac{w_i}{A_i}\right)^{1-\sigma} \cdot X_j P_j^{\sigma-1}$$

$$\Rightarrow \left(\frac{w_i}{A_i}\right)^{1-\sigma} = Y_i \Big/ \sum_j a_{ij}\, \tau_{ij}^{1-\sigma} X_j P_j^{\sigma-1} = \Pi_i^{1-\sigma}$$

$$\Rightarrow X_{ij}(\omega) = a_{ij}\, \tau_{ij}^{1-\sigma} \left(\frac{Y_i}{\Pi_i^{1-\sigma}}\right) \cdot \left(\frac{X_j}{P_j^{1-\sigma}}\right)$$

Welfare

$$\lambda_{ij} \equiv \frac{X_{ij}}{\sum_k X_{kj}} = \frac{a_{ij}\, t_{ij}^{1-\sigma} \cdot \left(\frac{w_i}{A_i}\right)^{1-\sigma}}{\sum_k a_{kj} \tau_{kj}^{1-\sigma} \left(\frac{w_k}{A_k}\right)^{1-\sigma}}$$

$$= \frac{a_{ij}\, t_{ij}^{1-\sigma} \left(\frac{w_i}{A_i}\right)^{1-\sigma}}{P_j^{1-\sigma}}$$

$$\lambda_{jj} = a_{jj} \left(\frac{w_j}{A_j}\right)^{1-\sigma} \Big/ P_j^{1-\sigma}$$

$$1/P_j = \lambda_{jj}^{\frac{1}{1-\sigma}} \cdot a_{jj}^{\frac{1}{\sigma-1}} \cdot \frac{A_j}{w_j}$$

$$U_j = \underset{\underset{W_j}{\parallel}}{\frac{X_j}{P_j}} = \frac{X_j}{w_j} \cdot \lambda_{jj}^{\frac{1}{1-\sigma}}\, a_{jj}^{\frac{1}{\sigma-1}} \cdot A_j$$

MC + Homog. Firms + CES demand.

Consumers:

$$U_j = \left(\sum_{i \in S} \int_{\Omega_i} q_{ij}(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \right)^{\frac{\sigma}{\sigma-1}}$$

demand:

$$q_{ij}(\omega) = p_{ij}(\omega)^{-\sigma} X_j P_j^{\sigma-1}$$

spending : $x_{ij}(\omega) = p_{ij}(\omega)^{1-\sigma} X_j P_j^{\sigma-1}$

$$P_j \equiv \left(\sum_{i \in S} \int_{\Omega_i} p_{ij}(\omega)^{1-\sigma} d\omega \right)^{\frac{1}{1-\sigma}}$$

$$X_{ij} = \int_{\Omega_i} x_{ij}(\omega) d\omega = X_j P_j^{1-\sigma} \int p_{ij}(\omega)^{1-\sigma} d\omega.$$

Firms:

$$\max_{\{p_j(\omega)\}_{j\in S}} \sum_{j\in S} \left(p_j(\omega) q_j(\omega) - w_i \frac{\tau_{ij}}{z_i} q_j(\omega) \right) - w_i f_i^e$$

$$\text{s.t. } q_j(\omega) = p_j(\omega)^{-\sigma} X_j P_j^{\sigma-1}$$

$$\max \sum_{j\in S} \left(p_j(\omega)^{1-\sigma} X_j P_j^{\sigma-1} - w_i \frac{\tau_{ij}}{z_i} p_j(\omega)^{-\sigma} X_j \cdot P_j^{\sigma-1} \right) - w_i f_i^e$$

$$p_{ij}(\omega) = \frac{\sigma}{\sigma-1} \frac{\tau_{ij} w_i}{z_i}$$

Without export entry costs, all firms will export in this model.

Gravity

$$X_{ij} = X_j P_j^{1-\sigma} \int_{\Omega_i} \left(\frac{\sigma}{\sigma-1} \frac{\tau_{ij} w_i}{z_i} \right)^{1-\sigma} d\omega$$

$$= \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \tau_{ij}^{1-\sigma} \left(\frac{w_i}{z_i}\right)^{1-\sigma} N_i X_j P_j^{1-\sigma}$$

Welfare

$$P_j^{1-\sigma} \equiv \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \sum_k \tau_{kj}^{1-\sigma} \left(\frac{w_k}{z_k}\right)^{1-\sigma} N_k$$

$$\lambda_{ij} \equiv \frac{X_{ij}}{\sum_k X_{kj}} = \frac{\tau_{ij}^{1-\sigma} \left(\frac{w_i}{z_i}\right)^{1-\sigma} N_i}{\sum \tau_{kj}^{1-\sigma} \left(\frac{w_k}{z_k}\right)^{1-\sigma} N_k}$$

$$= \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \frac{\tau_{ij}^{1-\sigma} \left(\frac{w_i}{z_i}\right)^{1-\sigma} N_i}{P_j^{1-\sigma}}$$

$$P_j = \frac{\sigma}{\sigma-1} \cdot \tau_{ij} \left(\frac{w_i}{z_i}\right) \cdot N_i^{\frac{1}{1-\sigma}} \cdot \lambda_{ij}^{\frac{1}{\sigma-1}}$$

$$u_j \neq \frac{w_j}{P_j} = \frac{\sigma-1}{\sigma} \cdot z_j \cdot N_j^{\frac{1}{\sigma-1}} \cdot \lambda_{jj}^{\frac{1}{1-\sigma}}$$

$$X_j \neq w_j L_j \qquad X_j = w_j L_j + \Pi_j$$

Free Entry Condition

So far, firms only differ in their products. No production heterogeneity.

Ricardian model

perfect competition. same goods, but different technology

2-goods (w), 2-country (i)

$$y^i(w) = z^i(w)\, l^i(w) \qquad i, w = 1, 2.$$

$$z^2(1) < z^1(1) \quad , \quad z^2(2) < z^1(2)$$

$$\underbrace{\frac{z^1(1)}{z^1(2)}}_{\substack{\text{CA of} \\ \text{cty 1 in 1}}} > \underbrace{\frac{z^2(1)}{z^2(2)}}_{\substack{\text{CA of} \\ \text{cty 2 in 1}}} \quad \text{CA in 1}$$

Consumer:

$$\max a(1) \log c^i(1) + a(2) \log c^i(2)$$

$$\text{s.t.} \quad p(1) c^i(1) + p(2) c^i(2) \le w^i \bar{l}^i$$

$\Rightarrow$

$$\frac{a(1)}{c^i(1)} = \lambda p(1) \qquad \frac{a(2)}{c^i(2)} = \lambda p(2)$$

$$p(1) c^i(1) \frac{1}{a(1)} = p(2) c^i(2) \cdot \frac{1}{a(2)}$$

$$\begin{cases} p(2) c^i(2) = \dfrac{a(2)}{a(1)} p(1) c^i(1) \\ p(1) c^i(1) + p(2) c^i(2) = w^i \bar{l}^i \end{cases}$$

Autarky:

perfect competition $\quad p^i(1) = \dfrac{w^i}{z^i(1)}$

$$p^i(2) = \frac{w^i}{z^i(2)}$$

country size plays no role here.

goods market clearing $y^i(w) = c^i(w)$

labor market clearing $l^i(1) + l^i(2) = \bar{l}^i$

$$\frac{w^i}{z^i(1)} c^i(1) \cdot \frac{a(2)}{a(1)} = \frac{w^i}{z^i(2)} \cdot c^i(2)$$

$$\frac{c^i(1)}{z^i(1)} \cdot \frac{a(2)}{a(1)} = \frac{c^i(2)}{z^i(2)}$$

$$\frac{y^i(1)}{z^i(1)} \cdot \frac{a(2)}{a(1)} = \frac{y^i(2)}{z^i(2)} \qquad \frac{z^i(1)\, l^i(1)}{z^i(1)} \frac{a(2)}{a(1)} = \frac{z^i(2)}{z^i(2)} \cdot l^i(2)$$

$$\frac{l^i(1)}{l^i(2)} = \frac{a(1)}{a(2)} \Rightarrow l^i(1) = \frac{a(1)}{a(1)+a(2)} \bar{l}^i$$

normalize $w^i = 1$, get all the allocations.

Free Trade

If no specilization, $p(1)$, $p(2)$

CM: $p(1) = \frac{w^1}{z^1(1)} = \frac{w^2}{z^2(1)}$, $p(2) = \frac{w^1}{z^1(2)} = \frac{w^2}{z^2(2)}$

$$\frac{z^1(1)}{z^2(1)} = \frac{w^1}{w^2}, \quad \frac{z^2(2)}{z^1(2)} = \frac{w^2}{w^1} \Rightarrow \frac{z^1(1)}{z^2(1)} = \frac{z^1(2)}{z^2(2)}$$

$$\Rightarrow \frac{z^1(1)}{z^1(2)} = \frac{z^2(1)}{z^2(2)} \quad \otimes$$

$$p(1) = \frac{w^1}{z^1(1)} < \frac{w^2}{z^2(1)} \qquad \frac{w^1}{z^1(2)} > \frac{w^2}{z^2(2)} = p(2)$$

$$\frac{z^2(1)}{z^1(1)} < \frac{w^2}{w^1} < \frac{z^2(2)}{z^1(2)}$$

incomplete specialization country sets the price

$$\frac{p(1)\, z^i(1)}{p(2)\, z^i(2)} = \frac{w^i}{w^i} = 1 . \qquad \frac{p(1)}{p(2)} = \frac{z^i(2)}{z^i(1)}$$

relative wage is set by common product.

PS.

1.

$$\max_{\{q(w)\}} U = \left(\int_\Omega q(w)^{\frac{\sigma-1}{\sigma}} dw \right)^{\frac{\sigma}{\sigma-1}}$$

s.t.

$$\int_\Omega p(w)\, q(w)\, dw \le X \qquad || \ \lambda$$

$$\mathcal{L} = \left(\int_\Omega q(w)^{\frac{\sigma-1}{\sigma}} dw \right)^{\frac{\sigma}{\sigma-1}} + \lambda \left(X - \int_\Omega p(w) q(w) dw \right)$$

$$\left(\int_\Omega q(w)^{\frac{\sigma-1}{\sigma}} dw \right)^{\frac{1}{\sigma-1}} \cdot q(w)^{-\frac{1}{\sigma}} = \lambda p(w)$$

$$\left(\frac{q(w)}{q(w')} \right)^{-\frac{1}{\sigma}} = \frac{p(w)}{p(w')}$$

$$\frac{q(w)}{q(w')} = \left(\frac{p(w')}{p(w)} \right)^{\sigma}$$

$$p(w')^{\sigma} \cdot q(w') = p(w)^{\sigma} \cdot q(w)$$

$$p(w') \cdot q(w') = p(w)^{\sigma} \cdot q(w) \cdot p(w')^{1-\sigma}$$

$$X = \int_{\Omega} p(w') q(w') dw' = p(w)^{\sigma} \cdot q(w) \cdot \int_{\Omega} p(w')^{1-\sigma} dw'$$

$$= p(w)^{\sigma} \cdot q(w) \cdot P^{1-\sigma}$$

$$q(w) = p(w)^{-\sigma} \cdot X \cdot P^{\sigma-1}$$

a) $P = \left(\int_{\Omega} p(w)^{1-\sigma} dw \right)^{\frac{1}{1-\sigma}}$

$$U = \left(\int_{\Omega} \left(p(w)^{-\sigma} \cdot X \cdot P^{\sigma-1} \right)^{\frac{\sigma-1}{\sigma}} dw \right)^{\frac{\sigma}{\sigma-1}}$$

$$= X \cdot P^{\sigma-1} \left(\int_{\Omega} p(w)^{1-\sigma} dw \right)^{\frac{\sigma}{\sigma-1}}$$

$$= X \cdot P^{\sigma-1} \cdot P^{-\sigma} = X \cdot P^{-1}$$

$$U = \frac{X}{P}$$

b) $q(w) = p(w)^{-\sigma} \cdot X \cdot P^{\sigma-1}$.

c) $\frac{\partial U}{\partial q(w)} = \lambda \cdot p(w)$ $\qquad \frac{\partial U / \partial q(w)}{\partial U / \partial q(w')} = \frac{p(w)}{p(w')}$

$a = b^{-\sigma}$

$$\frac{q(w)}{q(w')} = \left(\frac{p(w)}{p(w')}\right)^{-\sigma}$$

$$\frac{\partial\left(q(w)/q(w')\right)}{\partial\left(p(w)/p(w')\right)} = -\sigma \cdot \left(\frac{p(w)}{p(w')}\right)^{-\sigma-1}$$

$$\left(\frac{p(w)}{p(w')}\right)^{1+\sigma}$$

$$//$$

$$\frac{\partial \ln\left(q(w)/q(w')\right)}{\partial \ln\left(p(w)/p(w')\right)} = -\sigma \left(\frac{p(w)}{p(w')}\right)^{-\sigma-1} \cdot \frac{\frac{p(w)}{p(w')}}{q(w)/q(w')}$$

$$= -\sigma$$

$$\Leftrightarrow$$

$$\frac{\partial \ln\left(q(w)/q(w')\right)}{\partial \ln\left(\frac{\partial u/\partial q(w')}{\partial u/\partial q(w)}\right)} = \sigma$$

% in $\frac{p(w)}{p(w')}$ induces % in $\frac{q(w')}{q(w)}$.

Extension:

income elasticity:

$$\frac{\partial \ln(q(w))}{\partial \ln X} = \frac{\partial q(w)}{\partial X} \cdot \frac{X}{q(w)} = \frac{p(w)^{-\sigma} \cdot P^{\sigma-1} \cdot X}{p(w)^{-\sigma} \cdot P^{\sigma-1} \cdot X} = 1$$

d)

Case I : $\sigma \to \infty$ $\qquad \frac{\sigma-1}{\sigma} \to 1$

$$\lim_{\sigma\to\infty} U = \lim_{\sigma\to\infty} \int_\Omega q \, dw$$

linear case completely substitutable

Case II : $\sigma \to 1$, $\qquad \frac{\sigma-1}{\sigma} \to 0$

$$U = \left(\int q(w)^{\frac{\sigma-1}{\sigma}} dw\right)^{\frac{\sigma}{\sigma-1}}$$

$$\ln U = \frac{\sigma}{\sigma-1} \ln\left(\int q(w)^{\frac{\sigma-1}{\sigma}} dw\right)$$

$$\lim_{\sigma\to 1} \ln U = \lim_{\sigma\to 1} \frac{\sigma}{\sigma-1} \ln\left(\int q(w)^{\frac{\sigma-1}{\sigma}} dw\right)$$

$$\overset{\text{L'Hôpital}}{=} \lim_{\sigma\to 1} \frac{\ln\left(\int q(w)^{\frac{\sigma-1}{\sigma}} dw\right) + \sigma \dfrac{\int q(w)^{\frac{\sigma-1}{\sigma}} \ln q(w) \cdot \frac{1}{\sigma^2} dw}{\int q(w)^{\frac{\sigma-1}{\sigma}} dw}}{1}$$

$$= \lim_{\sigma\to 1} \left\{ \ln\left[\int q(w)^{\frac{\sigma-1}{\sigma}} dw\right] + \frac{\int q(w)^{\frac{\sigma-1}{\sigma}} \cdot \ln q(w) \, dw}{\sigma \int q(w)^{\frac{\sigma-1}{\sigma}} dw} \right\}$$

$$y = a^x$$
$$\ln y = x \ln a.$$
$$\frac{1}{y} dy = \ln a \, dx$$
$$\frac{dy}{dx} = y \ln a = a^x \ln a$$

$$= \ln 1 + \frac{\int \ln q(w) \, dw}{\sigma} = \int \ln q(w) \, dw$$

$$= \ln \prod_w q(w) \leftarrow \text{C-D}$$

case III : $\sigma \to 0$ $\quad \frac{\sigma-1}{\sigma} \to -\infty$

no substitution

$$\ln U = \frac{\sigma}{\sigma-1} \ln \left(\int q(w)^{\frac{\sigma-1}{\sigma}} dw \right)$$

$0 \cdot (-\infty)$

$$= \frac{\ln \left(\int q(w)^{\frac{\sigma-1}{\sigma}} dw \right)}{\frac{\sigma-1}{\sigma}}$$

$$\lim_{\frac{\sigma-1}{\sigma} \to -\infty} \ln U = \lim_{\rho \to -\infty} \frac{\ln \left(\int q(w)^{\rho} dw \right)}{\rho}$$

$\frac{0}{0}$

$$= \lim_{\rho \to -\infty} \frac{\int q(w)^{\rho} \cdot \ln q(w) \, dw}{\int q(w)^{\rho} dw}$$

define $q_{min} = \min \{ q(1), q(2), \dots q(w) \}$.

$$\lim_{\rho \to -\infty} \frac{\int \left(\frac{q(w)}{q_{min}} \right)^{\rho} \ln q(w) \, dw.}{\int \left(\frac{q(w)}{q_{min}} \right)^{\rho} dw}$$

$$= \frac{\ln q_{min}}{1} = \ln \left(\min \{ q(1), q(2), \dots q(w) \} \right)$$

$$\frac{q(w)}{q_{min}} \begin{cases} = 1 & \text{if } q(w) = q_{min} \\ > 1 & \text{if } q(w) \neq q_{min}. \end{cases}$$

$$\Rightarrow U = \min \{ q(1), q(2), \dots \}.$$

Leontif.