

STAD29 / STA 1007

Statistics for the Life and Social Sciences

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December 14, 2018

Section 1

Course Outline

Course and instructor

- ▶ Lecture: Wednesday 14:00-16:00 in HW 215. Optional computer lab Monday 16:00-17:00 in BV 498.
- ▶ Instructor: Ken Butler
- ▶ Office: IC 471.
- ▶ E-mail: butler@utsc.utoronto.ca
- ▶ Office hours: Monday 11:00-13:00. Also, Wednesday mornings good. I am often around. See if I'm in. Or make an appointment. E-mail always good.
- ▶ Course website: www.utsc.utoronto.ca/~butler/d29.
- ▶ Using Blackboard for assignments/grades only; using website for everything else.

Text, programs, prerequisites and exclusions

- ▶ There is no official text for this course. You may find <http://r4ds.had.co.nz/> helpful for R background.
- ▶ Prerequisites:
 - ▶ For undergrads: STAC32. Not negotiable.
 - ▶ For grad students, a first course in statistics, and some training in regression and ANOVA. The less you know, the more you'll have to catch up!
- ▶ This course is part of Applied Statistics minor.
- ▶ Exclusions: **this course is not for Math/Statistics/CS majors/minors.** It is for students in other fields who wish to learn some more advanced statistical methods. The exclusions in the Calendar reflect this.
- ▶ If you are in one of those programs, you won't get program credit for this course, **or for any future STA courses you take.**

Computing

- ▶ Computing: big part of the course, **not** optional. Demonstrate that you can use R to analyze data, and can critically interpret the output.
- ▶ For grad students who have not come through STAC32, I am happy to offer extra help to get you up to speed.

Computing and assessment

- ▶ Grading: (2 hour) midterm, (3 hour) final exam. Assignments most weeks, due Tuesday at 11:59pm. Graduate students (STA 1007) also required to complete a project using one or more of the techniques learned in class, on a dataset from their field of study. Projects due on the last day of classes.
- ▶ Assessment:

	STAD29	STA 1007
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Assignments	20%	20%
Midterm exam	30%	20%
Project	-	20%
Final exam	50%	40%

- ▶ Assessments missed *with documentation* will cause a re-weighting of other assessments of same type. No make-ups.
- ▶ You **must pass the final exam** to pass the course. If you fail the final exam but would otherwise have passed the course, you receive a grade of 45.

Plagiarism

- ▶ <http://www.utoronto.ca/academicintegrity/academicoffenses.html> defines academic offences at this university. Read it.
- ▶ Plagiarism is defined (at the end) as
The wrongful appropriation and purloining, and publication as one's own, of the ideas, or the expression of the ideas ... of another.
- ▶ The code and explanations that you write and hand in must be *yours and yours alone*.
- ▶ When you hand in work, it is implied that it is *your* work. Handing in work, with your name on it, that was actually done by someone else is an *academic offence*.
- ▶ If I am suspicious that anyone's work is plagiarized, I will take action.

Getting help

- ▶ The English Language Development Centre supports all students in developing better Academic English and critical thinking skills needed in academic communication. Make use of the personalized support in academic writing skills development. Details and sign-up information: <http://www.utsc.utoronto.ca/eld/>.
- ▶ Students with diverse learning styles and needs are welcome in this course. In particular, if you have a disability/health consideration that may require accommodations, please feel free to approach the AccessAbility Services Office as soon as possible. I will work with you and AccessAbility Services to ensure you can achieve your learning goals in this course. Enquiries are confidential. The UTSC AccessAbility Services staff are available by appointment to assess specific needs, provide referrals and arrange appropriate accommodations: (416) 287-7560 or by e-mail: ability@utsc.utoronto.ca.

What we (might) cover, part 1

Review of (multiple) regression

Logistic regression (ordinal/nominal response)

Survival analysis

Analysis of variance

Analysis of covariance

Multivariate ANOVA

Repeated measures by profile analysis

Discriminant analysis

Cluster analysis

What we (might) cover, part 2

Multidimensional scaling

Principal components

Exploratory factor analysis

Confirmatory factor analysis

Multiway frequency tables

Section 2

Review of (multiple) regression

Regression

- ▶ Use regression when one variable is an outcome (*response*, y).
- ▶ See if/how response depends on other variable(s), *explanatory*, x_1, x_2, \dots
- ▶ Can have *one or more than one* explanatory variable, but always one response.
- ▶ Assumes a *straight-line* relationship between response and explanatory.
- ▶ Ask:
 - ▶ *is there* a relationship between y and x 's, and if so, which ones?
 - ▶ what does the relationship look like?

Packages

```
library(MASS) # for Box-Cox, later
library(tidyverse)

## -- Attaching packages -----
tidyverse 1.2.1 --

## v ggplot2 3.1.0     v purrrr  0.2.5
## v tibble   1.4.2    v dplyr    0.7.8
## v tidyverse 0.8.1    v stringr  1.3.1
## v readr    1.1.1    v forcats  0.3.0

## -- Conflicts -----
tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()
## x dplyr::select() masks MASS::select()

library(broom)
```

A regression with one x

13 children, measure average total sleep time (ATST, mins) and age (years) for each. See if ATST depends on age. Data in `sleep.txt`, ATST then age. Read in data:

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/sleep.txt"
sleep=read_delim(my_url, " ")
## Parsed with column specification:
## cols(
##   atst = col_double(),
##   age = col_double()
## )
```

Check data

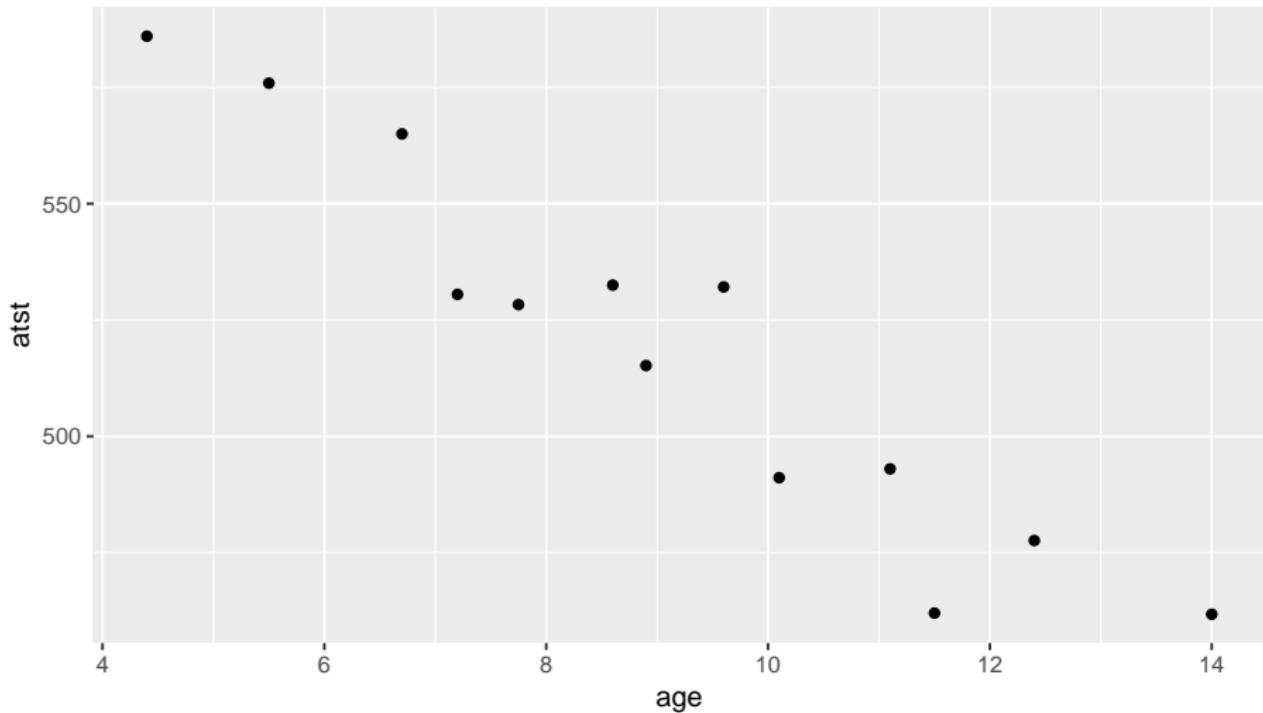
```
sleep

## # A tibble: 13 x 2
##       atst     age
##   <dbl> <dbl>
## 1    586    4.4
## 2    462.   14
## 3    491.   10.1
## 4    565    6.7
## 5    462    11.5
## 6    532.   9.6
## 7    478.   12.4
## 8    515.   8.9
## 9    493    11.1
## 10   528.   7.75
## 11   576.   5.5
## 12   532.   8.6
## 13   530.   7.2
```

and make scatter plot of ATST (response) vs. age (explanatory) using code overleaf:

The scatterplot

```
ggplot(sleep, aes(x=age, y=atst)) + geom_point()
```



Correlation

- ▶ Measures how well a straight line fits the data:

```
with(sleep, cor(atst, age))  
## [1] -0.9515469
```

- ▶ 1 is perfect upward trend, -1 is perfect downward trend, 0 is no trend.
- ▶ This one close to perfect downward trend.
- ▶ Can do correlations of whole data frame:

```
cor(sleep)  
##           atst          age  
## atst  1.0000000 -0.9515469  
## age   -0.9515469  1.0000000
```

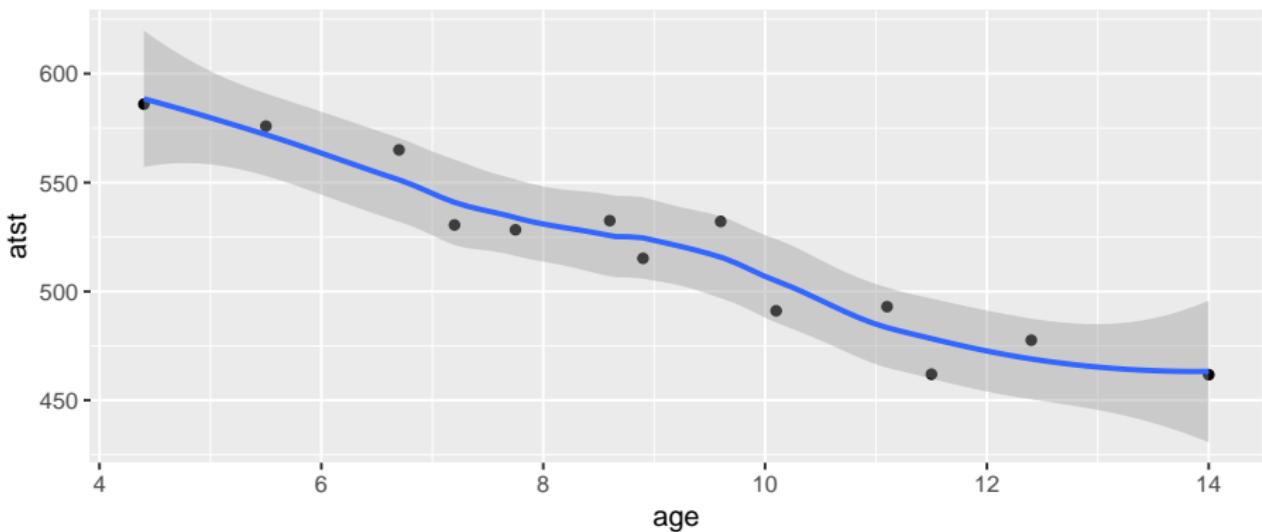
- ▶ Correlations of all possible pairs of variables.

Lowess curve

- ▶ Sometimes nice to guide the eye: is the trend straight, or not?
- ▶ Idea: *lowess curve*. “Locally weighted least squares”, not affected by outliers, not constrained to be linear.
- ▶ Lowess is a *guide*: even if straight line appropriate, may wiggle/bend a little. Looking for *serious* problems with linearity.
- ▶ Add lowess curve to plot using `geom_smooth`:

Plot with lowess curve

```
ggplot(sleep, aes(x=age, y=atst)) + geom_point() +  
  geom_smooth()  
  
## `geom_smooth()` using method = 'loess' and formula 'y ~  
x'
```



The regression

Scatterplot shows no obvious curve, and a pretty clear downward trend.

So we can run the regression:

```
sleep.1=lm(atst~age,data=sleep) ; summary(sleep.1)

##
## Call:
## lm(formula = atst ~ age, data = sleep)
##
## Residuals:
##     Min      1Q  Median      3Q     Max 
## -23.011   -9.365    2.372   6.770   20.411 
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)    
## (Intercept) 646.483    12.918   50.05 2.49e-14 ***
## age         -14.041     1.368  -10.26 5.70e-07 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1 
##
## Residual standard error: 13.15 on 11 degrees of freedom
## Multiple R-squared:  0.9054, Adjusted R-squared:  0.8968 
## F-statistic: 105.3 on 1 and 11 DF,  p-value: 5.7e-07
```

Conclusions

- ▶ The relationship appears to be a straight line, with a downward trend.
- ▶ F -tests for model as a whole and t -test for slope (same) both confirm this (P-value $5.7 \times 10^{-7} = 0.00000057$).
- ▶ Slope is -14 , so a 1-year increase in age goes with a 14-minute decrease in ATST on average.
- ▶ R-squared is correlation squared (when one x anyway), between 0 and 1 (1 good, 0 bad).
- ▶ Here R-squared is 0.9054, pleasantly high.

Doing things with the regression output

- ▶ Output from regression (and eg. t -test) is all right to look at, but hard to extract and re-use information from.
- ▶ Package `broom` extracts info from model output in way that can be used in pipe (later):

```
tidy(sleep.1)

## # A tibble: 2 x 5
##   term      estimate std.error statistic p.value
##   <chr>     <dbl>     <dbl>     <dbl>     <dbl>
## 1 (Intercept) 646.      12.9      50.0  2.49e-14
## 2 age        -14.0      1.37     -10.3  5.70e- 7

glance(sleep.1)

## # A tibble: 1 x 11
##   r.squared adj.r.squared sigma statistic p.value    df
##   <dbl>       <dbl>     <dbl>     <dbl>     <dbl> <int>
## 1 0.905       0.897     13.2      105.  5.70e-7    2
## # ... with 5 more variables: logLik <dbl>, AIC <dbl>,
## #   BIC <dbl>, deviance <dbl>, df.residual <int>
```

Broom part 2

```
augment(sleep.1) %>% slice(1:8)

## # A tibble: 8 x 9
##   atst    age .fitted .se.fit .resid   .hat .sigma .cooksdi
##   <dbl> <dbl>   <dbl>   <dbl>   <dbl>   <dbl>   <dbl>
## 1  586     4.4    585.    7.34    1.30  0.312    13.8  0.00320
## 2  462.    14     450.    7.68   11.8   0.341    13.0  0.319
## 3  491.    10.1   505.    3.92  -13.6   0.0887   13.0  0.0568
## 4  565     6.7    552.    4.87   12.6   0.137    13.1  0.0844
## 5  462     11.5   485.    4.95  -23.0   0.141    11.3  0.294
## 6  532.    9.6    512.    3.72   20.4   0.0801   12.0  0.114
## 7  478.    12.4   472.    5.85   5.23   0.198    13.7  0.0243
## 8  515.    8.9    522.    3.65  -6.32   0.0772   13.6  0.0105
## # ... with 1 more variable: .std.resid <dbl>
```

Useful for plotting residuals against an x -variable.

CI for mean response and prediction intervals

Once useful regression exists, use it for prediction:

- ▶ To get a single number for prediction at a given x , substitute into regression equation, eg. age 10: predicted ATST is $646.48 - 14.04(10) = 506$ minutes.
- ▶ To express uncertainty of this prediction:
 - ▶ *CI for mean response* expresses uncertainty about mean ATST for all children aged 10, based on data.
 - ▶ *Prediction interval* expresses uncertainty about predicted ATST for a new child aged 10 whose ATST not known. More uncertain.
- ▶ Also do above for a child aged 5.

Intervals

- ▶ Make new data frame with these values for age

```
my.age=c(10,5)
ages.new=tibble(age=my.age)
ages.new

## # A tibble: 2 x 1
##       age
##   <dbl>
## 1     10
## 2      5
```

- ▶ Feed into predict:

```
pc=predict(sleep.1,ages.new,interval="c")
pp=predict(sleep.1,ages.new,interval="p")
```

The intervals

Confidence intervals for mean response:

```
cbind(ages.new,pc)

##    age      fit      lwr      upr
## 1 10 506.0729 497.5574 514.5883
## 2 5 576.2781 561.6578 590.8984
```

Prediction intervals for new response:

```
cbind(ages.new,pp)

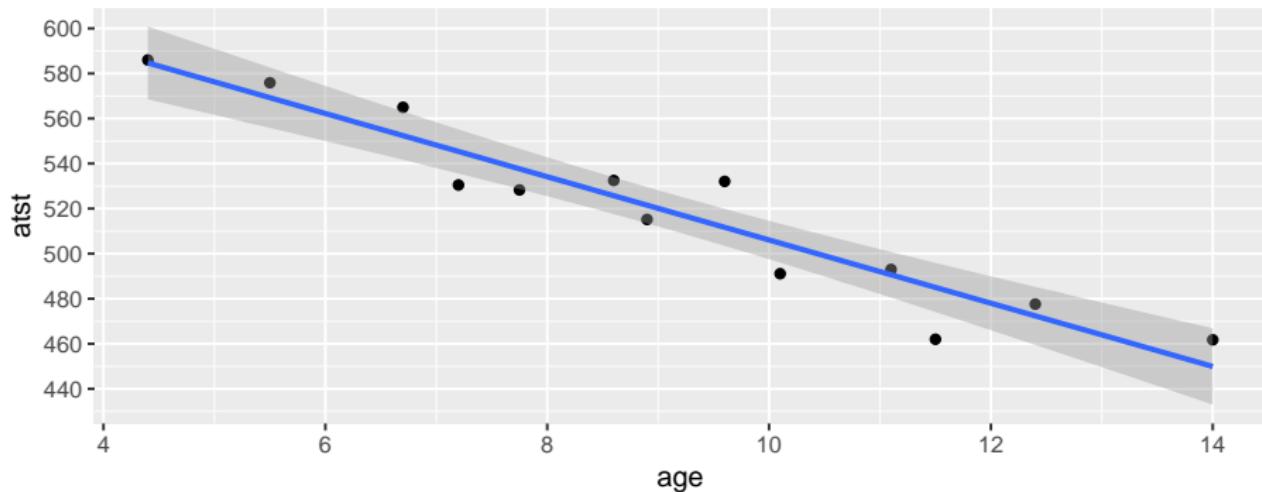
##    age      fit      lwr      upr
## 1 10 506.0729 475.8982 536.2475
## 2 5 576.2781 543.8474 608.7088
```

Comments

- ▶ Age 10 closer to centre of data, so intervals are both narrower than those for age 5.
- ▶ Prediction intervals bigger than CI for mean (additional uncertainty).
- ▶ Technical note: output from `predict` is R matrix, not data frame, so Tidyverse `bind_cols` does not work. Use base R `cbind`.

That grey envelope

```
ggplot(sleep, aes(x=age, y=atst)) + geom_point() +  
  geom_smooth(method="lm") +  
  scale_y_continuous(breaks=seq(420, 600, 20))
```



Marks confidence interval for mean for all x.

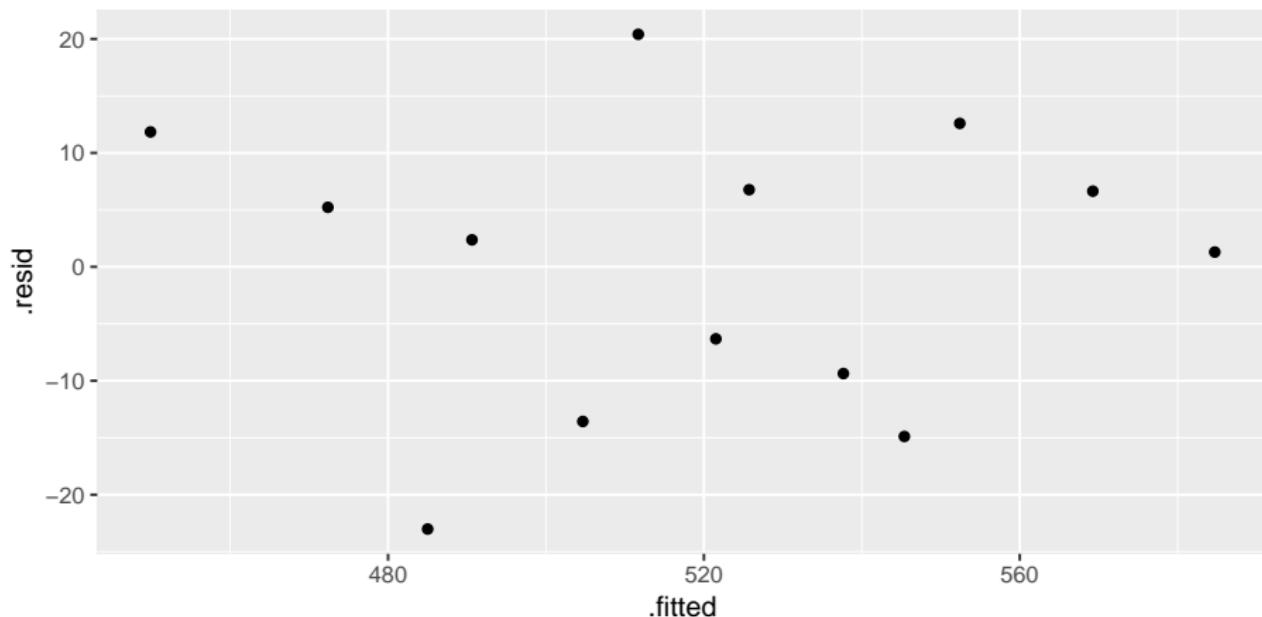
Diagnostics

How to tell whether a straight-line regression is appropriate?

- ▶ Before: check scatterplot for straight trend.
- ▶ After: plot *residuals* (observed minus predicted response) against predicted values. Aim: a plot with no pattern.

Output

```
ggplot(sleep.1,aes(x=.fitted,y=.resid))+geom_point()
```



Not much pattern here (is residual predictable from predicted? No).
Good, indicating regression appropriate.

An inappropriate regression

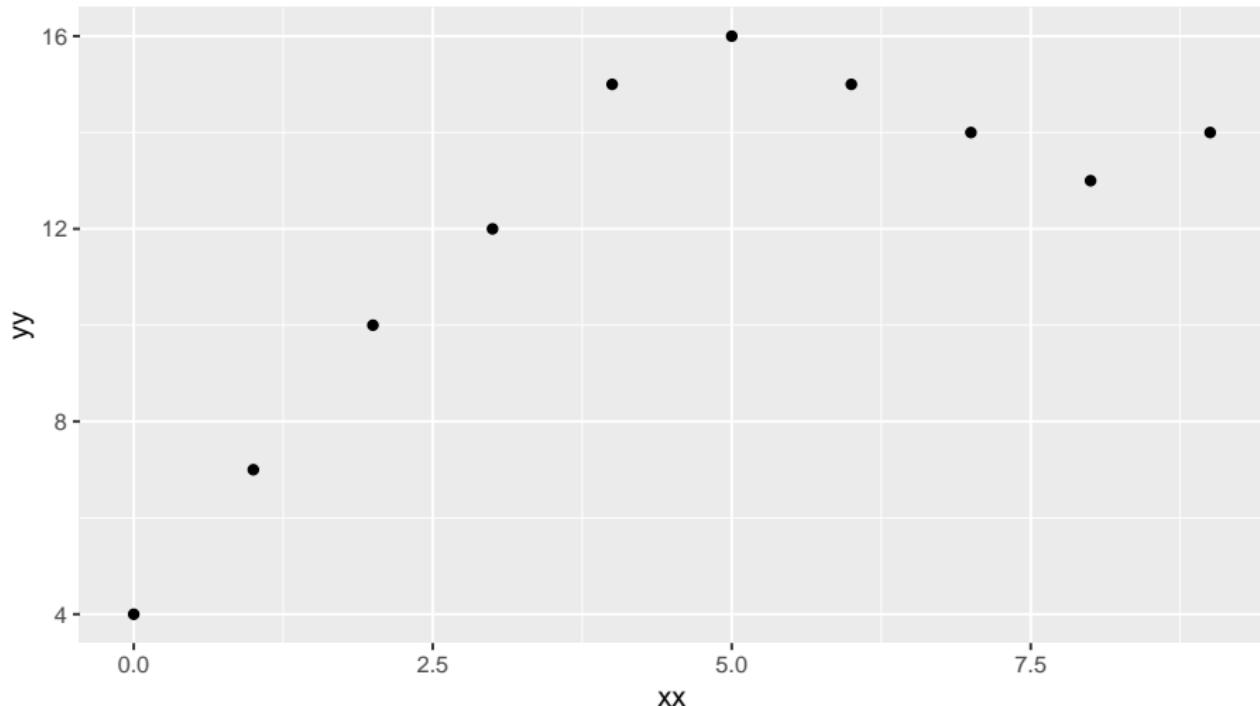
Different data:

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/curvy.txt"
curvy=read_delim(my_url, " ")

## Parsed with column specification:
## cols(
##   xx = col_integer(),
##   yy = col_integer()
## )
```

Scatterplot

```
ggplot(curvy, aes(x=xx, y=yy)) + geom_point()
```



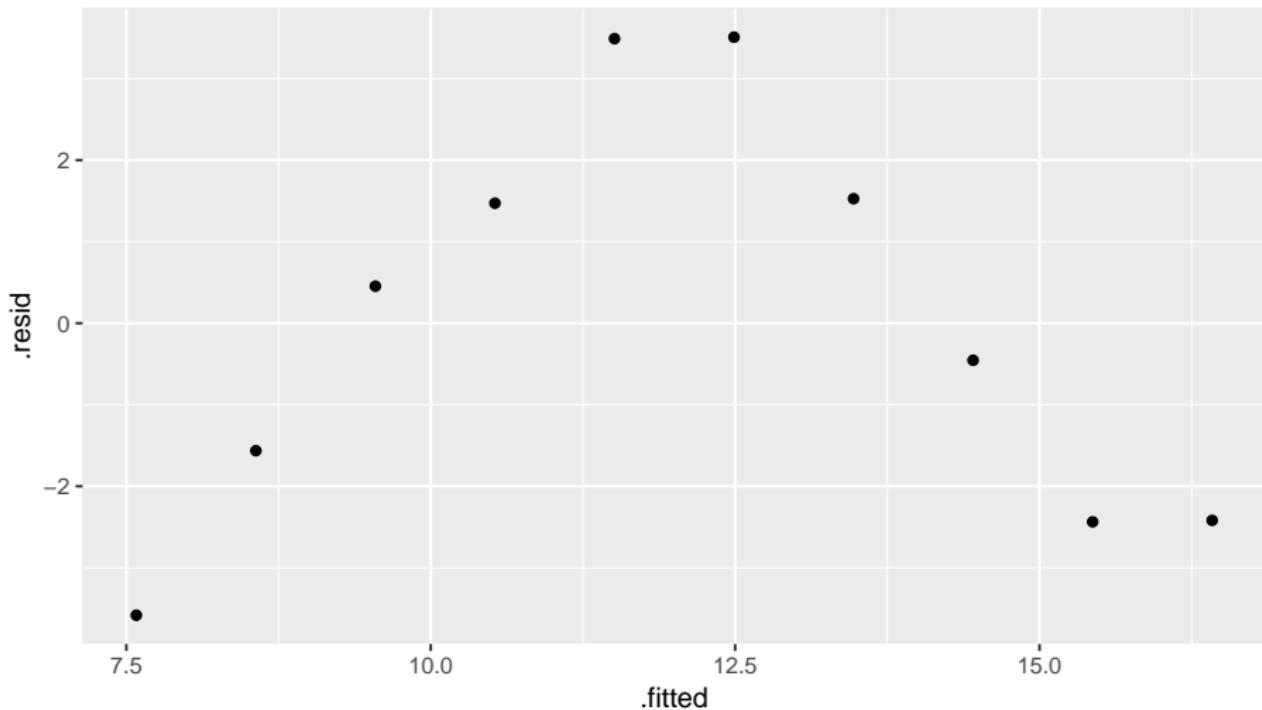
Regression line, anyway

```
curvy.1=lm(yy~xx,data=curvy) ; summary(curvy.1)

##
## Call:
## lm(formula = yy ~ xx, data = curvy)
##
## Residuals:
##     Min      1Q  Median      3Q     Max 
## -3.582 -2.204  0.000  1.514  3.509 
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)    
## (Intercept) 7.5818    1.5616   4.855  0.00126 ***
## xx          0.9818    0.2925   3.356  0.00998 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.657 on 8 degrees of freedom
## Multiple R-squared:  0.5848, Adjusted R-squared:  0.5329 
## F-statistic: 11.27 on 1 and 8 DF,  p-value: 0.009984
```

Residual plot

```
ggplot(curvy.1,aes(x=.fitted,y=.resid))+geom_point()
```



No good: fixing it up

- ▶ Residual plot has *curve*: middle residuals positive, high and low ones negative. Bad.
- ▶ Fitting a curve would be better. Try this:

```
curvy.2=lm(yy~xx+I(xx^2), data=curvy)
```

- ▶ Adding xx-squared term, to allow for curve.
- ▶ Another way to do same thing: specify how model *changes*:

```
curvy.2a=update(curvy.1, . ~ . + I(xx^2))
```

Regression 2

```
summary(curvy.2)

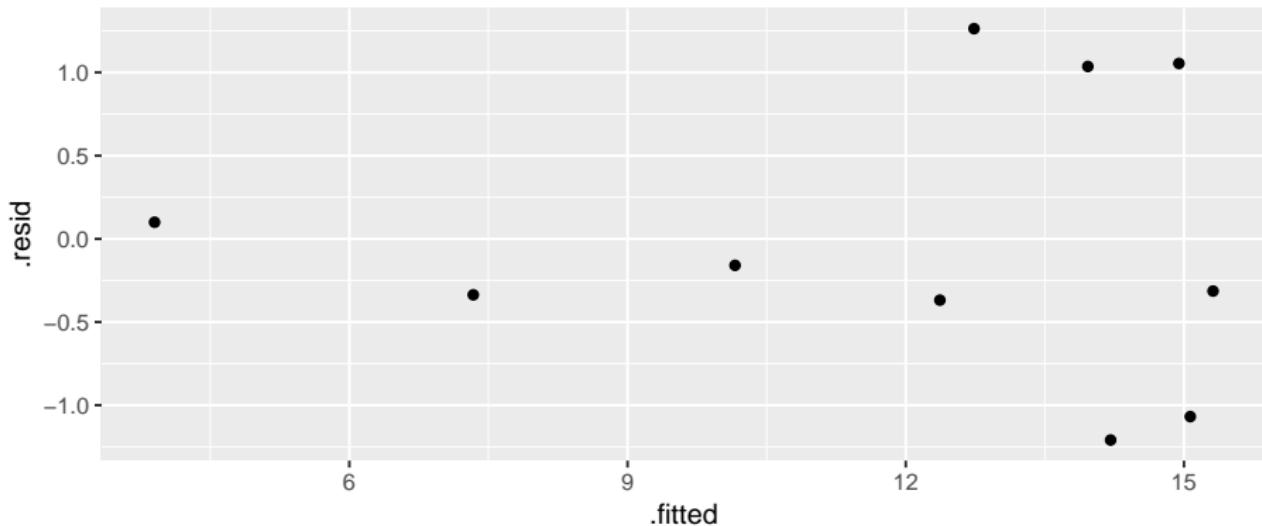
##
## Call:
## lm(formula = yy ~ xx + I(xx^2), data = curvy)
##
## Residuals:
##     Min      1Q  Median      3Q     Max 
## -1.2091 -0.3602 -0.2364  0.8023  1.2636 
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)    
## (Intercept) 3.90000   0.77312   5.045 0.001489 ***
## xx          3.74318   0.40006   9.357 3.31e-05 ***
## I(xx^2)    -0.30682   0.04279  -7.170 0.000182 ***  
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1 
##
## Residual standard error: 0.9833 on 7 degrees of freedom
## Multiple R-squared:  0.9502, Adjusted R-squared:  0.936 
## F-statistic: 66.83 on 2 and 7 DF,  p-value: 2.75e-05
```

Comments

- ▶ xx-squared term definitely significant (P-value 0.000182), so need this curve to describe relationship.
- ▶ Adding squared term has made R-squared go up from 0.5848 to 0.9502: great improvement.
- ▶ This is a definite curve!

The residual plot now

```
ggplot(curvy.2, aes(x=.fitted, y=.resid)) + geom_point()
```



No problems any more.

Another way to handle curves

- ▶ Above, saw that changing x (adding x^2) was a way of handling curved relationships.
- ▶ Another way: change y (transformation).
- ▶ Can guess how to change y , or might be theory:
 - ▶ example: relationship $y = ae^{bx}$ (exponential growth):
 - ▶ take logs to get $\ln y = \ln a + bx$.
 - ▶ Taking logs has made relationship linear ($\ln y$ as response).
- ▶ Or, *estimate* transformation, using Box-Cox method.

Box-Cox

- ▶ Install package MASS via `install.packages("MASS")` (only need to do *once*)
- ▶ Every R session you want to use something in MASS, type `library(MASS)`

Some made-up data

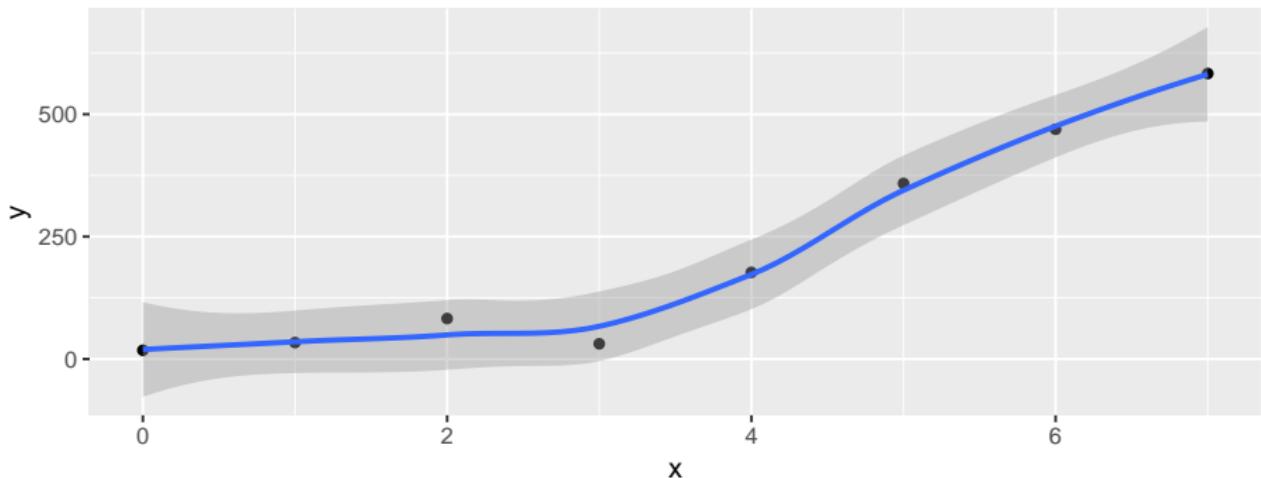
```
my_url="http://www.utsc.utoronto.ca/~butler/d29/madeup.csv"
madeup=read_csv(my_url)
madeup

## # A tibble: 8 x 3
##       row     x     y
##   <int> <int> <dbl>
## 1     1     0  17.9
## 2     2     1  33.6
## 3     3     2  82.7
## 4     4     3  31.2
## 5     5     4 177.
## 6     6     5 359.
## 7     7     6 469.
## 8     8     7 583.
```

Seems to be faster-than-linear growth, maybe exponential growth.
Scatterplot?

The scatterplot: faster than linear growth

```
ggplot(madeup, aes(x=x, y=y)) + geom_point() +  
  geom_smooth()  
  
## `geom_smooth()` using method = 'loess' and formula 'y ~  
x'
```

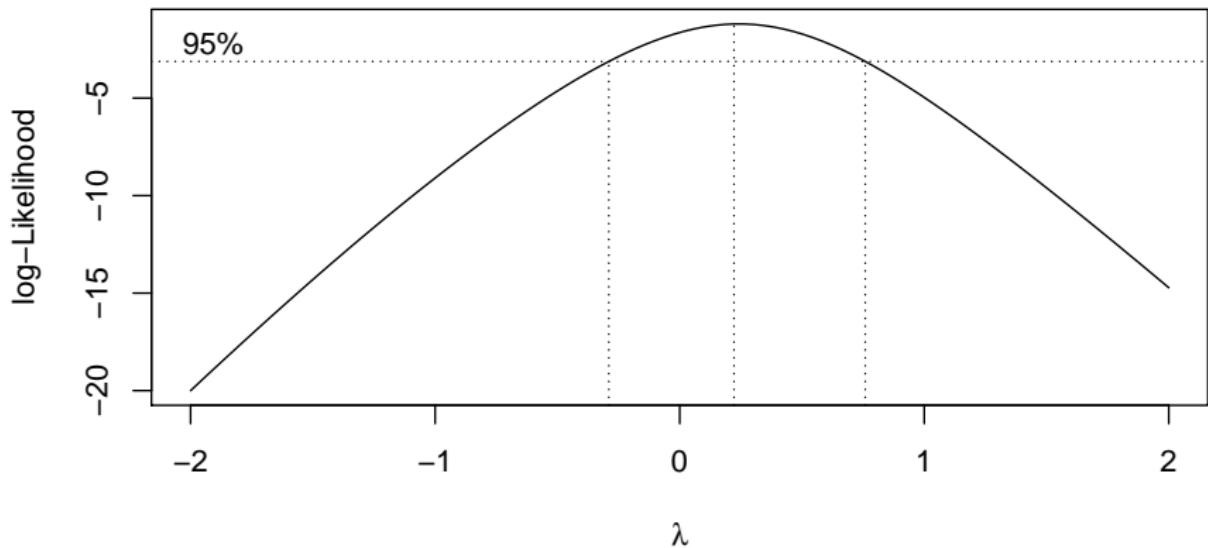


Running Box-Cox

- ▶ library(MASS) first.
- ▶ Feed boxcox a model formula with a squiggle in it, such as you would use for lm.
- ▶ Output: a graph (next page):

```
boxcox(y~x,data=madeup)
```

The Box-Cox output



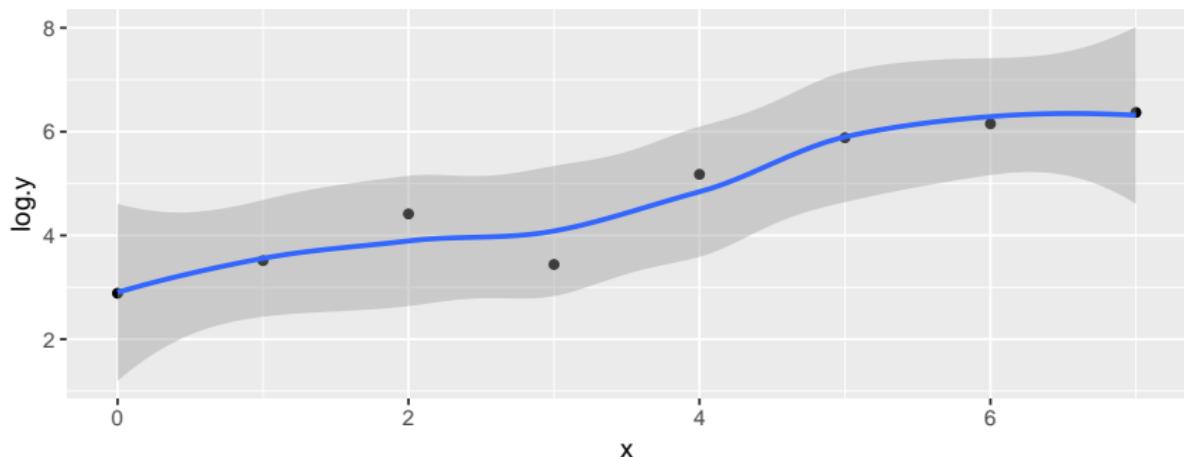
Comments

- ▶ λ (lambda) is the power by which you should transform y to get the relationship straight (straighter). Power 0 is “take logs”
- ▶ Middle dotted line marks best single value of λ (here about 0.1).
- ▶ Outer dotted lines mark 95% CI for λ , here -0.3 to 0.7 , approx.
(Rather uncertain about best transformation.)
- ▶ Any power transformation within the CI supported by data. In this case, log ($\lambda = 0$) and square root ($\lambda = 0.5$) good, but no transformation ($\lambda = 1$) not.
- ▶ Pick a “round-number” value of λ like $2, 1, 0.5, 0, -0.5, -1$. Here 0 and 0.5 good values to pick.

Did transformation straighten things?

- ▶ Calculate transformed y and plot against x . Here try log:

```
log.y=log(madeup$y)
ggplot(madeup,aes(x=x,y=log.y))+geom_point()+
  geom_smooth()
## `geom_smooth()` using method = 'loess' and formula
'y ~ x'
```



Multiple regression

- ▶ What if more than one x ? Extra issues:
 - ▶ Now one intercept and a slope for each x : how to interpret?
 - ▶ Which x -variables actually help to predict y ?
 - ▶ Different interpretations of “global” F -test and individual t -tests.
 - ▶ R-squared no longer correlation squared, but still interpreted as “higher better”.
- ▶ In `lm` line, add extra xs after `~`.
- ▶ Interpretation not so easy (and other problems that can occur).

Multiple regression example

Study of women and visits to health professionals, and how the number of visits might be related to other variables:

timedrs: number of visits to health professionals (over course of study)

phyheal: number of physical health problems

menheal: number of mental health problems

stress: result of questionnaire about number and type of life changes

timedrs response, others explanatory.

The data

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/regressx.txt"
visits=read_delim(my_url, " ")

## Parsed with column specification:
## cols(
##   subjno = col_integer(),
##   timedrs = col_integer(),
##   phyheal = col_integer(),
##   menheal = col_integer(),
##   stress = col_integer()
## )
```

Check data, fit multiple regression

```
visits
```

```
## # A tibble: 465 x 5
##   subjno timedrs phyheal menheal stress
##   <int>    <int>    <int>    <int>    <int>
## 1     1        1        5        8      265
## 2     2        2        3        4      415
## 3     3        3        0        3       92
## 4     4        4       13        2      241
## 5     5        5       15        3       86
## 6     6        6        3        5      247
## 7     7        7        2        5       13
## 8     8        8        0        4       12
## 9     9        9        7        5      269
## 10    10       10        4        3      391
## # ... with 455 more rows
```

```
visits.1=lm(timedrs~phyheal+menheal+stress,
            data=visits)
```

The regression

```
summary(visits.1)

##
## Call:
## lm(formula = timedrs ~ phyheal + menheal + stress, data = visits)
##
## Residuals:
##     Min      1Q  Median      3Q     Max 
## -14.792  -4.353  -1.815   0.902  65.886 
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)    
## (Intercept) -3.704848   1.124195  -3.296 0.001058 ** 
## phyheal      1.786948   0.221074   8.083 5.6e-15 *** 
## menheal     -0.009666   0.129029  -0.075 0.940318    
## stress       0.013615   0.003612   3.769 0.000185 *** 
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1 
##
## Residual standard error: 9.708 on 461 degrees of freedom
## Multiple R-squared:  0.2188, Adjusted R-squared:  0.2137 
## F-statistic: 43.03 on 3 and 461 DF,  p-value: < 2.2e-16
```

The slopes

Model as a whole strongly significant even though R-sq not very big (lots of data). At least one of the x's predicts `timedrs`.

```
tidy(visits.1)

## # A tibble: 4 x 5
##   term      estimate std.error statistic p.value
##   <chr>     <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept) -3.70     1.12     -3.30  1.06e- 3
## 2 phyheal      1.79     0.221     8.08  5.60e-15
## 3 menheal     -0.00967  0.129    -0.0749 9.40e- 1
## 4 stress       0.0136   0.00361    3.77  1.85e- 4
```

The physical health and stress variables definitely help to predict the number of visits, but *with those in the model* we don't need `menheal`.

However, look at prediction of `timedrs` from `menheal` by itself:

Just menheal

```
visits.2=lm(timedrs~menheal,data=visits) ; summary(visits.2)

##
## Call:
## lm(formula = timedrs ~ menheal, data = visits)
##
## Residuals:
##      Min       1Q   Median       3Q      Max 
## -13.826  -5.150  -2.818   1.177  72.513 
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)    
## (Intercept)  3.8159    0.8702   4.385 1.44e-05 ***
## menheal      0.6672    0.1173   5.688 2.28e-08 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 10.6 on 463 degrees of freedom
## Multiple R-squared:  0.06532, Adjusted R-squared:  0.0633 
## F-statistic: 32.35 on 1 and 463 DF,  p-value: 2.279e-08
```

menheal by itself

- ▶ menheal by itself *does* significantly help to predict timedrs.
- ▶ But the R-sq is much less (6.5% vs. 22%).
- ▶ So other two variables do a better job of prediction.
- ▶ With those variables in the regression (phyheal and stress), don't need menheal *as well*.

Investigating via correlation

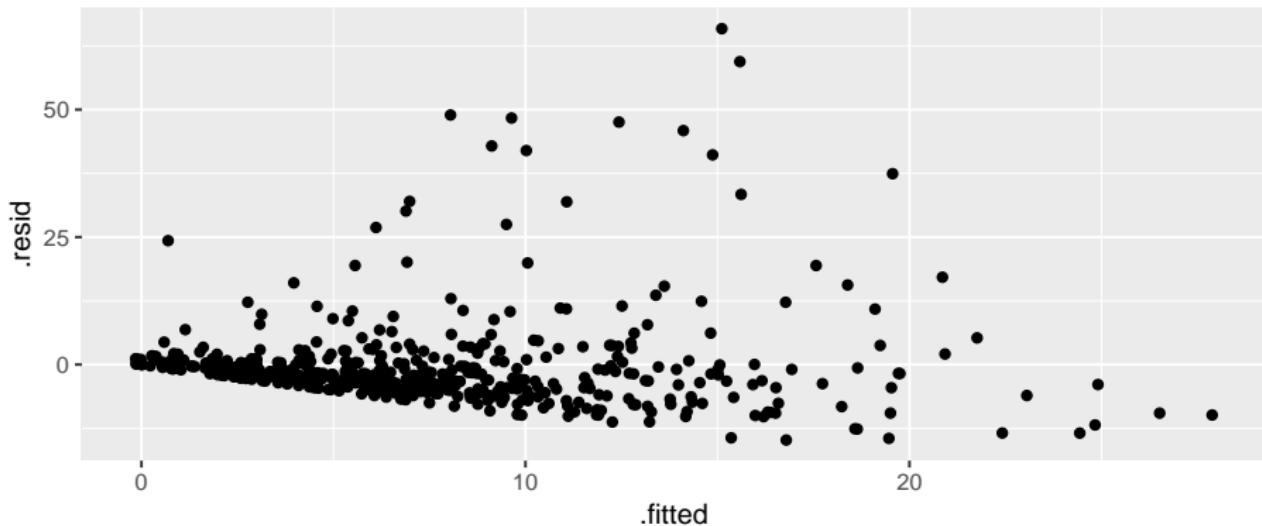
Leave out first column (subjno):

```
visits %>% select(-subjno) %>% cor()  
  
##          timedrs    phyheal    menheal      stress  
## timedrs 1.0000000 0.4395293 0.2555703 0.2865951  
## phyheal  0.4395293 1.0000000 0.5049464 0.3055517  
## menheal  0.2555703 0.5049464 1.0000000 0.3697911  
## stress   0.2865951 0.3055517 0.3697911 1.0000000
```

- ▶ phyheal most strongly correlated with timedrs.
- ▶ Not much to choose between other two.
- ▶ But menheal has higher correlation with phyheal, so not as much to *add* to prediction as stress.
- ▶ Goes to show things more complicated in multiple regression.

Residual plot (from timedrs on all)

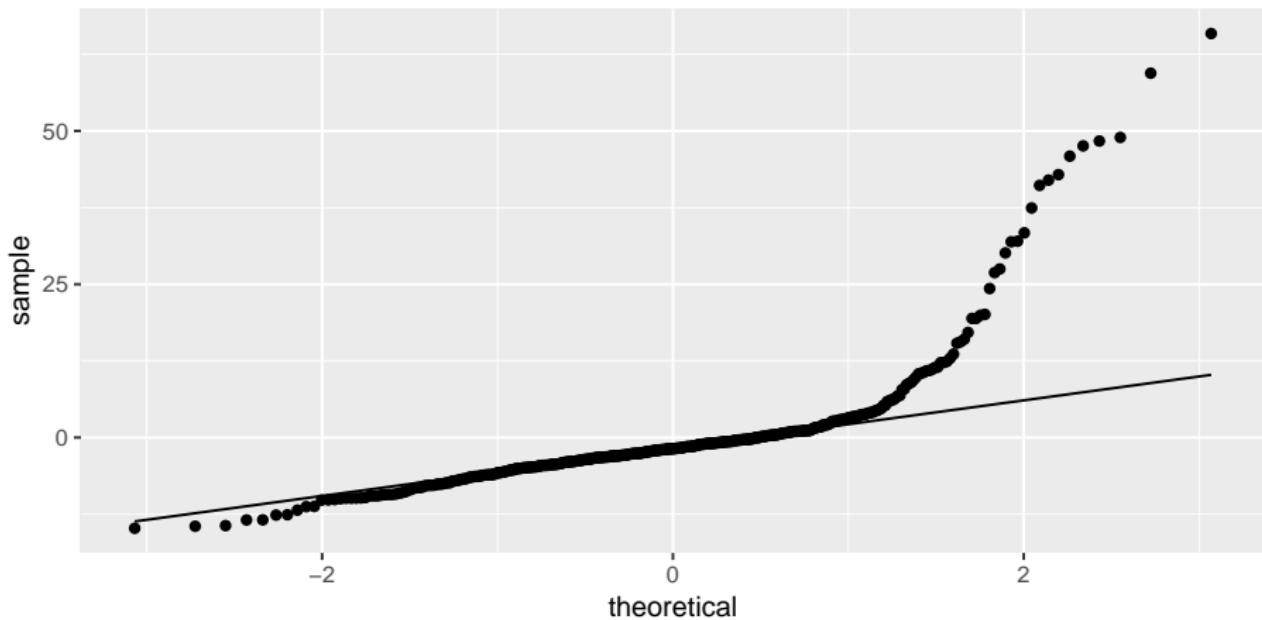
```
ggplot(visits.1, aes(x=.fitted, y=.resid)) + geom_point()
```



Apparently random. But...

Normal quantile plot of residuals

```
ggplot(visits.1, aes(sample=.resid))+stat_qq()+stat_qq_line()
```

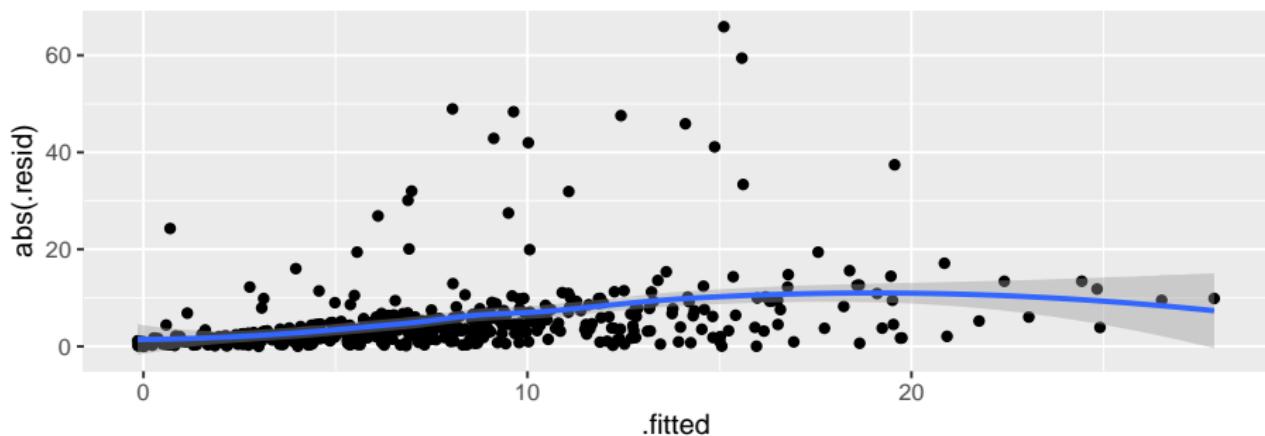


Absolute residuals

Is there trend in *size* of residuals (fan-out)? Plot *absolute value* of residual against fitted value:

```
ggplot(visits.1, aes(x=.fitted, y=abs(.resid)))+  
  geom_point() + geom_smooth()
```

```
## `geom_smooth()` using method = 'loess' and formula 'y ~  
x'
```



Comments

- ▶ On the normal quantile plot:
 - ▶ highest (most positive) residuals are way too high
 - ▶ distribution of residuals skewed to right (not normal at all)
- ▶ On plot of absolute residuals:
 - ▶ size of residuals getting bigger as fitted values increase
 - ▶ predictions getting more variable as fitted values increase
 - ▶ that is, predictions getting *less accurate* as fitted values increase, but predictions should be equally accurate all the way along.
- ▶ Both indicate problems with regression, of kind that transformation of response often fixes: that is, predict *function* of response `timedrs` instead of `timedrs` itself.

Fixing the problems

- ▶ Residuals not normal (skewed right), increase in size with fitted value.
- ▶ Sometimes residuals are very positive: observed a *lot* larger than predicted.
- ▶ Try *transforming* response: use log or square root of response. (Note that response is *count*, often skewed to right.)
- ▶ Try regression again, with transformed response instead of original one.
- ▶ Then check residual plot to see that it is OK now.

```
lgtime=with(visits, log(timedrs+1))
visits.3=lm(lgtime~phyheal+menheal+stress,
            data=visits)
```

- ▶ timedrs+1 because some timedrs values 0, can't take log of 0.
- ▶ Won't usually need to worry about this, but when response could be zero/negative, fix that before transformation.

Output

```
summary(visits.3)

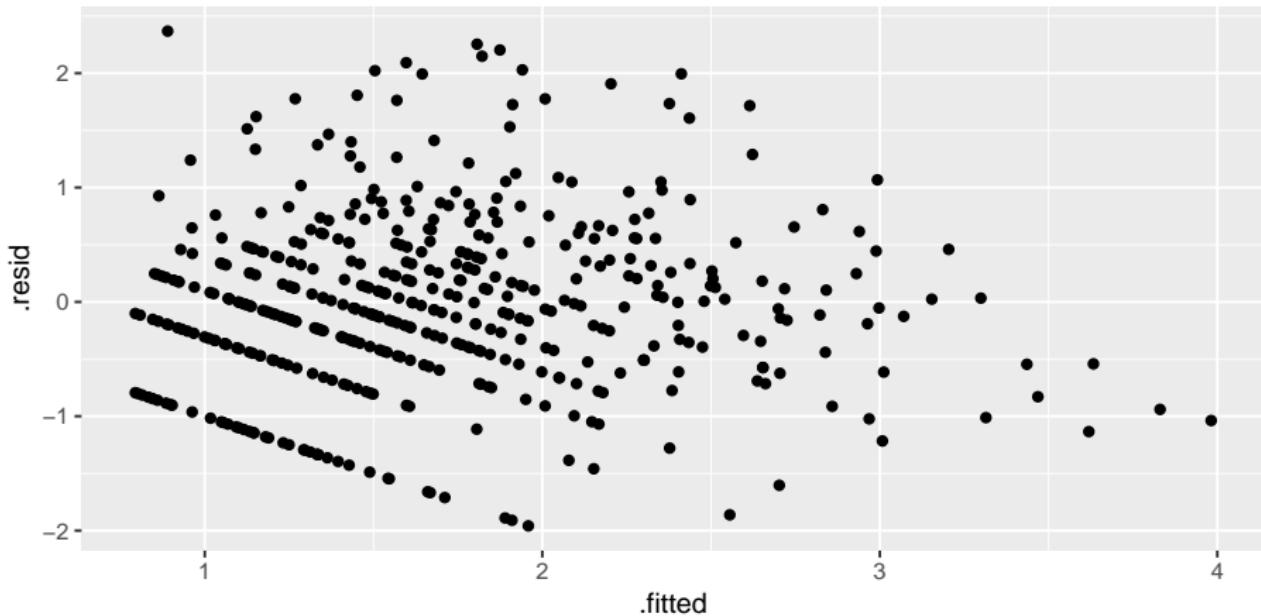
##
## Call:
## lm(formula = lgttime ~ phyheal + menheal + stress, data = visits)
##
## Residuals:
##       Min     1Q   Median     3Q    Max 
## -1.95865 -0.44076 -0.02331  0.42304  2.36797 
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)    
## (Intercept) 0.3903862  0.0882908  4.422 1.22e-05 ***
## phyheal     0.2019361  0.0173624 11.631 < 2e-16 ***
## menheal     0.0071442  0.0101335  0.705   0.481    
## stress      0.0013158  0.0002837  4.638 4.58e-06 ***
## ---      
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1 
##
## Residual standard error: 0.7625 on 461 degrees of freedom
## Multiple R-squared:  0.3682, Adjusted R-squared:  0.3641 
## F-statistic: 89.56 on 3 and 461 DF,  p-value: < 2.2e-16
```

Comments

- ▶ Model as a whole strongly significant again
- ▶ R-sq higher than before (37% vs. 22%) suggesting things more linear now
- ▶ Same conclusion re menheal: can take out of regression.
- ▶ Should look at residual plots (next pages). Have we fixed problems?

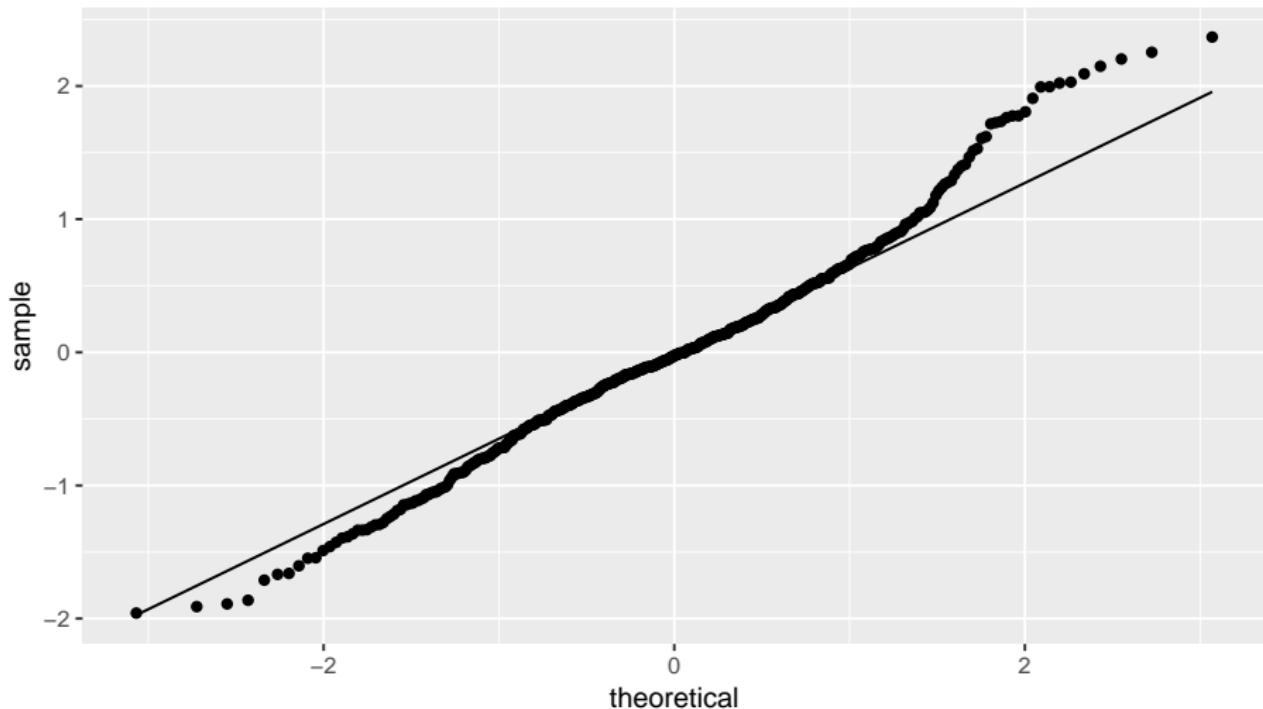
Residuals against fitted values

```
ggplot(visits.3, aes(x=.fitted, y=.resid)) +  
  geom_point()
```



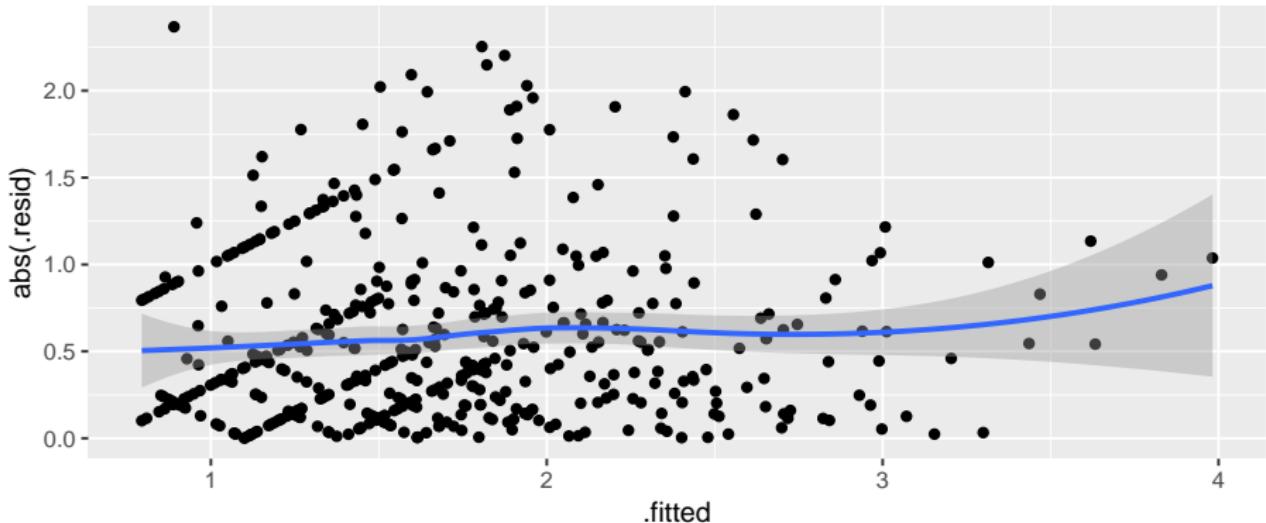
Normal quantile plot of residuals

```
ggplot(visits.3, aes(sample=.resid))+stat_qq()+stat_qq_line()
```



Absolute residuals against fitted

```
ggplot(visits.3, aes(x=.fitted, y=abs(.resid)))+  
  geom_point() + geom_smooth()  
  
## `geom_smooth()` using method = 'loess' and formula 'y ~  
x'
```



Comments

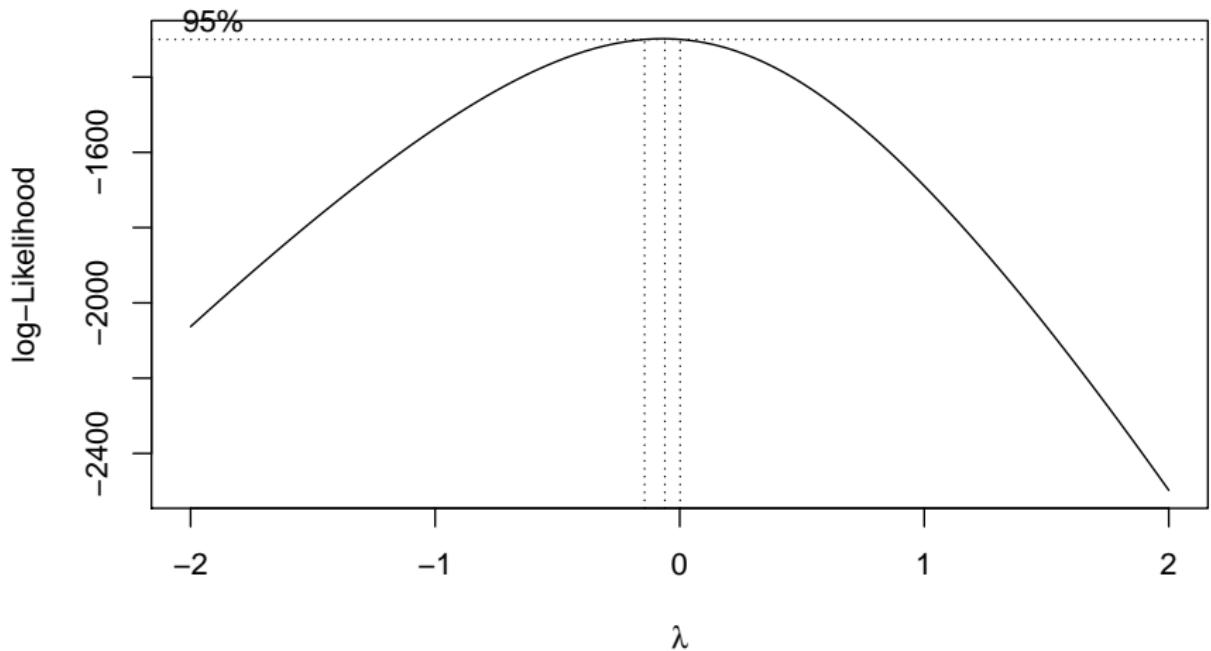
- ▶ Residuals vs. fitted looks a lot more random.
- ▶ Normal quantile plot looks a lot more normal (though still a little right-skewness)
- ▶ Absolute residuals: not so much trend (though still some).
- ▶ Not perfect, but much improved.

Box-Cox transformations

- ▶ Taking log of `timedrs` and having it work: lucky guess. How to find good transformation?
- ▶ Box-Cox again.
- ▶ Extra problem: some of `timedrs` values are 0, but Box-Cox expects all +. Note response for `boxcox`:

```
boxcox(timedrs+1~phyheal+menheal+stress, data=visitsp)
```

Try 1



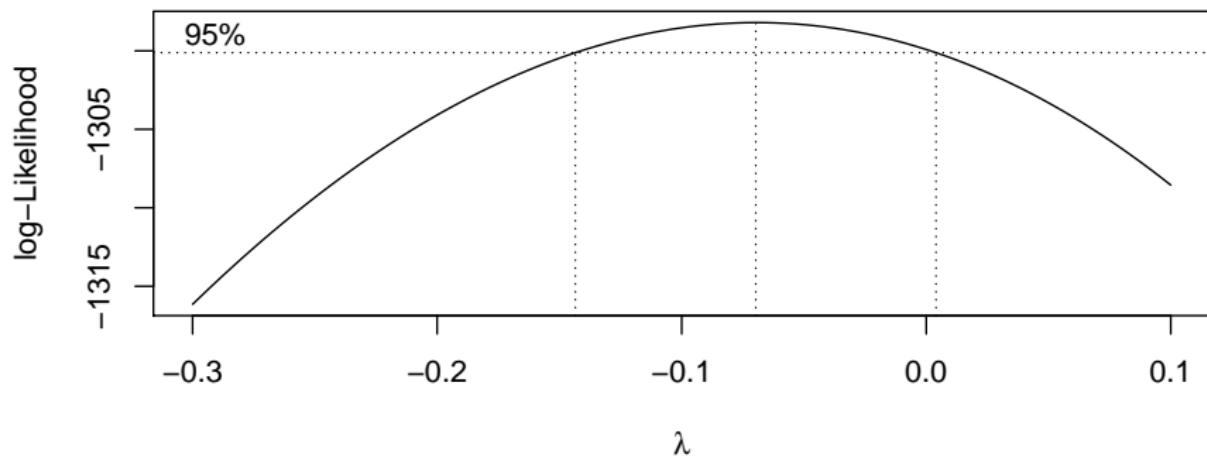
Comments on try 1

- ▶ Best: λ just less than zero.
- ▶ Hard to see scale.
- ▶ Focus on λ in $(-0.3, 0.1)$:

```
my.lambda=seq(-0.3,0.1,0.01)
my.lambda
## [1] -0.30 -0.29 -0.28 -0.27 -0.26 -0.25 -0.24 -0.23 -0.22
## [10] -0.21 -0.20 -0.19 -0.18 -0.17 -0.16 -0.15 -0.14 -0.13
## [19] -0.12 -0.11 -0.10 -0.09 -0.08 -0.07 -0.06 -0.05 -0.04
## [28] -0.03 -0.02 -0.01  0.00  0.01  0.02  0.03  0.04  0.05
## [37]  0.06  0.07  0.08  0.09  0.10
```

Try 2

```
boxcox(timedrs+1~phyheal+menheal+stress, lambda=my.lambda,  
       data=visits)
```



Comments

- ▶ Best: λ just about -0.07 .
- ▶ CI for λ about $(-0.14, 0.01)$.
- ▶ Only nearby round number: $\lambda = 0$, log transformation.
- ▶ So we made lucky guess with log before!

Testing more than one x at once

The *t*-tests test only whether one variable could be taken out of the regression you're looking at. To test significance of more than one variable at once, fit model with and without variables and use anova to compare fit of models:

```
visits.5=lm(lgtime~phyheal+menheal+stress,data=visits)
visits.6=lm(lgtime~stress,data=visits)
anova(visits.6,visits.5)

## Analysis of Variance Table
##
## Model 1: lgtime ~ stress
## Model 2: lgtime ~ phyheal + menheal + stress
##   Res.Df   RSS Df Sum of Sq    F    Pr(>F)
## 1     463 371.47
## 2     461 268.01  2     103.46 88.984 < 2.2e-16 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Results of tests

- ▶ Models don't fit equally well, so big one fits better.
- ▶ Or "taking both variables out makes the fit worse, so don't do it".
- ▶ Taking out those x 's is a mistake. Or putting them in is a good idea.

The punting data

Data set `punting.txt` contains 4 variables for 13 right-footed football kickers (punters): left leg and right leg strength (lbs), distance punted (ft), another variable called “fred”. Predict punting distance from other variables:

left	right	punt	fred
170	170	162.50	171
130	140	144.0	136
170	180	174.50	174
160	160	163.50	161
150	170	192.0	159
150	150	171.75	151
180	170	162.0	174
110	110	104.83	111
110	120	105.67	114
120	130	117.58	126
140	120	140.25	129
130	140	150.17	136
150	160	165.17	154

Reading in

- ▶ Separated by *multiple spaces* with *columns lined up*:

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/punting.txt"
punting=read_table(my_url)

## Parsed with column specification:
## cols(
##   left = col_integer(),
##   right = col_integer(),
##   punt = col_double(),
##   fred = col_integer()
## )
```

The data

punting

```
## # A tibble: 13 x 4
##       left   right   punt   fred
##     <int> <int> <dbl> <int>
## 1     170    170  162.    171
## 2     130    140  144     136
## 3     170    180  174.    174
## 4     160    160  164.    161
## 5     150    170  192     159
## 6     150    150  172.    151
## 7     180    170  162     174
## 8     110    110  105.    111
## 9     110    120  106.    114
## 10    120    130  118.    126
## 11    140    120  140.    129
## 12    130    140  150.    136
## 13    150    160  165.    154
```

Regression and output

```
punting.1=lm(punt~left+right+fred, data=punting)
summary(punting.1)

##
## Call:
## lm(formula = punt ~ left + right + fred, data = punting)
##
## Residuals:
##      Min       1Q   Median       3Q      Max 
## -14.9325 -11.5618 -0.0315  9.0415 20.0886 
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)    
## (Intercept) -4.6855    29.1172  -0.161   0.876    
## left         0.2679     2.1111   0.127   0.902    
## right        1.0524     2.1477   0.490   0.636    
## fred        -0.2672     4.2266  -0.063   0.951    
## 
## Residual standard error: 14.68 on 9 degrees of freedom
## Multiple R-squared:  0.7781, Adjusted R-squared:  0.7042 
## F-statistic: 10.52 on 3 and 9 DF,  p-value: 0.00267
```

Comments

- ▶ Overall regression strongly significant, R-sq high.
- ▶ None of the x's significant! Why?
- ▶ *t*-tests only say that you could take any one of the x's out without damaging the fit; doesn't matter which one.
- ▶ Explanation: look at *correlations*.

The correlations

```
cor(punting)

##           left      right      punt      fred
## left  1.0000000 0.8957224 0.8117368 0.9722632
## right 0.8957224 1.0000000 0.8805469 0.9728784
## punt  0.8117368 0.8805469 1.0000000 0.8679507
## fred  0.9722632 0.9728784 0.8679507 1.0000000
```

- ▶ All correlations are high: x's with punt (good) and with each other (bad, at least confusing).
- ▶ What to do? Probably do just as well to pick one variable, say right since kickers are right-footed.

Just right

```
punting.2=lm(punt~right,data=punting)
anova(punting.2,punting.1)

## Analysis of Variance Table
##
## Model 1: punt ~ right
## Model 2: punt ~ left + right + fred
##   Res.Df   RSS Df Sum of Sq      F Pr(>F)
## 1     11 1962.5
## 2     9 1938.2  2    24.263 0.0563 0.9456
```

No significant loss by dropping other two variables.

Comparing R-squareds

```
summary(punting.1)$r.squared  
## [1] 0.7781401  
  
summary(punting.2)$r.squared  
## [1] 0.7753629
```

Basically no difference. In regression (over), right significant:

Regression results

```
summary(punting.2)

##
## Call:
## lm(formula = punt ~ right, data = punting)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -15.7576 -11.0611   0.3656   7.8890  19.0423
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -3.6930    25.2649  -0.146   0.886
## right        1.0427     0.1692   6.162 7.09e-05 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 13.36 on 11 degrees of freedom
## Multiple R-squared:  0.7754, Adjusted R-squared:  0.7549
## F-statistic: 37.97 on 1 and 11 DF,  p-value: 7.088e-05
```

But...

- ▶ Maybe we got the *form* of the relationship with `left` wrong.
- ▶ Check: plot *residuals* from previous regression (without `left`) against `left`.
- ▶ Residuals here are “punting distance adjusted for right leg strength”.
- ▶ If there is some kind of relationship with `left`, we should include in model.
- ▶ Plot of residuals against original variable: `augment` from `broom`.

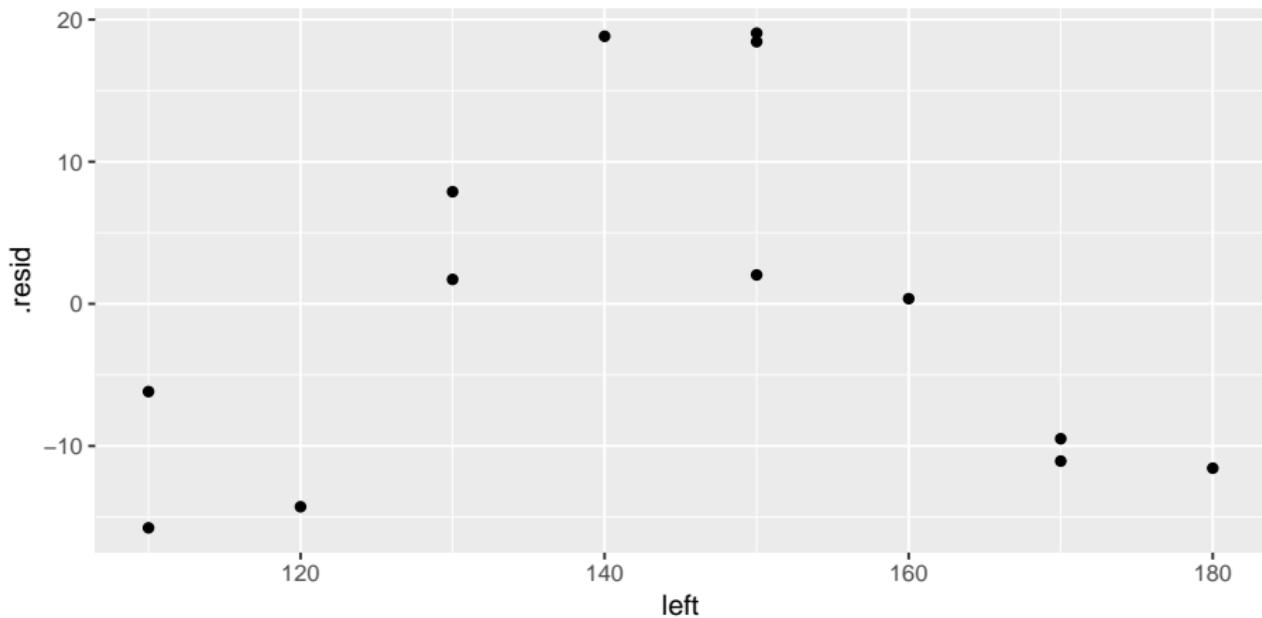
Augmenting punting.2

```
punting.2 %>% augment(punting) -> punting.2.aug
punting.2.aug %>% slice(1:8)

## # A tibble: 8 x 11
##   left right  punt fred .fitted .se.fit .resid   .hat
##   <int> <int> <dbl> <int>    <dbl>    <dbl>    <dbl>    <dbl>
## 1    170    170  162.    171    174.     5.29  -11.1   0.157
## 2    130    140  144     136    142.     3.93    1.72   0.0864
## 3    170    180  174.    174    184.     6.60   -9.49   0.244
## 4    160    160  164.    161    163.     4.25    0.366  0.101
## 5    150    170  192     159    174.     5.29   18.4   0.157
## 6    150    150  172.    151    153.     3.73   19.0   0.0778
## 7    180    170  162     174    174.     5.29  -11.6   0.157
## 8    110    110  105.    111    111.     7.38   -6.17   0.305
## # ... with 3 more variables: .sigma <dbl>, .cooksdi <dbl>,
## #   .std.resid <dbl>
```

Residuals against left

```
ggplot(punting.2.aug,aes(x=left,y=.resid))+  
  geom_point()
```



Comments

- ▶ There is a *curved* relationship with left.
- ▶ We should add left-squared to the regression (and therefore put left back in when we do that):

```
punting.3=lm(punt~left+I(left^2)+right,  
            data=punting)
```

Regression with left-squared

```
summary(punting.3)

##
## Call:
## lm(formula = punt ~ left + I(left^2) + right, data = punting)
##
## Residuals:
##       Min     1Q Median     3Q    Max 
## -11.3777 -5.3599  0.0459  4.5088 13.2669 
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)    
## (Intercept) -4.623e+02  9.902e+01 -4.669   0.00117 ** 
## left         6.888e+00  1.462e+00  4.710   0.00110 ** 
## I(left^2)   -2.302e-02  4.927e-03 -4.672   0.00117 ** 
## right        7.396e-01  2.292e-01  3.227   0.01038 *  
## ---      
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1 
##
## Residual standard error: 7.931 on 9 degrees of freedom
## Multiple R-squared:  0.9352, Adjusted R-squared:  0.9136 
## F-statistic: 43.3 on 3 and 9 DF,  p-value: 1.13e-05
```

Comments

- ▶ This was definitely a good idea (R-squared has clearly increased).
- ▶ We would never have seen it without plotting residuals from `punting.2` (without `left`) against `left`.
- ▶ Negative slope for `leftsq` means that increased left-leg strength only increases punting distance up to a point: beyond that, it decreases again.

Section 3

Logistic regression (ordinal/nominal response)

Logistic regression

- ▶ When response variable is measured/counted, regression can work well.
- ▶ But what if response is yes/no, lived/died, success/failure?
- ▶ Model *probability* of success.
- ▶ Probability must be between 0 and 1; need method that ensures this.
- ▶ *Logistic regression* does this. In R, is a *generalized linear model* with binomial "family": `glm(y~x,family="binomial")`
- ▶ Begin with simplest case.

Packages

```
library(MASS)
library(tidyverse)

## -- Attaching packages ----- tidyverse
1.2.1 --

## v ggplot2 3.1.0     v purrrr  0.2.5
## v tibble   1.4.2     v dplyr    0.7.8
## v tidyrr   0.8.1     v stringr  1.3.1
## v readr    1.1.1     v forcats  0.3.0

## -- Conflicts -----
tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()
## x dplyr::select() masks MASS::select()

library(broom)
library(nnet)
```

The rats, part 1

- ▶ Rats given dose of some poison; either live or die:

```
dose status  
0 lived  
1 died  
2 lived  
3 lived  
4 died  
5 died
```

- ▶ Read the data:

```
my_url = "http://www.utsc.utoronto.ca/~butler/d29/rat.txt"  
rats = read_delim(my_url, " ")  
  
## Parsed with column specification:  
## cols(  
##   dose = col_integer(),  
##   status = col_character()  
## )
```

Basic logistic regression

- Data:

```
rats  
## # A tibble: 6 x 2  
##   dose status  
##   <int> <chr>  
## 1     0 lived  
## 2     1 died  
## 3     2 lived  
## 4     3 lived  
## 5     4 died  
## 6     5 died
```

- Make response into a factor first:

```
rats2 = rats %>% mutate(status=factor(status))
```

- then fit model:

```
status.1 =  
  glm(status~dose,family="binomial",data=rats2)
```

Output

```
summary(status.1)

##
## Call:
## glm(formula = status ~ dose, family = "binomial", data = rats2)
##
## Deviance Residuals:
##       1        2        3        4        5        6
##  0.5835 -1.6254  1.0381  1.3234 -0.7880 -0.5835
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept) 1.6841    1.7979   0.937   0.349
## dose        -0.6736    0.6140  -1.097   0.273
##
## (Dispersion parameter for binomial family taken to be 1)
##
## Null deviance: 8.3178 on 5 degrees of freedom
## Residual deviance: 6.7728 on 4 degrees of freedom
## AIC: 10.773
##
## Number of Fisher Scoring iterations: 4
```

Interpreting the output

- ▶ Like (multiple) regression, get tests of significance of individual x 's
- ▶ Here not significant (only 6 observations).
- ▶ “Slope” for dose is negative, meaning that as dose increases, probability of event modelled (survival) decreases.

Output part 2: predicted survival probs

```
p=predict(status.1,type="response")
cbind(rats,p)

##      dose status          p
## 1      0   lived 0.8434490
## 2      1   died 0.7331122
## 3      2   lived 0.5834187
## 4      3   lived 0.4165813
## 5      4   died 0.2668878
## 6      5   died 0.1565510
```

The rats, more

- ▶ More realistic: more rats at each dose (say 10).
- ▶ Listing each rat on one line makes a big data file.
- ▶ Use format below: dose, number of survivors, number of deaths.

dose	lived	died
0	10	0
1	7	3
2	6	4
3	4	6
4	2	8
5	1	9

- ▶ 6 lines of data correspond to 60 actual rats.
- ▶ Saved in `rat2.txt`.

These data

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/rat2.txt"
rat2=read_delim(my_url, " ")

## Parsed with column specification:
## cols(
##   dose = col_integer(),
##   lived = col_integer(),
##   died = col_integer()
## )

## Warning in rbind(names(probs), probs_f): number of columns of result
## is not a multiple of vector length (arg 1)

## Warning: 5 parsing failures.

## # A tibble: 5 x 5
##   row col  expected actual file
##   <int> <chr> <chr>    <chr> <chr>
## 1 2 <NA>  3 columns 4 colu~ 'http://www.utsc.utoronto.~ file 2     3
## <NA>  3 columns 4 colu~ 'http://www.utsc.utoronto.~ row 3     4 <NA>  3
## columns 4 colu~ 'http://www.utsc.utoronto.~ col 4     5 <NA>  3 columns 4
## colu~ 'http://www.utsc.utoronto.~ expected 5     6 <NA>  3 columns 5
## colu~ 'http://www.utsc.utoronto.~

rat2

## # A tibble: 6 x 3
```

This logistic regression

```
response=with(rat2, cbind(lived,died))
rat2.1=glm(response~dose,family="binomial",
            data=rat2)
```

- ▶ Note construction of *two-column* response, #survivals in first column, #deaths in second.
- ▶ The response variable is an R matrix:

```
class(response)
## [1] "matrix"
```

Output

```
summary(rat2.1)
```

```
##  
## Call:  
## glm(formula = response ~ dose, family = "binomial", data = rat2)  
##  
## Deviance Residuals:  
##      1       2       3       4       5       6  
##  1.3421 -0.7916 -0.1034  0.1034  0.0389  0.1529  
##  
## Coefficients:  
##             Estimate Std. Error z value Pr(>|z|)  
## (Intercept)  2.3619    0.6719   3.515 0.000439 ***  
## dose        -0.9448    0.2351  -4.018 5.87e-05 ***  
## ---  
## Signif. codes:  
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1  
##  
## (Dispersion parameter for binomial family taken to be 1)  
##  
## Null deviance: 27.530 on 5 degrees of freedom  
## Residual deviance: 2.474 on 4 degrees of freedom  
## AIC: 18.94  
##  
## Number of Fisher Scoring iterations: 4
```

Predicted survival probs

```
p=predict(rat2.1,type="response")
cbind(rat2,p)

##      dose lived died          p
## 1      0     10    0 0.9138762
## 2      1      7    3 0.8048905
## 3      2      6    4 0.6159474
## 4      3      4    6 0.3840526
## 5      4      2    8 0.1951095
## 6      5      1    9 0.0861238
```

Comments

- ▶ Significant effect of dose.
- ▶ Effect of larger dose is to decrease survival probability (“slope” negative; also see in decreasing predictions.)

Multiple logistic regression

- ▶ With more than one x , works much like multiple regression.
- ▶ Example: study of patients with blood poisoning severe enough to warrant surgery. Relate survival to other potential risk factors.
- ▶ Variables, 1=present, 0=absent:
 - ▶ survival (death from sepsis=1), response
 - ▶ shock
 - ▶ malnutrition
 - ▶ alcoholism
 - ▶ age (as numerical variable)
 - ▶ bowel infarction
- ▶ See what relates to death.

Read in data

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/sepsis.txt"
sepsis=read_delim(my_url, " ")

## Parsed with column specification:
## cols(
##   death = col_integer(),
##   shock = col_integer(),
##   malnut = col_integer(),
##   alcohol = col_integer(),
##   age = col_integer(),
##   bowelinf = col_integer()
## )

## Warning in rbind(names(probs), probs_f): number of columns of result
## is not a multiple of vector length (arg 1)

## Warning: 106 parsing failures.

## # A tibble: 5 x 5
##   row     col      row     col   expected    actual    file
##   <int> <chr> <chr> <chr> <chr>          <chr>    file
## 1     1 <NA>   6 columns 8 colu~ 'http://www.utsc.utoronto.~ file 2     2
## <NA>   6 columns 8 colu~ 'http://www.utsc.utoronto.~ row 3     3 <NA>   6
## columns 8 colu~ 'http://www.utsc.utoronto.~ col 4     4 <NA>   6 columns 8
## colu~ 'http://www.utsc.utoronto.~ expected 5     5 <NA>   6 columns 8
## colu~ 'http://www.utsc.utoronto.~
```

The data

```
sepsis

## # A tibble: 106 x 6
##   death shock malnut alcohol    age bowelinf
##   <int>  <int>  <int>   <int> <int>     <int>
## 1     0      0      0       0    56        0
## 2     0      0      0       0    80        0
## 3     0      0      0       0    61        0
## 4     0      0      0       0    26        0
## 5     0      0      0       0    53        0
## 6     1      0      1       0    87        0
## 7     0      0      0       0    21        0
## 8     1      0      0       1    69        0
## 9     0      0      0       0    57        0
## 10    0      0      1       0    76        0
## # ... with 96 more rows
```

Fit model

```
sepsis.1=glm(death~shock+malnut+alcohol+age+
               bowelinf,family="binomial",
               data=sepsis)
```

Output part 1

```
tidy(sepsis.1)

## # A tibble: 6 x 5
##   term      estimate std.error statistic p.value
##   <chr>      <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept) -9.75     2.54     -3.84 0.000124
## 2 shock        3.67     1.16      3.15 0.00161
## 3 malnut       1.22     0.728    1.67 0.0948
## 4 alcohol      3.35     0.982    3.42 0.000635
## 5 age          0.0922   0.0303   3.04 0.00237
## 6 bowelinf     2.80     1.16      2.40 0.0162
```

- ▶ All P-values fairly small
- ▶ but malnut not significant: remove.

Removing malnut

```
sepsis.2=update(sepsis.1,.~.-malnut)
tidy(sepsis.2)

## # A tibble: 5 x 5
##   term      estimate std.error statistic p.value
##   <chr>      <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept) -8.89     2.32     -3.84 0.000124
## 2 shock        3.70     1.10      3.35 0.000797
## 3 alcohol       3.19     0.917     3.47 0.000514
## 4 age          0.0898    0.0292    3.07 0.00211
## 5 bowelinf      2.39     1.07      2.23 0.0260
```

- ▶ Everything significant now.

Comments

- ▶ Most of the original x 's helped predict death. Only `malnut` seemed not to add anything.
- ▶ Removed `malnut` and tried again.
- ▶ Everything remaining is significant (though `bowelinf` actually became *less* significant).
- ▶ All coefficients are *positive*, so having any of the risk factors (or being older) *increases* risk of death.

Predictions from model without “malnut”

- ▶ A few chosen at random:

```
sepsis.pred=predict(sepsis.2,type="response")
d=data.frame(sepsis,sepsis.pred)
myrows=c(4,1,2,11,32) ; slice(d,myrows)

##   death shock malnut alcohol age bowelinf sepsis.pred
## 1     0      0       0      0    26        0 0.001415347
## 2     0      0       0      0    56        0 0.020552383
## 3     0      0       0      0    80        0 0.153416834
## 4     1      0       0      1    66        1 0.931290137
## 5     1      0       0      1    49        0 0.213000997
```

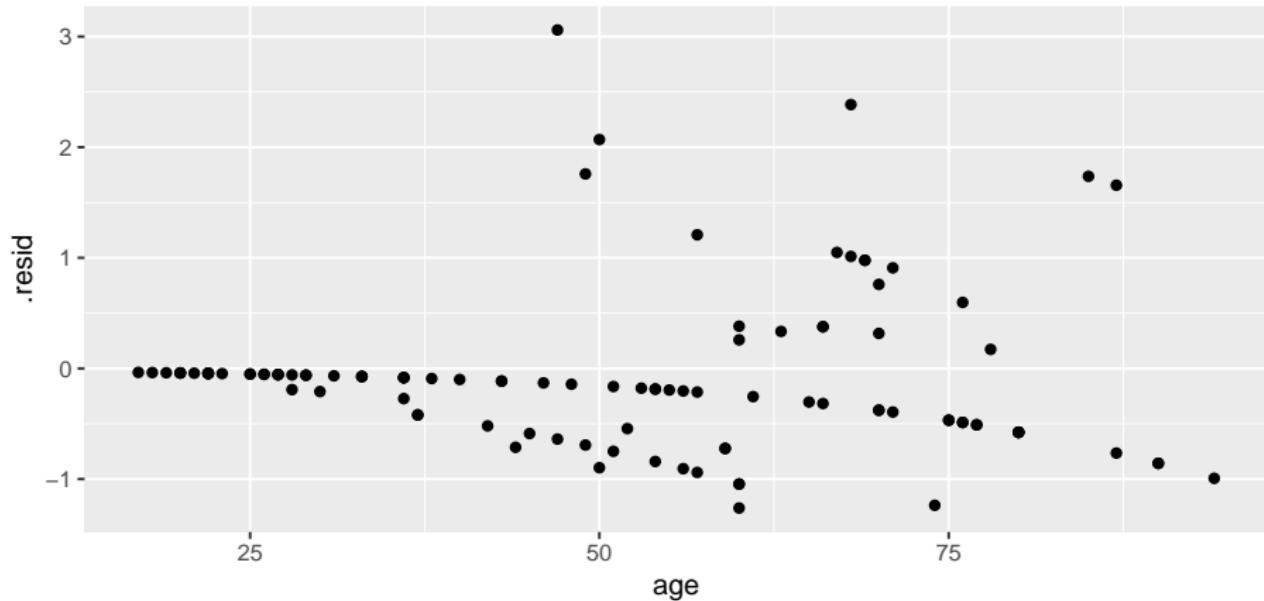
- ▶ Survival chances pretty good if no risk factors, though decreasing with age.
- ▶ Having more than one risk factor reduces survival chances dramatically.
- ▶ Usually good job of predicting survival; sometimes death predicted to survive.

Assessing proportionality of odds for age

- ▶ An assumption we made is that log-odds of survival depends linearly on age.
- ▶ Hard to get your head around, but basic idea is that survival chances go continuously up (or down) with age, instead of (for example) going up and then down.
- ▶ In this case, seems reasonable, but should check:

Residuals vs. age

```
ggplot(augment(sepsis.2), aes(x=age, y=.resid)) +  
  geom_point()
```



- ▶ No apparent problems overall.
- ▶ Confusing “line” across: no risk factors, survived.

Probability and odds

- ▶ For probability p , odds is $p/(1 - p)$. Examples:

Prob.	Odds	log-odds	in words
0.5	$0.5/0.5 = 1/1 = 1.00$	0.00	"even money"
0.1	$0.1/0.9 = 1/9 = 0.11$	-2.20	"9 to 1"
0.4	$0.4/0.6 = 1/1.5 = 0.67$	-0.41	"1.5 to 1"
0.8	$0.8/0.2 = 4/1 = 4.00$	1.39	"4 to 1 on"

- ▶ Gamblers use odds: if you win at 9 to 1 odds, get original stake back plus 9 times the stake.
- ▶ Probability has to be between 0 and 1
- ▶ Odds between 0 and infinity
- ▶ Log-odds can be anything: any log-odds corresponds to valid probability.

Odds ratio

- ▶ Suppose 90 of 100 men drank wine last week, but only 20 of 100 women.
- ▶ Prob of man drinking wine $90/100 = 0.9$, woman $20/100 = 0.2$.
- ▶ Odds of man drinking wine $0.9/0.1 = 9$, woman $0.2/0.8 = 0.25$.
- ▶ Ratio of odds is $9/0.25 = 36$.
- ▶ Way of quantifying difference between men and women: “odds of drinking wine 36 times larger for males than females”.

Sepsis data again

- ▶ Recall prediction of probability of death from risk factors:

```
sepsis.2.tidy=tidy(sepsis.2)
sepsis.2.tidy

## # A tibble: 5 x 5
##   term      estimate std.error statistic p.value
##   <chr>     <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept) -8.89     2.32     -3.84 0.000124
## 2 shock        3.70     1.10      3.35 0.000797
## 3 alcohol       3.19     0.917     3.47 0.000514
## 4 age          0.0898    0.0292    3.07 0.00211
## 5 bowelinf      2.39     1.07      2.23 0.0260
```

- ▶ Slopes in column estimate.

Multiplying the odds

- ▶ Can interpret slopes by taking “exp” of them. We ignore intercept.

```
cc=exp(sepsis.2.tidy$estimate)
data.frame(sepsis.2.tidy$term, expcoeff=round(cc, 2))

##    sepsis.2.tidy.term expcoeff
## 1      (Intercept)     0.00
## 2          shock     40.50
## 3        alcohol     24.19
## 4          age      1.09
## 5       bowelinf    10.88
```

- ▶ These say “how much do you *multiply* odds of death by for increase of 1 in corresponding risk factor?” Or, what is odds ratio for that factor being 1 (present) vs. 0 (absent)?
- ▶ Eg. being alcoholic vs. not increases odds of death by 24 times
- ▶ One year older multiplies odds by about 1.1 times. Over 40 years, about $1.09^{40} = 31$ times.

Odds ratio and relative risk

- ▶ **Relative risk** is ratio of probabilities.
- ▶ Above: 90 of 100 men (0.9) drank wine, 20 of 100 women (0.2).
- ▶ Relative risk $0.9/0.2=4.5$. (odds ratio was 36).
- ▶ When probabilities small, relative risk and odds ratio similar.
- ▶ Eg. prob of man having disease 0.02, woman 0.01.
- ▶ Relative risk $0.02/0.01 = 2$.

- ▶ Odds for men and for women:
- ▶ Odds ratio

($od1=0.02/0.98$)

```
## [1] 0.02040816
```

($od2=0.01/0.99$)

```
## [1] 0.01010101
```

$od1/od2$

```
## [1] 2.020408
```

▶ Very close to 2.

More than 2 response categories

- ▶ With 2 response categories, model the probability of one, and prob of other is one minus that. So doesn't matter which category you model.
- ▶ With more than 2 categories, have to think more carefully about the categories: are they
 - ▶ *ordered*: you can put them in a natural order (like low, medium, high)
 - ▶ *nominal*: ordering the categories doesn't make sense (like red, green, blue).
- ▶ R handles both kinds of response; learn how.

Ordinal response: the miners

- ▶ Model probability of being in given category or lower.
- ▶ Example: coal-miners often suffer disease pneumoconiosis. Likelihood of disease believed to be greater among miners who have worked longer.
- ▶ Severity of disease measured on categorical scale: 1 = none, 2 = moderate, 3 = severe.
- ▶ Data are frequencies:

Exposure	None	Moderate	Severe
5.8	98	0	0
15.0	51	2	1
21.5	34	6	3
27.5	35	5	8
33.5	32	10	9
39.5	23	7	8
46.0	12	6	10
51.5	4	2	5

Reading the data

Data in aligned columns with more than one space between, so:

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/miners-tab.txt"
freqs=read_table(my_url)

## Parsed with column specification:
## cols(
##   Exposure = col_double(),
##   None = col_integer(),
##   Moderate = col_integer(),
##   Severe = col_integer()
## )
```

The data

```
freqs
```

```
## # A tibble: 8 x 4
##   Exposure    None Moderate Severe
##   <dbl>     <int>     <int>    <int>
## 1      5.8      98        0        0
## 2      15       51        2        1
## 3     21.5      34        6        3
## 4     27.5      35        5        8
## 5     33.5      32       10        9
## 6     39.5      23        7        8
## 7      46       12        6       10
## 8     51.5       4        2        5
```

Tidying and row proportions

```
freqs %>% gather(Severity, Freq, None:Severe) %>%  
  group_by(Exposure) %>%  
  mutate(proportion=Freq/sum(Freq)) -> miners
```

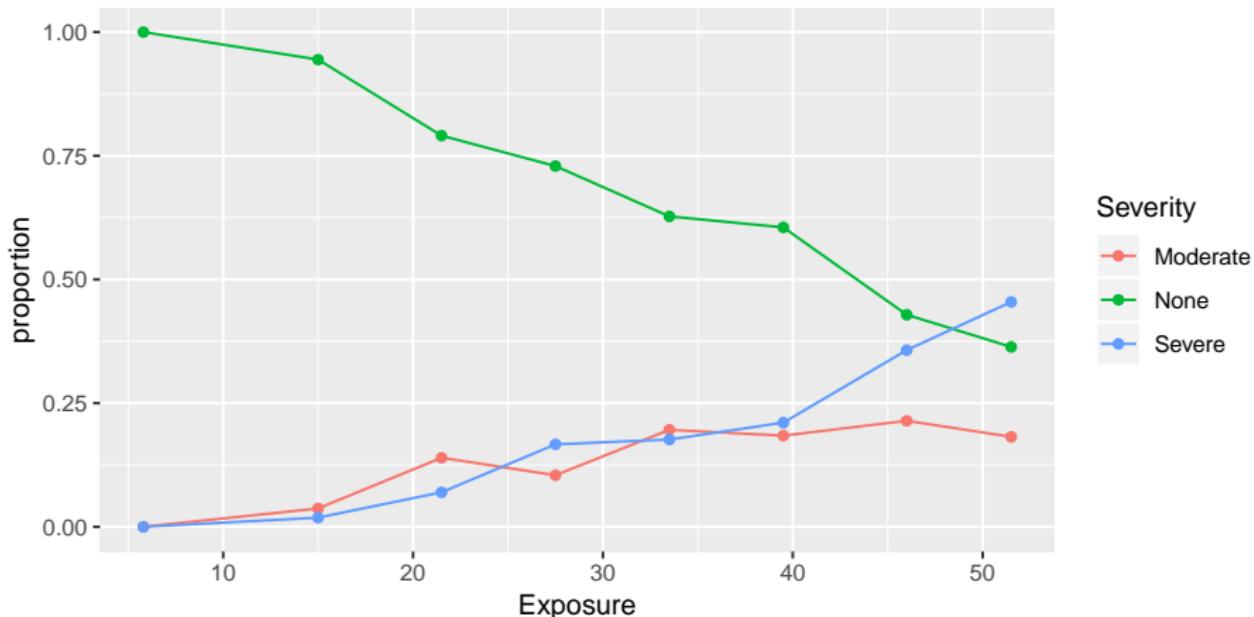
Result

miners

```
## # A tibble: 24 x 4
## # Groups:   Exposure [8]
##       Exposure Severity   Freq proportion
##       <dbl>    <chr>   <int>      <dbl>
## 1        5.8  None     98      1
## 2       15    None     51     0.944
## 3      21.5  None     34     0.791
## 4      27.5  None     35     0.729
## 5      33.5  None     32     0.627
## 6      39.5  None     23     0.605
## 7       46    None     12     0.429
## 8      51.5  None      4     0.364
## 9        5.8 Moderate     0      0
## 10      15   Moderate     2     0.0370
## # ... with 14 more rows
```

Plot proportions against exposure

```
ggplot(miners,aes(x=Exposure,y=proportion,  
colour=Severity))+geom_point() +geom_line()
```



Reminder of data setup

miners

```
## # A tibble: 24 x 4
## # Groups:   Exposure [8]
##       Exposure Severity   Freq proportion
##       <dbl>    <chr>     <int>      <dbl>
## 1      5.8  None        98      1
## 2     15   None        51      0.944
## 3    21.5  None        34      0.791
## 4    27.5  None        35      0.729
## 5    33.5  None        32      0.627
## 6    39.5  None        23      0.605
## 7     46   None        12      0.429
## 8    51.5  None         4      0.364
## 9      5.8 Moderate       0      0
## 10     15 Moderate        2      0.0370
## # ... with 14 more rows
```

Creating an ordered factor

- ▶ Problem: on plot, Severity categories in *wrong order*.
- ▶ First we need the different values in (text) Severity:

```
v=unique(miners$Severity)  
v  
## [1] "None"      "Moderate"   "Severe"
```

- ▶ These are in the right order.
- ▶ Now we make an ordered factor out of Severity with these as its levels:

```
miners = miners %>%  
  mutate(sev_ord=ordered(Severity,v))
```

New data frame

```
miners
```

```
## # A tibble: 24 x 5
## # Groups:   Exposure [8]
##       Exposure Severity Freq proportion sev_ord
##       <dbl>    <chr>   <int>        <dbl>    <ord>
## 1      5.8  None     98     1        None
## 2      15   None     51    0.944    None
## 3     21.5  None     34    0.791    None
## 4     27.5  None     35    0.729    None
## 5     33.5  None     32    0.627    None
## 6     39.5  None     23    0.605    None
## 7      46   None     12    0.429    None
## 8     51.5  None      4    0.364    None
## 9      5.8 Moderate     0     0        Moderate
## 10     15  Moderate     2    0.0370  Moderate
## # ... with 14 more rows
```

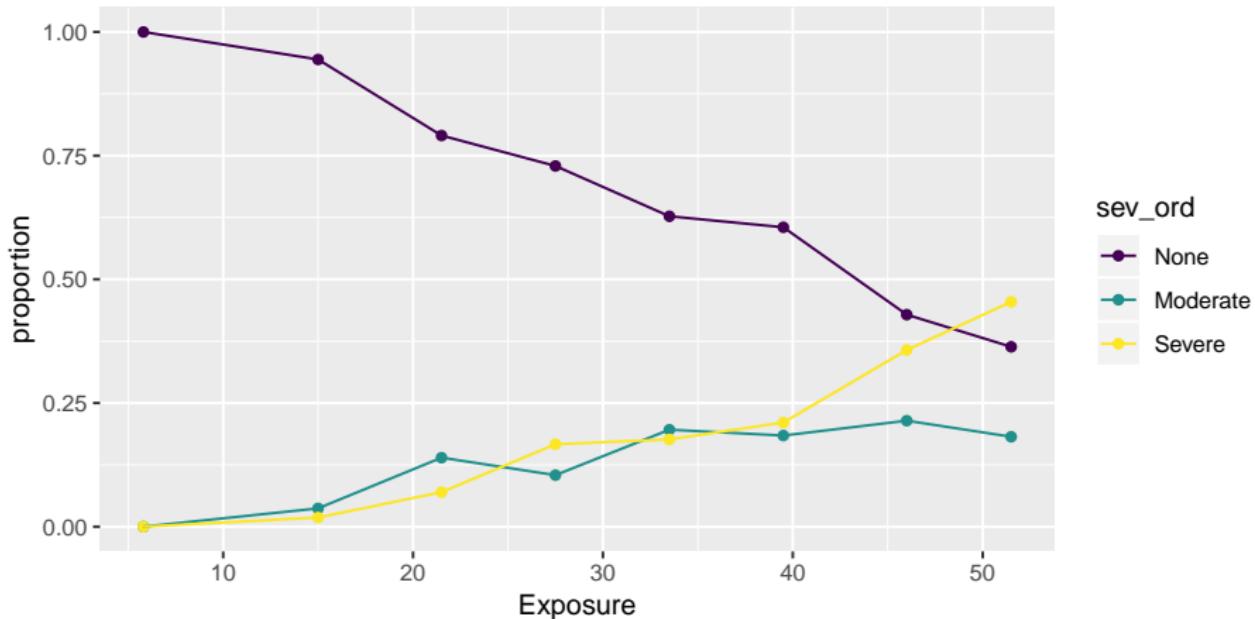
In the opposite order

Severe down to None:

```
miners %>% mutate(sev2=ordered(Severity,c(v[3],v[2],v[1])))  
  
## # A tibble: 24 x 6  
## # Groups:   Exposure [8]  
##       Exposure Severity   Freq proportion sev_ord    sev2  
##       <dbl>   <chr>     <int>        <dbl> <ord>    <ord>  
## 1      5.8  None      98      1.000  None    None  
## 2      15   None      51      0.944  None    None  
## 3     21.5  None      34      0.791  None    None  
## 4     27.5  None      35      0.729  None    None  
## 5     33.5  None      32      0.627  None    None  
## 6     39.5  None      23      0.605  None    None  
## 7      46   None      12      0.429  None    None  
## 8     51.5  None       4      0.364  None    None  
## 9      5.8 Moderate     0      0.000  Moderate Moderate  
## 10     15  Moderate     2      0.0370 Moderate Moderate  
## # ... with 14 more rows
```

Improved plot

```
ggplot(miners,aes(x=Exposure,y=proportion,  
colour=sev_ord))+geom_point()+geom_line()
```



Fitting ordered logistic model

Use function `polr` from package MASS. Like `glm`.

```
sev.1=polr(sev_ord~Exposure, weights=Freq,  
           data=miners)
```

Output: not very illuminating

```
summary(sev.1)

##
## Re-fitting to get Hessian

## Call:
## polr(formula = sev_ord ~ Exposure, data = miners, weights = Freq)
##
## Coefficients:
##             Value Std. Error t value
## Exposure 0.0959    0.01194   8.034
##
## Intercepts:
##             Value Std. Error t value
## None|Moderate 3.9558  0.4097   9.6558
## Moderate|Severe 4.8690  0.4411  11.0383
##
## Residual Deviance: 416.9188
## AIC: 422.9188
```

Does exposure have an effect?

Fit model without Exposure, and compare using anova. Note 1 for model with just intercept:

```
sev.0=polr(sev_ord~1,weights=Freq,data=miners)
anova(sev.0,sev.1)

## Likelihood ratio tests of ordinal regression models
##
## Response: sev_ord
##      Model Resid. df Resid. Dev   Test
## 1          1     369  505.1621
## 2 Exposure     368  416.9188 1 vs 2
##      Df LR stat. Pr(Chi)
## 1
## 2     1 88.24324       0
```

Exposure definitely has effect on severity of disease.

Another way

- ▶ What (if anything) can we drop from model with exposure?

```
drop1(sev.1,test="Chisq")  
  
## Single term deletions  
  
##  
## Model:  
## sev_ord ~ Exposure  
##          Df      AIC      LRT  Pr(>Chi)  
## <none>     422.92  
## Exposure   1 509.16 88.243 < 2.2e-16 ***  
## ---  
## Signif. codes:  
##  0 '***' 0.001 '**' 0.01 '*' 0.05  
##  '.' 0.1 ' ' 1
```

- ▶ Nothing. Exposure definitely has effect.

Predicted probabilities

Make new data frame out of all the exposure values (from original data frame), and predict from that:

```
sev.new=tibble(Exposure=freqs$Exposure)
pr=predict(sev.1,sev.new,type="p")
miners.pred=cbind(sev.new,pr)
miners.pred

##   Exposure      None   Moderate     Severe
## 1      5.8 0.9676920 0.01908912 0.01321885
## 2     15.0 0.9253445 0.04329931 0.03135614
## 3     21.5 0.8692003 0.07385858 0.05694115
## 4     27.5 0.7889290 0.11413004 0.09694093
## 5     33.5 0.6776641 0.16207145 0.16026444
## 6     39.5 0.5418105 0.20484198 0.25334756
## 7     46.0 0.3879962 0.22441555 0.38758828
## 8     51.5 0.2722543 0.21025011 0.51749563
```

Comments

- ▶ Model appears to match data: as exposure goes up, prob of None goes down, Severe goes up (sharply for high exposure).
- ▶ Like original data frame, this one nice to look at but *not tidy*. We want to make graph, so tidy it.
- ▶ Usual `gather`:

```
preds = miners.pred %>%  
  gather(Severity, probability, None:Severe)
```

Some of the gathered predictions

```
preds %>% slice(1:15)

##      Exposure Severity probability
## 1        5.8     None  0.96769203
## 2       15.0     None  0.92534455
## 3       21.5     None  0.86920028
## 4       27.5     None  0.78892903
## 5       33.5     None  0.67766411
## 6       39.5     None  0.54181046
## 7       46.0     None  0.38799618
## 8       51.5     None  0.27225426
## 9        5.8 Moderate 0.01908912
## 10      15.0 Moderate 0.04329931
## 11      21.5 Moderate 0.07385858
## 12      27.5 Moderate 0.11413004
## 13      33.5 Moderate 0.16207145
## 14      39.5 Moderate 0.20484198
## 15      46.0 Moderate 0.22441555
```

Plotting predicted and observed proportions

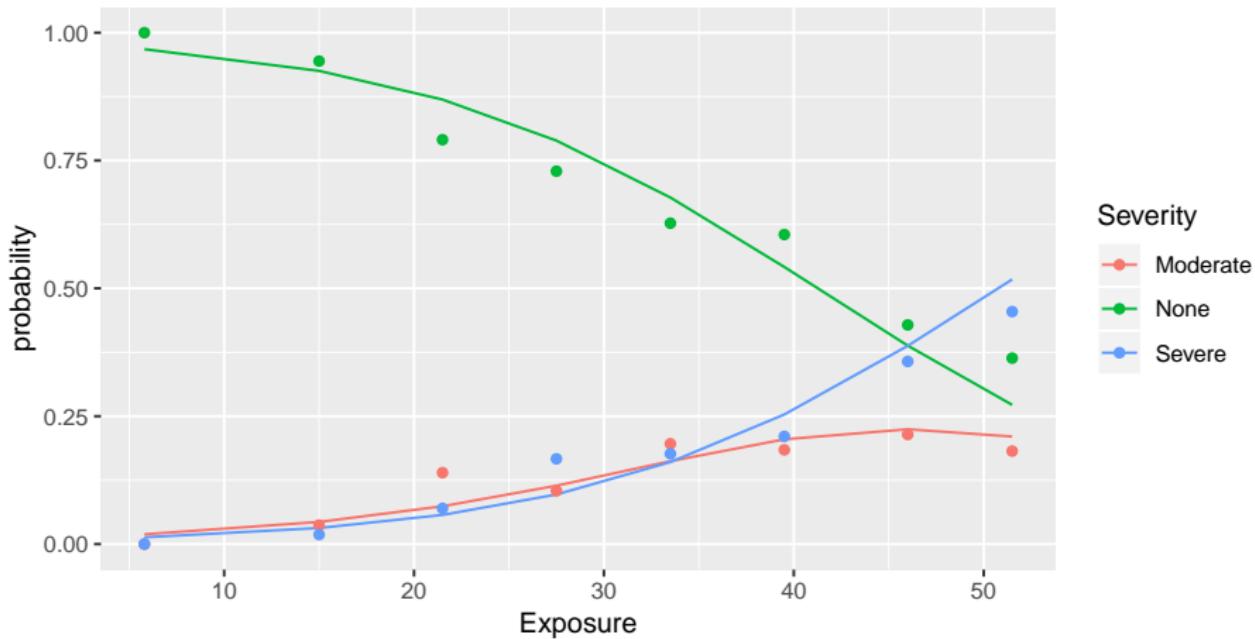
- ▶ Plot:
 - ▶ predicted probabilities, lines (shown) joining points (not shown)
 - ▶ data, just the points.
- ▶ Unfamiliar process: data from two *different* data frames:

```
g=ggplot(preds, aes(x=Exposure, y=probability,  
                     colour=Severity)) + geom_line() +  
  geom_point(data=miners, aes(y=proportion))
```

- ▶ Idea: final `geom_point` uses data in `miners` rather than `preds`,
`y`-variable for plot is proportion from that data frame, but
`x`-coordinate is `Exposure`, as it was before, and `colour` is `Severity`
as before. The final `geom_point` “inherits” from the first `aes` as
needed.
- ▶ Data conform to fitted relationship pretty well:

The plot

g



Unordered responses

- ▶ With unordered (nominal) responses, can use *generalized logit*.
- ▶ Example: 735 people, record age and sex (male 0, female 1), which of 3 brands of some product preferred.
- ▶ Data in `mlogit.csv` separated by commas (so `read_csv` will work):

```
my_url = "http://www.utsc.utoronto.ca/~butler/d29/mlogit.csv"
brandpref = read_csv(my_url)

## Parsed with column specification:
## cols(
##   brand = col_integer(),
##   sex = col_integer(),
##   age = col_integer()
## )
```

The data

```
brandpref
```

```
## # A tibble: 735 x 3
##   brand sex   age
##   <int> <int> <int>
## 1     1     0    24
## 2     2     0    26
## 3     3     0    26
## 4     4     1    27
## 5     5     1    27
## 6     6     3    27
## 7     7     0    27
## 8     8     0    27
## 9     9     1    27
## 10    10    0    27
## # ... with 725 more rows
```

Bashing into shape, and fitting model

- ▶ sex and brand not meaningful as numbers, so turn into factors:

```
brandpref = brandpref %>%  
  mutate(sex=factor(sex)) %>%  
  mutate(brand=factor(brand))
```

- ▶ We use multinom from package nnet. Works like polr.

```
brands.1=multinom(brand~age+sex,data=brandpref)  
  
## # weights: 12 (6 variable)  
## initial value 807.480032  
## iter 10 value 702.976983  
## final value 702.970704  
## converged
```

Can we drop anything?

- ▶ Unfortunately drop1 seems not to work:

```
drop1(brands.1,test="Chisq",trace=0)  
## trying - age  
## Error in if (trace) {: argument is not interpretable  
## as logical
```

- ▶ so fall back on fitting model without what you want to test, and comparing using anova.

Do age/sex help predict brand? 1/2

Fit models without each of age and sex:

```
brands.2=multinom(brand~age,data=brandpref)

## # weights: 9 (4 variable)
## initial value 807.480032
## iter 10 value 706.796323
## iter 10 value 706.796322
## final value 706.796322
## converged

brands.3=multinom(brand~sex,data=brandpref)

## # weights: 9 (4 variable)
## initial value 807.480032
## final value 791.861266
## converged
```

Do age/sex help predict brand? 2/2

```
anova(brands.2,brands.1)

## Likelihood ratio tests of Multinomial Models
##
## Response: brand
##      Model Resid. df Resid. Dev   Test     Df LR stat. Pr(Chi)
## 1      age     1466  1413.593
## 2 age + sex     1464  1405.941 1 vs 2      2 7.651236 0.02180495

anova(brands.3,brands.1)

## Likelihood ratio tests of Multinomial Models
##
## Response: brand
##      Model Resid. df Resid. Dev   Test     Df LR stat. Pr(Chi)
## 1      sex     1466  1583.723
## 2 age + sex     1464  1405.941 1 vs 2      2 177.7811       0
```

Do age/sex help predict brand? 3/3

- ▶ age definitely significant (second anova)
- ▶ sex seems significant also (first anova)
- ▶ Keep both.

Another way to build model

- ▶ Start from model with everything and run step:

```
step(brands.1, trace=0)

## trying - age
## trying - sex
## Call:
## multinom(formula = brand ~ age + sex, data = brandpref)
##
## Coefficients:
## (Intercept)      age      sex1
## 2   -11.77469 0.3682075 0.5238197
## 3   -22.72141 0.6859087 0.4659488
##
## Residual Deviance: 1405.941
## AIC: 1417.941
```

- ▶ Final model contains both age and sex so neither could be removed.

Predictions: all possible combinations

Create data frame with various age and sex:

```
ages=c(24,28,32,35,38)
sexes=factor(0:1)
new=crossing(age=ages, sex=sexes)
new

## # A tibble: 10 x 2
##       age   sex
##   <dbl> <fct>
## 1     24 0
## 2     24 1
## 3     28 0
## 4     28 1
## 5     32 0
## 6     32 1
## 7     35 0
## 8     35 1
## 9     38 0
## 10    38 1
```

Making predictions

```
p=predict(brands.1,new,type="probs")
probs=cbind(new,p)
```

The predictions

probs

```
##      age sex          1          2          3
## 1    24   0 0.94795822 0.05022928 0.001812497
## 2    24   1 0.91532076 0.08189042 0.002788820
## 3    28   0 0.79313204 0.18329690 0.023571058
## 4    28   1 0.69561789 0.27143910 0.032943012
## 5    32   0 0.40487271 0.40810321 0.187024082
## 6    32   1 0.29086347 0.49503135 0.214105181
## 7    35   0 0.13057819 0.39724053 0.472181272
## 8    35   1 0.08404134 0.43168592 0.484272746
## 9    38   0 0.02598163 0.23855071 0.735467663
## 10   38   1 0.01623089 0.25162197 0.732147148
```

- ▶ Young males (`sex=0`) prefer brand 1, but older males prefer brand 3.
- ▶ Females similar, but like brand 1 less and brand 2 more.

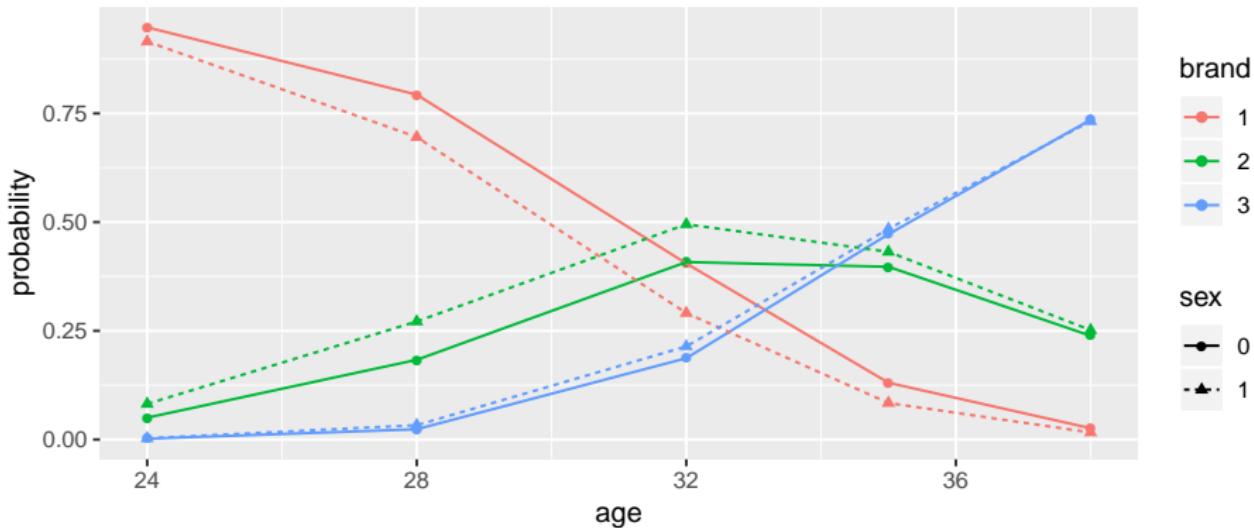
Making a plot

- ▶ Plot fitted probability against age, distinguishing brand by colour and gender by plotting symbol.
- ▶ Also join points by lines, and distinguish lines by gender.
- ▶ I thought about facetting, but this seems to come out clearer.
- ▶ First need tidy data frame, by familiar process:

```
probs.long = probs %>%  
  gather(brand,probability,-(age:sex))  
sample_n(probs.long,7) # 7 random rows  
  
##      age sex brand probability  
## 19    38   0     2  0.23855071  
## 8     35   1     1  0.08404134  
## 30    38   1     3  0.73214715  
## 3     28   0     1  0.79313204  
## 25    32   0     3  0.18702408  
## 5     32   0     1  0.40487271  
## 15    32   0     2  0.40810321
```

The plot

```
ggplot(probs.long, aes(x=age, y=probability,  
colour=brand, shape=sex))+  
  geom_point() + geom_line(aes(linetype=sex))
```



Digesting the plot

- ▶ Brand vs. age: younger people (of both genders) prefer brand 1, but older people (of both genders) prefer brand 3. (Explains significant age effect.)
- ▶ Brand vs. sex: females (dashed) like brand 1 less than males (solid), like brand 2 more (for all ages). more.
- ▶ Not much brand difference between genders (solid and dashed lines of same colours close), but enough to be significant.
- ▶ Model didn't include interaction, so modelled effect of gender on brand same for each age, modelled effect of age same for each gender.

Alternative data format

Summarize all people of same brand preference, same sex, same age on one line of data file with frequency on end:

```
1 0 24 1
1 0 26 2
1 0 27 4
1 0 28 4
1 0 29 7
1 0 30 3
...
```

Whole data set in 65 lines not 735! But how?

Getting alternative data format

```
brandpref %>%
  group_by(age,sex,brand) %>%
  summarize(Freq=n()) %>%
  ungroup() -> b
b %>% slice(1:6)

## # A tibble: 6 x 4
##       age   sex   brand Freq
##     <int> <fct> <fct> <int>
## 1     24     0      1     1
## 2     26     0      1     2
## 3     27     0      1     4
## 4     27     1      1     4
## 5     27     1      3     1
## 6     28     0      1     4
```

Fitting models, almost the same

- ▶ Just have to remember `weights` to incorporate frequencies.
- ▶ Otherwise `multinom` assumes you have just 1 obs on each line!
- ▶ Again turn (numerical) `sex` and `brand` into factors:

```
bf = b %>%
      mutate(sex=factor(sex)) %>%
      mutate(brand=factor(brand))
b.1=multinom(brand~age+sex,data=bf,weights=Freq)

## # weights: 12 (6 variable)
## initial value 807.480032
## iter 10 value 702.976983
## final value 702.970704
## converged

b.2=multinom(brand~age,data=bf,weights=Freq)

## # weights: 9 (4 variable)
## initial value 807.480032
## iter 10 value 706.796323
## iter 10 value 706.796322
## final value 706.796322
## converged
```

P-value for sex identical

```
anova(b.2,b.1)

## Likelihood ratio tests of Multinomial Models
##
## Response: brand
##      Model Resid. df Resid. Dev    Test     Df LR stat.    Pr(Chi)
## 1      age     126   1413.593
## 2 age + sex     124   1405.941 1 vs 2      2 7.651236 0.02180495
```

Same P-value as before, so we haven't changed anything important.

Including data on plot

- ▶ Everyone's age given as whole number, so maybe not too many different ages with sensible amount of data at each:

```
b %>% group_by(age) %>%
  summarize(total=sum(Freq))

## # A tibble: 14 x 2
##       age total
##   <int> <int>
## 1     24     1
## 2     26     2
## 3     27     9
## 4     28    15
## 5     29    19
## 6     30    23
## 7     31    40
## 8     32   333
## 9     33    55
## 10    34    64
## 11    35    35
## 12    36    85
## 13    37    22
## 14    38    32
```

Comments and next

- ▶ Not great (especially at low end), but live with it.
- ▶ Need proportions of frequencies in each brand for each age-gender combination. Mimic what we did for miners:

```
b %>%  
  group_by(age,sex) %>%  
  mutate(proportion=Freq/sum(Freq)) -> brands
```

Checking proportions for age 32

```
brands %>% filter(age==32)

## # A tibble: 6 x 5
## # Groups:   age, sex [2]
##       age  sex brand Freq proportion
##     <int> <fct> <fct> <int>      <dbl>
## 1     32  0     1      48      0.407
## 2     32  0     2      51      0.432
## 3     32  0     3      19      0.161
## 4     32  1     1      62      0.288
## 5     32  1     2     117      0.544
## 6     32  1     3      36      0.167
```

- ▶ First three proportions (males) add up to 1.
- ▶ Last three proportions (females) add up to 1.
- ▶ So looks like proportions of right thing.

Attempting plot

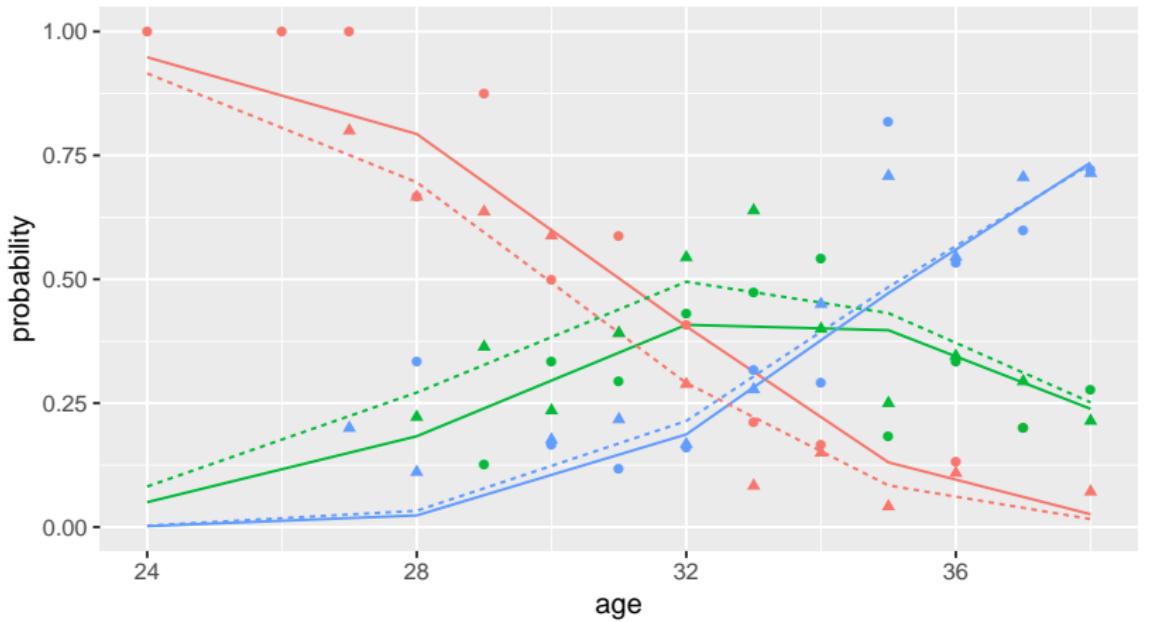
- ▶ Take code from previous plot and:
 - ▶ remove `geom_point` for fitted values
 - ▶ add `geom_point` with correct `data=` and `aes` to plot data.

```
g=ggplot(probs.long,aes(x=age,y=probability,  
colour=brand,shape=sex))+  
  geom_line(aes(linetype=sex))+  
  geom_point(data=brands,aes(y=proportion))
```

- ▶ Data seem to correspond more or less to fitted curves:

The plot

gg



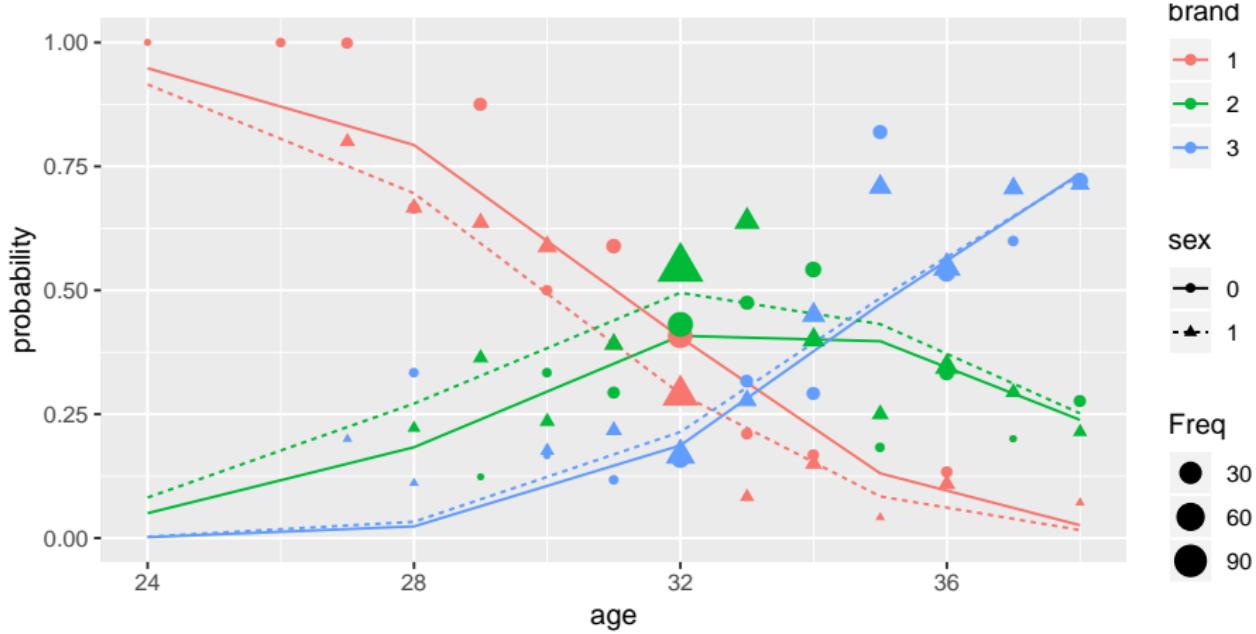
But...

- ▶ Some of the plotted points based on a lot of people, and some only a few.
- ▶ Idea: make the *size* of plotted point bigger if point based on a lot of people (in Freq).
- ▶ Hope that larger points then closer to predictions.
- ▶ Code:

```
g=ggplot(probs.long,aes(x=age,y=probability,  
    colour=brand,shape=sex))+  
    geom_line(aes(linetype=sex))+  
    geom_point(data=brands,  
        aes(y=proportion,size=Freq))
```

The plot

gg



Trying interaction between age and gender

```
b.4=update(b.1,.~.+age:sex)

## # weights: 15 (8 variable)
## initial value 807.480032
## iter 10 value 704.811229
## iter 20 value 702.582802
## final value 702.582761
## converged

anova(b.1,b.4)

## Likelihood ratio tests of Multinomial Models
##
## Response: brand
##          Model Resid. df Resid. Dev   Test    Df
## 1      age + sex     124  1405.941
## 2 age + sex + age:sex     122  1405.166 1 vs 2      2
## LR stat. Pr(Chi)
## 1
## 2 0.7758861 0.678451
```

- ▶ No evidence that effect of age on brand preference differs for the two genders.

Section 4

Survival analysis

Survival analysis

- ▶ So far, have seen:
 - ▶ response variable counted or measured (regression)
 - ▶ response variable categorized (logistic regression)
- and have predicted response from explanatory variables.
- ▶ But what if response is time until event (eg. time of survival after surgery)?
- ▶ Additional complication: event might not have happened at end of study (eg. patient still alive). But knowing that patient has “not died yet” presumably informative. Such data called *censored*.
- ▶ Enter *survival analysis*, in particular the “Cox proportional hazards model”.
- ▶ Explanatory variables in this context often called *covariates*.

Example: still dancing?

- ▶ 12 women who have just started taking dancing lessons are followed for up to a year, to see whether they are still taking dancing lessons, or have quit. The “event” here is “quit”.
- ▶ This might depend on:
 - ▶ a treatment (visit to a dance competition)
 - ▶ woman's age (at start of study).
- ▶ Data:

Months	Quit	Treatment	Age
1	1	0	16
2	1	0	24
2	1	0	18
3	0	0	27
4	1	0	25
7	1	1	26
8	1	1	36
10	1	1	38
10	0	1	45
12	1	1	47

About the data

- ▶ months and quit are kind of combined response:
 - ▶ Months is number of months a woman was actually observed dancing
 - ▶ quit is 1 if woman quit, 0 if still dancing at end of study.
- ▶ Treatment is 1 if woman went to dance competition, 0 otherwise.
- ▶ Fit model and see whether Age or Treatment have effect on survival.
- ▶ Want to do predictions for probabilities of still dancing as they depend on whatever is significant, and draw plot.

The code

- ▶ Install packages `survival` and `survminer` if not done.
- ▶ Load `survival`, `survminer`, `broom` and `tidyverse` packages, read data (column-aligned):

```
library(tidyverse)
library(survival)
library(survminer)
library(broom)
my_url="http://www.utsc.utoronto.ca/~butler/d29/dancing.txt"
dance=read_table(my_url)
```

The data

dance

```
## # A tibble: 12 x 4
##   Months Quit Treatment Age
##   <int> <int>     <int> <int>
## 1     1     1         0     16
## 2     2     2         1     24
## 3     2     2         1     18
## 4     3     3         0     27
## 5     4     4         1     25
## 6     5     5         1     21
## 7    11    11        1     55
## 8     7     7         1     26
## 9     8     8         1     36
## 10    10    10        1     38
## 11    10    10        0     45
## 12    12    12        1     47
```

Examine response and fit model

- ▶ Response variable (has to be outside data frame):

```
mth=with(dance, Surv(Months, Quit))  
mth  
## [1] 1 2 2 3+ 4 5 11 7 8 10 10+ 12
```

- ▶ Then fit model, predicting mth from explanatories:

```
dance.1=coxph(mth~Treatment+Age, data=dance)
```

Output looks a lot like regression

```
summary(dance.1)

## Call:
## coxph(formula = mth ~ Treatment + Age, data = dance)
##
##      n= 12, number of events= 10
##
##              coef  exp(coef) se(coef)      z Pr(>|z|)
## Treatment -4.44915   0.01169  2.60929 -1.705   0.0882 .
## Age        -0.36619   0.69337  0.15381 -2.381   0.0173 *
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
##          exp(coef) exp(-coef) lower .95 upper .95
## Treatment    0.01169     85.554 7.026e-05    1.9444
## Age         0.69337     1.442 5.129e-01    0.9373
##
## Concordance= 0.964  (se = 0.039 )
## Rsquare= 0.836  (max possible= 0.938 )
## Likelihood ratio test= 21.68  on 2 df,  p=2e-05
## Wald test           = 5.67  on 2 df,  p=0.06
## Score (logrank) test = 14.75  on 2 df,  p=6e-04
```

Conclusions

- ▶ Use $\alpha = 0.10$ here since not much data.
- ▶ Three tests at bottom like global F-test. Consensus that something predicts survival time (whether or not dancer quit and how long it took).
- ▶ Age (definitely), Treatment (marginally) both predict survival time.

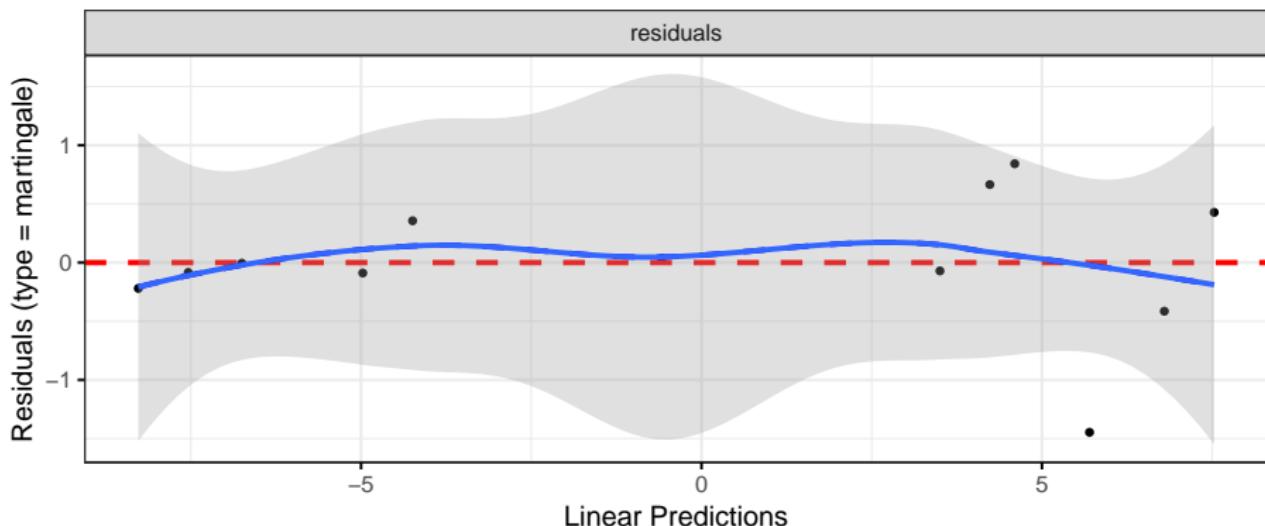
Model checking

- ▶ With regression, usually plot residuals against fitted values.
- ▶ Not quite same here (nonlinear model), but “martingale residuals” should have no pattern vs. “linear predictor”.
- ▶ `ggcoxdiagnostics` from package `survminer` makes plot, to which we add smooth. If smooth trend more or less straight across, model OK.
- ▶ Martingale residuals can go very negative, so won’t always look normal.

Martingale residual plot for dance data

```
ggcoxdiagnostics(dance.1)+geom_smooth(se=F)
```

```
## `geom_smooth()` using method = 'loess' and formula 'y ~ x'
```



This looks good (with only 12 points).

Predicted survival probs

The function we use is called `survfit`, though actually works rather like `predict`.

First create a data frame of values to predict from. We'll do all combos of ages 20 and 40, treatment and not, using `crossing` to get all the combos:

```
treatments=c(0,1)
ages=c(20,40)
dance.new=crossing(Treatment=treatments, Age=ages)
dance.new

## # A tibble: 4 x 2
##   Treatment   Age
##       <dbl> <dbl>
## 1         0     20
## 2         0     40
## 3         1     20
## 4         1     40
```

The predictions

One prediction *for each time* for each combo of age and treatment:

```
s=survfit(dance.1,newdata=dance.new,data=dance)
summary(s)

## Call: survfit(formula = dance.1, newdata = dance.new, data = dance)
##
##    time n.risk n.event survival1 survival2 survival3 survival4
##      1      12      1 8.76e-01 1.000e+00 9.98e-01 1.000
##      2      11      2 3.99e-01 9.99e-01 9.89e-01 1.000
##      4       8      1 1.24e-01 9.99e-01 9.76e-01 1.000
##      5       7      1 2.93e-02 9.98e-01 9.60e-01 1.000
##      7       6      1 2.96e-323 6.13e-01 1.70e-04 0.994
##      8       5      1 0.00e+00 2.99e-06 1.35e-98 0.862
##     10      4      1 0.00e+00 3.61e-20 0.00e+00 0.593
##     11      2      1 0.00e+00 0.00e+00 0.00e+00 0.000
##     12      1      1 0.00e+00 0.00e+00 0.00e+00 0.000
```

```
t(dance.new)
```

```
## [,1] [,2] [,3] [,4]
## Treatment 0 0 1 1
## Age 20 40 20 40
```

`dance.new` transposed (flipped around) shows which combo the four lists of survival probabilities belong to.

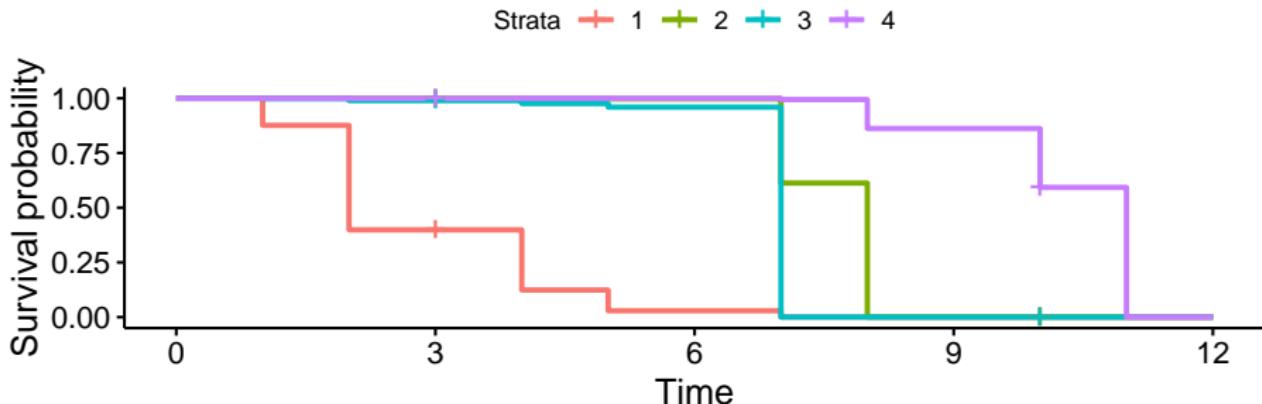
Conclusions from predicted probs

- ▶ Older women more likely to be still dancing than younger women (compare “profiles” for same treatment group).
- ▶ Effect of treatment seems to be to increase prob of still dancing (compare “profiles” for same age for treatment group vs. not)
- ▶ Would be nice to see this on a graph. This is `ggsurvplot` from package `survminer`:

```
g=ggsurvplot(s,conf.int=F)
```

Plotting survival probabilities

gg



Stratum	Age	Treatment
1	20	no
2	20	yes
3	40	no
4	40	yes

Discussion

- ▶ Survivor curve farther to the right is better (better chance of surviving longer).
- ▶ Best is age 40 with treatment, worst age 20 without.
- ▶ Appears to be:
 - ▶ age effect (40 better than 20)
 - ▶ treatment effect (treatment better than not)
- ▶ In analysis, treatment effect only marginally significant.

A more realistic example: lung cancer

- ▶ When you load in an R package, get data sets to illustrate functions in the package.
- ▶ One such is `lung`. Data set measuring survival in patients with advanced lung cancer.
- ▶ Along with survival time, number of “performance scores” included, measuring how well patients can perform daily activities.
- ▶ Sometimes high good, but sometimes bad!
- ▶ Variables below, from the help file data set (`?lung`).

The variables

Format

inst: Institution code

time: Survival time in days

status: censoring status 1=censored, 2=dead

age: Age in years

sex: Male=1 Female=2

ph.ecog: ECOG performance score (0=good 5=dead)

ph.karno: Karnofsky performance score (bad=0-good=100) rated by physician

pat.karno: Karnofsky performance score as rated by patient

meal.cal: Calories consumed at meals

wt.loss: Weight loss in last six months

Uh oh, missing values

```
lung %>% slice(1:16)
```

```
##   inst time status age sex ph.ecog ph.karno pat.karno meal.cal wt.loss
## 1     3 306      2  74   1       1      90       100     1175     NA
## 2     3 455      2  68   1       0      90       90      1225     15
## 3     3 1010     1  56   1       0      90       90      NA      15
## 4     5 210      2  57   1       1      90       60      1150     11
## 5     1 883      2  60   1       0     100       90      NA      0
## 6    12 1022     1  74   1       1      50       80      513      0
## 7     7 310      2  68   2       2      70       60      384      10
## 8    11 361      2  71   2       2      60       80      538      1
## 9     1 218      2  53   1       1      70       80      825      16
## 10    7 166      2  61   1       2      70       70      271      34
## 11    6 170      2  57   1       1      80       80     1025      27
## 12   16 654      2  68   2       2      70       70      NA      23
## 13   11 728      2  68   2       1      90       90      NA      5
## 14   21  71      2  60   1      NA      60       70     1225     32
## 15   12 567      2  57   1       1      80       70     2600     60
## 16    1 144      2  67   1       1      80       90      NA      15
```

A closer look

```
summary(lung)
```

```
##      inst          time        status         age         sex
##  Min.   : 1.00   Min.   : 5.0   Min.   :1.000   Min.   :39.00   Min.   :1.000
##  1st Qu.: 3.00   1st Qu.: 166.8  1st Qu.:1.000   1st Qu.:56.00   1st Qu.:1.000
##  Median :11.00   Median : 255.5  Median :2.000   Median :63.00   Median :1.000
##  Mean    :11.09   Mean    : 305.2  Mean    :1.724   Mean    :62.45   Mean    :1.395
##  3rd Qu.:16.00   3rd Qu.: 396.5  3rd Qu.:2.000   3rd Qu.:69.00   3rd Qu.:2.000
##  Max.    :33.00   Max.    :1022.0  Max.    :2.000   Max.    :82.00   Max.    :2.000
##  NA's    :1
##      ph.ecog        ph.karno       pat.karno       meal.cal       wt.loss
##  Min.   :0.0000   Min.   : 50.00   Min.   : 30.00   Min.   : 96.0   Min.   :-24.000
##  1st Qu.:0.0000   1st Qu.: 75.00   1st Qu.: 70.00   1st Qu.:635.0   1st Qu.:  0.000
##  Median :1.0000   Median : 80.00   Median : 80.00   Median :975.0   Median :  7.000
##  Mean    :0.9515   Mean    : 81.94   Mean    : 79.96   Mean    :928.8   Mean    : 9.832
##  3rd Qu.:1.0000   3rd Qu.: 90.00   3rd Qu.: 90.00   3rd Qu.:1150.0  3rd Qu.: 15.750
##  Max.    :3.0000   Max.    :100.00   Max.    :100.00   Max.    :2600.0  Max.    : 68.000
##  NA's    :1         NA's   :1         NA's   :3         NA's   :47        NA's   :14
```

Remove any obs with any missing values

```
cc=complete.cases(lung)
lung %>% filter(cc) -> lung.complete
lung.complete %>%
  select(meal.cal:wt.loss) %>% head(10)

##      meal.cal wt.loss
## 1      1225     15
## 2      1150     11
## 3       513      0
## 4       384     10
## 5       538      1
## 6       825     16
## 7       271     34
## 8      1025     27
## 9      2600     60
## 10     1150    -5
```

Missing values seem to be gone.

Check!

```
summary(lung.complete)
```

```
##      inst          time        status         age         sex
##  Min.   : 1.00   Min.   : 5.0   Min.   :1.000   Min.   :39.00   Min.   :1.000
##  1st Qu.: 3.00   1st Qu.: 174.5  1st Qu.:1.000   1st Qu.:57.00   1st Qu.:1.000
##  Median :11.00   Median : 268.0  Median :2.000   Median :64.00   Median :1.000
##  Mean   :10.71   Mean   : 309.9  Mean   :1.719   Mean   :62.57   Mean   :1.383
##  3rd Qu.:15.00   3rd Qu.: 419.5  3rd Qu.:2.000   3rd Qu.:70.00   3rd Qu.:2.000
##  Max.   :32.00   Max.   :1022.0  Max.   :2.000   Max.   :82.00   Max.   :2.000
##      ph.ecog       ph.karno      pat.karno     meal.cal      wt.loss
##  Min.   :0.0000   Min.   : 50.00   Min.   : 30.00   Min.   : 96.0   Min.   :-24.000
##  1st Qu.:0.0000   1st Qu.: 70.00   1st Qu.: 70.00   1st Qu.: 619.0  1st Qu.: 0.000
##  Median :1.0000   Median : 80.00   Median : 80.00   Median : 975.0  Median : 7.000
##  Mean   :0.9581   Mean   : 82.04   Mean   : 79.58   Mean   : 929.1  Mean   : 9.719
##  3rd Qu.:1.0000   3rd Qu.: 90.00   3rd Qu.: 90.00   3rd Qu.:1162.5 3rd Qu.: 15.000
##  Max.   :3.0000   Max.   :100.00   Max.   :100.00   Max.   :2600.0  Max.   : 68.000
```

No missing values left.

Model 1: use everything except inst

```
str(lung.complete)

## 'data.frame': 167 obs. of 10 variables:
## $ inst      : num  3 5 12 7 11 1 7 6 12 22 ...
## $ time      : num  455 210 1022 310 361 ...
## $ status     : num  2 2 1 2 2 2 2 2 2 2 ...
## $ age       : num  68 57 74 68 71 53 61 57 57 70 ...
## $ sex       : num  1 1 1 2 2 1 1 1 1 1 ...
## $ ph.ecog   : num  0 1 1 2 2 1 2 1 1 1 ...
## $ ph.karno  : num  90 90 50 70 60 70 70 80 80 90 ...
## $ pat.karno: num  90 60 80 60 80 80 70 80 70 100 ...
## $ meal.cal  : num  1225 1150 513 384 538 ...
## $ wt.loss   : num  15 11 0 10 1 16 34 27 60 -5 ...
```

```
resp=with(lung.complete, Surv(time, status==2))
lung.1=coxph(resp~.-inst-time-status,
             data=lung.complete)
```

“Dot” means “all the other variables”.

summary of model 1: too tiny to see!

```
summary(lung.1)

## Call:
## coxph(formula = resp ~ . - inst - time - status, data = lung.complete)
##
## n= 167, number of events= 120
##
##          coef  exp(coef)   se(coef)      z Pr(>|z|)
## age     1.080e-02 1.011e+00 1.160e-02  0.931  0.35168
## sex    -5.536e-01 5.749e-01 2.016e-01 -2.746  0.00603 **
## ph.ecog 7.395e-01 2.095e+00 2.250e-01  3.287  0.00101 **
## ph.karno 2.244e-02 1.023e+00 1.123e-02  1.998  0.04575 *
## pat.karno -1.207e-02 9.880e-01 8.116e-03 -1.488  0.13685
## meal.cal  2.835e-05 1.000e+00 2.594e-04  0.109  0.91298
## wt.loss   -1.420e-02 9.859e-01 7.766e-03 -1.828  0.06748 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
##          exp(coef) exp(-coef) lower .95 upper .95
## age      1.0109    0.9893   0.9881    1.0341
## sex      0.5749    1.7395   0.3872    0.8534
## ph.ecog   2.0950    0.4773   1.3479    3.2560
## ph.karno   1.0227    0.9778   1.0004    1.0455
## pat.karno   0.9880    1.0121   0.9724    1.0038
## meal.cal    1.0000    1.0000   0.9995    1.0005
## wt.loss     0.9859    1.0143   0.9710    1.0010
##
## Concordance= 0.653  (se = 0.029 )
## Rsquare= 0.155  (max possible= 0.998 )
## Likelihood ratio test= 28.16 on 7 df,  p=2e-04
## Wald test           = 27.5 on 7 df,  p=3e-04
## Score (logrank) test = 28.31 on 7 df,  p=2e-04
```

Overall significance

The three tests of overall significance:

```
glance(lung.1)[c(4,6,8)]  
  
## # A tibble: 1 x 3  
##   p.value.log p.value.sc p.value.wald  
##       <dbl>      <dbl>      <dbl>  
## 1     0.000205  0.000193  0.000271
```

All strongly significant. *Something* predicts survival.

Coefficients for model 1

```
tidy(lung.1) %>% select(term, p.value) %>% arrange(p.value)

## # A tibble: 7 x 2
##   term      p.value
##   <chr>     <dbl>
## 1 ph.ecog  0.00101
## 2 sex       0.00603
## 3 ph.karno  0.0457
## 4 wt.loss   0.0675
## 5 pat.karno 0.137
## 6 age       0.352
## 7 meal.cal  0.913
```

- ▶ Model as a whole significant (strongly)
- ▶ sex and ph.ecog definitely significant
- ▶ age, pat.karno and meal.cal definitely not
- ▶ others in between

Model 2

```
lung.2=update(lung.1, .~.-age-pat.karno-meal.cal)
tidy(lung.2) %>% select(term,p.value)

## # A tibble: 4 x 2
##   term      p.value
##   <chr>     <dbl>
## 1 sex      0.00409
## 2 ph.ecog  0.000112
## 3 ph.karno 0.101
## 4 wt.loss  0.108
```

- ▶ Compare with first model:

```
anova(lung.2, lung.1)

## Analysis of Deviance Table
## Cox model: response is resp
## Model 1: ~ sex + ph.ecog + ph.karno + wt.loss
## Model 2: ~ (inst + time + status + age + sex + ph.ecog + ph.karno +
##              loglik Chisq Df P(>|Chi|)
## 1 -495.67
## 2 -494.03 3.269 3       0.352
```

- ▶ No harm in taking out those variables.

Model 3, and last

Take out ph.karno and wt.loss as well.

```
lung.3=update(lung.2,.~.-ph.karno-wt.loss)
tidy(lung.3) %>% select(term,estimate,p.value)

## # A tibble: 2 x 3
##   term      estimate  p.value
##   <chr>      <dbl>    <dbl>
## 1 sex       -0.510  0.00958
## 2 ph.ecog    0.483  0.000266

anova(lung.3, lung.2)

## Analysis of Deviance Table
## Cox model: response is resp
## Model 1: ~ sex + ph.ecog
## Model 2: ~ sex + ph.ecog + ph.karno + wt.loss
## loglik  Chisq Df P(>|Chi|)
## 1 -498.38
## 2 -495.67 5.4135  2  0.06675 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Commentary

- ▶ OK (just) to take out those two covariates.
- ▶ Both remaining variables strongly significant.
- ▶ Effect on survival time:
 - ▶ Higher value of sex (female) has *negative* effect on event (death).
 - ▶ Higher value of ph.ecog has *positive* effect on death.
 - ▶ i. e. being female or having lower ph.ecog score has positive effect on survival.
- ▶ Picture?

Plotting survival probabilities

- ▶ Create new data frame of values to predict for, then predict:

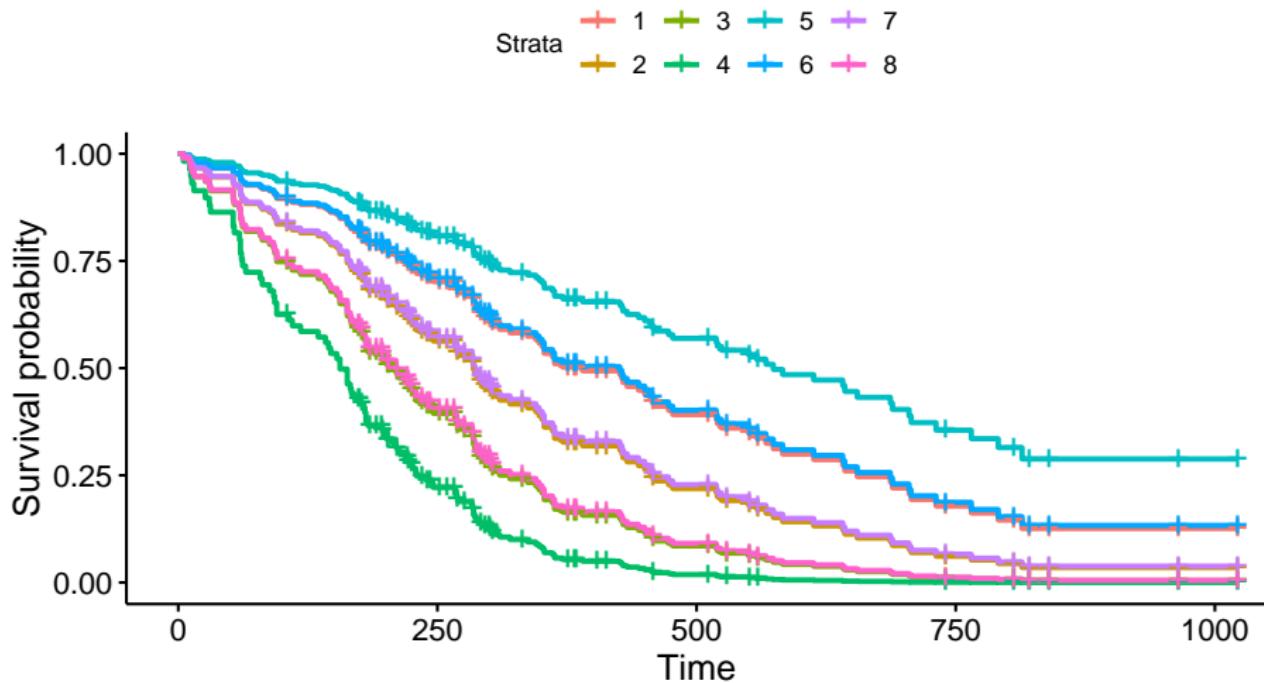
```
sexes=c(1,2)
ph.ecogs=0:3
lung.new=crossing(sex=sexes,ph.ecog=ph.ecogs)
lung.new

## # A tibble: 8 x 2
##       sex ph.ecog
##   <dbl>    <int>
## 1     1        0
## 2     1        1
## 3     1        2
## 4     1        3
## 5     2        0
## 6     2        1
## 7     2        2
## 8     2        3

s=survfit(lung.3,data=lung.complete,newdata=lung.new)
```

The plot

```
ggsurvplot(s, conf.int=F)
```



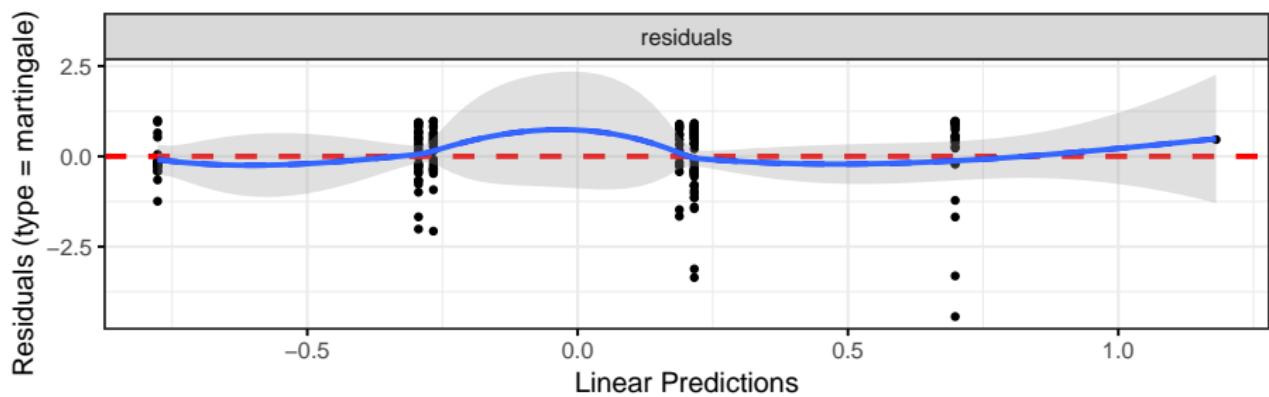
Discussion of survival curves

- ▶ Best survival is teal-blue curve, stratum 5, females with (ph.ecog) score 0.
- ▶ Next best: blue, stratum 6, females with score 1, and red, stratum 1, males score 0.
- ▶ Worst: green, stratum 4, males score 3.
- ▶ For any given ph.ecog score, females have better predicted survival than males.
- ▶ For both genders, a lower score associated with better survival.
- ▶ sex coeff in model 3 negative, so being higher sex value (female) goes with *less* hazard of dying.
- ▶ ph.ecog coeff in model 3 positive, so higher ph.ecog score goes with *more* hazard of dying
- ▶ Two coeffs about same size, so being male rather than female corresponds to 1-point increase in ph.ecog score. Note how survival curves come in 3 pairs plus 2 odd

Martingale residuals for this model

```
ggcoxdiagnostics(lung.3)+geom_smooth(se=F)
```

```
## `geom_smooth()` using method = 'loess' and formula 'y ~ x'
```



No problems here.

When the Cox model fails

- ▶ Invent some data where survival is best at middling age, and worse at high *and* low age:

```
age=seq(20,60,5)
survtime=c(10,12,11,21,15,20,8,9,11)
stat=c(1,1,1,1,0,1,1,1,1)
d=tibble(age,survtime,stat)
y=with(d, Surv(survtime,stat))
```

- ▶ Small survival time 15 in middle was actually censored, so would have been longer if observed.

Fit Cox model

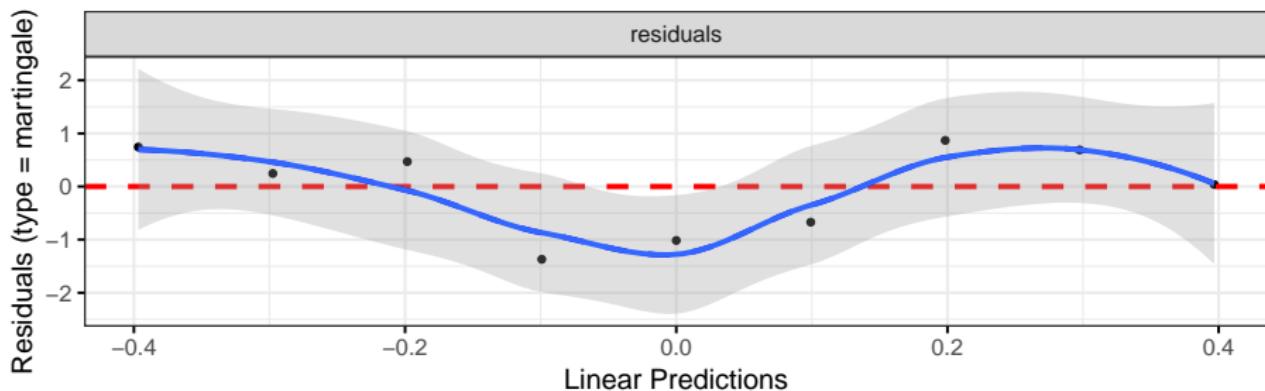
```
y.1=coxph(y~age,data=d)
summary(y.1)

## Call:
## coxph(formula = y ~ age, data = d)
##
##      n= 9, number of events= 8
##
##              coef exp(coef) se(coef)     z Pr(>|z|)
## age  0.01984   1.02003  0.03446  0.576    0.565
##
##          exp(coef) exp(-coef) lower .95 upper .95
## age      1.02      0.9804   0.9534    1.091
##
## Concordance= 0.545  (se = 0.105 )
## Rsquare= 0.036  (max possible= 0.926 )
## Likelihood ratio test= 0.33  on 1 df,    p=0.6
## Wald test             = 0.33  on 1 df,    p=0.6
## Score (logrank) test = 0.33  on 1 df,    p=0.6
```

Martingale residuals

```
ggcoxdiagnostics(y.1)+geom_smooth(se=F)
```

```
## `geom_smooth()` using method = 'loess' and formula 'y ~ x'
```



Down-and-up indicates incorrect relationship between age and survival.
Add age-squared term.

Attempt 2

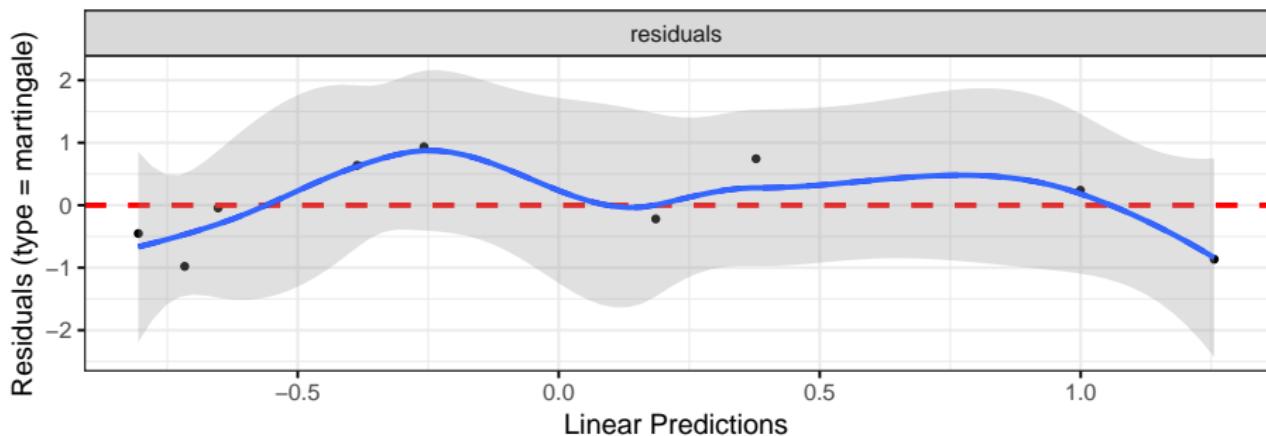
```
y.2=coxph(y~age+I(age^2),data=d)
summary(y.2)

## Call:
## coxph(formula = y ~ age + I(age^2), data = d)
##
##      n= 9, number of events= 8
##
##              coef exp(coef)    se(coef)      z Pr(>|z|)
## age      -0.380184  0.683736  0.241617 -1.573   0.1156
## I(age^2)  0.004832  1.004844  0.002918  1.656   0.0977 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
##          exp(coef) exp(-coef) lower .95 upper .95
## age        0.6837     1.4626    0.4258     1.098
## I(age^2)   1.0048     0.9952    0.9991     1.011
##
## Concordance= 0.758  (se = 0.123 )
## Rsquare= 0.304  (max possible= 0.926 )
## Likelihood ratio test= 3.26  on 2 df,  p=0.2
## Wald test           = 3.16  on 2 df,  p=0.2
## Score (logrank) test = 3.75  on 2 df,  p=0.2
```

Martingale residuals this time

```
ggcoxdiagnostics(y.2)+geom_smooth(se=F)
```

```
## `geom_smooth()` using method = 'loess' and formula 'y ~ x'
```



Not great, but less problematic than before.

Section 5

Analysis of variance

Analysis of variance

- ▶ Analysis of variance used with:
 - ▶ counted/measured response
 - ▶ categorical explanatory variable(s)
 - ▶ that is, data divided into groups, and see if response significantly different among groups
 - ▶ or, see whether knowing group membership helps to predict response.
- ▶ Typically two stages:
 - ▶ F -test to detect *any* differences among/due to groups
 - ▶ if F -test significant, do *multiple comparisons* to see which groups significantly different from which.
 - ▶ Need special multiple comparisons method because just doing (say) two-sample t -tests on each pair of groups gives too big a chance of finding “significant” differences by accident.

Packages

These:

```
library(tidyverse)  
library(broom)
```

Example: Pain threshold and hair colour

- ▶ Do people with different hair colour have different abilities to deal with pain?
- ▶ Men and women of various ages divided into 4 groups by hair colour: light and dark blond, light and dark brown.
- ▶ Each subject given a pain sensitivity test resulting in pain threshold score: higher score is higher pain tolerance.
- ▶ 19 subjects altogether.

The data

In hairpain.txt:

hair pain	darkblond 43
lightblond 62	lightbrown 42
lightblond 60	lightbrown 50
lightblond 71	lightbrown 41
lightblond 55	lightbrown 37
lightblond 48	darkbrown 32
darkblond 63	darkbrown 39
darkblond 57	darkbrown 51
darkblond 52	darkbrown 30
darkblond 41	darkbrown 35

Summarizing the groups

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/hairpain.txt"
hairpain=read_delim(my_url, " ")

## Parsed with column specification:
## cols(
##   hair = col_character(),
##   pain = col_integer()
## )

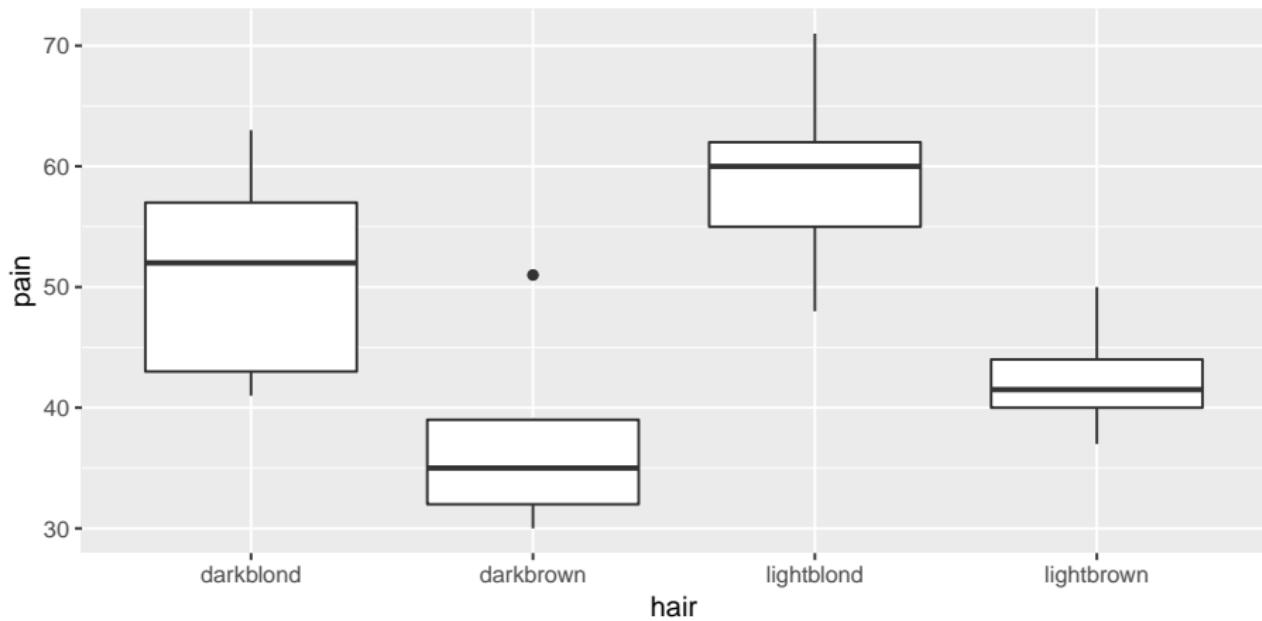
hairpain %>% group_by(hair) %>%
  summarize( n=n(),
             xbar=mean(pain),
             s=sd(pain))

## # A tibble: 4 x 4
##   hair          n    xbar     s
##   <chr>     <int> <dbl> <dbl>
## 1 darkblond     5   51.2  9.28
## 2 darkbrown     5   37.4  8.32
## 3 lightblond    5   59.2  8.53
## 4 lightbrown    4   42.5  5.45
```

Brown-haired people seem to have lower pain tolerance.

Boxplot

```
ggplot(hairpain, aes(x=hair, y=pain))+geom_boxplot()
```



Assumptions

- ▶ Data should be:
 - ▶ normally distributed within each group
 - ▶ same spread for each group
- ▶ darkbrown group has upper outlier (suggests not normal)
- ▶ darkblond group has smaller IQR than other groups.
- ▶ But, groups *small*.
- ▶ Shrug shoulders and continue for moment.

Testing equality of SDs

- ▶ via **Levene's test** in package car:

```
car::leveneTest(pain~hair, data=hairpain)

## Warning in leveneTest.default(y = y, group = group, ...):
## group coerced to factor.

## Levene's Test for Homogeneity of Variance (center = median)
##          Df F value Pr(>F)
## group    3  0.3927   0.76
##          15
```

- ▶ No evidence (at all) of difference among group SDs.
- ▶ Possibly because groups *small*.

Analysis of variance

```
hairpain.1=aov(pain~hair,data=hairpain)
summary(hairpain.1)

##           Df Sum Sq Mean Sq F value    Pr(>F)
## hair         3   1361   453.6   6.791 0.00411 **
## Residuals   15   1002    66.8
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- ▶ P-value small: the mean pain tolerances for the four groups are *not* all the same.
- ▶ Which groups differ from which, and how?

Multiple comparisons

- ▶ Which groups differ from which? Multiple comparisons method. Lots.
- ▶ Problem: by comparing all the groups with each other, doing many tests, have large chance to (possibly incorrectly) reject H_0 : groups have equal means.
- ▶ 4 groups: 6 comparisons (1 vs 2, 1 vs 3, ..., 3 vs 4). 5 groups: 10 comparisons. Thus 6 (or 10) chances to make mistake.
- ▶ Get “familywise error rate” of 0.05 (whatever), no matter how many comparisons you’re doing.
- ▶ My favourite: Tukey, or “honestly significant differences”: how far apart might largest, smallest group means be (if actually no differences). Group means more different: significantly different.

Tukey

► TukeyHSD:

```
TukeyHSD(hairpain.1)
```

```
##      Tukey multiple comparisons of means
##      95% family-wise confidence level
##
## Fit: aov(formula = pain ~ hair, data = hairpain)
##
## $hair
##          diff      lwr      upr    p adj
## darkbrown-darkblond -13.8 -28.696741 1.0967407 0.0740679
## lightblond-darkblond   8.0 -6.896741 22.8967407 0.4355768
## lightbrown-darkblond  -8.7 -24.500380 7.1003795 0.4147283
## lightblond-darkbrown  21.8  6.903259 36.6967407 0.0037079
## lightbrown-darkbrown   5.1 -10.700380 20.9003795 0.7893211
## lightbrown-lightblond -16.7 -32.500380 -0.8996205 0.0366467
```

The old-fashioned way

- ▶ List group means in order
- ▶ Draw lines connecting groups that are *not* significantly different:

darkbrown	lightbrown	darkblond	lightblond
37.4	42.5	51.2	59.2

- ▶ lightblond significantly higher than everything except darkblond (at $\alpha = 0.05$).
- ▶ darkblond in middle ground: not significantly less than lightblond, not significantly greater than darkbrown and lightbrown.
- ▶ More data might resolve this.
- ▶ Looks as if blond-haired people do have higher pain tolerance, but not completely clear.

Some other multiple-comparison methods

- ▶ Work any time you do k tests at once (not just ANOVA).
- ▶ **Bonferroni:** multiply all P-values by k .
- ▶ **Holm:** multiply smallest P-value by k , next-smallest by $k - 1$, etc.
- ▶ False discovery rate: multiply smallest P-value by $k/1$, 2nd-smallest by $k/2$, ..., i -th smallest by k/i .
- ▶ Stop after non-rejection.

Example

- ▶ P-values 0.005, 0.015, 0.03, 0.06 (4 tests all done at once) Use $\alpha = 0.05$.
- ▶ Bonferroni:
 - ▶ Multiply all P-values by 4 (4 tests).
 - ▶ Reject only 1st null.
- ▶ Holm:
 - ▶ Times smallest P-value by 4: $0.005 * 4 = 0.020 < 0.05$, reject.
 - ▶ Times next smallest by 3: $0.015 * 3 = 0.045 < 0.05$, reject.
 - ▶ Times next smallest by 2: $0.03 * 2 = 0.06 > 0.05$, do not reject. Stop.
- ▶ False discovery rate:
 - ▶ Times smallest P-value by 4: $0.005 * 4 = 0.02 < 0.05$: reject.
 - ▶ Times second smallest by 4/2: $0.015 * 4/2 = 0.03 < 0.05$, reject.
 - ▶ Times third smallest by 4/3: $0.03 * 4/3 = 0.04 < 0.05$, reject.
 - ▶ Times fourth smallest by 4/4: $0.06 * 4/4 = 0.06 > 0.05$, do not reject.

pairwise.t.test

```
attach(hairpain)
pairwise.t.test(pain,hair,p.adj="none")

##
##  Pairwise comparisons using t tests with pooled SD
##
## data: pain and hair
##
##      darkblond darkbrown lightblond
## darkbrown  0.01748   -       -
## lightblond 0.14251   0.00075   -
## lightbrown 0.13337   0.36695   0.00817
##
## P value adjustment method: none

pairwise.t.test(pain,hair,p.adj="holm")

##
##  Pairwise comparisons using t tests with pooled SD
##
## data: pain and hair
##
##      darkblond darkbrown lightblond
## darkbrown  0.0699   -       -
## lightblond 0.4001   0.0045   -
## lightbrown 0.4001   0.4001   0.0408
##
## P value adjustment method: holm
```

```
pairwise.t.test(pain,hair,p.adj="fdr")

##
##  Pairwise comparisons using t tests with pooled SD
##
## data: pain and hair
##
##      darkblond darkbrown lightblond
## darkbrown  0.0350   -       -
## lightblond 0.1710   0.0045   -
## lightbrown 0.1710   0.3670   0.0245
##
## P value adjustment method: fdr

pairwise.t.test(pain,hair,p.adj="bon")

##
##  Pairwise comparisons using t tests with pooled SD
##
## data: pain and hair
##
##      darkblond darkbrown lightblond
## darkbrown  0.1049   -       -
## lightblond 0.8550   0.0045   -
## lightbrown 0.8002   1.0000   0.0490
##
## P value adjustment method: bonferroni
```

Comments

- ▶ P-values all adjusted upwards from “none”.
- ▶ Required because 6 tests at once.
- ▶ Highest P-values for Bonferroni: most “conservative”.
- ▶ Prefer Tukey or FDR or Holm.
- ▶ Tukey only applies to ANOVA, not to other cases of multiple testing.

Rats and vitamin B

- ▶ What is the effect of dietary vitamin B on the kidney?
- ▶ A number of rats were randomized to receive either a B-supplemented diet or a regular diet.
- ▶ Desired to control for initial size of rats, so classified into size classes lean and obese.
- ▶ After 20 weeks, rats' kidneys weighed.
- ▶ Variables:
 - ▶ Response: kidneyweight (grams).
 - ▶ Explanatory: diet, ratsize.
- ▶ Read in data:

```
my_url = "http://www.utsc.utoronto.ca/~butler/d29/vitaminb.txt"
vitaminb = read_delim(my_url, " ")
## Parsed with column specification:
## cols(
```

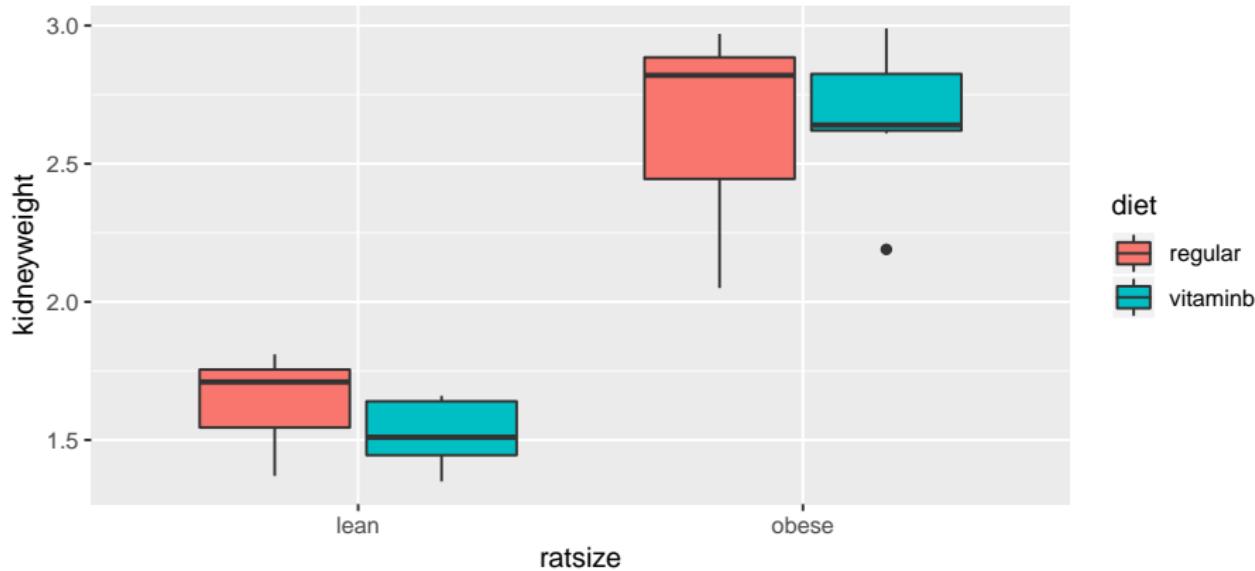
The data

```
vitaminb
```

```
## # A tibble: 28 x 3
##   ratsize diet      kidneyweight
##   <chr>    <chr>        <dbl>
## 1 lean     regular     1.62
## 2 lean     regular     1.8
## 3 lean     regular     1.71
## 4 lean     regular     1.81
## 5 lean     regular     1.47
## 6 lean     regular     1.37
## 7 lean     regular     1.71
## 8 lean     vitaminb   1.51
## 9 lean     vitaminb   1.65
## 10 lean    vitaminb   1.45
## # ... with 18 more rows
```

Grouped boxplot

```
ggplot(vitaminb,aes(x=ratsize,y=kidneyweight,  
fill=diet))+geom_boxplot()
```



What's going on?

- ▶ Calculate group means:

```
summary = vitaminb %>% group_by(ratsize,diet) %>%  
  summarize(mean=mean(kidneyweight))  
  
summary  
  
## # A tibble: 4 x 3  
## # Groups:   ratsize [?]  
##   ratsize diet      mean  
##   <chr>    <chr>    <dbl>  
## 1 lean     regular   1.64  
## 2 lean     vitaminb  1.53  
## 3 obese    regular   2.64  
## 4 obese    vitaminb  2.67
```

- ▶ Rat size: a large and consistent effect.
- ▶ Diet: small/no effect (compare same rat size, different diet).
- ▶ Effect of rat size *same* for each diet: no interaction.

ANOVA with interaction

```
vitaminb.1=aov(kidneyweight~ratsize*diet,
  data=vitaminb)
summary(vitaminb.1)

##                                Df Sum Sq Mean Sq F value    Pr(>F)
## ratsize                  1  8.068   8.068 141.179 1.53e-11 ***
## diet                     1  0.012   0.012   0.218     0.645
## ratsize:diet      1  0.036   0.036   0.638     0.432
## Residuals            24  1.372   0.057
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1
```

Significance/nonsignificance as we expected. Note no significant interaction (can be removed).

Interaction plot

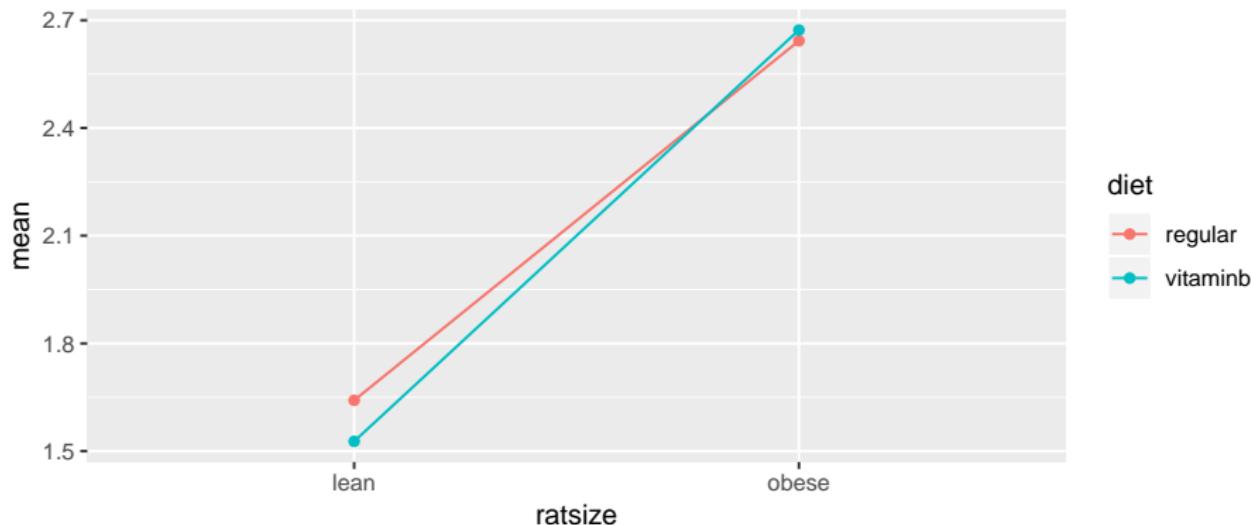
- ▶ Plot mean of response variable against one of the explanatory, using other one as groups. Start from summary:

```
g=ggplot(summary,aes(x=ratsize,y=mean,  
colour=diet,group=diet))+  
geom_point()+geom_line()
```

- ▶ For this, have to give *both* group and colour.

The interaction plot

gg



Lines basically parallel, indicating no interaction.

Take out interaction

```
vitaminb.2=update(vitaminb.1,.~.-ratsize:diet)
summary(vitaminb.2)

##           Df  Sum Sq Mean Sq F value    Pr(>F)
## ratsize      1  8.068   8.068 143.256 7.59e-12 ***
## diet         1  0.012   0.012   0.221     0.643
## Residuals   25  1.408   0.056
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- ▶ No Tukey for diet: not significant.
- ▶ No Tukey for ratsize: only two sizes, and already know that obese rats have larger kidneys than lean ones.
- ▶ Bottom line: diet has no effect on kidney size once you control for size of rat.

The auto noise data

In 1973, the President of Texaco cited an automobile filter developed by Associated Octel Company as effective in reducing pollution. However, questions had been raised about the effects of filter silencing. He referred to the data included in the report (and below) as evidence that the silencing properties of the Octel filter were at least equal to those of standard silencers.

```
my_url="http://www.utsc.utoronto.ca/~butler/c32/autonoise.txt"
autonoise=read_table(my_url)

## Error in open.connection(con, "rb"): HTTP error 403.
```

The data

```
autonoise
```

```
## Error in eval(expr, envir, enclos): object 'autonoise'  
not found
```

Making boxplot

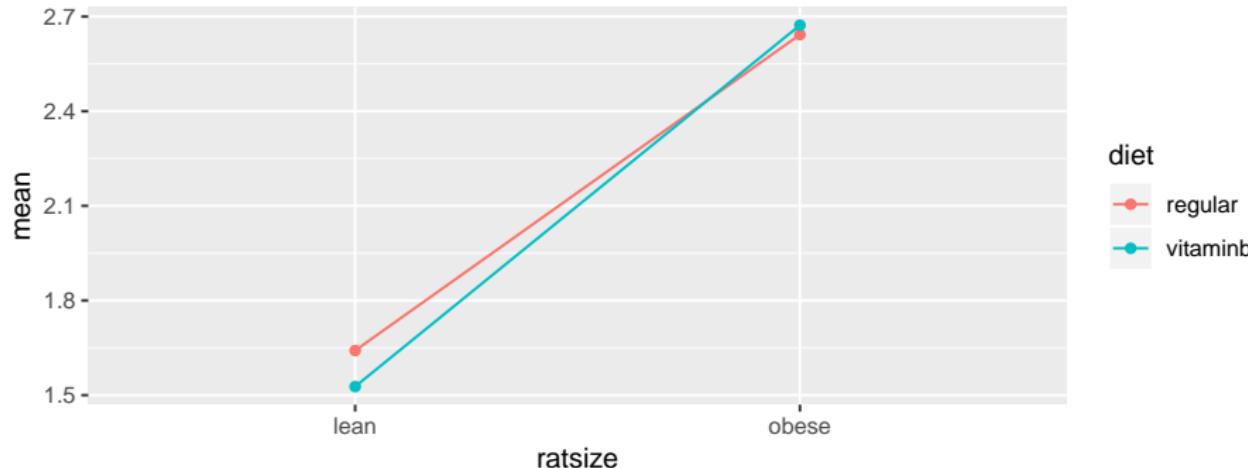
- ▶ Make a boxplot, but have combinations of filter type and engine size.
- ▶ Use grouped boxplot again, thus:

```
g = autonoise %>%
  ggplot(aes(x=size, y=noise, fill=type)) +
  geom_boxplot()

## Error in eval(lhs, parent, parent): object
'autonoise' not found
```

The boxplot

gg



- ▶ Difference in engine noise between Octel and standard is larger for medium engine size than for large or small.
- ▶ Some evidence of differences in spreads (ignore for now).

ANOVA

```
autonoise.1=aov(noise~size*type,data=autonoise)
## Error in terms.formula(formula, "Error", data = data): object
'autonoise' not found
summary(autonoise.1)
## Error in summary(autonoise.1): object 'autonoise.1' not found
```

- ▶ The interaction is significant, as we suspected from the boxplots.
- ▶ The within-group spreads don't look very equal, but only based on 6 obs each.

Tukey: ouch!

```
autonoise.2=TukeyHSD(autonoise.1)
## Error in TukeyHSD(autonoise.1): object 'autonoise.1' not found
autonoise.2$`size:type`
## Error in eval(expr, envir, enclos): object 'autonoise.2' not found
```

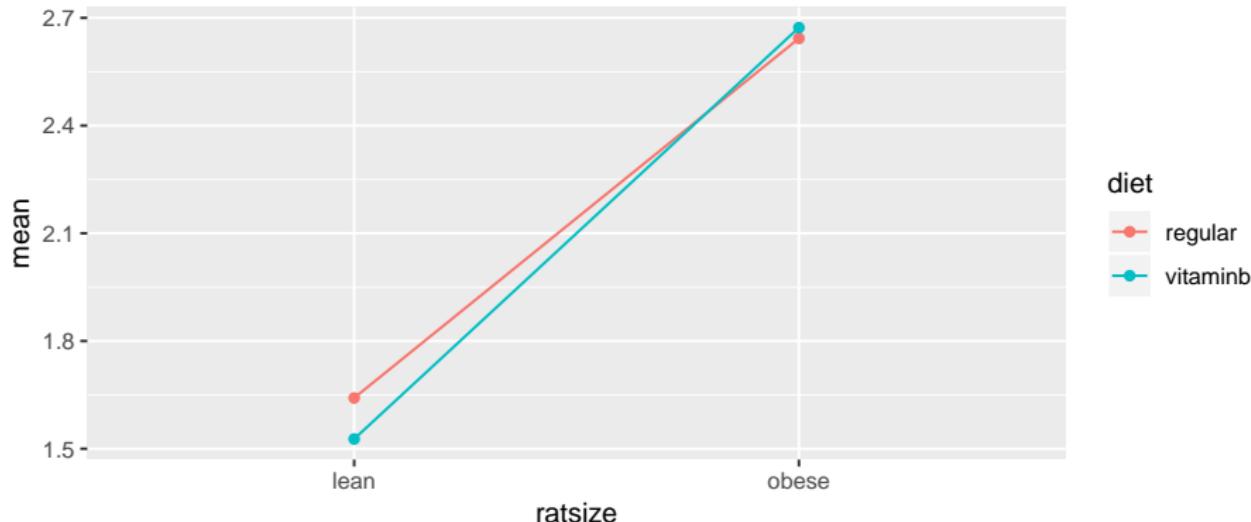
Interaction plot

- ▶ This time, don't have summary of mean noise for each size-type combination.
- ▶ One way is to compute summaries (means) first, and feed into ggplot as in vitamin B example.
- ▶ Or, have ggplot compute them for us, thus:

```
g=ggplot(autonoise,aes(x=size,y=noise,  
    colour=type,group=type))+  
    stat_summary(fun.y=mean,geom="point") +  
    stat_summary(fun.y=mean,geom="line")  
  
## Error in ggplot(autonoise, aes(x = size, y = noise,  
colour = type, group = type)): object 'autonoise' not  
found
```

Interaction plot

gg



The lines are definitely *not* parallel, showing that the effect of type is different for medium-sized engines than for others.

If you don't like that...

...then compute the means first, in a pipeline:

```
autonoise %>% group_by(size,type) %>%
  summarize(mean_noise=mean(noise)) %>%
  ggplot(aes(x=size,y=mean_noise,group=type,
    colour=type))+geom_point()+geom_line()

## Error in eval(lhs, parent, parent): object 'autonoise' not
found
```

Simple effects for auto noise example

- ▶ In auto noise example, weren't interested in all comparisons between car size and filter type combinations.
- ▶ Wanted to demonstrate (lack of) difference between filter types *for each car type*.
- ▶ These are called **simple effects** of one variable (filter type) conditional on other variable (car type).
- ▶ To do this, pull out just the data for small cars, compare noise for the two filter types. Then repeat for medium and large cars. (Three one-way ANOVAs.)

Do it using dplyr tools

- ▶ Small cars:

```
autonoise %>% filter(size=="S") %>%  
  aov(noise~type,data=.) %>% summary()  
  
## Error in eval(lhs, parent, parent): object  
'autonoise' not found
```

- ▶ No filter difference for small cars.
- ▶ For Medium, change S to M and repeat.

Simple effect of filter type for medium cars

```
autonoise %>% filter(size=="M") %>%  
  aov(noise~type,data=.) %>% summary()  
  
## Error in eval(lhs, parent, parent): object 'autonoise' not  
found
```

- ▶ There *is* an effect of filter type for medium cars. Look at means to investigate:

```
autonoise %>% filter(size=="M") %>%  
  group_by(type) %>% summarize(m=mean(noise))  
  
## Error in eval(lhs, parent, parent): object 'autonoise' not found
```

Medium and large cars

- ▶ Octel filters produce *less* noise for medium cars.
- ▶ Large cars:

```
autonoise %>% filter(size=="L") %>%  
  aov(noise~type,data=.) %>% summary()  
  
## Error in eval(lhs, parent, parent): object  
'autonoise' not found
```

- ▶ No significant difference again.
- ▶ Or use glance from broom:

```
autonoise %>% filter(size=="L") %>%  
  aov(noise~type,data=.) %>% glance()  
  
## Error in eval(lhs, parent, parent): object 'autonoise' not found
```

All at once, using split/apply/combine

The “split” part:

```
autonoise %>% group_by(size) %>%  
  nest()  
  
## Error in eval(lhs, parent, parent): object 'autonoise'  
not found
```

Now have *three* rows, with the data frame for each size encoded as *one element* of this data frame.

Apply

- ▶ Write function to do aov on a data frame with columns noise and type, returning P-value:

```
aov_pval=function(x) {  
  noise.1=aov(noise~type,data=x)  
  gg=glance(noise.1)  
  gg$p.value  
}
```

- ▶ Test it:

```
autonoise %>% filter(size=="L") %>%  
  aov_pval()  
  
## Error in eval(lhs, parent, parent): object  
'autonoise' not found
```

- ▶ Check.

Combine

- ▶ Apply this function to each of the nested data frames (one per engine size):

```
autonoise %>% group_by(size) %>%
  nest() %>%
  mutate(p_val=map dbl(data,aov_pval))

## Error in eval(lhs, parent, parent): object
'autonoise' not found
```

- ▶ `map dbl` because `aov_pval` returns a decimal number (a `dbl`). Investigate what happens if you use `map` instead.

Tidy up

- The data column was stepping-stone to getting answer. Don't need it any more:

```
simple_effects = autonoise %>% group_by(size) %>%
  nest() %>%
  mutate(p_val=map_dbl(data,aov_pval)) %>%
  select(-data)

## Error in eval(lhs, parent, parent): object 'autonoise' not
found

simple_effects

## Error in eval(expr, envir, enclos): object
'simple_effects' not found
```

Simultaneous tests

- ▶ When testing simple effects, doing several tests at once. (In this case, 3.)
- ▶ Have to adjust P-values for this. Eg. Holm:

```
simple_effects %>%
  arrange(p_val) %>%
  mutate(multiplier=4-row_number()) %>%
  mutate(p_val_adj=p_val*multiplier)

## Error in eval(lhs, parent, parent): object
'simple_effects' not found
```

- ▶ No change in rejection decisions.
- ▶ Octel filters sig. better in terms of noise for medium cars, and not sig. different for other sizes.
- ▶ Octel filters never significantly worse than standard ones.

Confidence intervals

- ▶ Perhaps better way of assessing simple effects: look at *confidence intervals* rather than tests.
- ▶ Gives us sense of accuracy of estimation, and thus whether non-significance might be lack of power: “absence of evidence is not evidence of absence”.
- ▶ Works here because *two* filter types, using `t.test` for each engine type.
- ▶ Want to show that the Octel filter is equivalent to or better than the standard filter, in terms of engine noise.

Equivalence and noninferiority

- ▶ Known as “equivalence testing” in medical world. A good read:
<http://www.ncbi.nlm.nih.gov/pmc/articles/PMC3019319/>.
Basic idea: decide on size of difference δ that would be considered “equivalent”, and if CI entirely inside $\pm\delta$, have evidence in favour of equivalence.
- ▶ We really want to show that the Octel filters are “no worse” than the standard one: that is, equivalent *or better* than standard filters.
- ▶ Such a “noninferiority test” done by checking that upper limit of CI, new minus old, is *less* than δ . (This requires careful thinking about (i) which way around the difference is and (ii) whether a higher or lower value is better.)

CI for small cars

Same idea as for simple effect test:

```
autonoise %>% filter(size=="S") %>%  
  t.test(noise~type,data=.) %>% .[["conf.int"]]  
  
## Error in eval(lhs, parent, parent): object 'autonoise'  
not found
```

CI for medium cars

```
autonoise %>% filter(size=="M") %>%
  t.test(noise~type,data=.) %>% .[["conf.int"]]

## Error in eval(lhs, parent, parent): object 'autonoise'
not found
```

CI for large cars

```
autonoise %>% filter(size=="L") %>%
  t.test(noise~type,data=.) %>% .[["conf.int"]]

## Error in eval(lhs, parent, parent): object 'autonoise'
not found
```

Or, all at once: split/apply/combine

```
ci_func=function(x) {  
  tt=t.test(noise~type,data=x)  
  tt$conf.int  
}  
cis = autonoise %>%  
  group_by(size) %>% nest() %>%  
  mutate(ci=map(data,ci_func)) %>%  
  unnest(ci)  
  
## Error in eval(lhs, parent, parent): object 'autonoise' not found
```

```
cis  
  
## Error in eval(expr, envir,  
enclos): object 'cis' not found
```

- ▶ Function to get CI of difference in noise means for types of filter on input data frame

- ▶ Group by size, nest (mini-df per size)
- ▶ Calculate CI for each thing in data (ie. each size). map: CI is two numbers long
- ▶ unnest ci column to see two numbers in each CI.

ClIs and noninferiority test

- ▶ Suppose we decide that a 20 dB difference would be considered equivalent. (I have no idea whether that is reasonable.)
- ▶ Intervals:

```
cis %>% mutate(hilo=rep(c("lower", "upper"), 3)) %>%
  spread(hilo, ci)

## Error in eval(lhs, parent, parent): object 'cis' not found
```

- ▶ In all cases, upper limit of CI is less than 20 dB. The Octel filters are “noninferior” to the standard ones.
- ▶ Caution: we did 3 procedures at once again. The true confidence level is not 95%. (Won’t worry about that here.)

Contrasts in ANOVA

- ▶ Sometimes, don't want to compare *all* groups, only *some* of them.
- ▶ Might be able to specify these comparisons ahead of time; other comparisons of no interest.
- ▶ Wasteful to do ANOVA and Tukey.

Example: chainsaw kickback

- ▶ From <http://www.ohio.edu/plantbio/staff/mccarthy/quantmet/lectures/ANOVA2.pdf>.
- ▶ Forest manager concerned about safety of chainsaws issued to field crew. 4 models of chainsaws, measure “kickback” (degrees of deflection) for 5 of each:

A	B	C	D

42	28	57	29
17	50	45	29
24	44	48	22
39	32	41	34
43	61	54	30

- ▶ So far, standard 1-way ANOVA: what differences are there among models?

chainsaw kickback (2)

- ▶ But: models A and D are designed to be used at home, while models B and C are industrial models.
- ▶ Suggests these comparisons of interest:
 - ▶ home vs. industrial
 - ▶ the two home models A vs. D
 - ▶ the two industrial models B vs. C.
- ▶ Don't need to compare *all* the pairs of models.

What is a contrast?

- ▶ Contrast is a linear combination of group means.
- ▶ Notation: μ_A for (population) mean of group A , and so on.
- ▶ In example, compare two home models: $H_0 : \mu_A - \mu_D = 0$.
- ▶ Compare two industrial models: $H_0 : \mu_B - \mu_C = 0$.
- ▶ Compare average of two home models vs. average of two industrial models: $H_0 : \frac{1}{2}(\mu_A + \mu_D) - \frac{1}{2}(\mu_B + \mu_C) = 0$ or
 $H_0 : 0.5\mu_A - 0.5\mu_B - 0.5\mu_C + 0.5\mu_D = 0$.
- ▶ Note that coefficients of contrasts add to 0, and right-hand side is 0.

Contrasts in R

- ▶ Comparing two home models A and D ($\mu_A - \mu_D = 0$):

```
c.home=c(1,0,0,-1)
```

- ▶ Comparing two industrial models B and C ($\mu_B - \mu_C = 0$):

```
c.industrial=c(0,1,-1,0)
```

- ▶ Comparing home average vs. industrial average
($0.5\mu_A - 0.5\mu_B - 0.5\mu_C + 0.5\mu_D = 0$):

```
c.home.ind=c(0.5,-0.5,-0.5,0.5)
```

Orthogonal contrasts

- ▶ What happens if we multiply the contrast coefficients one by one?

```
c.home*c.industrial
```

```
## [1] 0 0 0 0
```

```
c.home*c.home.ind
```

```
## [1] 0.5 0.0 0.0 -0.5
```

```
c.industrial*c.home.ind
```

```
## [1] 0.0 -0.5 0.5 0.0
```

- ▶ in each case, the results **add up to zero**. Such contrasts are called **orthogonal**.

Orthogonal contrasts (2)

- ▶ Compare these:

```
c1=c(1,-1,0)
```

```
c1
```

```
## [1] 1 -1 0
```

```
c2=c(0,1,-1)
```

```
c2
```

```
## [1] 0 1 -1
```

```
c1*c2
```

```
## [1] 0 -1 0
```

Does not add up to zero, so c1 and c2 are *not* orthogonal.

- ▶ Orthogonal contrasts are much easier to deal with.
- ▶ Can use non-orthogonal contrasts, but much more trouble (and beyond us).

Starting the analysis

```
my_url="http://www.utsc.utoronto.ca/~butler/c32/chainsaw.txt"
chain.wide=read_table(my_url)

## Error in open.connection(con, "rb"): HTTP error 403.

chain.wide

## Error in eval(expr, envir, enclos): object 'chain.wide' not found
```

Tidying

Need all the kickbacks in *one* column:

```
chain=gather(chain.wide,model,kickback,A:D,  
            factor_key=T)  
  
## Error in gather(chain.wide, model, kickback, A:D,  
factor_key = T): object 'chain.wide' not found
```

Starting the analysis (2)

The proper data frame, displayed in two pieces:

```
chain %>% slice(1:10)  
## Error in eval(lhs,  
parent, parent): object  
'chain' not found
```

```
chain %>% slice(11:20)  
## Error in eval(lhs,  
parent, parent): object  
'chain' not found
```

Setting up contrasts

```
m=cbind(c.home,c.industrial,c.home.ind)
m

##      c.home c.industrial c.home.ind
## [1,]     1          0       0.5
## [2,]     0          1      -0.5
## [3,]     0         -1      -0.5
## [4,]    -1          0       0.5

contrasts(chain$model)=m

## Error in contrasts(chain$model) = m: object 'chain' not
found
```

ANOVA as regression

Now run ANOVA *as if regression*:

```
chain.1=lm(kickback~model,data=chain)

## Error in is.data.frame(data): object 'chain' not found

summary(chain.1)

## Error in summary(chain.1): object 'chain.1' not found
```

Conclusions

```
tidy(chain.1) %>% select(term,p.value)  
## Error in tidy(chain.1): object 'chain.1' not found
```

- ▶ Two home models not sig. diff. (P-value 0.51)
- ▶ Two industrial models not sig. diff. (P-value 0.34)
- ▶ Home, industrial models *are* sig. diff. (P-value 0.0032).

Means by model

- ▶ The means:

```
chain %>% group_by(model) %>%
  summarize(mean.kick=mean(kickback)) %>%
  arrange(desc(mean.kick))

## Error in eval(lhs, parent, parent): object 'chain'
not found
```

- ▶ Home models A & D have less kickback than industrial ones B & C.
- ▶ Makes sense because industrial users should get training to cope with additional kickback.

Section 6

Analysis of covariance

Analysis of covariance

- ▶ ANOVA: explanatory variables categorical (divide data into groups)
- ▶ traditionally, analysis of covariance has categorical x 's plus one numerical x ("covariate") to be adjusted for.
- ▶ `lm` handles this too.
- ▶ Simple example: two treatments (drugs) (a and b), with before and after scores.
 - ▶ Does knowing before score and/or treatment help to predict after score?
 - ▶ Is after score different by treatment/before score?

Data

Treatment, before, after:

a 5 20

a 10 23

a 12 30

a 9 25

a 23 34

a 21 40

a 14 27

a 18 38

a 6 24

a 13 31

b 7 19

b 12 26

b 27 33

b 24 35

b 18 30

b 22 31

b 26 34

b 21 28

b 14 23

b 9 22

Packages

tidyverse and broom:

```
library(tidyverse)
```

```
## -- Attaching packages
```

```
----- tidyverse
```

```
1.2.1 --
```

```
## v ggplot2 3.1.0      v purrrr  0.2.5
```

```
## v tibble  1.4.2      v dplyr    0.7.8
```

```
## v tidyrr   0.8.1     v stringr  1.3.1
```

```
## v readr    1.1.1     vforcats  0.3.0
```

```
## -- Conflicts
```

```
-----  
tidyverse_conflicts() --
```

```
## x dplyr::filter() masks stats::filter()
```

```
## x dplyr::lag()   masks stats::lag()
```

```
library(broom)
```

Making a plot

```
my_url="http://www.utsc.utoronto.ca/~butler/c32/ancova.txt"
prepost=read_delim(my_url, " ")

## Error in open.connection(con, "rb"): HTTP error 403.

glimpse(prepost)

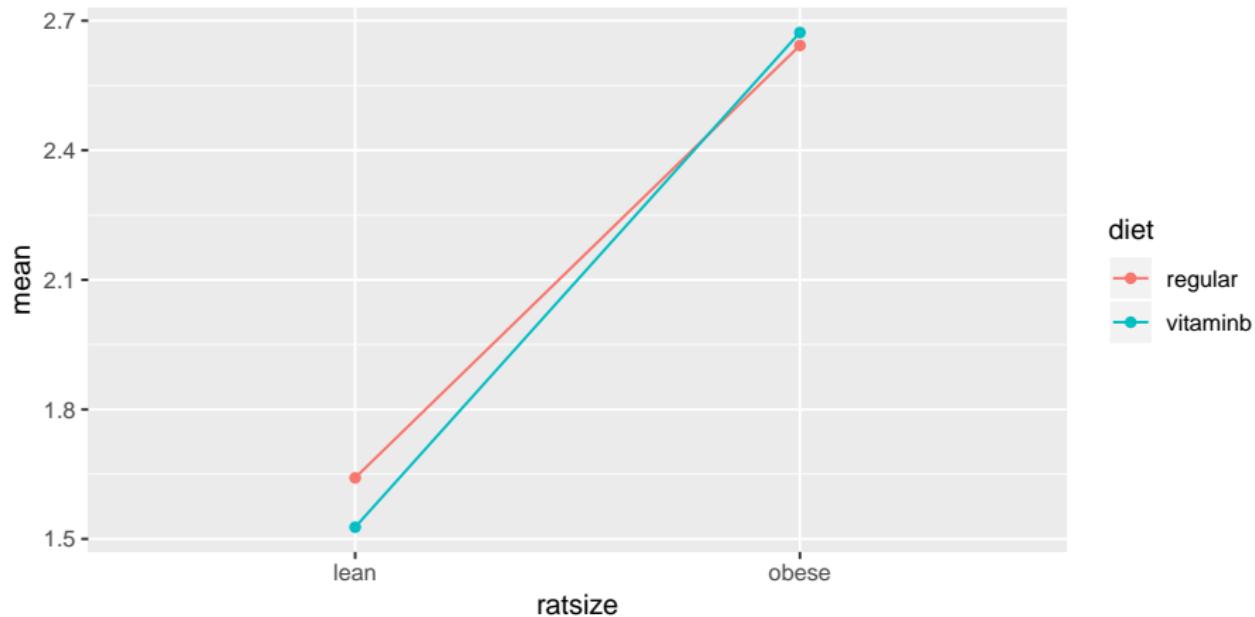
## Error in glimpse(prepost): object 'prepost' not found

g=ggplot(prepost,aes(x=before,y=after,colour=drug))+
  geom_point()

## Error in ggplot(prepost, aes(x = before, y = after, colour =
drug)): object 'prepost' not found
```

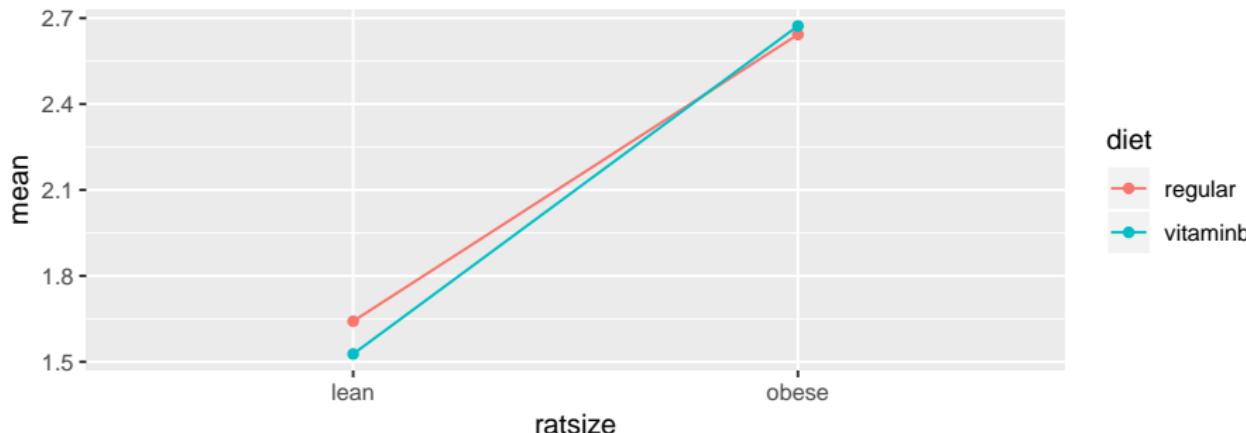
The plot

gg



Comments

g



- ▶ As before score goes up, after score goes up.
- ▶ Red points (drug A) generally above blue points (drug B), for comparable before score.
- ▶ Suggests before score effect *and* drug effect.

The means

```
prepost %>% group_by(drug) %>%
  summarize(before_mean=mean(before),
            after_mean=mean(after)
  )
## Error in eval(lhs, parent, parent): object 'prepost' not found
```

- ▶ Mean “after” score slightly higher for treatment A.
- ▶ Mean “before” score much higher for treatment B.
- ▶ Greater *improvement* on treatment A.

Testing for interaction

```
prepost.1=lm(after~before*drug,data=prepost)
## Error in is.data.frame(data): object 'prepost' not found
anova(prepost.1)
## Error in anova(prepost.1): object 'prepost.1' not found
```

- ▶ Interaction not significant. Will remove later.

Predictions, with interaction included

Make combinations of before score and drug:

```
new=crossing(  
    before=c(5,15,25),  
    drug=c("a","b"))  
  
new  
  
## # A tibble: 6 x 2  
##   before drug  
##   <dbl> <chr>  
## 1      5 a  
## 2      5 b  
## 3     15 a  
## 4     15 b  
## 5     25 a  
## 6     25 b
```

```
pred=predict(prepost.1,new)  
  
## Error in predict(prepost.1,  
## new): object 'prepost.1' not  
## found  
  
preds=bind_cols(new,pred)  
  
## Error in dots_values(...):  
## object 'pred' not found  
  
preds  
  
##   Exposure Severity probability  
## 1      5.8      None  0.9676920  
## 2     15.0      None  0.9253445  
## 3     21.5      None  0.8692002  
## 4     27.5      None  0.7889290  
## 5     33.5      None  0.6776641  
## 6     39.5      None  0.5418104  
## 7     46.0      None  0.3879961  
## 8     51.5      None  0.2722542
```

Making a plot with lines for each drug

```
g=ggplot(prepost,  
         aes(x=before,y=after,colour=drug))+  
  geom_point() +  
  geom_line(data=preds,aes(y=pred))  
  
## Error in ggplot(prepost, aes(x = before, y = after,  
colour = drug)): object 'prepost' not found
```

- ▶ Last line could (more easily) be

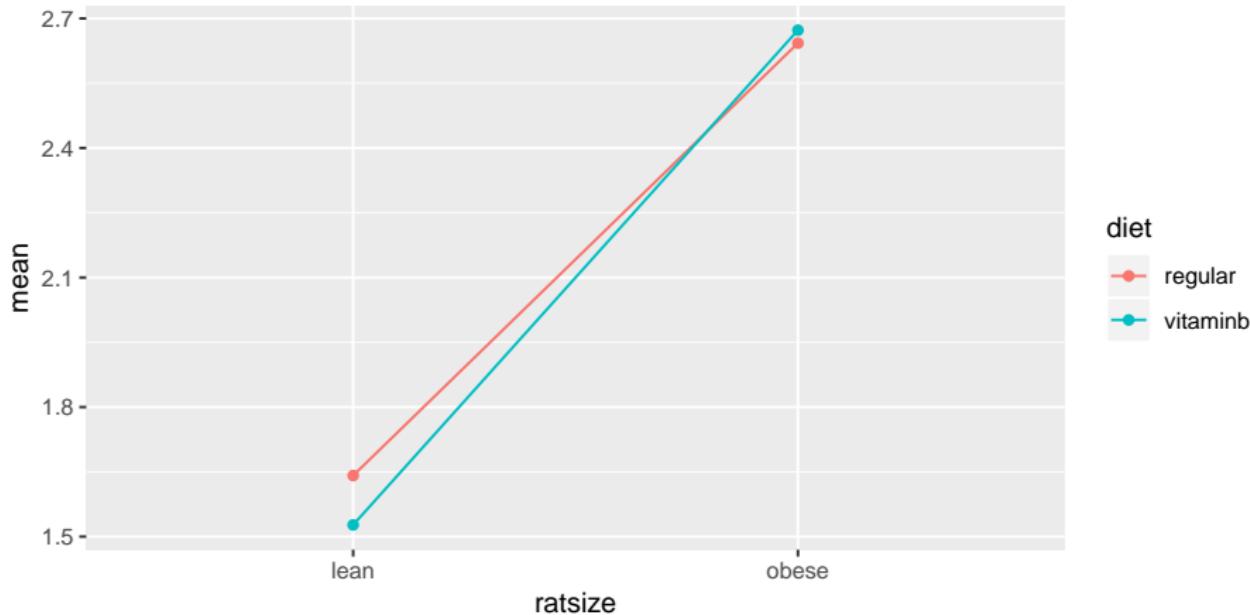
```
geom_smooth(method="lm",se=F)
```

which would work here, but not for later plot.

- ▶ Here, final line:
 - ▶ joins points by lines *for different data set* (preds rather than prepost),
 - ▶ *different y* (pred rather than after),
 - ▶ but same x (x=before inherited from first aes).

The plot

g



- ▶ Lines almost parallel, but not quite.
- ▶ Non-parallelism (interaction) not significant.

Taking out interaction

```
prepost.2=update(prepost.1,.~.-before:drug)  
## Error in update(prepost.1, . ~ . - before:drug): object  
'prepost.1' not found  
  
anova(prepost.2)  
## Error in anova(prepost.2): object 'prepost.2' not found
```

- ▶ Take out non-significant interaction.
- ▶ before and drug strongly significant.
- ▶ Do predictions again and plot them.

Predicted values again (no-interaction model)

```
pred=predict(prepost.2,new)
```

```
## Error in predict(prepost.2, new): object 'prepost.2' not found
```

```
preds=bind_cols(new,pred)
```

```
## Error in dots_values(...): object 'pred' not found
```

```
preds
```

```
##      Exposure Severity probability
```

```
## 1      5.8     None  0.96769203
```

```
## 2     15.0     None  0.92534455
```

```
## 3     21.5     None  0.86920028
```

```
## 4     27.5     None  0.78892903
```

```
## 5     33.5     None  0.67766411
```

```
## 6     39.5     None  0.54181046
```

```
## 7     46.0     None  0.38799618
```

```
## 8     51.5     None  0.27225426
```

```
## 9      5.8 Moderate 0.01908912
```

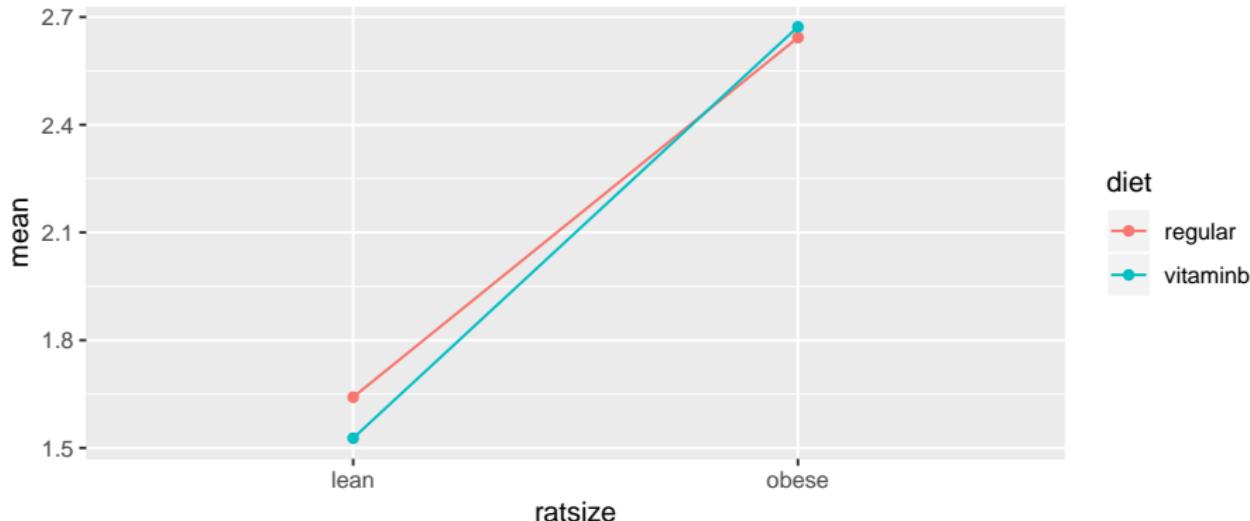
Making a plot, again

```
g=ggplot(prepost,  
         aes(x=before,y=after,colour=drug))+  
  geom_point() +  
  geom_line(data=preds,aes(y=pred))  
  
## Error in ggplot(prepost, aes(x = before, y = after,  
colour = drug)): object 'prepost' not found
```

Exactly same as before, but using new predictions.

The no-interaction plot of predicted values

gg



Lines now *parallel*. No-interaction model forces them to have the same slope.

Different look at model output

- ▶ `anova(prepost.2)` tests for significant effect of before score and of drug, but doesn't help with interpretation.
- ▶ `summary(prepost.2)` views as regression with slopes:

```
summary(prepost.2)
```

```
## Error in summary(prepost.2): object 'prepost.2' not found
```

Understanding those slopes

```
tidy(prepost.2)
```

```
## Error in tidy(prepost.2): object 'prepost.2' not found
```

- ▶ before ordinary numerical variable; drug categorical.
- ▶ lm uses first category druga as baseline.
- ▶ Intercept is prediction of after score for before score 0 and *drug A*.
- ▶ before slope is predicted change in after score when before score increases by 1 (usual slope)
- ▶ Slope for drugb is *change* in predicted after score for being on drug B rather than drug A. Same for *any* before score (no interaction).
- ▶ In summary(prepost.1), before:drugb would be change in *slope* for being on drug B rather than A.

Summary

- ▶ ANCOVA model: fits different regression line for each group, predicting response from covariate.
- ▶ ANCOVA model with interaction between factor and covariate allows different slopes for each line.
- ▶ Sometimes those lines can cross over!
- ▶ If interaction not significant, take out. Lines then parallel.
- ▶ With parallel lines, groups have consistent effect regardless of value of covariate.

Section 7

Multivariate ANOVA

Multivariate analysis of variance

- ▶ Standard ANOVA has just one response variable.
- ▶ What if you have more than one response?
- ▶ Try an ANOVA on each response separately.
- ▶ But might miss some kinds of interesting dependence between the responses that distinguish the groups.

Packages

```
library(car)

## Loading required package: carData

library(tidyverse)

## -- Attaching packages
----- tidyverse
1.2.1 --

## v ggplot2 3.1.0     v purrrr  0.2.5
## v tibble   1.4.2     v dplyr    0.7.8
## v tidyverse 0.8.1     v stringr  1.3.1
## v readr    1.1.1     vforcats  0.3.0

## -- Conflicts
-----  
tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()
## x dplyr::recode() masks car::recode()
```

Small example

- ▶ Measure yield and seed weight of plants grown under 2 conditions: low and high amounts of fertilizer.
- ▶ Data (fertilizer, yield, seed weight):

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/manova1.tx  
hilo=read_delim(my_url, " ")  
  
## Parsed with column specification:  
## cols(  
##   fertilizer = col_character(),  
##   yield = col_integer(),  
##   weight = col_integer()  
## )
```

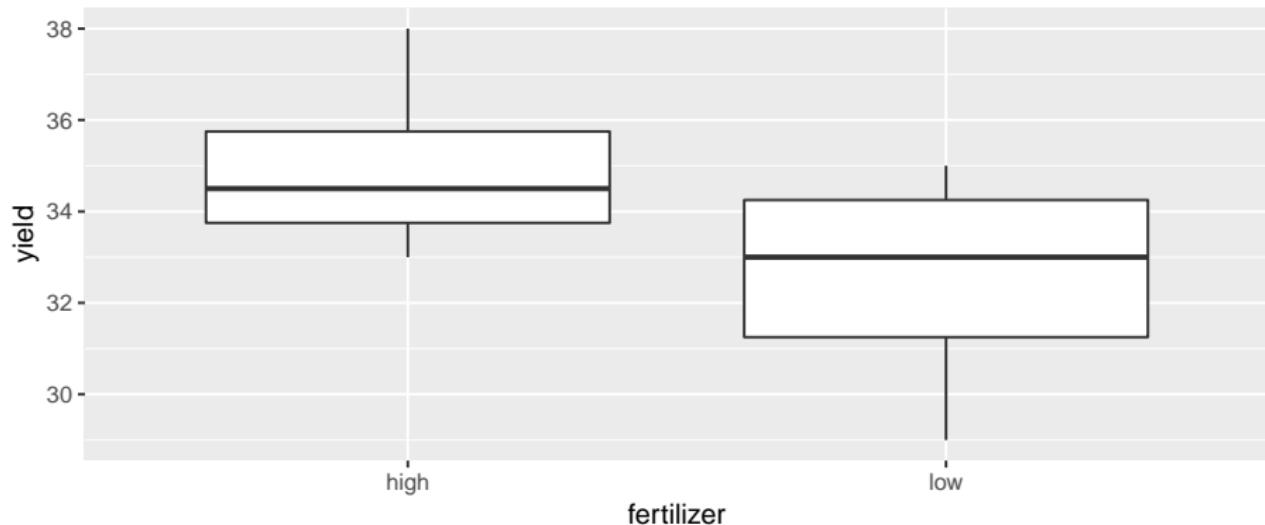
- ▶ 2 responses, yield and seed weight.

The data

```
hilo  
## # A tibble: 8 x 3  
##   fertilizer yield weight  
##   <chr>      <int>   <int>  
## 1 low          34     10  
## 2 low          29     14  
## 3 low          35     11  
## 4 low          32     13  
## 5 high         33     14  
## 6 high         38     12  
## 7 high         34     13  
## 8 high         35     14
```

Boxplot for yield for each fertilizer group

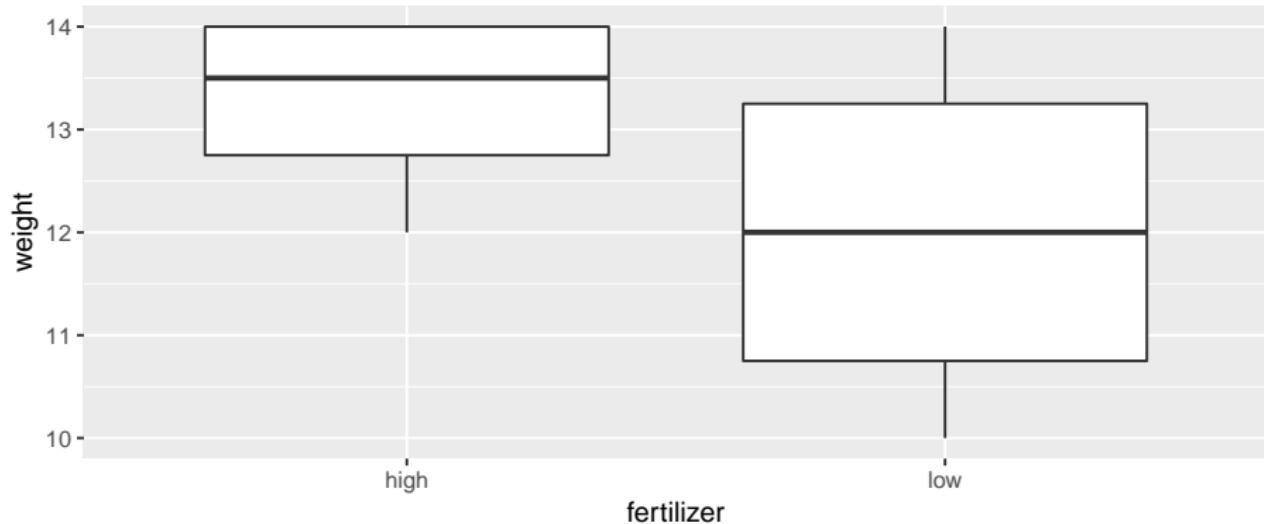
```
ggplot(hilo,aes(x=fertilizer,y=yield))+geom_boxplot()
```



Yields overlap for fertilizer groups.

Boxplot for weight for each fertilizer group

```
ggplot(hilo,aes(x=fertilizer,y=weight))+geom_boxplot()
```



Weights overlap for fertilizer groups.

ANOVAs for yield and weight

```
hilo.y=aov(yield~fertilizer,data=hilo)
summary(hilo.y)

##           Df Sum Sq Mean Sq F value Pr(>F)
## fertilizer   1   12.5   12.500   2.143  0.194
## Residuals    6   35.0    5.833

hilo.w=aov(weight~fertilizer,data=hilo)
summary(hilo.w)

##           Df Sum Sq Mean Sq F value Pr(>F)
## fertilizer   1   3.125   3.125   1.471  0.271
## Residuals    6  12.750   2.125
```

Neither response depends significantly on fertilizer. But...

Plotting both responses at once

Have two response variables (not more), so can plot the response variables against *each other*, labelling points by which fertilizer group they're from.

```
g=ggplot(hilo,aes(x=yield,y=weight,  
colour=fertilizer))+geom_point()
```

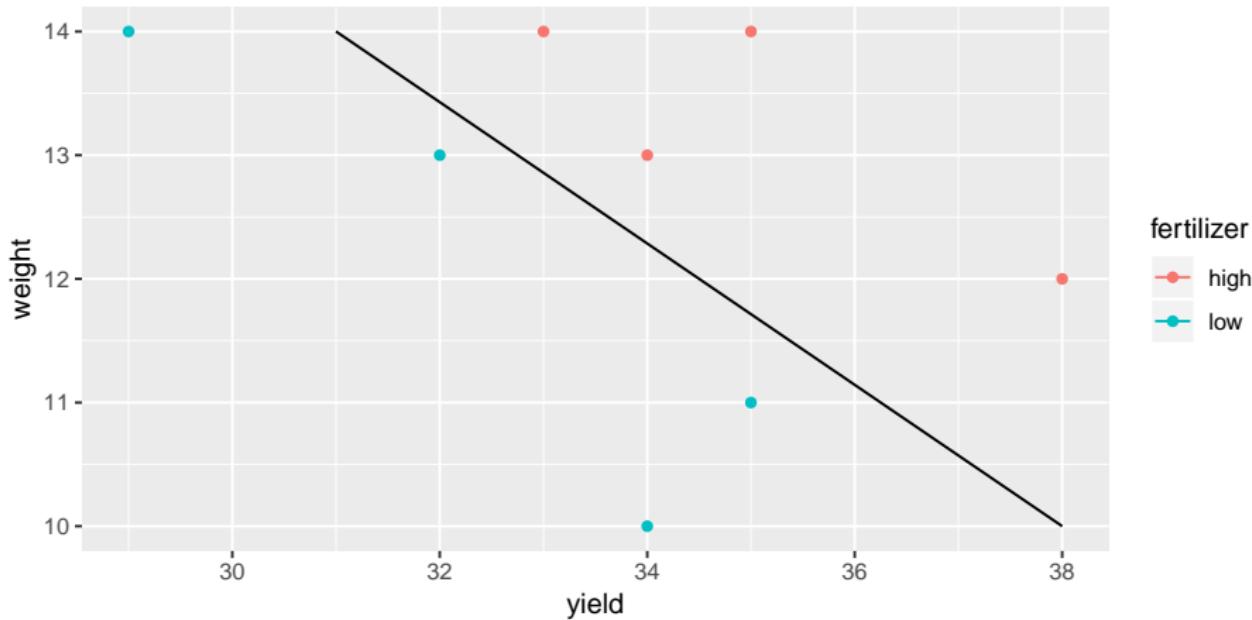
Want line through points (31, 14) and (38, 10) (why? Later):

```
d=tribble(  
  ~line_x, ~line_y,  
  31, 14,  
  38, 10)  
g=g+geom_line(data=d,aes(x=line_x,y=line_y,  
colour=NULL))
```

Fitting regression line through points in d. Adding to previous ggplot, so geom_smooth inherits colour from first one. This data frame has no fertilizer (previous colour), so have to unset.

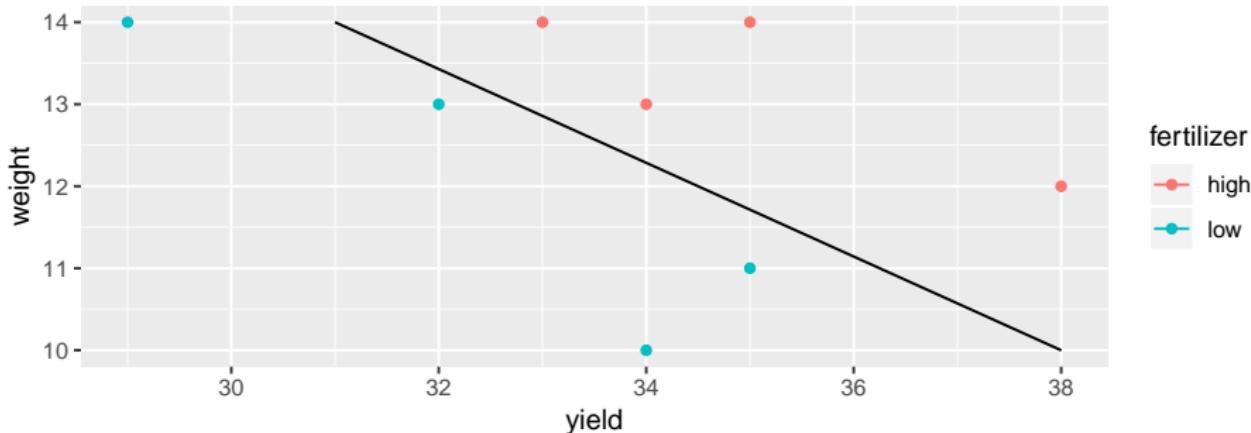
The plot

gg



MANOVA

g



- ▶ High-fertilizer plants have both yield and weight high.
- ▶ True even though no sig difference in yield or weight individually.
- ▶ Drew line separating highs from lows on plot.

MANOVA finds multivariate differences

- ▶ Is difference found by diagonal line significant? MANOVA finds out.

```
response=with(hilo,cbind(yield,weight))
hilo.1=manova(response~fertilizer,data=hilo)
summary(hilo.1)

##           Df Pillai approx F num Df den Df Pr(>F)
## fertilizer  1 0.80154   10.097      2      5 0.01755 *
## Residuals   6
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- ▶ Yes! Difference between groups is *diagonally*, not just up/down (weight) or left-right (yield). The *yield-weight combination* matters.

Strategy

- ▶ Create new response variable by gluing together columns of responses, using `cbind`.
- ▶ Use `manova` with new response, looks like `lm` otherwise.
- ▶ With more than 2 responses, cannot draw graph. What then?
- ▶ If MANOVA test significant, cannot use Tukey. What then?
- ▶ Use *discriminant analysis* (of which more later).

Another way to do MANOVA

Install (once) and load package car:

```
library(car)
```

Another way...

```
hilo.2.lm=lm(response~fertilizer,data=hilo)
hilo.2=Manova(hilo.2.lm)
hilo.2

##
## Type II MANOVA Tests: Pillai test statistic
##          Df test stat approx F num Df den Df Pr(>F)
## fertilizer  1  0.80154   10.097      2      5 0.01755 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- ▶ Same result as small-m manova.
- ▶ Manova will also do *repeated measures*, coming up later.

Another example: peanuts

- ▶ Three different varieties of peanuts (mysteriously, 5, 6 and 8) planted in two different locations.
- ▶ Three response variables: `y`, `smk` and `w`.

```
my_url = "http://www.utsc.utoronto.ca/~butler/d29/peanuts.txt"  
(peanuts.orig = read_delim(my_url, " "))
```

```
## # A tibble: 12 x 6  
##   obs location variety     y   smk     w  
##   <int>    <int>   <int> <dbl> <dbl> <dbl>  
## 1     1        1      1  195.  153.  51.4  
## 2     2        2      1  194.  168.  53.7  
## 3     3        3      2  190.  140.  55.5  
## 4     4        4      2  180.  121.  44.4  
## 5     5        5      1  203.  157.  49.8  
## 6     6        6      1  196.  166.  45.8  
## 7     7        7      2  203.  166.  60.4  
## 8     8        8      2  198.  162.  54.1  
## 9     9        9      1  194.  164.  57.8  
## 10   10       10      1  187.  165.  58.6  
## 11   11       11      2  202.  167.  65  
## 12   12       12      2  200.  174.  67.2
```

Setup for analysis

```
peanuts = peanuts.orig %>%
  mutate(location=factor(location),
        variety=factor(variety))
response=with(peanuts, cbind(y,smk,w))
head(response)

##           y     smk      w
## [1,] 195.3 153.1 51.4
## [2,] 194.3 167.7 53.7
## [3,] 189.7 139.5 55.5
## [4,] 180.4 121.1 44.4
## [5,] 203.0 156.8 49.8
## [6,] 195.9 166.0 45.8
```

Analysis (using Manova)

```
peanuts.1=lm(response~location*variety,data=peanuts)
peanuts.2=Manova(peanuts.1)
peanuts.2

##
## Type II MANOVA Tests: Pillai test statistic
##                               Df test stat approx F num Df den Df   Pr(>F)
## location                  1   0.89348   11.1843      3     4 0.020502 *
## variety                   2   1.70911   9.7924      6    10 0.001056 **
## location:variety         2   1.29086   3.0339      6    10 0.058708 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- ▶ Interaction not quite significant, but main effects are.
- ▶ Combined response variable (y, smk, w) definitely depends on location and on variety
- ▶ Weak dependence of (y, smk, w) on the location-variety *combination*.
- ▶ Understanding that dependence beyond our scope right now.

Section 8

Repeated measures by profile analysis

Repeated measures by profile analysis

- ▶ More than one response *measurement* for each subject. Might be
 - ▶ measurements of the same thing at different times
 - ▶ measurements of different but related things
- ▶ Generalization of matched pairs (“matched triples”, etc.).
- ▶ Variation: each subject does several different treatments at different times (called *crossover design*).
- ▶ Expect measurements on same subject to be correlated, so assumptions of independence will fail.
- ▶ Called *repeated measures*. Different approaches, but *profile analysis* uses Manova (set up right way).
- ▶ Another approach uses *mixed models* (random effects).

Packages

```
library(car)

## Loading required package: carData

library(tidyverse)

## -- Attaching packages
----- tidyverse
1.2.1 --

## v ggplot2 3.1.0     v purrrr  0.2.5
## v tibble   1.4.2     v dplyr    0.7.8
## v tidyverse 0.8.1     v stringr  1.3.1
## v readr    1.1.1     vforcats  0.3.0

## -- Conflicts
-----

tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()
## x dplyr::recode() masks car::recode()
```

Example: histamine in dogs

- ▶ 8 dogs take part in experiment.
- ▶ Dogs randomized to one of 2 different drugs.
- ▶ Response: log of blood concentration of histamine 0, 1, 3 and 5 minutes after taking drug. (Repeated measures.)
- ▶ Data in dogs.txt, column-aligned.

Read in data

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/dogs.txt"
dogs=read_table(my_url)

## Parsed with column specification:
## cols(
##   dog = col_character(),
##   drug = col_character(),
##   x = col_character(),
##   lh0 = col_double(),
##   lh1 = col_double(),
##   lh3 = col_double(),
##   lh5 = col_double()
## )
```

Setting things up

```
dogs
```

```
## # A tibble: 8 x 7
##   dog    drug      x    lh0    lh1    lh3    lh5
##   <chr> <chr> <chr> <dbl> <dbl> <dbl> <dbl>
## 1 A     Morphine N     -3.22 -1.61 -2.3   -2.53
## 2 B     Morphine N     -3.91 -2.81 -3.91 -3.91
## 3 C     Morphine N     -2.66  0.34 -0.73 -1.43
## 4 D     Morphine N     -1.77 -0.56 -1.05 -1.43
## 5 E     Trimethaphan N    -3.51 -0.48 -1.17 -1.51
## 6 F     Trimethaphan N    -3.51  0.05 -0.31 -0.51
## 7 G     Trimethaphan N    -2.66 -0.19  0.07 -0.22
## 8 H     Trimethaphan N    -2.41  1.14  0.72  0.21
```

```
response=with(dogs, cbind(lh0,lh1,lh3,lh5))
```

```
dogs.lm=lm(response~drug, data=dogs)
```

The repeated measures MANOVA

Get list of response variable names; we call them times. Save in data frame.

```
times=colnames(response)
times.df=data.frame(times)
dogs.manova=Manova(dogs.lm, idata=times.df,
    idesign=~times)
dogs.manova

##
## Type II Repeated Measures MANOVA Tests: Pillai test statistic
##          Df test stat approx F num Df den Df   Pr(>F)
## (Intercept) 1  0.76347  19.3664      1     6 0.004565 ***
## drug         1  0.34263   3.1272      1     6 0.127406
## times        1  0.94988  25.2690      3     4 0.004631 ***
## drug:times   1  0.89476  11.3362      3     4 0.020023 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Interaction significant. Pattern of response over time different for the two drugs.

Wide and long format

- ▶ Want to investigate interaction.
- ▶ But data frame has several observations per line (“wide format”):

```
dogs %>% print(n=5)

## # A tibble: 8 x 7
##   dog   drug      x     lh0    lh1    lh3    lh5
##   <chr> <chr>    <chr> <dbl> <dbl> <dbl> <dbl>
## 1 A     Morphine N     -3.22 -1.61 -2.3   -2.53
## 2 B     Morphine N     -3.91 -2.81 -3.91 -3.91 
## 3 C     Morphine N     -2.66  0.34 -0.73 -1.43 
## 4 D     Morphine N     -1.77 -0.56 -1.05 -1.43 
## 5 E     Trimethaphan N    -3.51 -0.48 -1.17 -1.51
## # ... with 3 more rows
```

- ▶ Plotting works with data in “long format”: one response per line.
- ▶ The responses are log-histamine at different times, labelled lh-something. Call them all lh and put them in one column, with the time they belong to labelled.

Running gather, try 1

```
dogs %>% gather(time, lh, lh0:lh5) %>% print(n=12)

## # A tibble: 32 x 5
##       dog    drug     x   time     lh
##       <chr> <chr> <chr> <chr> <dbl>
## 1     A Morphine    N   lh0    -3.22
## 2     B Morphine    N   lh0    -3.91
## 3     C Morphine    N   lh0    -2.66
## 4     D Morphine    N   lh0    -1.77
## 5     E Trimethaphan N   lh0    -3.51
## 6     F Trimethaphan N   lh0    -3.51
## 7     G Trimethaphan N   lh0    -2.66
## 8     H Trimethaphan N   lh0    -2.41
## 9     A Morphine    N   lh1    -1.61
## 10    B Morphine    N   lh1    -2.81
## 11    C Morphine    N   lh1     0.34
## 12    D Morphine    N   lh1    -0.56
## # ... with 20 more rows
```

Getting the times

Not quite right: for the times, we want just the numbers, not the letters lh every time. Want new variable containing just number in time: parse_number.

```
dogs %>% gather(timex, lh, lh0:lh5) %>%
  mutate(time=parse_number(timex)) %>% print(n=10)

## # A tibble: 32 x 6
##   dog    drug      x     timex     lh   time
##   <chr> <chr>    <chr> <chr> <dbl> <dbl>
## 1 A     Morphine N     lh0    -3.22    0
## 2 B     Morphine N     lh0    -3.91    0
## 3 C     Morphine N     lh0    -2.66    0
## 4 D     Morphine N     lh0    -1.77    0
## 5 E     Trimethaphan N    lh0    -3.51    0
## 6 F     Trimethaphan N    lh0    -3.51    0
## 7 G     Trimethaphan N    lh0    -2.66    0
## 8 H     Trimethaphan N    lh0    -2.41    0
## 9 A     Morphine  N    lh1    -1.61    1
## 10 B    Morphine  N    lh1    -2.81    1
## # ... with 22 more rows
```

What I did differently

- ▶ I realized that `gather` was going to produce something like `lh1`, which I needed to do something further with, so this time I gave it a temporary name `timex`.
- ▶ This enabled me to use the name `time` for the actual numeric time.
- ▶ This works now, so next save into a new data frame `dogs.long`.

Saving the chained results

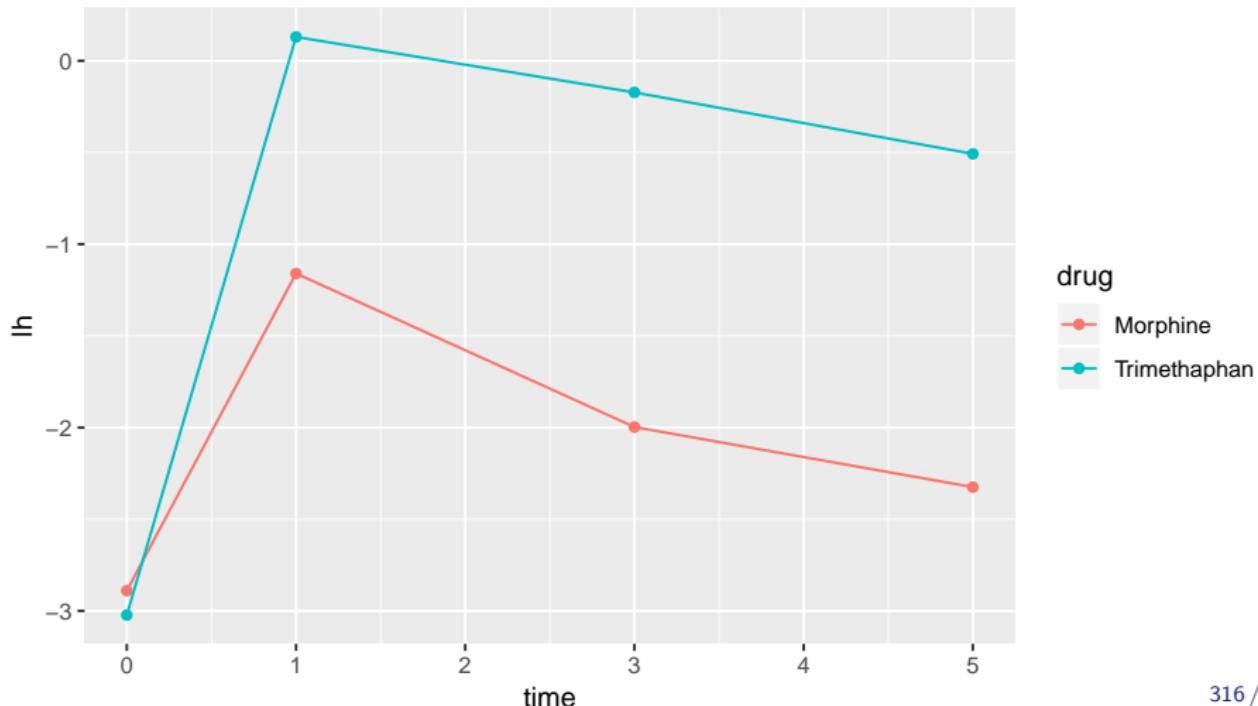
```
dogs.long = dogs %>% gather(timex, lh, lh0:lh5) %>%
  mutate(time=parse_number(timex))
```

This says:

- ▶ Take data frame `dogs`, and then:
- ▶ Combine the columns `lh0` through `lh5` into one column called `lh`, with the column that each `lh` value originally came from labelled by `timex`, and then:
- ▶ Pull out numeric values in `timex`, saving in `time` and then:
- ▶ save the result in a data frame `dogs.long`.

Interaction plot

```
ggplot(dogs.long, aes(x=time, y=lh,  
                      colour=drug, group=drug)) +  
  stat_summary(fun.y=mean, geom="point") +  
  stat_summary(fun.y=mean, geom="line")
```



Comments

- ▶ Plot mean 1h value at each time, joining points on same drug by lines.
- ▶ drugs same at time 0
- ▶ after that, Trimethaphan higher than Morphine.
- ▶ Effect of drug not consistent over time: significant interaction.

Take out time zero

- ▶ Lines on interaction plot would then be parallel, and so interaction should no longer be significant.
- ▶ Go back to original “wide” dogs data frame.

```
response=with(dogs, cbind(lh1, lh3, lh5)) # excluding time zero
dogs.lm=lm(response~drug, data=dogs)
times=colnames(response)
times.df=data.frame(times)
dogs.manova=Manova(dogs.lm, idata=times.df,
                    idesign=~times)
```

Results and comments

```
dogs.manova

##
## Type II Repeated Measures MANOVA Tests: Pillai test statistic
##          Df test stat approx F num Df den Df   Pr(>F)
## (Intercept) 1  0.54582  7.2106     1      6 0.036281 *
## drug         1  0.44551  4.8207     1      6 0.070527 .
## times        1  0.85429 14.6569     2      5 0.008105 **
## drug:times   1  0.43553  1.9289     2      5 0.239390
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- ▶ Correct: interaction no longer significant.
- ▶ Significant effect of time.
- ▶ Drug effect not quite significant (some variety among dogs within drug).

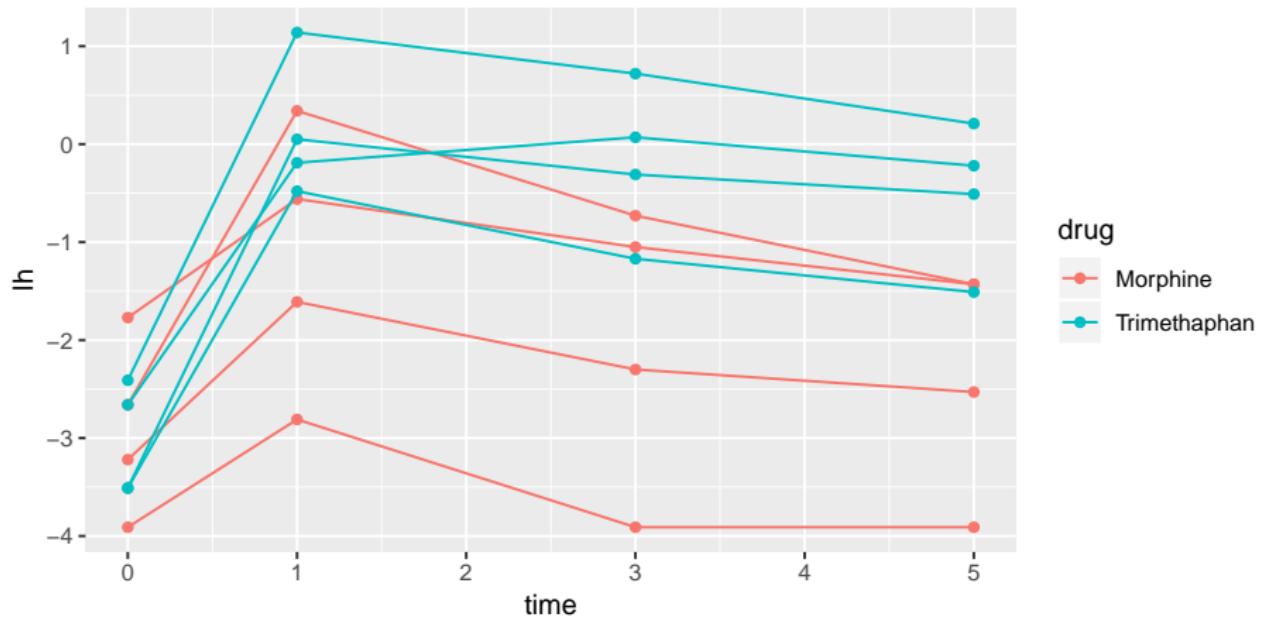
Is the non-significant drug effect reasonable?

- ▶ Plot *actual data*: lh against days, labelling observations by drug: “spaghetti plot”.
- ▶ Uses long data frame (confusing, yes I know):
- ▶ Plot (time,lh) points coloured by drug
- ▶ and connecting measurements for each *dog* by lines.
- ▶ This time, we want group=dog (want the measurements for each *dog* joined by lines), but colour=drug:

```
g=ggplot(dogs.long,aes(x=time,y=lh,  
colour=drug,group=dog)) +  
geom_point() + geom_line()
```

The spaghetti plot

g



Comments

- ▶ For each dog over time, there is a strong increase and gradual decrease in log-histamine. This explains the significant time effect.
- ▶ The pattern is more or less the same for each dog, regardless of drug. This explains the non-significant interaction.
- ▶ Most of the trimethaphan dogs (blue) have higher log-histamine throughout (time 1 and after), and some of the morphine dogs have lower.
- ▶ *But* two of the morphine dogs have log-histamine profiles like the trimethaphan dogs. This ambiguity is probably why the drug effect is not quite significant.

The exercise data

- ▶ 30 people took part in an exercise study.
- ▶ Each subject was randomly assigned to one of two diets ("low fat" or "non-low fat") and to one of three exercise programs ("at rest", "walking", "running").
- ▶ There are $2 \times 3 = 6$ experimental treatments, and thus each one is replicated $30/6 = 5$ times.
- ▶ Nothing unusual so far.
- ▶ However, each subject had their pulse rate measured at three different times (1, 15 and 30 minutes after starting their exercise), so have repeated measures.

Reading the data

Separated by *tabs*:

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/exercise.txt"
exercise.long=read_tsv(my_url)
```

Parsed with column specification:

```
## cols(
##   id = col_integer(),
##   diet = col_character(),
##   exertype = col_character(),
##   pulse = col_integer(),
##   time = col_character()
## )
```

The data

```
exercise.long %>% print(n=8)

## # A tibble: 90 x 5
##       id diet     exerotype pulse time
##   <int> <chr>    <chr>      <int> <chr>
## 1     1 nonlowfat atrest      85 min01
## 2     1 nonlowfat atrest      85 min15
## 3     1 nonlowfat atrest      88 min30
## 4     2 nonlowfat atrest      90 min01
## 5     2 nonlowfat atrest      92 min15
## 6     2 nonlowfat atrest      93 min30
## 7     3 nonlowfat atrest      97 min01
## 8     3 nonlowfat atrest      97 min15
## # ... with 82 more rows
```

- ▶ This is “long format”, which is usually what we want.
- ▶ But for repeated measures analysis, we want *wide* format!
- ▶ “undo” gather: spread.

Making wide format

- ▶ Spread needs three things: a data frame, a column that is going to be split, and the column to make the values out of:

```
exercise.wide=spread(exercise.long,time,pulse)
```

```
exercise.wide %>% print(n=6)
```

```
## # A tibble: 30 x 6
##       id diet     exertype min01 min15 min30
##   <int> <chr>    <chr>    <int>  <int>  <int>
## 1     1 nonlowfat atrest      85      85      88
## 2     2 nonlowfat atrest      90      92      93
## 3     3 nonlowfat atrest      97      97      94
## 4     4 nonlowfat atrest      80      82      83
## 5     5 nonlowfat atrest      91      92      91
## 6     6 lowfat      atrest      83      83      84
## # ... with 24 more rows
```

- ▶ See how we would normally gather min01, min15, min30 into one column called pulse labelled by the number of minutes? But Manova needs it the other way.

Setting up the repeated-measures analysis

- ▶ Make a response variable consisting of min01, min15, min30:

```
response=with(exercise.wide,  
             cbind(min01, min15, min30))
```

- ▶ Predict that from diet and exertype and interaction using lm:

```
exercise.1=lm(response~diet*exertype,  
               data=exercise.wide)
```

- ▶ Run this through Manova:

```
times=colnames(response)  
times.df=data.frame(times)  
exercise.2=Manova(exercise.1, idata=times.df,  
                  idesign=~times)
```

Results

```
exercise.2
```

```
##  
## Type II Repeated Measures MANOVA Tests: Pillai test statistic  
##  
## Df test stat approx F num Df den Df Pr(>F)  
## (Intercept) 1 0.99767 10296.7 1 24 < 2.2e-16 ***  
## diet 1 0.37701 14.5 1 24 0.0008483 ***  
## exertype 2 0.79972 47.9 2 24 4.166e-09 ***  
## diet:exertype 2 0.28120 4.7 2 24 0.0190230 *  
## times 1 0.78182 41.2 2 23 2.491e-08 ***  
## diet:times 1 0.25153 3.9 2 23 0.0357258 *  
## exertype:times 2 0.83557 8.6 4 48 2.538e-05 ***  
## diet:exertype:times 2 0.51750 4.2 4 48 0.0054586 **  
## ---  
## Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- ▶ Three-way interaction significant, so cannot remove anything.
- ▶ Pulse rate depends on diet and exercise type *combination*, and *that* is different for each time.

Making some graphs

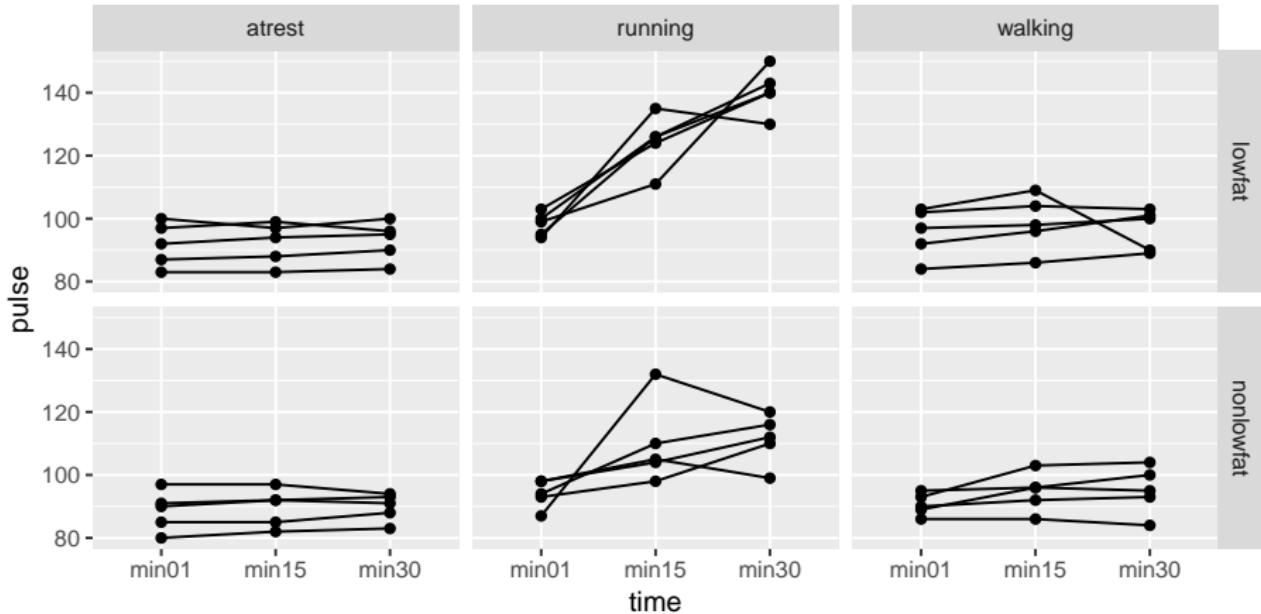
- ▶ Three-way interactions are difficult to understand. To make an attempt, look at some graphs.
- ▶ Plot time trace of pulse rates for each individual, joined by lines, and make *separate* plots for each diet-exertype combo.
- ▶ ggplot again. Using *long* data frame:

```
g=ggplot(exercise.long, aes(x=time, y=pulse,  
    group=id))+geom_point()+geom_line()  
    facet_grid(diet~exertype)
```

- ▶ `facet_grid(diet~exertype)`: do a separate plot for each combination of diet and exercise type, with diets going down the page and exercise types going across. (Graphs are usually landscape, so have the factor `exertype` with more levels going across.)

The graph(s)

5



Comments on graphs

- ▶ For subjects who were at rest, no change in pulse rate over time, for both diet groups.
- ▶ For walking subjects, not much change in pulse rates over time. Maybe a small increase on average between 1 and 15 minutes.
- ▶ For both running groups, an overall increase in pulse rate over time, but the increase is stronger for the lowfat group.
- ▶ No consistent effect of diet over all exercise groups.
- ▶ No consistent effect of exercise type over both diet groups.
- ▶ No consistent effect of time over all diet-exercise type combos.

“Simple effects” of diet for the subjects who ran

- ▶ Looks as if there is only any substantial time effect for the runners. For them, does diet have an effect?
- ▶ Pull out only the runners from the wide data:

```
runners.wide = exercise.wide %>%  
  filter(exertype=="running")
```

- ▶ Create response variable and do MANOVA. Some of this looks like before, but I have different data now:

```
response=with(runners.wide, cbind(min01,min15,min30))  
runners.1=lm(response~diet, data=runners.wide)  
times=colnames(response)  
times.df=data.frame(times)  
runners.2=Manova(runners.1, idata=times.df,  
                 idesign=~times)
```

Results

```
runners.2

##
## Type II Repeated Measures MANOVA Tests: Pillai test statistic
##           Df test stat approx F num Df den Df     Pr(>F)
## (Intercept) 1  0.99912   9045.3      1      8 1.668e-13 ***
## diet         1  0.84986     45.3       1      8 0.0001482 ***
## times        1  0.92493     43.1       2      7 0.0001159 ***
## diet:times   1  0.68950      7.8       2      7 0.0166807 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- ▶ The diet by time interaction is still significant (at $\alpha = 0.05$): the effect of time on pulse rates is different for the two diets.
- ▶ At $\alpha = 0.01$, the interaction is not significant, and then we have only two (very) significant main effects of diet and time.

How is the effect of diet different over time?

- ▶ Table of means. Only I need long data for this, so make it (in a pipe):

```
summ = runners.wide %>%  
  gather(time,pulse,min01:min30) %>%  
  group_by(time,diet) %>%  
  summarize(mean=mean(pulse),  
            sd=sd(pulse))
```

- ▶ Result of `summarize` is data frame, so can save it (and do more with it if needed).

Understanding diet-time interaction

- The summary:

```
summ

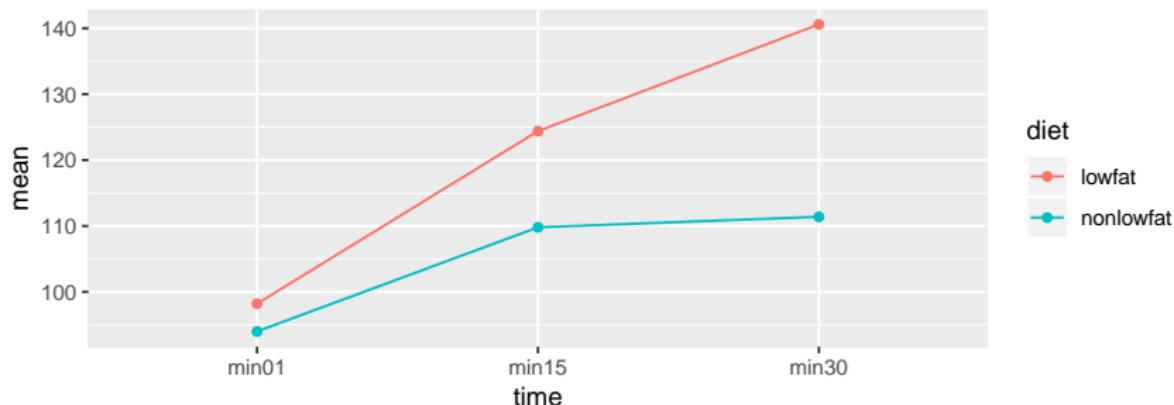
## # A tibble: 6 x 4
## # Groups:   time [?]
##   time   diet     mean     sd
##   <chr>  <chr>    <dbl>   <dbl>
## 1 min01 lowfat    98.2   3.70
## 2 min01 nonlowfat 94     4.53
## 3 min15 lowfat    124.   8.62
## 4 min15 nonlowfat 110.   13.1
## 5 min30 lowfat    141.   7.20
## 6 min30 nonlowfat 111.   7.92
```

- Pulse rates at any given time higher for lowfat (diet effect),
- Pulse rates increase over time of exercise (time effect),
- but the *amount by which pulse rate higher* for a diet depends on time: diet by time interaction.

Interaction plot

- We went to trouble of finding means by group, so making interaction plot is now mainly easy:

```
ggplot(summ,aes(x=time,y=mean,colour=diet,  
group=diet))+geom_point()+geom_line()
```



- The lines are not parallel, so there is interaction between diet and time.

Section 9

Discriminant analysis

Discriminant analysis

- ▶ ANOVA and MANOVA: predict a (counted/measured) response from group membership.
- ▶ Discriminant analysis: predict group membership based on counted/measured variables.
- ▶ Covers same ground as logistic regression (and its variations), but emphasis on classifying observed data into correct groups.
- ▶ Does so by searching for linear combination of original variables that best separates data into groups (canonical variables).
- ▶ Assumption here that groups are known (for data we have). If trying to “best separate” data into unknown groups, see *cluster analysis*.
- ▶ Examples: revisit seed yield and weight data, peanut data, professions/activities data; remote-sensing data.

Packages

```
library(MASS)
```

```
library(tidyverse)
```

```
## -- Attaching packages
```

```
----- tidyverse 1.2.1
```

```
--
```

```
## v ggplot2 3.1.0      v purrrr  0.2.5
```

```
## v tibble   1.4.2      v dplyr    0.7.8
```

```
## v tidyverse 0.8.1      v stringr  1.3.1
```

```
## v readr    1.1.1      vforcats  0.3.0
```

```
## -- Conflicts
```

```
tidyverse_conflicts() --
```

```
## x dplyr::filter() masks stats::filter()
```

```
## x dplyr::lag()   masks stats::lag()
```

```
## x dplyr::select() masks MASS::select()
```

```
library(ggrepel)
```

```
library(ggbiplot)
```

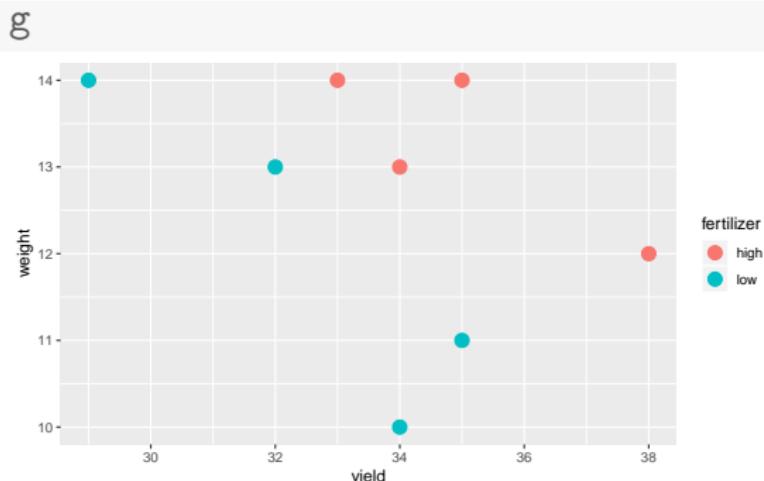
```
## Loading required package: plyr
```

About select

- ▶ Both dplyr (in tidyverse) and MASS have a function called select, and *they do different things*.
- ▶ How do you know which select is going to get called?
- ▶ With library, the one loaded *last* is visible, and others are not.
- ▶ Thus we can access the select in dplyr but not the one in MASS. If we wanted that one, we'd have to say MASS::select.
- ▶ This is why I loaded MASS before tidyverse. If I had done it the other way around, the tidyverse select, which I want to use, would have been the invisible one.

Example 1: seed yields and weights

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/manova1.txt"
hilo=read_delim(my_url, " ")
g=ggplot(hilo,aes(x=yield,y=weight,
  colour=fertilizer))+geom_point(size=4)
```



Recall data from MANOVA:
needed a multivariate analysis to find difference in seed yield and weight based on whether they were high or low fertilizer.

Basic discriminant analysis

```
hilo.1=lda(fertilizer~yield+weight,data=hilo)
```

- ▶ Uses lda from package MASS.
- ▶ “Predicting” group membership from measured variables.

Output

```
hilo.1

## Call:
## lda(fertilizer ~ yield + weight, data = hilo)
##
## Prior probabilities of groups:
## high   low
## 0.5   0.5
##
## Group means:
##      yield weight
## high  35.0 13.25
## low   32.5 12.00
##
## Coefficients of linear discriminants:
##                 LD1
## yield -0.7666761
## weight -1.2513563
```

Things to take from output

- ▶ Group means: high-fertilizer plants have (slightly) higher mean yield and weight than low-fertilizer plants.
- ▶ “Coefficients of linear discriminants”: LD1 , LD2 ,...are scores constructed from observed variables that best separate the groups.
 - ▶ For any plant, get LD1 score by taking -0.76 times yield plus -1.25 times weight, add up, standardize.
 - ▶ Understand by pretending all variables standardized (mean 0, + above mean, - below mean). If yield and weight high (above average), contribute a + to LD1 score, so LD1 *negative*. If yield and weight low (think -), LD1 score *positive*.
 - ▶ High-fertilizer plants have higher yield and weight, thus negative LD1 score. Low-fertilizer plants have low yield and weight, thus positive LD1 score.
 - ▶ One LD1 score for each observation. Plot with actual groups.

How many linear discriminants?

- ▶ Number of variables
- ▶ Number of groups *minus 1*
- ▶ Smaller of these
- ▶ Seed yield and weight: 2 variables, 2 groups, $\min(2, 2 - 1) = 1$.

Getting LD scores

Feed output from LDA into predict:

```
hilo.pred=predict(hilo.1)
```

Component x contains LD score(s), here in descending order:

```
d = cbind(hilo,hilo.pred$x) %>% arrange(desc(LD1))
d

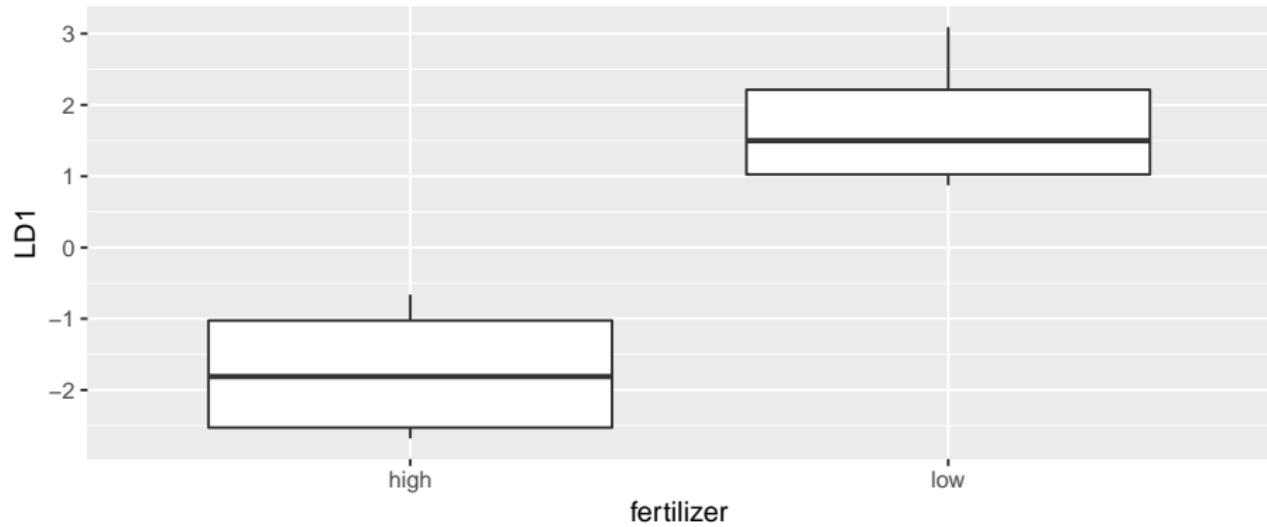
##   fertilizer yield weight      LD1
## 1       low    34     10  3.0931414
## 2       low    29     14  1.9210963
## 3       low    35     11  1.0751090
## 4       low    32     13  0.8724245
## 5      high    34     13 -0.6609276
## 6      high    33     14 -1.1456079
## 7      high    38     12 -2.4762756
## 8      high    35     14 -2.6789600
```

High fertilizer have yield and weight high, negative LD1 scores.

Plotting LD1 scores

With one LD score, plot against (true) groups, eg. boxplot:

```
ggplot(d,aes(x=fertilizer,y=LD1))+geom_boxplot()
```



Potentially misleading

- ▶ These are like regression slopes:

```
hilo.1$scaling
```

```
## LD1
## yield -0.7666761
## weight -1.2513563
```

- ▶ Reflect change in LD1 score for 1-unit change in variables.
- ▶ But one-unit change in variables might not be comparable:

```
summary(hilo)
```

```
## fertilizer          yield          weight
## Length:8           Min.   :29.00    Min.   :10.00
## Class  :character  1st Qu.:32.75   1st Qu.:11.75
## Mode   :character  Median :34.00   Median :13.00
##                   Mean   :33.75   Mean   :12.62
##                   3rd Qu.:35.00   3rd Qu.:14.00
##                   Max.   :38.00   Max.   :14.00
```

- ▶ Here, IQRs *identical*, so 1-unit change in each variable means same thing

What else is in hilo.pred?

```
names(hilo.pred)
## [1] "class"      "posterior"   "x"
```

- ▶ class: predicted fertilizer level (based on values of yield and weight).
- ▶ posterior: predicted probability of being low or high fertilizer given yield and weight.

Predictions and predicted groups

...based on yield and weight:

```
cbind(hilo,predicted=hilo.pred$class)

##   fertilizer yield weight predicted
## 1      low     34     10      low
## 2      low     29     14      low
## 3      low     35     11      low
## 4      low     32     13      low
## 5    high     33     14     high
## 6    high     38     12     high
## 7    high     34     13     high
## 8    high     35     14     high

table(obs=hilo$fertilizer,pred=hilo.pred$class)

##      pred
## obs   high low
##   high     4   0
##   low      0   4
```

Understanding the predicted groups

- ▶ Each predicted fertilizer level is exactly same as observed one (perfect prediction).
- ▶ Table shows no errors: all values on top-left to bottom-right diagonal.

Posterior probabilities

show how clear-cut the classification decisions were:

```
pp=round(hilo.pred$posterior,4)
d=cbind(hilo,hilo.pred$x,pp)
d

##   fertilizer yield weight      LD1    high    low
## 1       low     34     10  3.0931414 0.0000 1.0000
## 2       low     29     14  1.9210963 0.0012 0.9988
## 3       low     35     11  1.0751090 0.0232 0.9768
## 4       low     32     13  0.8724245 0.0458 0.9542
## 5      high     33     14 -1.1456079 0.9818 0.0182
## 6      high     38     12 -2.4762756 0.9998 0.0002
## 7      high     34     13 -0.6609276 0.9089 0.0911
## 8      high     35     14 -2.6789600 0.9999 0.0001
```

Only obs. 7 has any doubt: yield low for a high-fertilizer, but high weight makes up for it.

Example 2: the peanuts

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/peanuts.txt"
peanuts=read_delim(my_url, " ")
peanuts

## # A tibble: 12 x 6
##       obs location variety     y   smk     w
##   <int>    <int>    <int> <dbl> <dbl> <dbl>
## 1     1        1        1     5  195.  153.  51.4
## 2     2        2        1     5  194.  168.  53.7
## 3     3        3        2     5  190.  140.  55.5
## 4     4        4        2     5  180.  121.  44.4
## 5     5        5        1     6  203.  157.  49.8
## 6     6        6        1     6  196.  166.  45.8
## 7     7        7        2     6  203.  166.  60.4
## 8     8        8        2     6  198.  162.  54.1
## 9     9        9        1     8  194.  164.  57.8
## 10   10       10       1     8  187.  165.  58.6
## 11   11       11       2     8  202.  167.  65
## 12   12       12       2     8  200.  174.  67.2
```

Recall: location and variety both significant in MANOVA. Make combo of them (over):

Location-varietyp combos

```
peanuts %>% unite(combo,c(variety,location)) ->
  peanuts.combo
peanuts.combo

## # A tibble: 12 x 5
##       obs  combo      y    smk     w
##   <int> <chr> <dbl> <dbl> <dbl>
## 1     1 5_1    195.  153.  51.4
## 2     2 5_1    194.  168.  53.7
## 3     3 5_2    190.  140.  55.5
## 4     4 5_2    180.  121.  44.4
## 5     5 6_1    203   157.  49.8
## 6     6 6_1    196.  166.  45.8
## 7     7 6_2    203.  166.  60.4
## 8     8 6_2    198.  162.  54.1
## 9     9 8_1    194.  164.  57.8
## 10   10 8_1   187   165.  58.6
## 11   11 8_2   202.  167.  65
## 12   12 8_2   200   174.  67.2
```

Discriminant analysis

```
peanuts.1=lda(combo~y+smk+w, data=peanuts.combo)
peanuts.1$scaling

##           LD1          LD2          LD3
## y    -0.4027356 -0.02967881  0.18839237
## smk -0.1727459  0.06794271 -0.09386294
## w     0.5792456  0.16300221  0.07341123

peanuts.1$svd

## [1] 6.141323 2.428396 1.075589
```

- ▶ Now 3 LDs (3 variables, 6 groups, $\min(3, 6 - 1) = 3$).
- ▶ First: relationship of LDs to original variables. Look for coeffs far from zero: here,
 - ▶ high LD1 mainly high w or low y.
 - ▶ high LD2 mainly high w.
- ▶ svd values show relative importance of LDs: LD1 much more important than LD2.

Group means by variable

```
peanuts.1$means
```

```
##          y      smk      w
## 5_1 194.80 160.40 52.55
## 5_2 185.05 130.30 49.95
## 6_1 199.45 161.40 47.80
## 6_2 200.15 163.95 57.25
## 8_1 190.25 164.80 58.20
## 8_2 200.75 170.30 66.10
```

- ▶ 5_2 clearly smallest on y, smk, near smallest on w
- ▶ 8_2 clearly biggest on smk, w, also largest on y
- ▶ 8_1 large on w, small on y.
- ▶ scaling links LDs with original variables, means links original variables with groups.
- ▶ Implies: link between groups and LDs.

The predictions and misclassification

```
peanuts.pred=predict(peanuts.1)
table(obs=peanuts.combo$combo,
      pred=peanuts.pred$class)

##          pred
## obs      5_1 5_2 6_1 6_2 8_1 8_2
##   5_1      2   0   0   0   0   0
##   5_2      0   2   0   0   0   0
##   6_1      0   0   2   0   0   0
##   6_2      1   0   0   1   0   0
##   8_1      0   0   0   0   2   0
##   8_2      0   0   0   0   0   2
```

Actually classified very well. Only one 6_2 classified as a 5_1, rest all correct.

Posterior probabilities

```
pp=round(peanuts.pred$posterior,2)
peanuts.combo %>% select(-c(y,smk,w)) %>%
  cbind(.,pred=peanuts.pred$class,pp)

##      obs  combo pred  5_1  5_2  6_1  6_2  8_1  8_2
## 1      1    5_1   5_1 0.69    0    0 0.31 0.00 0.00
## 2      2    5_1   5_1 0.73    0    0 0.27 0.00 0.00
## 3      3    5_2   5_2 0.00    1    0 0.00 0.00 0.00
## 4      4    5_2   5_2 0.00    1    0 0.00 0.00 0.00
## 5      5    6_1   6_1 0.00    0    1 0.00 0.00 0.00
## 6      6    6_1   6_1 0.00    0    1 0.00 0.00 0.00
## 7      7    6_2   6_2 0.13    0    0 0.87 0.00 0.00
## 8      8    6_2   5_1 0.53    0    0 0.47 0.00 0.00
## 9      9    8_1   8_1 0.02    0    0 0.02 0.75 0.21
## 10    10    8_1   8_1 0.00    0    0 0.00 0.99 0.01
## 11    11    8_2   8_2 0.00    0    0 0.00 0.03 0.97
## 12    12    8_2   8_2 0.00    0    0 0.00 0.06 0.94
```

Some doubt about which combo each plant belongs in, but not too much. The one misclassified plant was a close call.

Discriminant scores, again

- ▶ How are discriminant scores related to original variables?
- ▶ Construct data frame with original data and discriminant scores side by side:

```
peanuts.1$scaling
```

	LD1	LD2	LD3
## y	-0.4027356	-0.02967881	0.18839237
## smk	-0.1727459	0.06794271	-0.09386294
## w	0.5792456	0.16300221	0.07341123

```
lds=round(peanuts.pred$x,2)
mm=with(peanuts.combo,
        data.frame(combo,y,smk,w,lds))
```

- ▶ LD1 positive if w large and/or y small.
- ▶ LD2 positive if w large.
- ▶ But, what if y, smk, w differ in spread?

Discriminant scores for data

mm

```
##      combo      y     smk      w     LD1     LD2     LD3
## 1      5_1 195.3 153.1 51.4 -1.42 -1.01  0.26
## 2      5_1 194.3 167.7 53.7 -2.20  0.38 -1.13
## 3      5_2 189.7 139.5 55.5  5.56 -1.10  0.79
## 4      5_2 180.4 121.1 44.4  6.06 -3.89 -0.05
## 5      6_1 203.0 156.8 49.8 -6.08 -1.25  1.25
## 6      6_1 195.9 166.0 45.8 -7.13 -1.07 -1.24
## 7      6_2 202.7 166.1 60.4 -1.43  1.12  1.10
## 8      6_2 197.6 161.8 54.1 -2.28 -0.05  0.08
## 9      8_1 193.5 164.5 57.8  1.05  0.86 -0.67
## 10     8_1 187.0 165.1 58.6  4.02  1.22 -1.90
## 11     8_2 201.5 166.8 65.0  1.60  1.95  1.15
## 12     8_2 200.0 173.8 67.2  2.27  2.83  0.37
```

- ▶ Obs. 5 and 6 have most negative LD1: large y, small w.
- ▶ Obs. 4 has most negative LD2: small w.

Predict typical LD1 scores

First and third quartiles for three response variables (reading down):

```
quartiles = peanuts %>% select(y:w) %>%
  map_df(quantile, c(0.25,0.75))
quartiles

## # A tibble: 2 x 3
##       y     smk      w
##   <dbl> <dbl> <dbl>
## 1 193.  156.   51
## 2 200.  166.   59.0

new=with(quartiles,crossing(y,smk,w))
```

The combinations

```
new

## # A tibble: 8 x 3
##       y     smk      w
##   <dbl> <dbl> <dbl>
## 1 193.  156.   51
## 2 193.  156.  59.0
## 3 193.  166.   51
## 4 193.  166.  59.0
## 5 200.  156.   51
## 6 200.  156.  59.0
## 7 200.  166.   51
## 8 200.  166.  59.0

pp=predict(peanuts.1,new)
```

Predicted typical LD1 scores

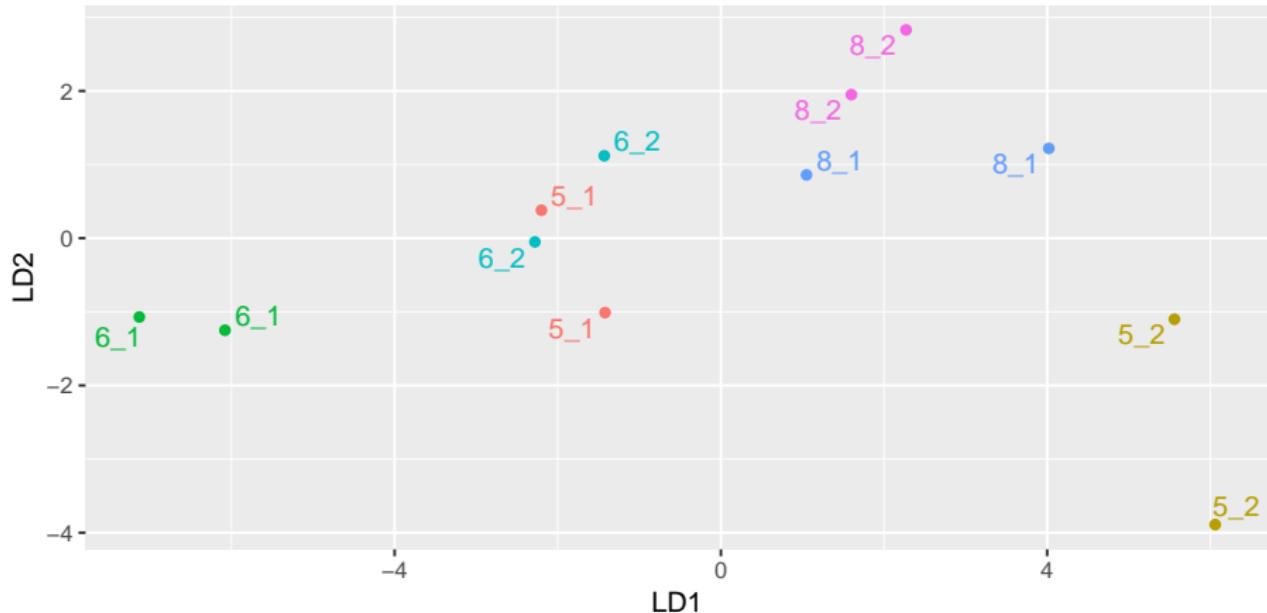
```
cbind(new,pp$x) %>% arrange(LD1)

##          y      smk      w       LD1       LD2       LD3
## 1 200.375 166.275 51.00 -5.9688625 -0.3330095 -0.04523828
## 2 200.375 155.875 51.00 -4.1723048 -1.0396138  0.93093630
## 3 192.550 166.275 51.00 -2.8174566 -0.1007728 -1.51940856
## 4 200.375 166.275 59.05 -1.3059358  0.9791583  0.54572212
## 5 192.550 155.875 51.00 -1.0208989 -0.8073770 -0.54323399
## 6 200.375 155.875 59.05  0.4906219  0.2725540  1.52189670
## 7 192.550 166.275 59.05  1.8454701  1.2113950 -0.92844817
## 8 192.550 155.875 59.05  3.6420278  0.5047907  0.04772641
```

- ▶ Very negative LD1 score with large y and small w
- ▶ smk doesn't contribute much to LD1
- ▶ Very positive LD1 score with small y and large w .
- ▶ Same as we saw from Coefficients of Linear Discriminants.

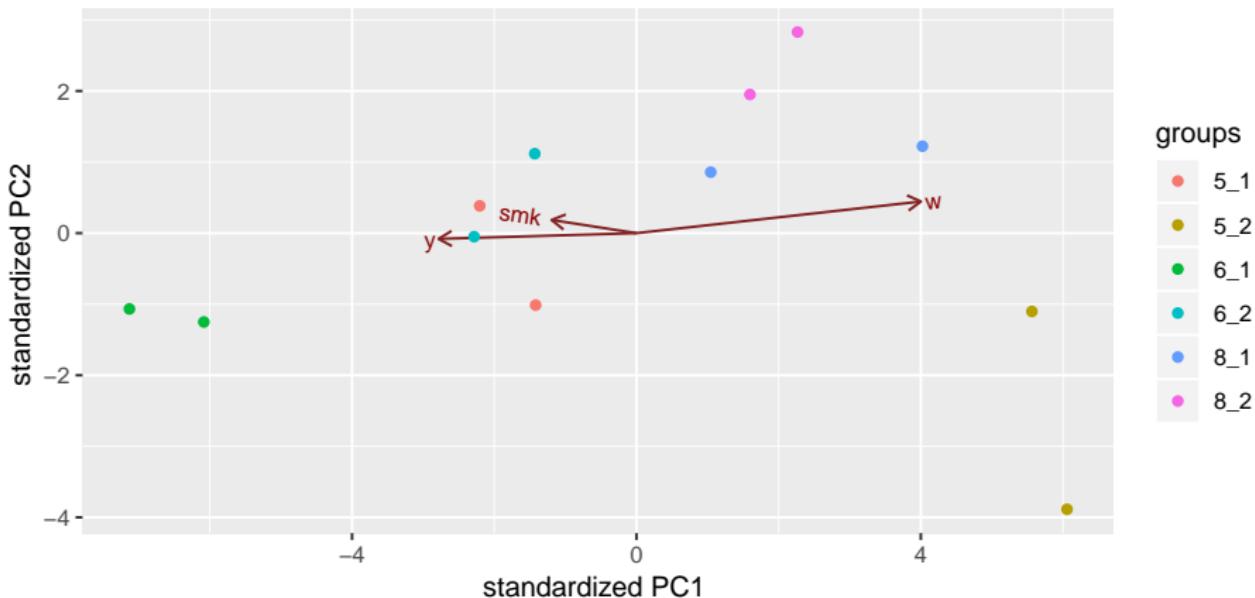
Plot LD1 vs. LD2, labelling by combo

```
g=ggplot(mm,aes(x=LD1,y=LD2,colour=combo,  
label=combo))+geom_point() +  
geom_text_repel() + guides(colour=F) ; g
```



“Bi-plot” from ggbiplot

```
ggbiplot(peanuts.1,  
groups=factor(peanuts.combo$combo))
```



Installing ggbiplot

- ▶ ggbiplot not on CRAN, so usual `install.packages` will not work.
- ▶ Install package `devtools` first (once):

```
install.packages("devtools")
```

- ▶ Then install `ggbiplot` (once):

```
library(devtools)  
install_github("vqv/ggbiplot")
```

Cross-validation

- ▶ So far, have predicted group membership from same data used to form the groups — dishonest!
- ▶ Better: *cross-validation*: form groups from all observations *except one*, then predict group membership for that left-out observation.
- ▶ No longer cheating!
- ▶ Illustrate with peanuts data again.

Misclassifications

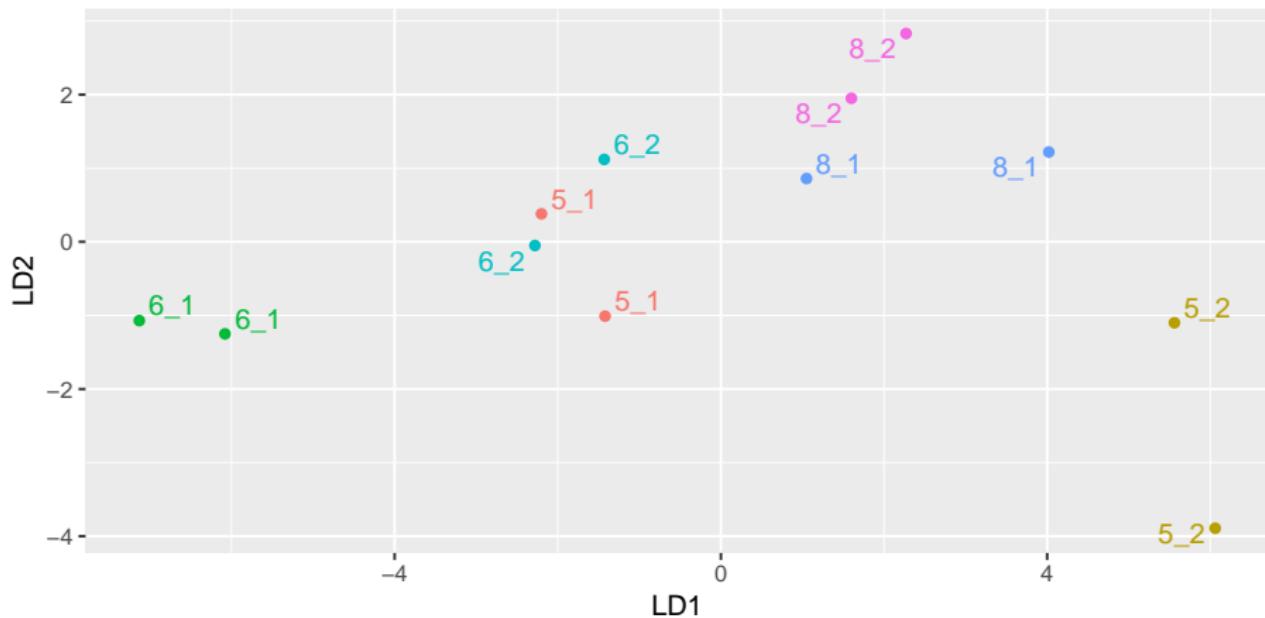
- ▶ Fitting and prediction all in one go:

```
peanuts.cv=lda(combo~y+smk+w,  
    data=peanuts.combo, CV=T)  
  
table(obs=peanuts.combo$combo,  
      pred=peanuts.cv$class)  
  
##          pred  
##   obs  5_1 5_2 6_1 6_2 8_1 8_2  
## 5_1    0  0  0  2  0  0  
## 5_2    0  1  0  0  1  0  
## 6_1    0  0  2  0  0  0  
## 6_2    1  0  0  1  0  0  
## 8_1    0  1  0  0  0  1  
## 8_2    0  0  0  0  0  2
```

- ▶ Some more misclassification this time.

Repeat of LD plot

gg



Posterior probabilities

```
pp=round(peanuts.cv$posterior,3)
data.frame(obs=peanuts.combo$combo,
           pred=peanuts.cv$class,pp)

##      obs pred  X5_1 X5_2  X6_1  X6_2  X8_1  X8_2
## 1  5_1   6_2 0.162 0.00 0.000 0.838 0.000 0.000
## 2  5_1   6_2 0.200 0.00 0.000 0.799 0.000 0.000
## 3  5_2   8_1 0.000 0.18 0.000 0.000 0.820 0.000
## 4  5_2   5_2 0.000 1.00 0.000 0.000 0.000 0.000
## 5  6_1   6_1 0.194 0.00 0.669 0.137 0.000 0.000
## 6  6_1   6_1 0.000 0.00 1.000 0.000 0.000 0.000
## 7  6_2   6_2 0.325 0.00 0.000 0.667 0.001 0.008
## 8  6_2   5_1 0.821 0.00 0.000 0.179 0.000 0.000
## 9  8_1   8_2 0.000 0.00 0.000 0.000 0.000 1.000
## 10 8_1   5_2 0.000 1.00 0.000 0.000 0.000 0.000
## 11 8_2   8_2 0.001 0.00 0.000 0.004 0.083 0.913
## 12 8_2   8_2 0.000 0.00 0.000 0.000 0.167 0.833
```

Why more misclassification?

- ▶ When predicting group membership for one observation, only uses the *other one* in that group.
- ▶ So if two in a pair are far apart, or if two groups overlap, great potential for misclassification.
- ▶ Groups 5_1 and 6_2 overlap.
- ▶ 5_2 closest to 8_1s looks more like an 8_1 than a 5_2 (other one far away).
- ▶ 8_1s relatively far apart and close to other things, so one appears to be a 5_2 and the other an 8_2.

Example 3: professions and leisure activities

- ▶ 15 individuals from three different professions (politicians, administrators and belly dancers) each participate in four different leisure activities: reading, dancing, TV watching and skiing. After each activity they rate it on a 0–10 scale.
- ▶ Some of the data:

bellydancer 7 10 6 5

bellydancer 8 9 5 7

bellydancer 5 10 5 8

politician 5 5 5 6

politician 4 5 6 5

admin 4 2 2 5

admin 7 1 2 4

admin 6 3 3 3

- ▶ How can we best use the scores on the activities to predict a person's profession?
- ▶ Or, what combination(s) of scores best separate data into profession

Discriminant analysis

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/profile.txt"
active=read_delim(my_url, " ")
active.1=lda(job~reading+dance+tv+ski, data=active)
active.1$svd

## [1] 9.856638 3.434555

active.1$scaling

##           LD1       LD2
## reading -0.01297465  0.4748081
## dance   -0.95212396  0.4614976
## tv      -0.47417264 -1.2446327
## ski     0.04153684  0.2033122
```

- ▶ Two discriminants, first fair bit more important than second.
- ▶ LD1 depends (negatively) most on dance, a bit on tv.
- ▶ LD2 depends mostly on tv.

Misclassification

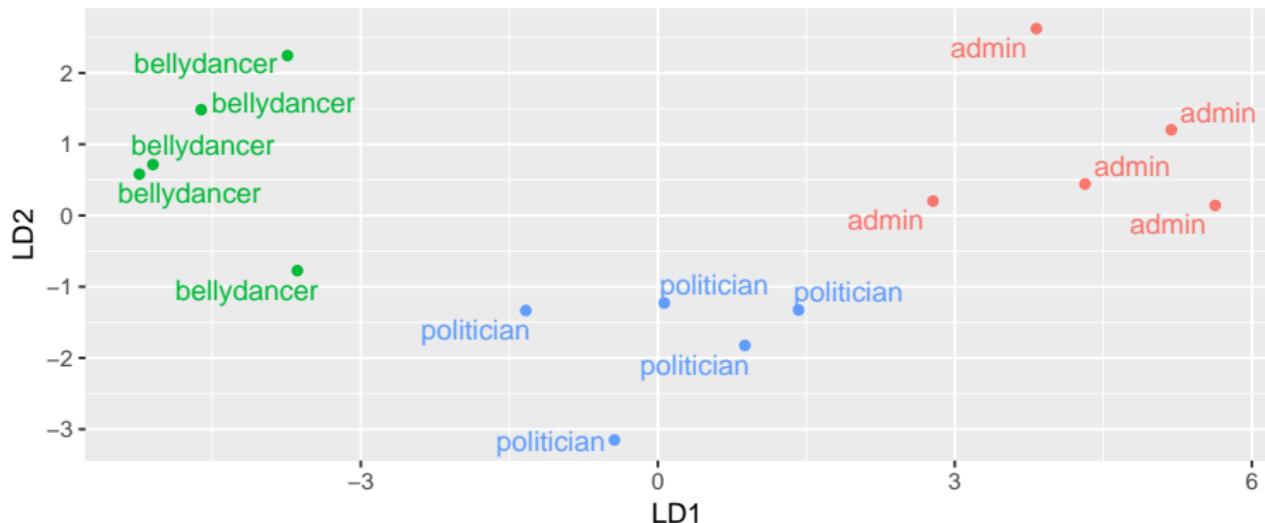
```
active.pred=predict(active.1)
table(obs=active$job,pred=active.pred$class)

##          pred
##   obs      admin bellydancer politician
##   admin      5        0         0
##   bellydancer 0        5         0
##   politician  0        0         5
```

Everyone correctly classified.

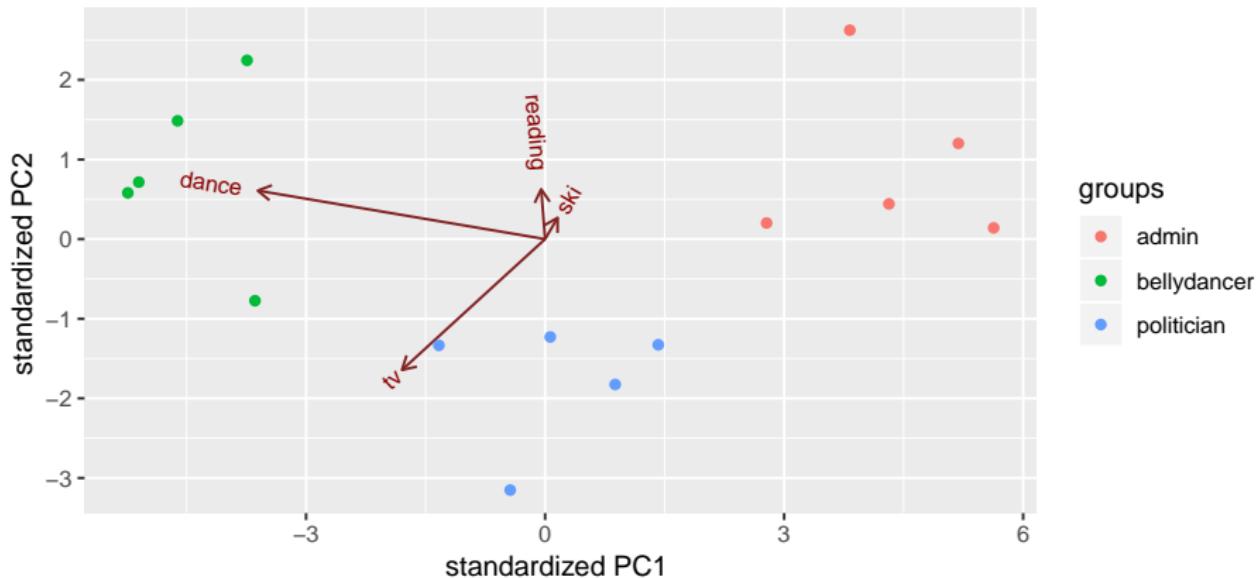
Plotting LDs

```
mm=data.frame(job=active$job,active.pred$x, person=1:15)
g=ggplot(mm,aes(x=LD1,y=LD2,
  colour=job,label=job))+geom_point()+
  geom_text_repel() + guides(colour=F) ; g
```



Biplot

```
ggbiplot(active.1,groups=active$job)
```



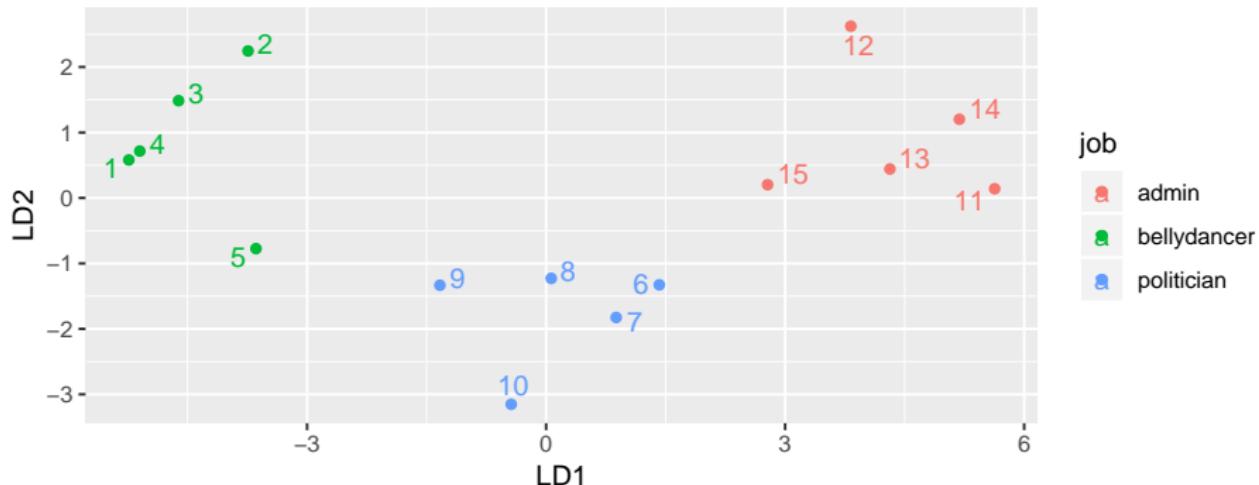
Comments on plot

- ▶ Groups well separated: bellydancers top left, administrators top right, politicians lower middle.
- ▶ Bellydancers most negative on LD1: like dancing most.
- ▶ Administrators most positive on LD1: like dancing least.
- ▶ Politicians most negative on LD2: like TV-watching most.

Plotting individual persons

Make label be identifier of person. Now need legend:

```
ggplot(mm,aes(x=LD1,y=LD2,  
colour=job,label=person))+geom_point() +  
geom_text_repel()
```



Posterior probabilities

```
pp=round(active.pred$posterior,3)
data.frame(obs=active$job,pred=active.pred$class,pp)

##          obs      pred admin bellydancer politician
## 1  bellydancer bellydancer 0.000      1.000     0.000
## 2  bellydancer bellydancer 0.000      1.000     0.000
## 3  bellydancer bellydancer 0.000      1.000     0.000
## 4  bellydancer bellydancer 0.000      1.000     0.000
## 5  bellydancer bellydancer 0.000      0.997     0.003
## 6  politician   politician 0.003      0.000     0.997
## 7  politician   politician 0.000      0.000     1.000
## 8  politician   politician 0.000      0.000     1.000
## 9  politician   politician 0.000      0.002     0.998
## 10 politician   politician 0.000      0.000     1.000
## 11      admin      admin 1.000      0.000     0.000
## 12      admin      admin 1.000      0.000     0.000
## 13      admin      admin 1.000      0.000     0.000
## 14      admin      admin 1.000      0.000     0.000
## 15      admin      admin 0.982      0.000     0.018
```

Not much doubt.

Cross-validating the jobs-activities data

Recall: no need for predict. Just pull out class and make a table:

```
active.cv=lda(job~reading+dance+tv+ski,  
    data=active, CV=T)  
table(obs=active$job, pred=active.cv$class)
```

		pred		
##	## obs	admin	bellydancer	politician
##	## admin	5	0	0
##	## bellydancer	0	4	1
##	## politician	0	0	5

This time one of the bellydancers was classified as a politician.

and look at the posterior probabilities

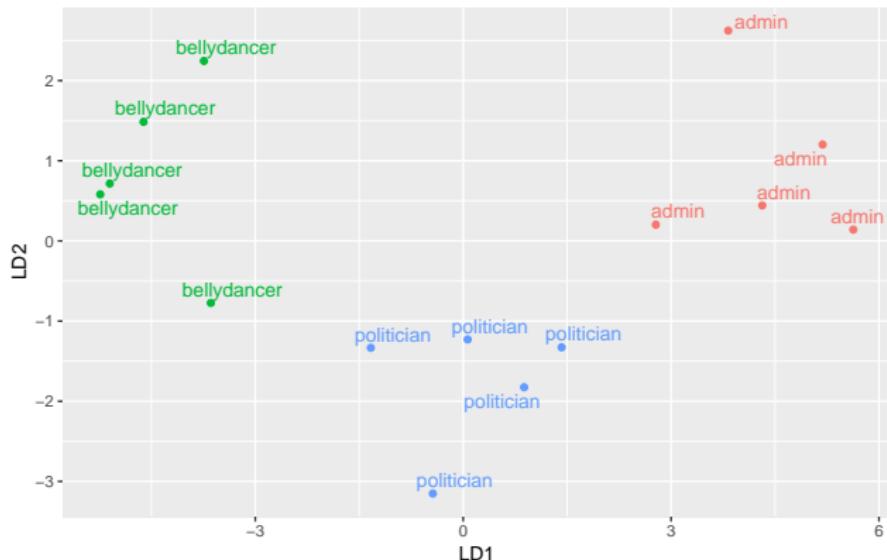
picking out the ones where things are not certain:

```
pp=round(active.cv$posterior,3)
data.frame(obs=active$job,pred=active.cv$class,pp) %>%
  mutate(max=pmax(admin,bellydancer,politician)) %>%
  filter(max<0.9995)

##           obs      pred admin bellydancer politician   max
## 1 bellydancer politician 0.000      0.001    0.999 0.999
## 2 politician politician 0.006      0.000    0.994 0.994
## 3 politician politician 0.001      0.000    0.999 0.999
## 4 politician politician 0.000      0.009    0.991 0.991
## 5       admin      admin 0.819      0.000    0.181 0.819
```

- ▶ Bellydancer was “definitely” a politician!
- ▶ One of the administrators might have been a politician too.

Why did things get misclassified?



- ▶ Go back to plot of discriminant scores:
- ▶ one bellydancer much closer to the politicians,
- ▶ one administrator a bit closer to the politicians.

Example 4: remote-sensing data

- ▶ View 38 crops from air, measure 4 variables x_1-x_4 .
- ▶ Go back and record what each crop was.
- ▶ Can we use the 4 variables to distinguish crops?

Reading in

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/remote-sensing.csv"
crops=read_table(my_url)

## Parsed with column specification:
## cols(
##   crop = col_character(),
##   x1 = col_integer(),
##   x2 = col_integer(),
##   x3 = col_integer(),
##   x4 = col_integer(),
##   cr = col_character()
## )
```

Starting off: number of LDs

```
crops.lda=lda(crop~x1+x2+x3+x4,data=crops)
crops.lda$svd
## [1] 2.2858251 1.1866352 0.6394041 0.2303634
```

- ▶ 4 LDs (four variables, six groups).
- ▶ 1st one important, maybe 2nd as well.

Connecting original variables and LDs

```
crops.lda$means
```

```
##           x1        x2        x3        x4
## Clover    46.36364 32.63636 34.18182 36.63636
## Corn      15.28571 22.71429 27.42857 33.14286
## Cotton    34.50000 32.66667 35.00000 39.16667
## Soybeans  21.00000 27.00000 23.50000 29.66667
## Sugarbeets 31.00000 32.16667 20.00000 40.50000
```

```
round(crops.lda$scaling, 3)
```

```
##       LD1     LD2     LD3     LD4
## x1 -0.061  0.009 -0.030 -0.015
## x2 -0.025  0.043  0.046  0.055
## x3  0.016 -0.079  0.020  0.009
## x4  0.000 -0.014  0.054 -0.026
```

- ▶ Links groups to original variables to LDs.

LD1 and LD2

```
round(crops.lda$scaling,3)

##          LD1      LD2      LD3      LD4
## x1 -0.061  0.009 -0.030 -0.015
## x2 -0.025  0.043  0.046  0.055
## x3  0.016 -0.079  0.020  0.009
## x4  0.000 -0.014  0.054 -0.026
```

- ▶ LD1 mostly x1 (minus), so clover low on LD1, corn high.
- ▶ LD2 x3 (minus), x2 (plus), so sugarbeets should be high on LD2.

Predictions

- ▶ Thus:

```
crops.pred=predict(crops.lda)
table(obs=crops$crop,pred=crops.pred$class)

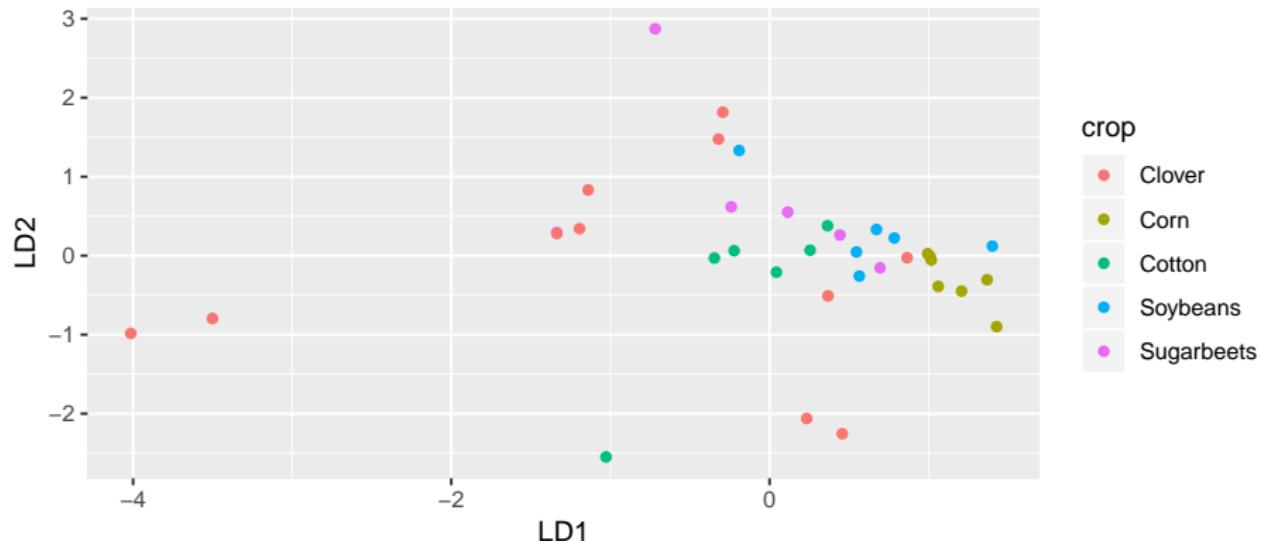
##          pred
## obs      Clover Corn Cotton Soybeans Sugarbeets
## Clover        6   0     3       0        2
## Corn          0   6     0       1        0
## Cotton        3   0     1       2        0
## Soybeans      0   1     1       3        1
## Sugarbeets    1   1     0       2        2
```

- ▶ Not very good, eg. only 6 of 11 Clover classified correctly.
- ▶ Set up for plot:

```
mm=data.frame(crop=crops$crop,crops.pred$x)
```

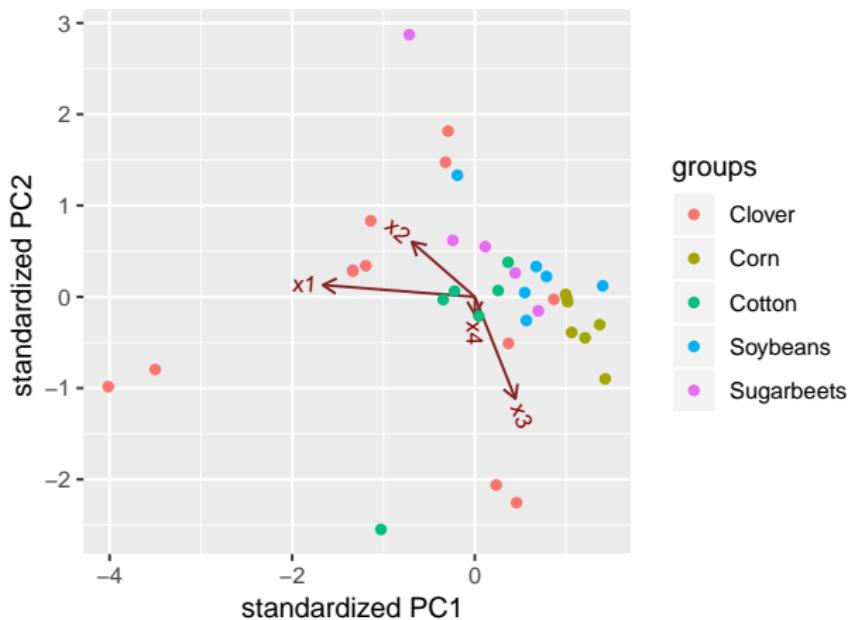
Plotting the LDs

```
ggplot(mm, aes(x=LD1, y=LD2, colour=crop)) +  
  geom_point()
```



Biplot

```
ggbiplot(crops.lda, groups=crops$crop)
```



Comments

- ▶ Corn high on LD1 (right).
- ▶ Clover all over the place, but mostly low on LD1 (left).
- ▶ Sugarbeets tend to be high on LD2.
- ▶ Cotton tends to be low on LD2.
- ▶ Very mixed up.

Try removing Clover

- ▶ the dplyr way:

```
crops %>% filter(crop!="Clover") -> crops2  
crops2.lda=lda(crop~x1+x2+x3+x4, data=crops2)
```

- ▶ LDs for crops2 will be different from before.
- ▶ Concentrate on plot and posterior probs.

```
crops2.pred=predict(crops2.lda)  
mm=data.frame(crop=crops2$crop, crops2.pred$x)
```

lda output

Different from before:

```
crops2.lda$means
```

```
##           x1         x2         x3         x4
## Corn      15.28571 22.71429 27.42857 33.14286
## Cotton    34.50000 32.66667 35.00000 39.16667
## Soybeans  21.00000 27.00000 23.50000 29.66667
## Sugarbeets 31.00000 32.16667 20.00000 40.50000
```

```
crops2.lda$svd
```

```
## [1] 3.3639389 1.6054750 0.4180292
```

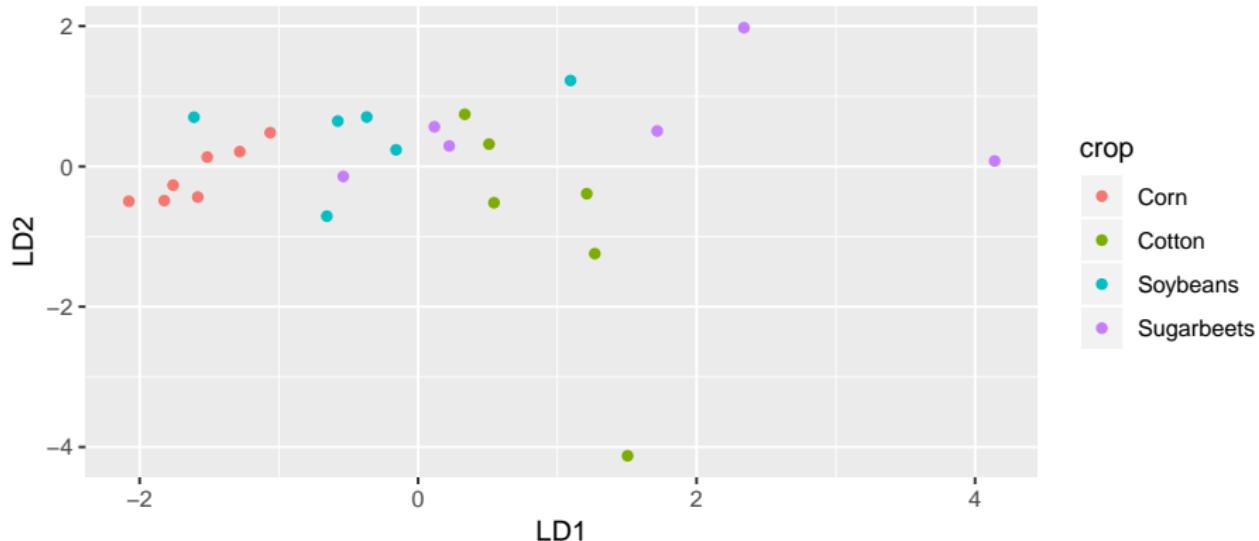
```
crops2.lda$scaling
```

```
##           LD1         LD2         LD3
## x1  0.14077479  0.007780184 -0.0312610362
## x2  0.03006972  0.007318386  0.0085401510
## x3 -0.06363974 -0.099520895 -0.0005309869
## x4 -0.00677414 -0.035612707  0.0577718649
```

Plot

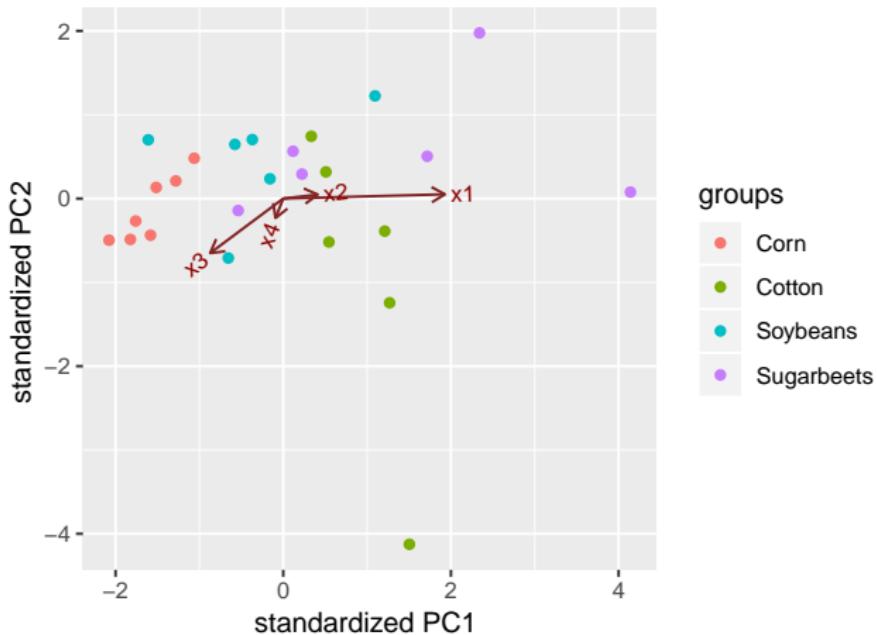
A bit more clustered:

```
ggplot(mm, aes(x=LD1, y=LD2, colour=crop)) +  
  geom_point()
```



Biplot

```
ggbiplot(crops2.lda, groups=crops2$crop)
```



Quality of classification

```
table(obs=crops2$crop,pred=crops2.pred$class)

##           pred
## obs          Corn Cotton Soybeans Sugarbeets
## Corn         6     0      1        0
## Cotton       0     4      2        0
## Soybeans     2     0      3        1
## Sugarbeets   0     0      3        3
```

Better.

Posterior probs, the wrong ones

```
post=round(crops2.pred$posterior,3)
data.frame(obs=crops2$crop,pred=crops2.pred$class,post) %>%
  filter(obs!=pred)

##          obs      pred   Corn Cotton Soybeans Sugarbeets
## 1       Corn    Soybeans 0.443  0.034     0.494    0.029
## 2   Soybeans  Sugarbeets 0.010  0.107     0.299    0.584
## 3   Soybeans        Corn 0.684  0.009     0.296    0.011
## 4   Soybeans        Corn 0.467  0.199     0.287    0.047
## 5       Cotton    Soybeans 0.056  0.241     0.379    0.324
## 6       Cotton    Soybeans 0.066  0.138     0.489    0.306
## 7 Sugarbeets    Soybeans 0.381  0.146     0.395    0.078
## 8 Sugarbeets    Soybeans 0.106  0.144     0.518    0.232
## 9 Sugarbeets    Soybeans 0.088  0.207     0.489    0.216
```

- ▶ These were the misclassified ones, but the posterior probability of being correct was not usually too low.
- ▶ The correctly-classified ones are not very clear-cut either.

MANOVA

Began discriminant analysis as a followup to MANOVA. Do our variables significantly separate the crops (excluding Clover)?

```
response=with(crops2, cbind(x1,x2,x3,x4))
crops2.manova=manova(response~crop, data=crops2)
summary(crops2.manova)

##           Df Pillai approx F num Df den Df Pr(>F)
## crop       3 0.9113    2.1815     12      60 0.02416 *
## Residuals 21
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Yes, at least one of the crops differs (in means) from the others. So it is worth doing this analysis.

We did this the wrong way around, though!

The right way around

- ▶ *First*, do a MANOVA to see whether any of the groups differ significantly on any of the variables.
- ▶ *If the MANOVA is significant*, do a discriminant analysis in the hopes of understanding how the groups are different.
- ▶ For remote-sensing data (without Clover):
 - ▶ LD1 a fair bit more important than LD2 (definitely ignore LD3).
 - ▶ LD1 depends mostly on x_1 , on which Cotton was high and Corn was low.
- ▶ Discriminant analysis in MANOVA plays the same kind of role that Tukey does in ANOVA.

Section 10

Cluster analysis

Cluster Analysis

- ▶ One side-effect of discriminant analysis: could draw picture of data (if 1st 2s LDs told most of story) and see which individuals “close” to each other.
- ▶ Discriminant analysis requires knowledge of groups.
- ▶ Without knowledge of groups, use *cluster analysis*: see which individuals close, which groups suggested by data.
- ▶ Idea: see how individuals group into “clusters” of nearby individuals.
- ▶ Base on “dissimilarities” between individuals.
- ▶ Or base on standard deviations and correlations between variables (assesses dissimilarity behind scenes).

Packages

```
library(MASS) # for lda later
library(tidyverse)

## -- Attaching packages ----- tidyverse
1.2.1 --
## v ggplot2 3.1.0     v purrrr  0.2.5
## v tibble   1.4.2     v dplyr    0.7.8
## v tidyverse 0.8.1    v stringr  1.3.1
## v readr    1.1.1     vforcats  0.3.0

## -- Conflicts -----
tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()
## x dplyr::select() masks MASS::select()

library(ggrepel)
```

One to ten in 11 languages

	English	Norwegian	Danish	Dutch	German
1	one	en	en	een	eins
2	two	to	to	twee	zwei
3	three	tre	tre	drie	drei
4	four	fire	fire	vier	vier
5	five	fem	fem	vijf	funf
6	six	seks	seks	zes	sechs
7	seven	sju	syv	zeven	sieben
8	eight	atte	otte	acht	acht
9	nine	ni	ni	negen	neun
10	ten	ti	ti	tien	zehn

One to ten

	French	Spanish	Italian	Polish	Hungarian	Finnish
1	un	uno	uno	jeden	egy	yksi
2	deux	dos	due	dwa	ketto	kaksi
3	trois	tres	tre	trzy	harom	kolme
4	quatre	cuatro	quattro	cztery	negy	nelja
5	cinq	cinco	cinque	piec	öt	viisi
6	six	seis	sei	szesc	hat	kuusi
7	sept	siete	sette	siedem	het	seitseman
8	huit	ocho	otto	osiem	nyolc	kahdeksan
9	neuf	nueve	nove	dziewiec	kilenc	yhdeksan
10	dix	diez	dieci	dziesiec	tiz	kymmenen

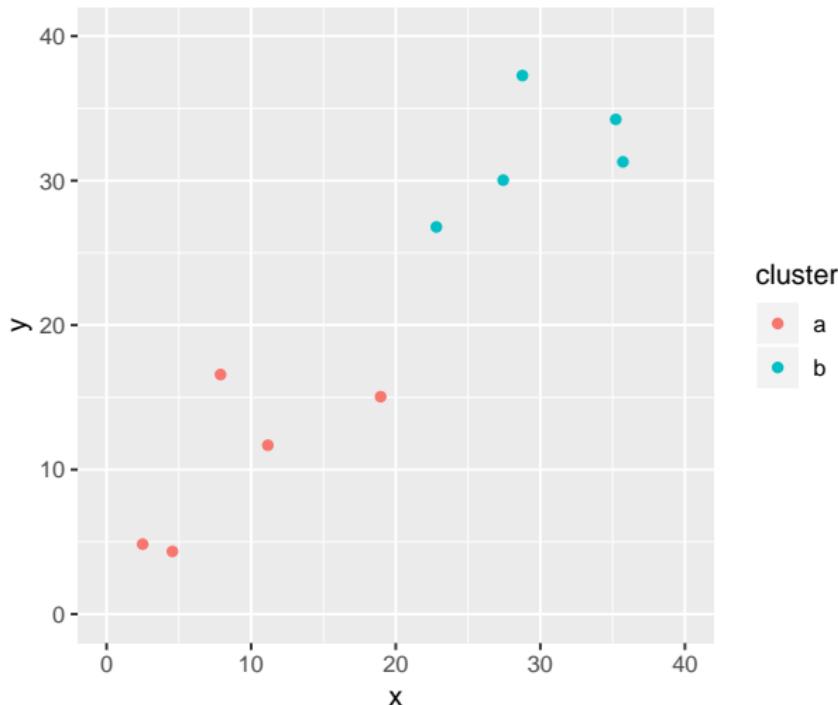
Dissimilarities and languages example

- ▶ Can define dissimilarities how you like (whatever makes sense in application).
- ▶ Sometimes defining “similarity” makes more sense; can turn this into dissimilarity by subtracting from some maximum.
- ▶ Example: numbers 1–10 in various European languages. Define similarity between two languages by counting how often the same number has a name starting with the same letter (and dissimilarity by how often number has names starting with different letter).
- ▶ Crude (doesn’t even look at most of the words), but see how effective.

Two kinds of cluster analysis

- ▶ Looking at process of forming clusters (of similar languages): **hierarchical cluster analysis** (`hclust`).
 - ▶ Start with each individual in cluster by itself.
 - ▶ Join “closest” clusters one by one until all individuals in one cluster.
 - ▶ How to define closeness of two *clusters*? Not obvious, investigate in a moment.
- ▶ Know how many clusters: which division into that many clusters is “best” for individuals? **K-means clustering** (`kmeans`).

Two made-up clusters



How to measure distance between set of red points and set of blue ones?

Single-linkage distance

Find the red point and the blue point that are closest together:

```
## Error in distances[i, j] <- dd: incorrect number of  
subscripts on matrix  
  
## Error in apply(distances, 1, min): dim(X) must have a  
positive length  
  
## Error in apply(distances, 2, min): dim(X) must have a  
positive length  
  
## Error in ` [.data.frame`(a, wml, ): object 'wml' not  
found  
  
## Error in fortify(data): object 'closest' not found
```

Single-linkage distance between 2 clusters is distance between their closest points.

Complete linkage

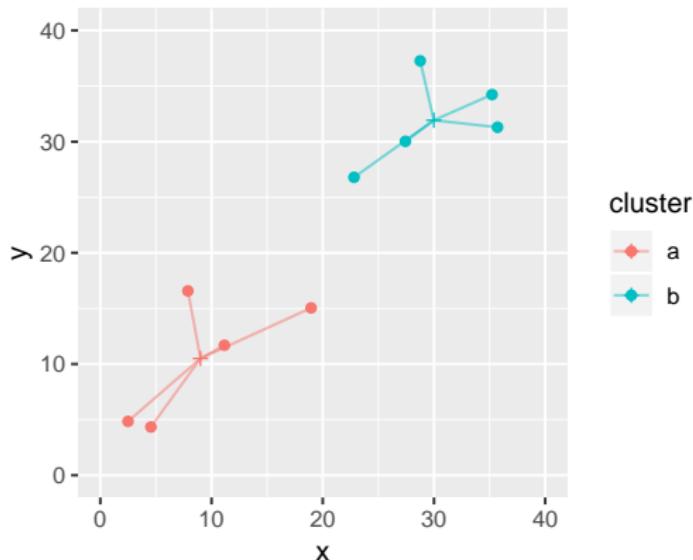
Find the red and blue points that are farthest apart:

```
## Error in apply(distances, 1, max): dim(X) must have a  
positive length  
  
## Error in apply(distances, 2, max): dim(X) must have a  
positive length  
  
## Error in ` [.data.frame`(a, wml, ): object 'wml' not  
found  
  
## Error in fortify(data): object 'closest' not found
```

Complete-linkage distance is distance between farthest points.

Ward's method

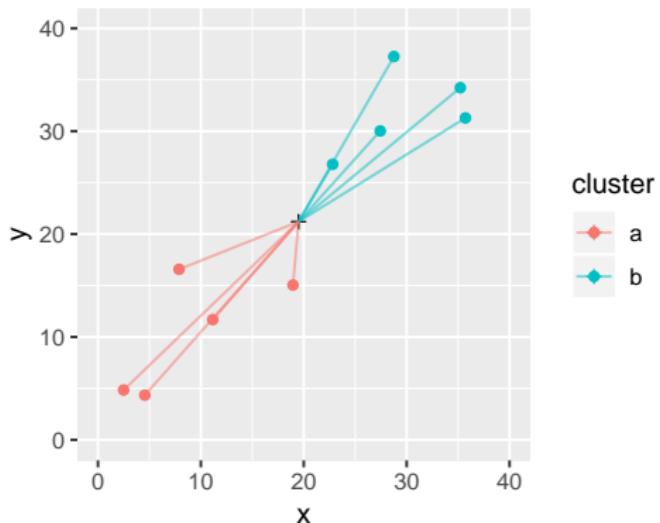
Work out mean of each cluster and join point to its mean:



- (i) Work out sum of squared distances of points from means.

Ward's method part 2

Now imagine combining the two clusters and working out overall mean.
Join each point to this mean:



- (ii) Calc sum of squared distances of points to combined mean.

Ward's method part 3

- ▶ (ii) will be bigger than (i) (points closer to own cluster mean than combined mean).
- ▶ Ward's distance is (ii) minus (i).
- ▶ Think of as “cost” of combining clusters:
 - ▶ if clusters close together, (ii) only a little larger than (i)
 - ▶ if clusters far apart, (ii) a lot larger than (i) (as in example).

Hierarchical clustering revisited

- ▶ Single linkage, complete linkage, Ward are ways of measuring closeness of clusters.
- ▶ Use them, starting with each observation in own cluster, to repeatedly combine two closest clusters until all points in one cluster.
- ▶ They will give different answers (clustering stories).
- ▶ Single linkage tends to make “stringy” clusters because clusters can be very different apart from two closest points.
- ▶ Complete linkage insists on whole clusters being similar.
- ▶ Ward tends to form many small clusters first.

Dissimilarity data in R

Dissimilarities for language data were how many number names had *different* first letter:

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/languages.txt"
number.d=read_table(my_url)
number.d

## # A tibble: 11 x 12
##   la     en     no     dk     nl     de     fr     es     it
##   <chr> <int> <int> <int> <int> <int> <int> <int> <int>
## 1 en      0      2      2      7      6      6      6      6
## 2 no      2      0      1      5      4      6      6      6
## 3 dk      2      1      0      6      5      6      5      5
## 4 nl      7      5      6      0      5      9      9      9
## 5 de      6      4      5      5      0      7      7      7
## 6 fr      6      6      6      9      7      0      2      1
## 7 es      6      6      5      9      7      2      0      1
## 8 it      6      6      5      9      7      1      1      0
## 9 pl      7      7      6     10      8      5      3      4
## 10 hu     9      8      8      8      9     10     10     10
## 11 fi     9      9      9      9      9      9      9      8
## # ... with 3 more variables: pl <int>, hu <int>, fi <int>
```

Making a distance object

```
d = number.d %>%
  select(-la) %>%
  as.dist()

d

##      en no dk nl de fr es it pl hu
## no  2
## dk  2  1
## nl  7  5  6
## de  6  4  5  5
## fr  6  6  6  9  7
## es  6  6  5  9  7  2
## it  6  6  5  9  7  1  1
## pl  7  7  6 10  8  5  3  4
## hu  9  8  8  8  9 10 10 10 10
## fi  9  9  9  9  9  9  9  8  9  8

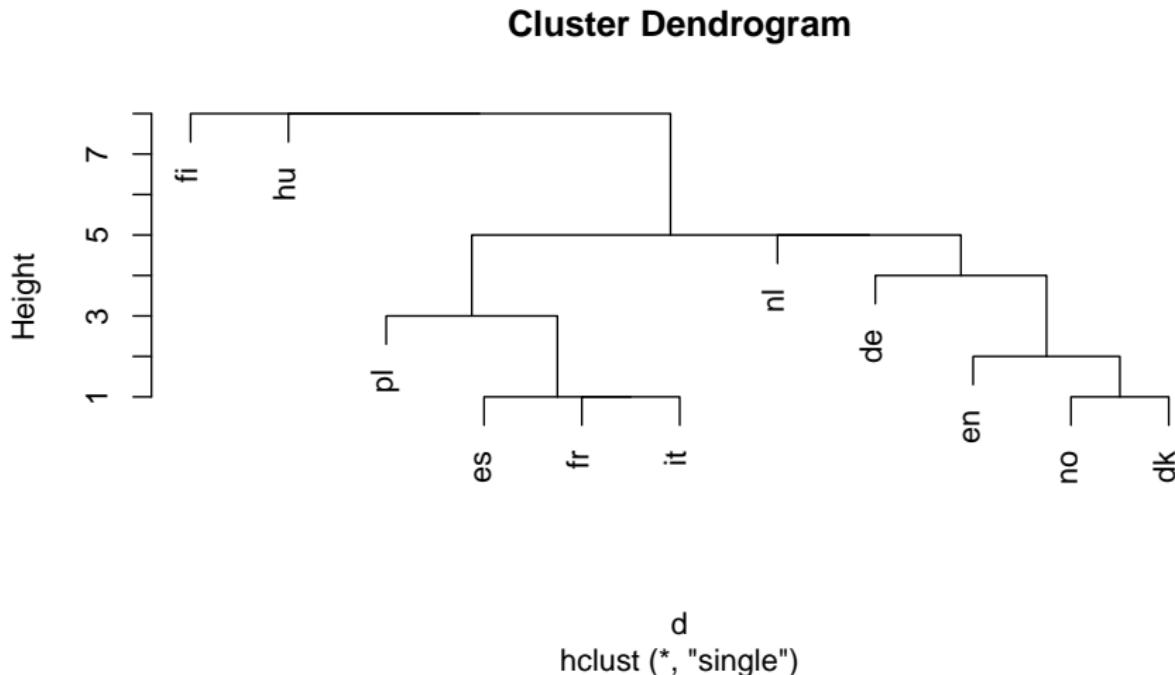
class(d)

## [1] "dist"
```

Cluster analysis and dendrogram

```
d.hc=hclust(d,method="single")
```

```
plot(d.hc)
```



Comments

- ▶ Tree shows how languages combined into clusters.
- ▶ First (bottom), Spanish, French, Italian joined into one cluster, Norwegian and Danish into another.
- ▶ Later, English joined to Norse languages, Polish to Romance group.
- ▶ Then German, Dutch make a Germanic group.
- ▶ Finally, Hungarian and Finnish joined to each other and everything else.

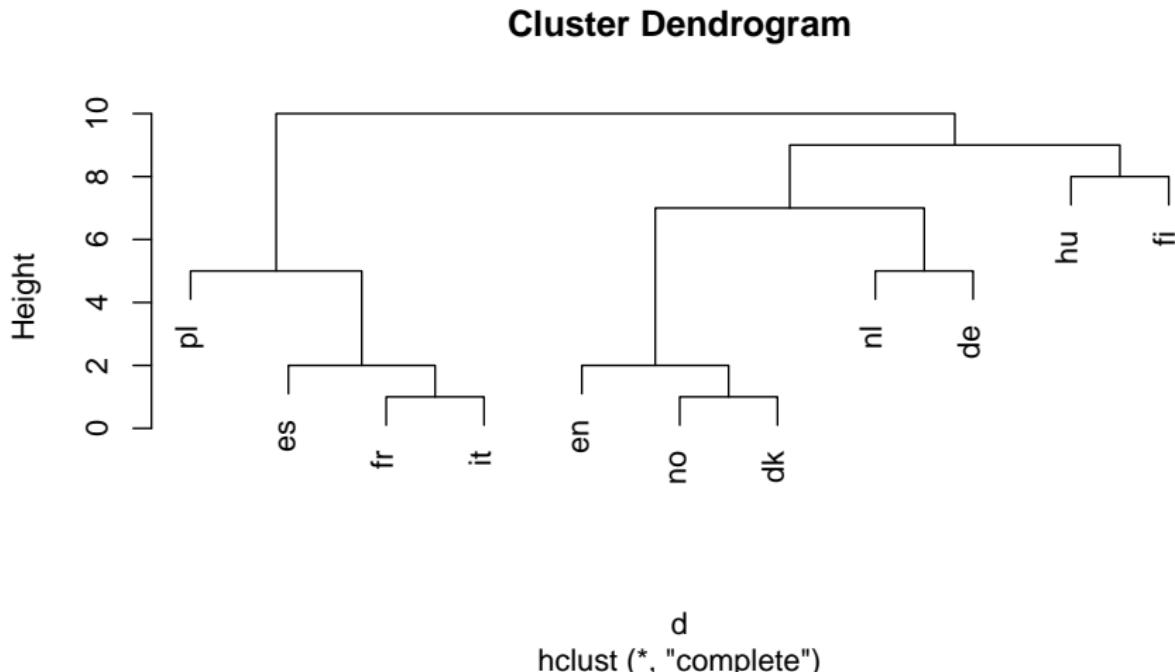
Clustering process

```
d.hc$labels  
## [1] "en" "no" "dk" "nl"  
## [5] "de" "fr" "es" "it"  
## [9] "pl" "hu" "fi"  
  
d.hc$merge  
## [,1] [,2]  
## [1,] -2 -3  
## [2,] -6 -8  
## [3,] -7 2  
## [4,] -1 1  
## [5,] -9 3  
## [6,] -5 4  
## [7,] -4 6  
## [8,] 5 7  
## [9,] -10 8  
## [10,] -11 9
```

- ▶ Lines of `merge` show what was combined
- ▶ First, languages 2 and 3 (`no` and `dk`)
- ▶ Then languages 6 and 8 (`fr` and `it`)
- ▶ Then #7 combined with cluster formed at step 2 (`es` joined to `fr` and `it`).
- ▶ Then `en` joined to `no` and `dk` ...
- ▶ Finally `fi` joined to all others.

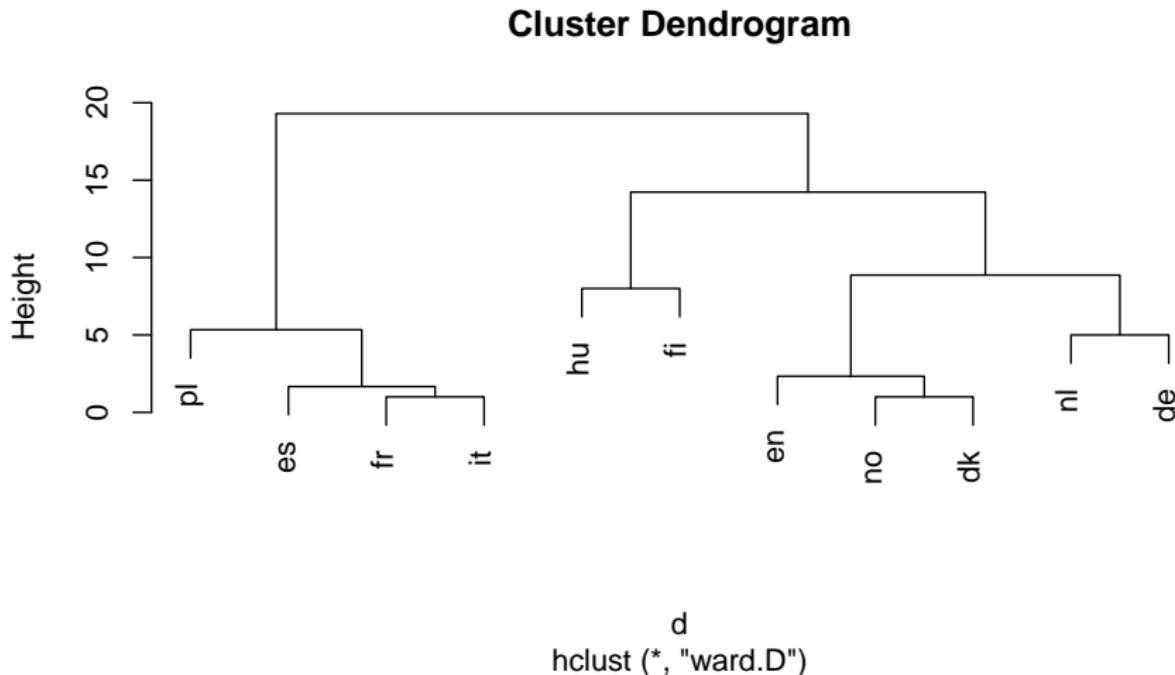
Complete linkage

```
d.hc=hclust(d,method="complete")  
plot(d.hc)
```



Ward

```
d.hc=hclust(d,method="ward.D")  
plot(d.hc)
```



Chopping the tree

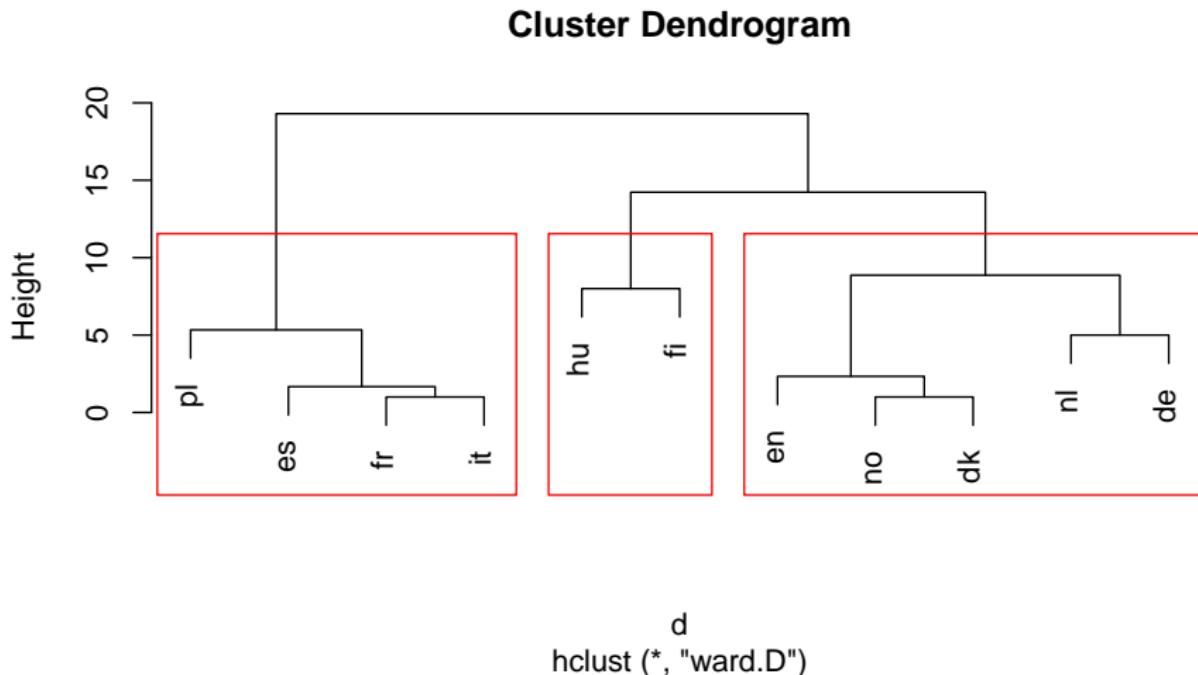
- ▶ Three clusters (from Ward) looks good:

```
cutree(d.hc, 3)

## en no dk nl de fr es it
## 1 1 1 1 1 2 2 2
## pl hu fi
## 2 3 3
```

Drawing those clusters on the tree

```
plot(d.hc)
rect.hclust(d.hc, 3)
```



Comparing single-linkage and Ward

- ▶ In Ward, Dutch and German get joined earlier (before joining to Germanic cluster).
- ▶ Also Hungarian and Finnish get combined earlier.

Making those dissimilarities

Original data:

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/one-ten.txt"
lang=read_delim(my_url, " ")
lang

## # A tibble: 10 x 11
##   en    no    dk    nl    de    fr    es    it    pl
##   <chr> <chr> <chr> <chr> <chr> <chr> <chr> <chr> <chr>
## 1 one   en    en    een   eins  un    uno   uno   jeden
## 2 two   to    to    twee  zwei  deux  dos    due   dwa
## 3 three tre   tre   drie  drei  trois tres  tre   trzy
## 4 four  fire  fire  vier  vier  quatre cuat~ quat~ cztery
## 5 five  fem   fem   vijf  funf  cinq   cinco cinq~ piec
## 6 six   seks  seks  zes   sechs six   seis  sei   szesc
## 7 seven sju   syv   zeven sieben sept  siete sette siedem
## 8 eight atte  otte  acht  acht  huit  ocho   otto  osiem
## 9 nine  ni    ni    negen neun  neuf  nueve nove dziew~
## 10 ten   ti   ti    tien  zehn  dix   diez  dieci dzies~
## # ... with 2 more variables: hu <chr>, fi <chr>
```

It would be a lot easier to extract the first letter if the number names were all in one column.

Tidy, and extract first letter

```
lang.long = lang %>% mutate(number=row_number()) %>%
  gather(language,name,-number) %>%
  mutate(first=str_sub(name,1,1))
lang.long %>% print(n=12)

## # A tibble: 110 x 4
##   number language name   first
##       <int>    <chr>   <chr> <chr>
## 1       1      en     one    o
## 2       2      en     two    t
## 3       3      en    three   t
## 4       4      en    four    f
## 5       5      en    five    f
## 6       6      en    six     s
## 7       7      en   seven   s
## 8       8      en   eight   e
## 9       9      en   nine    n
## 10    10      en   ten     t
## 11    11      no     en     e
## 12    12      no     to     t
## # ... with 98 more rows
```

Calculating dissimilarity

- ▶ Suppose we wanted dissimilarity between English and Norwegian. It's the number of first letters that are different.
- ▶ First get the lines for English:

```
english = lang.long %>% filter(language=="en")
english

## # A tibble: 10 x 4
##   number language name  first
##       <int>    <chr>   <chr> <chr>
## 1       1      en    one    o
## 2       2      en    two    t
## 3       3      en   three   t
## 4       4      en    four    f
## 5       5      en    five    f
## 6       6      en    six     s
## 7       7      en   seven   s
## 8       8      en   eight   e
## 9       9      en   nine    n
## 10      10     en   ten     t
```

And then the lines for Norwegian

```
norwegian = lang.long %>% filter(language=="no")
norwegian

## # A tibble: 10 x 4
##   number language name first
##   <int>    <chr>   <chr> <chr>
## 1     1      no    en    e
## 2     2      no    to    t
## 3     3      no   tre   t
## 4     4      no   fire   f
## 5     5      no   fem   f
## 6     6      no   seks  s
## 7     7      no   sju   s
## 8     8      no  atte   a
## 9     9      no    ni    n
## 10    10     no    ti    t
```

And now we want to put them side by side, matched by number. This is what `left_join` does. (A “join” is a lookup of values in one table using another.)

The join

```
english %>% left_join(norwegian, by="number")  
  
## # A tibble: 10 x 7  
##   number language.x name.x first.x language.y name.y first.y  
##   <int> <chr>     <chr>  <chr>    <chr>    <chr>  <chr>  
## 1      1 en        one     o      no       en      e  
## 2      2 en        two     t      no       to      t  
## 3      3 en        three   t      no       tre     t  
## 4      4 en        four    f      no       fire    f  
## 5      5 en        five    f      no       fem     f  
## 6      6 en        six     s      no       seks    s  
## 7      7 en        seven   s      no       sju    s  
## 8      8 en        eight   e      no       atte   a  
## 9      9 en        nine    n      no       ni     n  
## 10     10 en       ten    t      no       ti     t
```

first.x is 1st letter of English word, first.y 1st letter of Norwegian word.

Counting the different ones

```
english %>% left_join(norwegian, by="number") %>%
  mutate(different=(first.x!=first.y)) %>%
  summarize(diff=sum(different))

## # A tibble: 1 x 1
##       diff
##   <int>
## 1     2
```

Words for 1 and 8 start with different letter; rest are same.

Function to do this for any two languages

```
countdiff=function(lang.1,lang.2,d) {  
  lang1d=d %>% filter(language==lang.1)  
  lang2d=d %>% filter(language==lang.2)  
  lang1d %>% left_join(lang2d, by="number") %>%  
    mutate(different=(first.x!=first.y)) %>%  
    summarize(diff=sum(different)) %>%  
    pull(diff)  
}
```

Test:

```
countdiff("en","no",lang.long)  
## [1] 2
```

For all pairs of languages?

- ▶ First need all the languages:

```
languages=names(lang)
languages

## [1] "en" "no" "dk" "nl" "de" "fr" "es" "it" "pl"
## [10] "hu" "fi"
```

- ▶ and then all *pairs* of languages:

```
pairs=crossing(lang=languages, lang2=languages) %>% print(n=12)

## # A tibble: 121 x 2
##       lang    lang2
##       <chr>   <chr>
## 1 de      de
## 2 de      dk
## 3 de      en
## 4 de      es
## 5 de      fi
## 6 de      fr
## 7 de      hu
## 8 de      it
## 9 de      nl
## 10 de     no
## 11 de     pl
## 12 de     fi
## 13 de     fr
## 14 de     es
## 15 de     nl
## 16 de     no
## 17 de     pl
## 18 de     hu
## 19 de     it
## 20 de     en
## 21 de     dk
## 22 de     en
## 23 de     es
## 24 de     fi
## 25 de     fr
## 26 de     hu
## 27 de     it
## 28 de     nl
## 29 de     no
## 30 de     pl
## 31 de     fi
## 32 de     fr
## 33 de     es
## 34 de     nl
## 35 de     no
## 36 de     pl
## 37 de     hu
## 38 de     it
## 39 de     en
## 40 de     dk
## 41 de     en
## 42 de     es
## 43 de     fi
## 44 de     fr
## 45 de     hu
## 46 de     it
## 47 de     nl
## 48 de     no
## 49 de     pl
## 50 de     fi
## 51 de     fr
## 52 de     es
## 53 de     nl
## 54 de     no
## 55 de     pl
## 56 de     hu
## 57 de     it
## 58 de     en
## 59 de     dk
## 60 de     en
## 61 de     es
## 62 de     fi
## 63 de     fr
## 64 de     hu
## 65 de     it
## 66 de     nl
## 67 de     no
## 68 de     pl
## 69 de     fi
## 70 de     fr
## 71 de     es
## 72 de     nl
## 73 de     no
## 74 de     pl
## 75 de     hu
## 76 de     it
## 77 de     en
## 78 de     dk
## 79 de     en
## 80 de     es
## 81 de     fi
## 82 de     fr
## 83 de     hu
## 84 de     it
## 85 de     nl
## 86 de     no
## 87 de     pl
## 88 de     fi
## 89 de     fr
## 90 de     es
## 91 de     nl
## 92 de     no
## 93 de     pl
## 94 de     hu
## 95 de     it
## 96 de     en
## 97 de     dk
## 98 de     en
## 99 de     es
## 100 de    fi
## 101 de    fr
## 102 de    hu
## 103 de    it
## 104 de    nl
## 105 de    no
## 106 de    pl
## 107 de    fi
## 108 de    fr
## 109 de    es
## 110 de    nl
## 111 de    no
## 112 de    pl
## 113 de    hu
## 114 de    it
## 115 de    en
## 116 de    dk
## 117 de    en
## 118 de    es
## 119 de    fi
## 120 de    fr
## 121 de    hu
```

Run countdiff for all those language pairs

```
thediffs = pairs %>%
  mutate(diff=map2_int(lang,lang2,countdiff,lang.long)) %>%
  print(n=12)

## # A tibble: 121 x 3
##       lang   lang2  diff
##       <chr>  <chr> <int>
## 1 de      de        0
## 2 de      dk        5
## 3 de      en        6
## 4 de      es        7
## 5 de      fi        9
## 6 de      fr        7
## 7 de      hu        9
## 8 de      it        7
## 9 de      nl        5
## 10 de     no        4
## 11 de     pl        8
## 12 dk     de        5
## # ... with 109 more rows
```

Make square table of these

```
thediiffs %>% spread(lang2,diff)

## # A tibble: 11 x 12
##   lang    de    dk    en    es    fi    fr    hu    it
##   <chr> <int> <int> <int> <int> <int> <int> <int> <int>
## 1 de      0     5     6     7     9     7     9     7
## 2 dk      5     0     2     5     9     6     8     5
## 3 en      6     2     0     6     9     6     9     6
## 4 es      7     5     6     0     9     2     10    1
## 5 fi      9     9     9     9     0     9     8     9
## 6 fr      7     6     6     2     9     0     10    1
## 7 hu      9     8     9     10    8     10    0     10
## 8 it      7     5     6     1     9     1     10    0
## 9 nl      5     6     7     9     9     9     8     9
## 10 no     4     1     2     6     9     6     8     6
## 11 pl     8     6     7     3     9     5     10    4
## # ... with 3 more variables: nl <int>, no <int>, pl <int>
```

and that was where we began.

Another example

Birth, death and infant mortality rates for 97 countries (variables not dissimilarities):

24.7	5.7	30.8	Albania	12.5	11.9	14.4	Bulgaria
13.4	11.7	11.3	Czechoslovakia	12	12.4	7.6	Former_E._Germany
11.6	13.4	14.8	Hungary	14.3	10.2	16	Poland
13.6	10.7	26.9	Romania	14	9	20.2	Yugoslavia
17.7	10	23	USSR	15.2	9.5	13.1	Byelorussia_SSR
13.4	11.6	13	Ukrainian_SSR	20.7	8.4	25.7	Argentina
46.6	18	111	Bolivia	28.6	7.9	63	Brazil
23.4	5.8	17.1	Chile	27.4	6.1	40	Columbia
32.9	7.4	63	Ecuador	28.3	7.3	56	Guyana

...

- ▶ Want to find groups of similar countries (and how many groups, which countries in each group).
- ▶ Tree would be unwieldy with 97 countries.
- ▶ More automatic way of finding given number of clusters?

Reading in

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/birthrate.txt"
vital=read_table(my_url)

## Parsed with column specification:
## cols(
##   birth = col_double(),
##   death = col_double(),
##   infant = col_double(),
##   country = col_character()
## )
```

The data

vital

```
## # A tibble: 97 x 4
##   birth death infant country
##   <dbl> <dbl>  <dbl> <chr>
## 1 24.7   5.7   30.8 Albania
## 2 13.4   11.7  11.3 Czechoslovakia
## 3 11.6   13.4  14.8 Hungary
## 4 13.6   10.7  26.9 Romania
## 5 17.7   10     23    USSR
## 6 13.4   11.6  13    Ukrainian_SSR
## 7 46.6   18     111   Bolivia
## 8 23.4   5.8   17.1 Chile
## 9 32.9   7.4   63    Ecuador
## 10 34.8   6.6   42    Paraguay
## # ... with 87 more rows
```

Standardizing

- ▶ Infant mortality rate numbers bigger than others, consequence of measurement scale (arbitrary).
- ▶ Standardize (numerical) columns of data frame to have mean 0, SD 1, done by `scale`.

```
vital.s = vital %>% mutate_if(is.numeric,scale)
```

Three clusters

Pretend we know 3 clusters is good. Take off the 4th column (of countries) and run kmeans on the resulting data frame, asking for 3 clusters:

```
vital.km3 = vital.s %>% select(-4) %>% kmeans(3)
names(vital.km3)

## [1] "cluster"      "centers"       "totss"
## [4] "withinss"     "tot.withinss" "betweenss"
## [7] "size"         "iter"          "ifault"
```

A lot of output, so look at these individually.

What's in the output?

- ▶ Cluster sizes:

```
vital.km3$size  
## [1] 29 44 24
```

- ▶ Cluster centres:

```
vital.km3$centers  
##           birth      death     infant  
## 1  0.4737967 -0.4878149  0.2466440  
## 2 -0.9593341 -0.4322350 -0.8904328  
## 3  1.1862748  1.3818738  1.3344318
```

- ▶ Cluster 2 has lower than average rates on everything; cluster 3 has much higher than average.

Cluster sums of squares and membership

```
vital.km3$withinss  
## [1] 14.96356 25.13922 26.78049
```

Cluster 1 compact relative to others (countries in cluster 1 more similar).

```
vital.km3$cluster  
## [1] 2 2 2 2 2 2 3 2 1 1 2 3 2 2 2 2 2 2 2 2 2 2 2 2 3 1 2 2 1 1 3  
## [29] 2 1 2 1 1 2 2 1 1 1 3 3 1 1 3 3 1 3 3 3 1 2 2 2 2 2 2 1  
## [57] 1 1 1 2 2 2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 2 3 2 1 1 3 1 2 1  
## [85] 3 3 3 3 1 3 3 3 3 3 1 3 3
```

The cluster membership for each of the 97 countries.

Store countries and clusters to which they belong

```
vital.3=tibble(country=vital.s$country,  
                 cluster=vital.km3$cluster)
```

Next, which countries in which cluster?

Write function to extract them:

```
get_countries=function(i,d) {  
  d %>% filter(cluster==i) %>% pull(country)  
}
```

Cluster membership: cluster 2

```
get_countries(2,vital.3)
```

```
## [1] "Albania"           "Czechoslovakia"  
## [3] "Hungary"          "Romania"  
## [5] "USSR"              "Ukrainian_SSR"  
## [7] "Chile"              "Uruguay"  
## [9] "Finland"            "France"  
## [11] "Greece"             "Italy"  
## [13] "Norway"             "Spain"  
## [15] "Switzerland"        "Austria"  
## [17] "Canada"              "Israel"  
## [19] "Kuwait"              "China"  
## [21] "Korea"                "Singapore"  
## [23] "Thailand"            "Bulgaria"  
## [25] "Former_E._Germany" "Poland"  
## [27] "Yugoslavia"         "Byelorussia_SSR"  
## [29] "Argentina"            "Venezuela"  
## [31] "Belgium"              "Denmark"  
## [33] "Germany"              "Ireland"  
## [35] "Netherlands"         "Portugal"  
## [37] "Sweden"                "U.K."  
## [39] "Japan"                  "U.S.A."  
## [41] "Bahrain"                "United_Arab_Emirates"  
## [43] "Hong_Kong"              "Sri_Lanka"
```

Cluster 3

```
get_countries(3,vital.3)

## [1] "Bolivia"      "Mexico"       "Afghanistan"
## [4] "Bangladesh"   "Gabon"        "Ghana"
## [7] "Namibia"       "Sierra_Leone" "Swaziland"
## [10] "Uganda"        "Zaire"        "Cambodia"
## [13] "Nepal"         "Angola"       "Congo"
## [16] "Ethiopia"      "Gambia"       "Malawi"
## [19] "Mozambique"    "Nigeria"     "Somalia"
## [22] "Sudan"         "Tanzania"    "Zambia"
```

Cluster 1

```
get_countries(1,vital.3)

## [1] "Ecuador"      "Paraguay"      "Iran"
## [4] "Oman"         "Turkey"        "India"
## [7] "Mongolia"     "Pakistan"     "Algeria"
## [10] "Botswana"     "Egypt"        "Libya"
## [13] "Morocco"      "South_Africa" "Zimbabwe"
## [16] "Brazil"        "Columbia"     "Guyana"
## [19] "Peru"          "Iraq"          "Jordan"
## [22] "Lebanon"       "Saudi_Arabia" "Indonesia"
## [25] "Malaysia"      "Philippines"   "Vietnam"
## [28] "Kenya"         "Tunisia"
```

Problem!

- ▶ `kmeans` uses randomization. So result of one run might be different from another run.
- ▶ Example: just run again on 3 clusters, table of results:

```
vital.km3a=vital.s %>% select(-4) %>% kmeans(3)
table(first=vital.km3$cluster,
      second=vital.km3a$cluster)

##      second
## first  1  2  3
##      1  1  0 28
##      2  0 40  4
##      3 24  0  0
```

- ▶ Clusters are similar but *not same*.
- ▶ Solution: `nstart` option on `kmeans` runs that many times, takes best. Should be same every time:

```
vital.km3b = vital.s %>% select(-4) %>%
  kmeans(3,nstart=20)
```

How many clusters?

- ▶ Three was just a guess.
- ▶ Idea: try a whole bunch of #clusters (say 2–20), obtain measure of goodness of fit for each, make plot.
- ▶ Appropriate measure is `tot.withinss`.
- ▶ Use loop to run `kmeans` for each #clusters, keep track of `tot.withinss`.

Function to get tot.withinss

...for an input number of clusters, taking only numeric columns of input data frame:

```
ss=function(i,d) {  
  km = d %>% select_if(is.numeric) %>%  
    kmeans(i,nstart=20)  
  km$tot.withinss  
}
```

Note: writing function to be as general as possible, so that we can re-use it later.

Constructing within-cluster SS

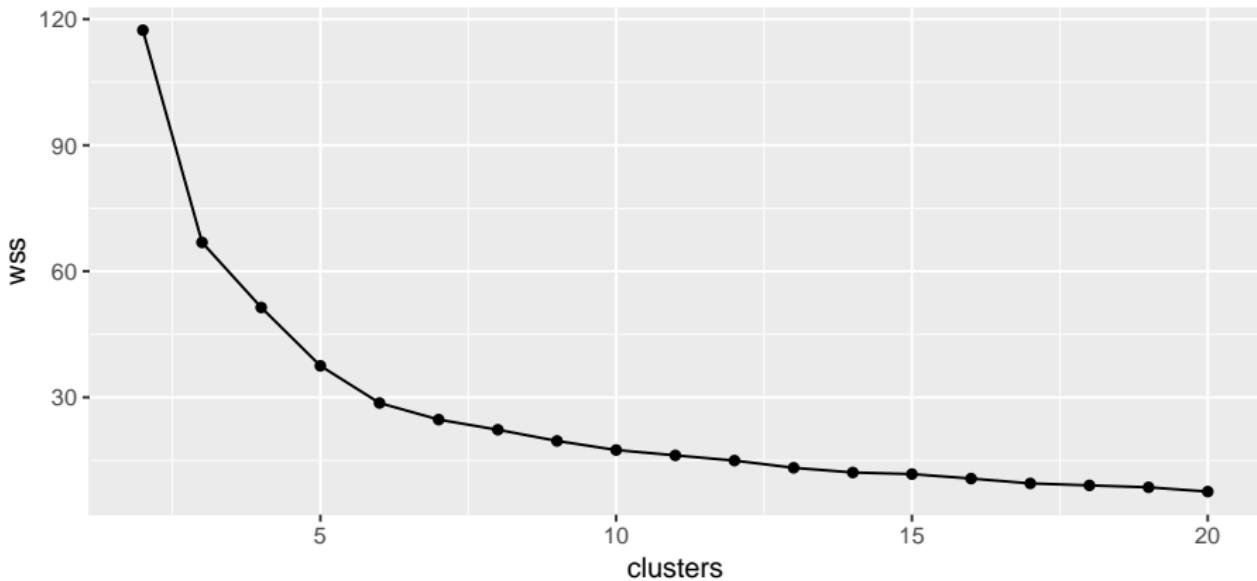
Make a data frame with desired numbers of clusters, and fill it with the total within-group sums of squares. “For each number of clusters, run ss”, so `map dbl`.

```
ssd = tibble(clusters=2:20) %>%
  mutate(wss=map_dbl(clusters,ss,vital.s)) %>%
  print(n=10)

## # A tibble: 19 x 2
##   clusters     wss
##       <int>  <dbl>
## 1         2 117.
## 2         3  66.9
## 3         4  51.4
## 4         5  37.5
## 5         6  28.7
## 6         7  24.7
## 7         8  22.3
## 8         9  19.6
## 9        10  17.5
## 10        11  16.2
## # ... with 9 more rows
```

Scree plot

```
ggplot(ssd,aes(x=clusters,y=wss))+geom_point()+
  geom_line()
```



Interpreting scree plot

- ▶ Lower wss better.
- ▶ But lower for larger #clusters, harder to explain.
- ▶ Compromise: low-ish wss and low-ish #clusters.
- ▶ Look for “elbow” in plot.
- ▶ Idea: this is where wss decreases fast then slow.
- ▶ On our plot, small elbow at 6 clusters. Try this many clusters.

Six clusters, using nstart

```
vital.km6 = vital.s %>% select(-4) %>%  
  kmeans(6, nstart=20)  
vital.km6$size  
## [1] 24   8  30  15  18   2  
  
vital.km6$centers  
  
##          birth      death     infant  
## 1  0.4160993 -0.5169988  0.2648754  
## 2  1.3043848  2.1896567  1.9470306  
## 3 -1.1737104 -0.1856375 -0.9534370  
## 4 -0.4357690 -1.1438599 -0.7281108  
## 5  1.2092406  0.7441347  1.0278003  
## 6 -0.2199722  2.1116577 -0.4544435  
  
vital.6=tibble(country=vital.s$country,  
                cluster=vital.km6$cluster)
```

Cluster 1

Below-average death rate, though other rates a little higher than average:

```
get_countries(1,vital.6)
```

```
## [1] "Ecuador"      "Paraguay"     "Oman"  
## [4] "Turkey"       "India"        "Mongolia"  
## [7] "Pakistan"     "Algeria"      "Egypt"  
## [10] "Libya"        "Morocco"     "South_Africa"  
## [13] "Zimbabwe"     "Brazil"       "Guyana"  
## [16] "Peru"          "Iraq"         "Jordan"  
## [19] "Lebanon"       "Saudi_Arabia" "Indonesia"  
## [22] "Philippines"   "Vietnam"     "Tunisia"
```

Cluster 2

High on everything:

```
get_countries(2,vital.6)
## [1] "Afghanistan"   "Sierra_Leone"  "Angola"
## [4] "Ethiopia"       "Gambia"        "Malawi"
## [7] "Mozambique"     "Somalia"
```

Cluster 3

Low on everything, though death rate close to average:

```
get_countries(3,vital.6)

## [1] "Czechoslovakia"      "Hungary"
## [3] "Romania"            "USSR"
## [5] "Ukrainian_SSR"       "Uruguay"
## [7] "Finland"             "France"
## [9] "Greece"              "Italy"
## [11] "Norway"              "Spain"
## [13] "Switzerland"          "Austria"
## [15] "Canada"               "Bulgaria"
## [17] "Former_E._Germany"   "Poland"
## [19] "Yugoslavia"           "Byelorussia_SSR"
## [21] "Belgium"              "Denmark"
## [23] "Germany"              "Ireland"
## [25] "Netherlands"          "Portugal"
## [27] "Sweden"                "U.K."
## [29] "Japan"                 "U.S.A."
```

Cluster 4

Low on everything, especially death rate:

```
get_countries(4,vital.6)
```

```
## [1] "Albania"                 "Chile"  
## [3] "Israel"                  "Kuwait"  
## [5] "China"                   "Singapore"  
## [7] "Thailand"                "Argentina"  
## [9] "Columbia"                "Venezuela"  
## [11] "Bahrain"                 "United_Arab_Emirates"  
## [13] "Hong_Kong"               "Malaysia"  
## [15] "Sri_Lanka"
```

Cluster 5

Higher than average on everything, though not the highest:

```
get_countries(5,vital.6)

## [1] "Bolivia"      "Iran"        "Bangladesh"
## [4] "Botswana"     "Gabon"       "Ghana"
## [7] "Namibia"      "Swaziland"    "Uganda"
## [10] "Zaire"        "Cambodia"    "Nepal"
## [13] "Congo"        "Kenya"       "Nigeria"
## [16] "Sudan"        "Tanzania"    "Zambia"
```

Cluster 6

Very high death rate, just below average on all else:

```
get_countries(6,vital.6)  
## [1] "Mexico" "Korea"
```

Comparing our 3 and 6-cluster solutions

```
table(three=vital.km3$cluster,six=vital.km6$cluster)

##       six
## three  1  2  3  4  5  6
##      1 24  0  0  2  3  0
##      2  0  0 30 13  0  1
##      3  0  8  0  0 15  1
```

Compared to 3-cluster solution:

- ▶ most of cluster 1 gone to (new) cluster 1
- ▶ cluster 2 split into clusters 3 and 4 (two types of “richer” countries)
- ▶ cluster 3 split into clusters 2 and 5 (two types of “poor” countries, divided by death rate).
- ▶ cluster 6 (Mexico and Korea) was split before.

Getting a picture from kmeans

- ▶ Use multidimensional scaling (later)
- ▶ Use discriminant analysis on clusters found, treating them as “known” groups.

MANOVA and discriminant analysis

- ▶ Go back to 1st 3 columns of vital.s (variables, standardized), plus cf (cluster as factor). clus (6 clusters).
- ▶ First, do they actually differ by group? (MANOVA):

```
v = vital.s %>% select(-4) %>% as.matrix()
cf = as.factor(vital.km6$cluster)
vital.manova=manova(v~cf)
summary(vital.manova)

##                  Df Pillai approx F num Df den Df
## cf            5 1.9215    32.427      15     273
## Residuals  91
##                  Pr(>F)
## cf            < 2.2e-16 ***
## Residuals
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Oh yes.

Discriminant analysis

- ▶ So what makes the groups different?
- ▶ Uses package MASS (loaded):

```
vital.lda=lda(cf~birth+death+infant, data=vital.s)
vital.lda$svd

## [1] 21.687195 8.851811 1.773006

vital.lda$scaling

##           LD1          LD2          LD3
## birth  2.6879695  1.1224202  1.9483853
## death  0.6652712 -2.7213044  0.6049358
## infant 2.1111801  0.7650912 -2.3542296
```

- ▶ LD1 is some of everything, but not so much death rate (high=poor, low=rich).
- ▶ LD2 mainly death rate, high or low.

To make a plot

- Get predictions first:

```
vital.pred=predict(vital.lda)
d=data.frame(country=vital.s$country,
             cluster=vital.km6$cluster,vital.pred$x)
glimpse(d)

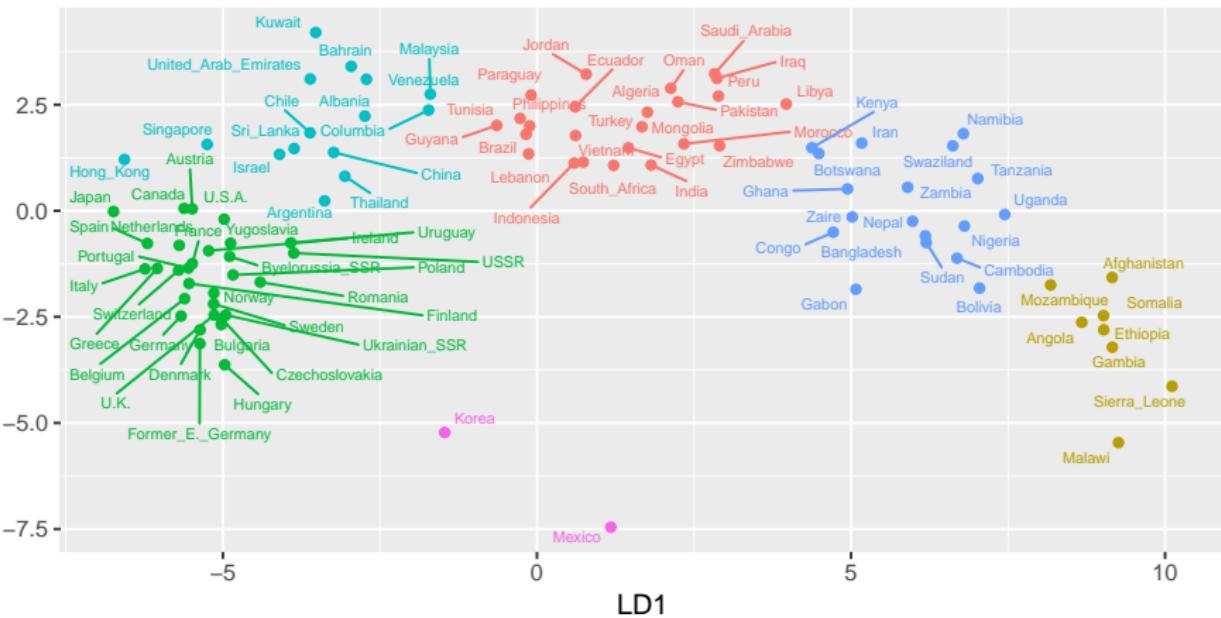
## Observations: 97
## Variables: 5
## $ country <fct> Albania, Czechoslovakia, Hung...
## $ cluster <int> 4, 3, 3, 3, 3, 3, 5, 4, 1, 1, ...
## $ LD1      <dbl> -2.74034473, -5.01874312, -4....
## $ LD2      <dbl> 2.2311427, -2.5427640, -3.629...
## $ LD3      <dbl> -0.086392118, 0.067491502, -0...
```

- d contains country names, cluster memberships and discriminant scores. Plot LD1 against LD2, colouring points by cluster and labelling by country:

```
g=ggplot(d,aes(x=LD1,y=LD2,colour=factor(cluster),
                 label=country))+geom_point()+
  geom_text_repel(size=2)+guides(colour=F)
```

The plot

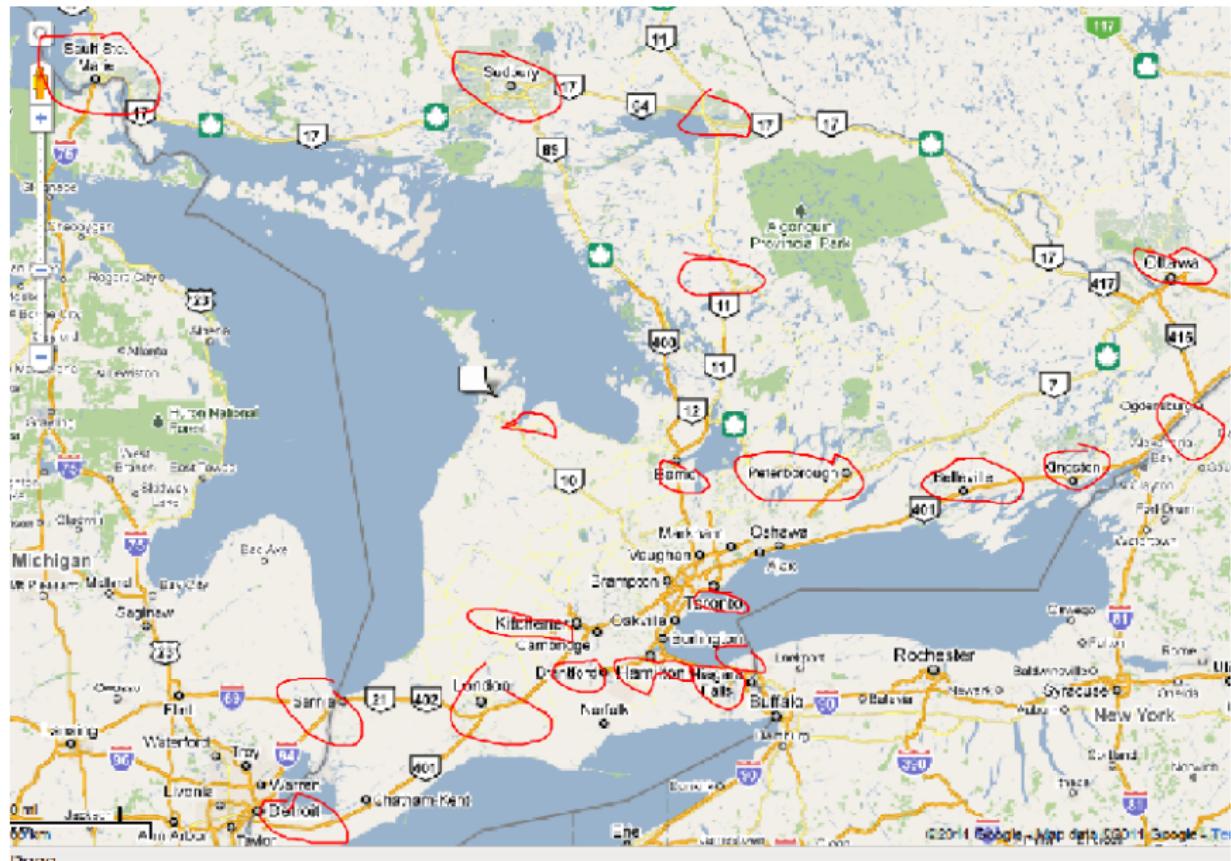
g



Final example: a hockey league

- ▶ An Ontario hockey league has teams in 21 cities. How can we arrange those teams into 4 geographical divisions?
- ▶ Distance data in spreadsheet.
- ▶ Take out spaces in team names.
- ▶ Save as “text/csv”.
- ▶ Distances, so back to `hclust`.

A map

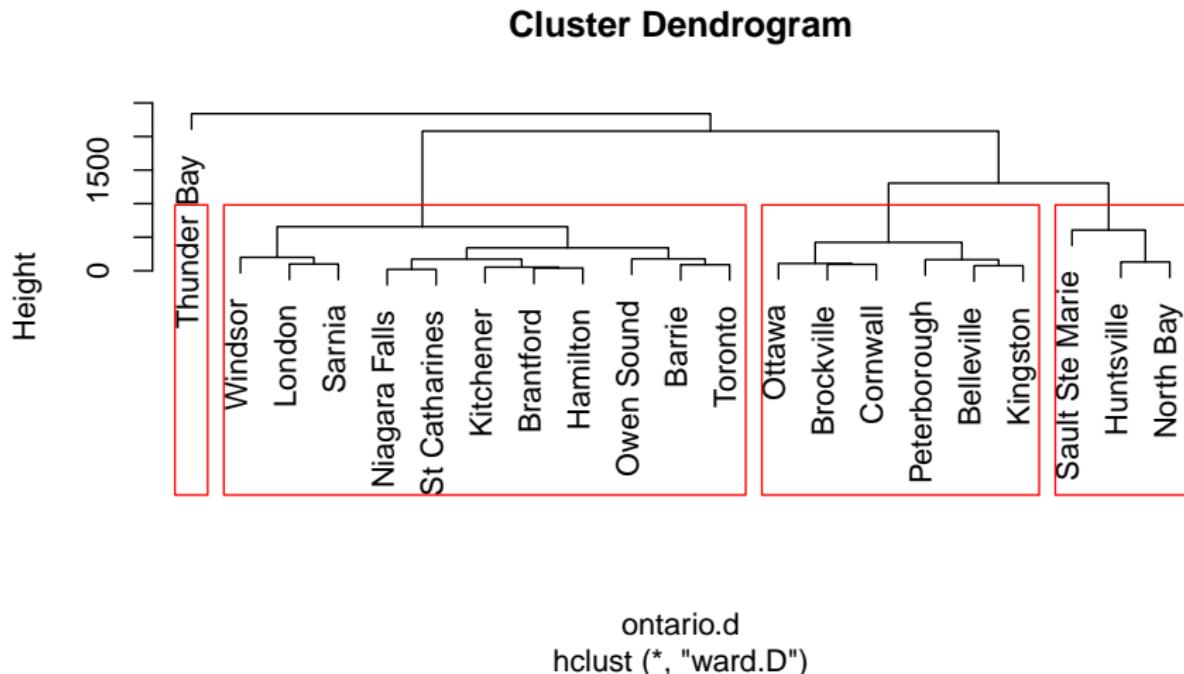


Attempt 1

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/ontario-road-
distances.csv"
ontario=read_csv(my_url)
ontario.d = ontario %>% select(-1) %>% as.dist()
ontario.hc=hclust(ontario.d,method="ward.D")
```

Plot, with 4 clusters

```
plot(ontario.hc)
rect.hclust(ontario.hc, 4)
```

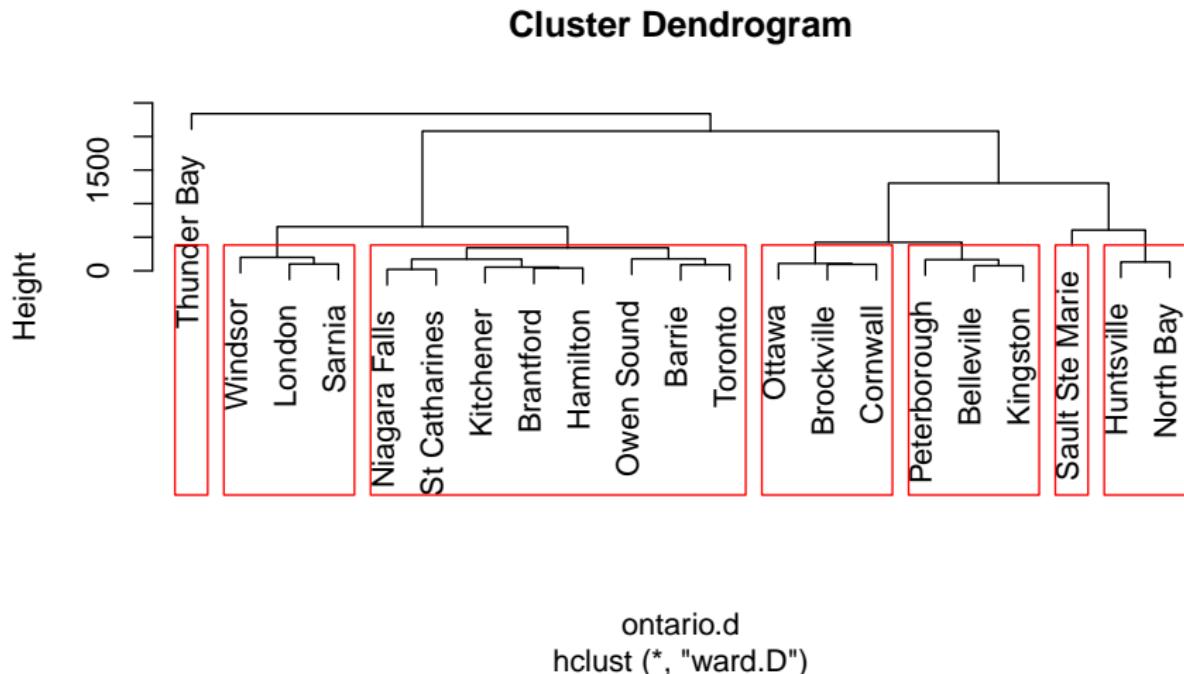


Comments

- ▶ Can't have divisions of 1 team!
- ▶ "Southern" divisions way too big!
- ▶ Try splitting into more. I found 7 to be good:

Seven clusters

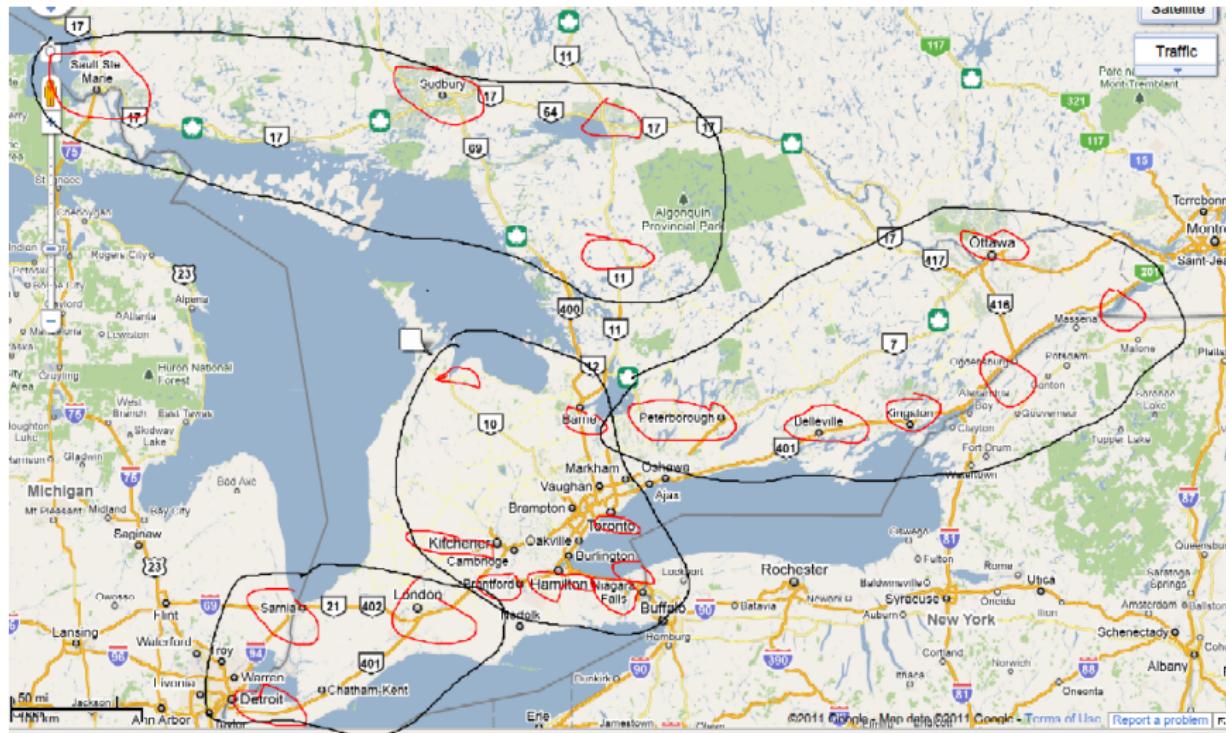
```
plot(ontario.hc)
rect.hclust(ontario.hc, 7)
```



Divisions now

- ▶ I want to put Huntsville and North Bay together with northern teams.
- ▶ I'll put the Eastern teams together. Gives:
 - ▶ North: Sault Ste Marie, Sudbury, Huntsville, North Bay
 - ▶ East: Brockville, Cornwall, Ottawa, Peterborough, Belleville, Kingston
 - ▶ West: Windsor, London, Sarnia
 - ▶ Central: Owen Sound, Barrie, Toronto, Niagara Falls, St Catharines, Brantford, Hamilton, Kitchener
- ▶ Getting them same size beyond us!

Another map



Section 11

Multidimensional scaling

Multidimensional Scaling

- ▶ Have distances between individuals.
- ▶ Want to draw a picture (map) in 2 dimensions showing individuals so that distances (or order of distances) as close together as possible. (Or maybe 3 with rgl.)
- ▶ If want to preserve actual distances, called *metric multidimensional scaling* (in R, cmdscale).
- ▶ If only want to preserve order of distances, called *non-metric multidimensional scaling* (in R, isoMDS in package MASS).
- ▶ Metric scaling has solution that can be worked out exactly.
- ▶ Non-metric only has iterative solution.
- ▶ Assess quality of fit, see whether use of resulting map is reasonable. (Try something obviously 3-dimensional and assess its failure.)

Packages

The usual, plus a new one:

```
library(MASS)
library(tidyverse)
library(ggrepel)
library(ggmap)
library(shapes)
```

Metric scaling: European cities

CSV file `europe.csv` contains road distances (in km) between 16 European cities. Can we reproduce a map of Europe from these distances?

Read in data:

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/europe.csv"
europe=read_csv(my_url)

## Warning: Missing column names filled in: 'X1' [1]

## Parsed with column specification:
## cols(
##   X1 = col_character(),
##   Amsterdam = col_integer(),
##   Athens = col_integer(),
##   Barcelona = col_integer(),
##   Berlin = col_integer(),
##   Cologne = col_integer(),
##   Copenhagen = col_integer(),
##   Edinburgh = col_integer(),
##   Geneva = col_integer(),
##   London = col_integer(),
##   Madrid = col_integer(),
##   Marseille = col_integer(),
##   Munich = col_integer(),
##   Paris = col_integer(),
##   Prague = col_integer()
```

The data

europe

```
## # A tibble: 16 x 17
##   X1     Amsterdam Athens Barcelona Berlin Cologne Copenhagen
##   <chr>    <int>    <int>    <int>    <int>    <int>    <int>
## 1 Amst~        0    3082    1639     649     280     904
## 2 Athe~     3082        0    3312    2552    2562    3414
## 3 Barc~     1639    3312        0    1899    1539    2230
## 4 Berl~      649    2552    1899        0     575     743
## 5 Colo~      280    2562    1539     575        0     730
## 6 Cope~      904    3414    2230     743     730        0
## 7 Edin~     1180    3768    2181    1727    1206    1864
## 8 Gene~     1014    2692      758    1141     765    1531
## 9 Lond~      494    3099    1512    1059     538    1196
## 10 Madr~    1782    3940      628    2527    1776    2597
## 11 Mars~    1323    2997      515    1584    1208    1914
## 12 Muni~     875    2210    1349      604     592    1204
## 13 Paris     515    3140    1125    1094     508    1329
## 14 Prag~     973    2198    1679      354     659    1033
## 15 Rome     1835    2551    1471    1573    1586    2352
## 16 Vien~     1196    1886    1989      666     915    1345
## # ... with 10 more variables: Edinburgh <int>, Geneva <int>,
## #   London <int>, Madrid <int>, Marseille <int>, Munich <int>,
## #   Paris <int>, Prague <int>, Rome <int>, Vienna <int>
```

Multidimensional scaling

- ▶ Create distance object first using all but first column of europe.
europe has distances in it already, so make into dist with as.dist.
- ▶ Then run multidimensional scaling and look at result:

```
europe.d = europe %>% select(-1) %>% as.dist()  
europe.scale=cmdscale(europe.d)  
head(europe.scale)
```

	[,1]	[,2]
## Amsterdam	-348.162277	528.2657
## Athens	2528.610410	-509.5208
## Barcelona	-695.970779	-984.6093
## Berlin	384.178025	634.5239
## Cologne	5.153446	356.7230
## Copenhagen	-187.104072	1142.5926

- ▶ This is a matrix of x and y coordinates.

As a data frame; make picture

We know how to plot data frames, so make one first.

```
europre_coord = europe.scale %>% as_tibble() %>%
    mutate(city=europe$City) %>% print(n=12)

## Warning: Unknown or uninitialized column: 'City'.

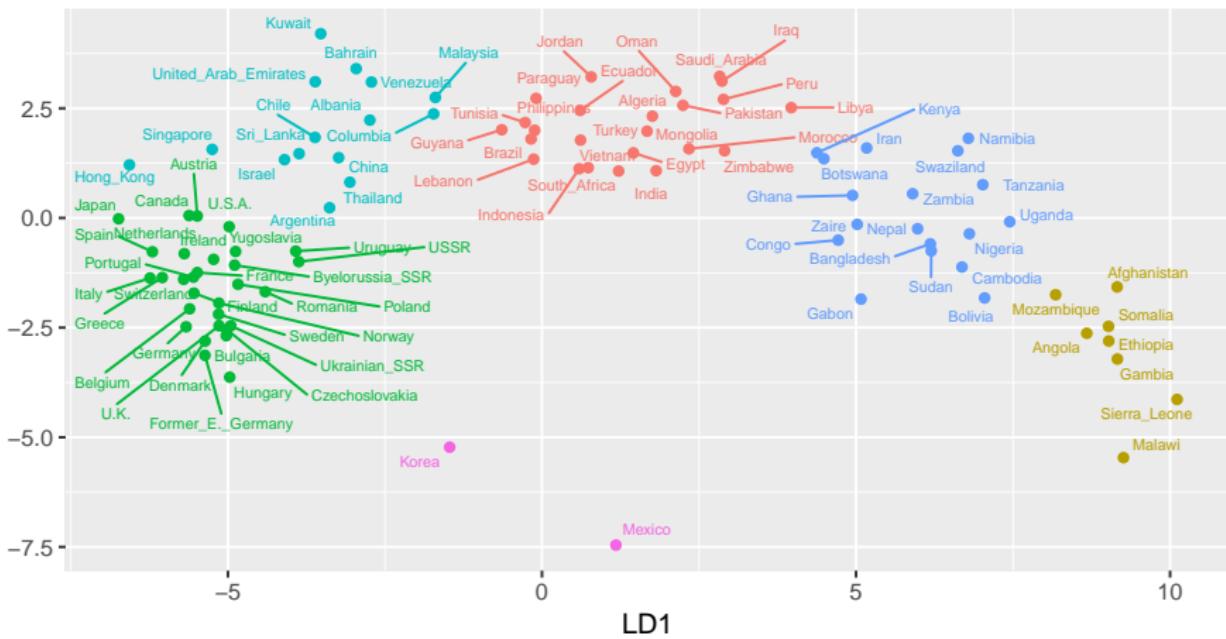
## Error in mutate_impl(.data, dots): Column `city` is of unsupported type NULL

g = ggplot(europe_coord, aes(x=V1,y=V2,label=city))+
    geom_point() + geom_text_repel()

## Error in ggplot(europe_coord, aes(x = V1, y = V2, label = city)): object
'europe_coord' not found
```

The map

g



Making a function

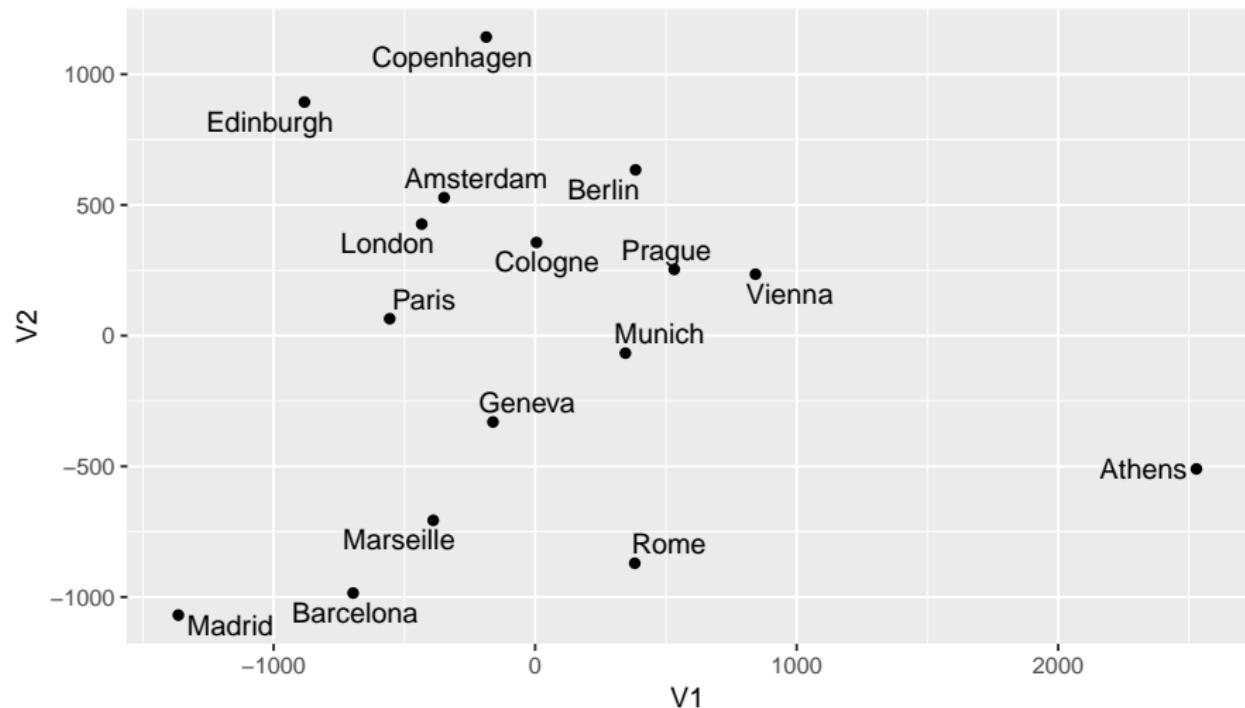
- Idea: given input distance matrix (as stored in a CSV file), output a map (like the one on the previous page).

```
mds_map=function(filename) {  
  x=read_csv(filename)  
  dist = x %>% select_if(is.numeric) %>%  
    as.dist()  
  x.scale=cmdscale(dist) # this is a matrix  
  x_coord = x.scale %>%  
    as_tibble() %>%  
    mutate(place=row.names(x.scale))  
  ggplot(x_coord, aes(x=V1,y=V2,label=place))+  
    geom_point()+geom_text_repel()+  
    coord_fixed()  
}
```

- Use `select_if` to pick out all the numerical columns (no text), whichever they are.
- `x.scale` is matrix with no column headers. Turn into data frame, acquires headers V1 and V2.
- Get place names from `cmdscale`'s output

Does it work?

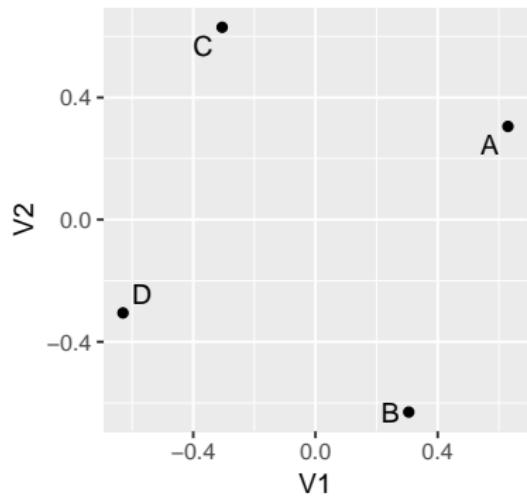
```
mds_map("europe.csv")
```



A square

- ▶ The data, in `square.csv`:

x,A	,B	,C	,D
A,0	,1	,1	,1.4
B,1	,0	,1.4	,1
C,1	,1.4	,0	,1
D,1.4	,1	,1	,0



- ▶ The map (on right):

```
mds_map("square.csv")
```

Drawing a map of the real Europe

- ▶ Works with package ggmap.
- ▶ First find latitudes and longitudes of our cities, called *geocoding*:

```
latlong = geocode(europe$City)
latlong = bind_cols(city=europe$City, latlong)
latlong %>% print(n=6)
```

```
## # A tibble: 16 x 3
##   city           lon     lat
##   <chr>        <dbl>  <dbl>
## 1 Amsterdam    4.90   52.4
## 2 Athens       23.7   38.0
## 3 Barcelona    2.17   41.4
## 4 Berlin        13.4   52.5
## 5 Cologne      6.96   50.9
## 6 Copenhagen  12.6   55.7
## # ... with 10 more rows
```

- ▶ Just so you know, there is a limit of 2500 queries per day (it queries Google Maps).

Making the map

- ▶ Get a map of Europe from Google Maps (specify what you want a map of any way you can in Google Maps). This one centres the map on the city shown and zooms it so all the cities appear (I had to experiment):

```
map=get_map("Memmingen DE",zoom=5)
```

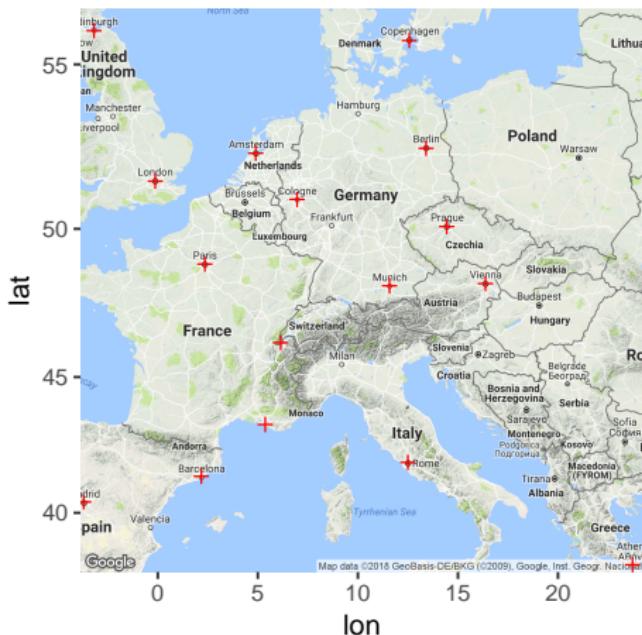
- ▶ Plot the map with ggmap. This is ggplot, so add anything to it that you would add to a ggplot, such as cities we want to show:

```
g2=ggmap(map)+  
  geom_point(data=latlong,aes(x=lon,y=lat),  
             shape=3,colour="red")
```

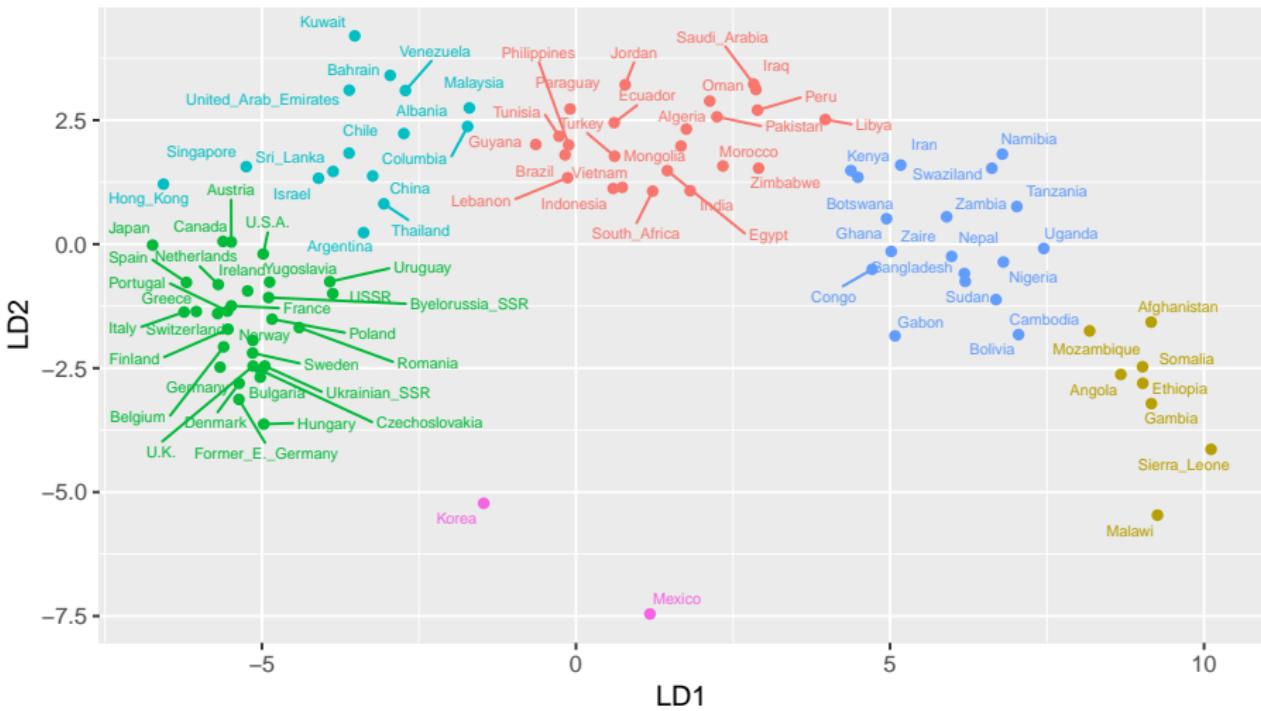
- ▶ We don't have a default data frame or aes for our geom_point, so have to specify one.

The real Europe with our cities

g2



Compare our scaling map



Comments

- ▶ North-south not quite right: Edinburgh and Copenhagen on same latitude, also Amsterdam and Berlin; Athens should be south of Rome.
- ▶ Rotating clockwise by about 45 degrees should fix that.
- ▶ General point: MDS only uses distances, so answer can be “off” by rotation (as here) or reflection (flipping over, say exchanging west and east while leaving north and south same).

Exploring the map by plotting in 3 dimensions

- ▶ Package `rgl` makes 3D plots.
- ▶ We have to fake up a 3rd dimension (by setting all its values to 1).
- ▶ Try this code:

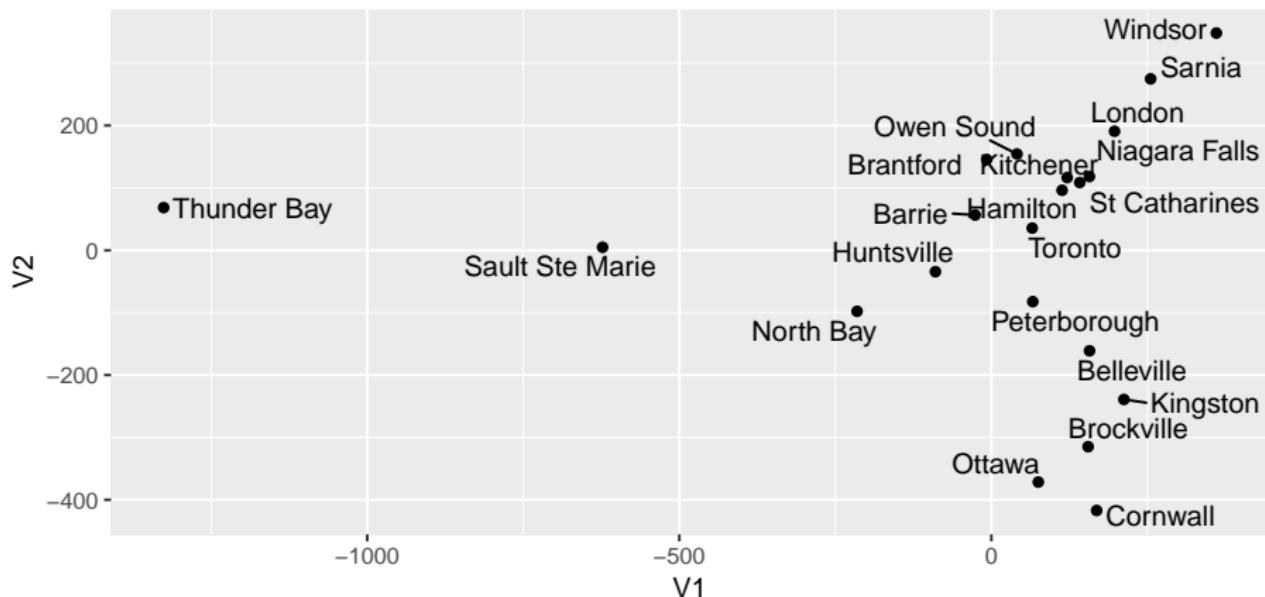
```
library(rgl)
es.2=cbind(europe.scale,1)
plot3d(es.2,zlim=c(-1000,1000))
text3d(es.2,text=d$city)
```

- ▶ Opens a graphics window with the cities plotted and named.
- ▶ Click and hold left mouse button to rotate plot. “Rotate away” 3rd dimension to get a possible map (that preserves distances).

Ontario, the same way

...using our function:

```
g=mds_map("ontario-road-distances.csv") ; g
```



Removing points

- ▶ Messy: have to find which rows and columns contain those cities, then remove just those rows and columns.
- ▶ Better:
 - ▶ "tidy" the distance matrix
 - ▶ then remove rows we don't need
 - ▶ then "untidy" it again
 - ▶ save into .csv file
- ▶ Illustrate with square data first (easier to see).

Square data

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/square.csv"
square=read_csv(my_url)
square

## # A tibble: 4 x 5
##   x     A     B     C     D
##   <chr> <dbl> <dbl> <dbl> <dbl>
## 1 A     0     1     1     1.4
## 2 B     1     0     1.4    1
## 3 C     1     1.4    0     1
## 4 D     1.4    1     1     0
```

Make tidy

```
square %>% gather(point,distance,-1)
```

```
## # A tibble: 16 x 3
##       x     point distance
##   <chr> <chr>     <dbl>
## 1 A     A         0
## 2 B     A         1
## 3 C     A         1
## 4 D     A         1.4
## 5 A     B         1
## 6 B     B         0
## 7 C     B         1.4
## 8 D     B         1
## 9 A     C         1
## 10 B    C         1.4
## 11 C    C         0
## 12 D    C         1
## 13 A    D         1.4
## 14 B    D         1
## 15 C    D         1
## 16 D    D         0
```

Remove all references to point C

In column x or point:

```
square %>% gather(point,distance,-1) %>%
  filter(x != "C", point != "C")

## # A tibble: 9 x 3
##   x     point distance
##   <chr> <chr>    <dbl>
## 1 A     A         0
## 2 B     A         1
## 3 D     A         1.4
## 4 A     B         1
## 5 B     B         0
## 6 D     B         1
## 7 A     D         1.4
## 8 B     D         1
## 9 D     D         0
```

Put back as distance matrix

and save as .csv when we are happy:

```
noc = square %>% gather(point,distance,-1) %>%
  filter(x != "C", point != "C") %>%
  spread(point, distance)

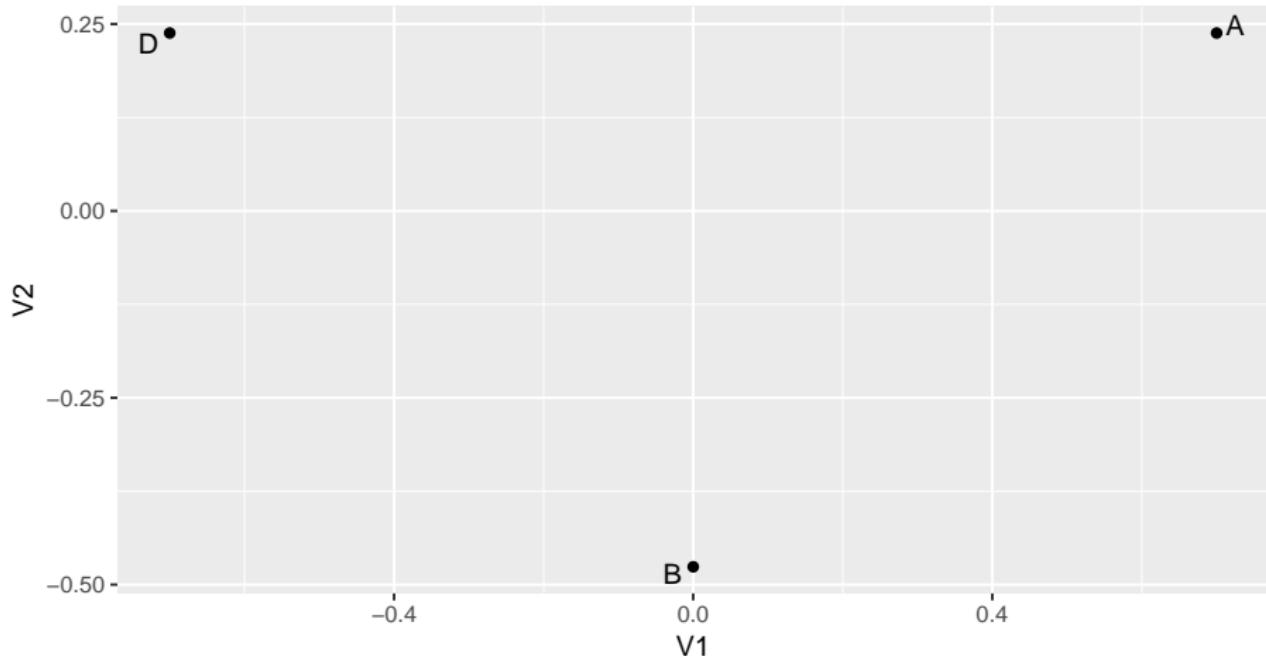
noc

## # A tibble: 3 x 4
##   x       A     B     D
##   <chr> <dbl> <dbl> <dbl>
## 1 A       0     1    1.4
## 2 B       1     0     1
## 3 D      1.4    1     0

noc %>% write_csv("no-c.csv")
```

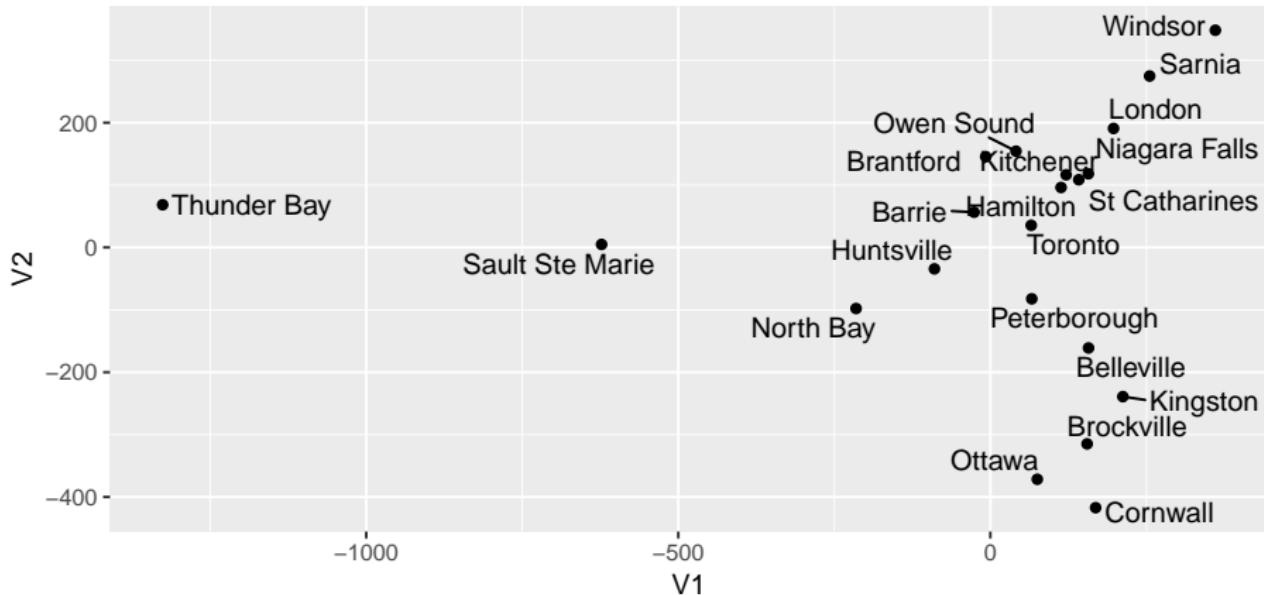
Make map of square-without-C

```
mds_map("no-c.csv")
```



Back to Ontario

g



Get rid of Thunder Bay and Sault Ste Marie.

Tidy. remove. untidy

```
my_url = "http://www.utsc.utoronto.ca/~butler/d29/ontario-road-distances.csv"
ontario2 = read_csv(my_url) %>%
  gather(place,distance,-1) %>%
  filter(x != "Thunder Bay",
         place != "Thunder Bay",
         x != "Sault Ste Marie",
         place != "Sault Ste Marie") %>%
  spread(place, distance) %>%
  write_csv("southern-ontario.csv")

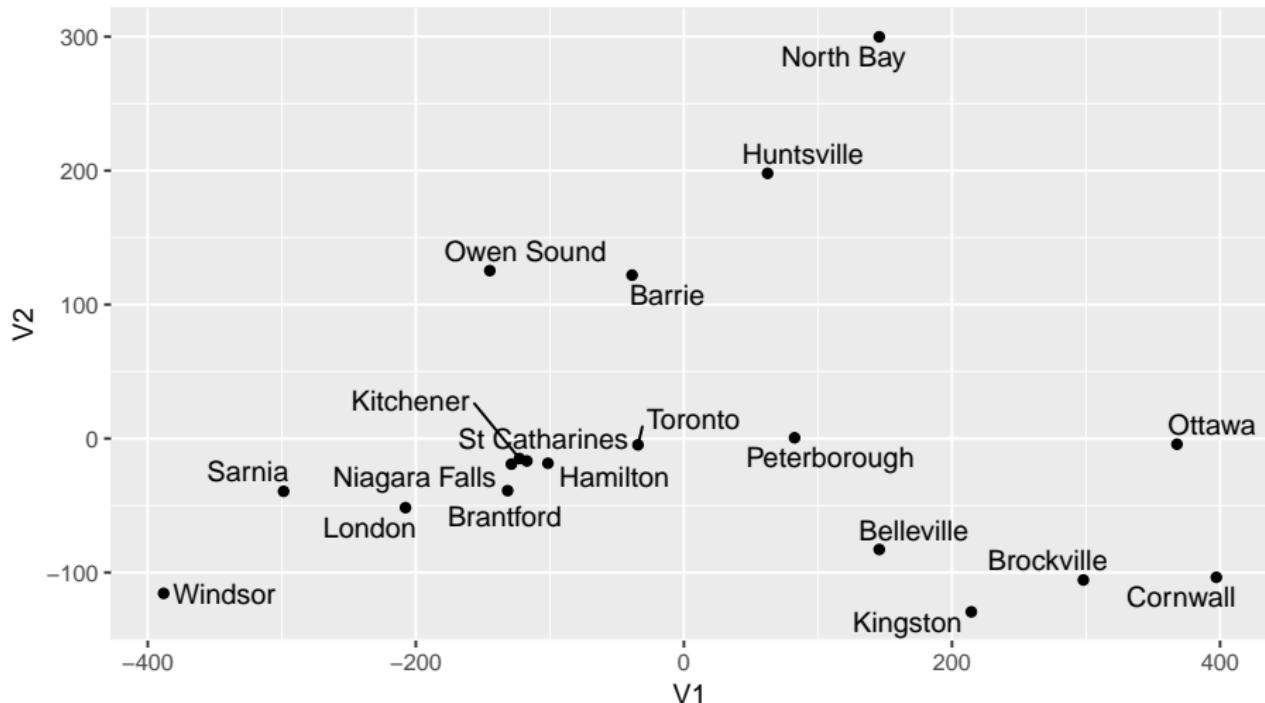
## Warning in (~x != "Thunder Bay") & ~place != "Thunder Bay": longer
object length is not a multiple of shorter object length

## Warning in (~x != "Thunder Bay") & ~place != "Thunder Bay" & ~x !=
"Sault Ste Marie": longer object length is not a multiple of shorter
object length

## Error: Duplicate identifiers for rows (1, 2, 3, 4, 5, 6, 7, 8, 9, 10,
11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21), (22, 23, 24, 25, 26, 27,
28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42), (43, 44,
45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62,
63), (64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79,
80, 81, 82, 83, 84), (85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96,
97, 98, 99, 100, 101, 102, 103, 104, 105), (106, 107, 108, 109, 110), 497/700
```

Map of Southern Ontario

```
g = mds_map("southern-ontario.csv") ; g
```



Came out geographically about right.

What about that cluster of points?

- ▶ Plot looks generally good, but what about that cluster of points?
- ▶ “Zoom in” on area between -150 and -100 on x axis, -50 to 0 on y axis.
- ▶ Code below overrides the `coord_fixed` we had before.

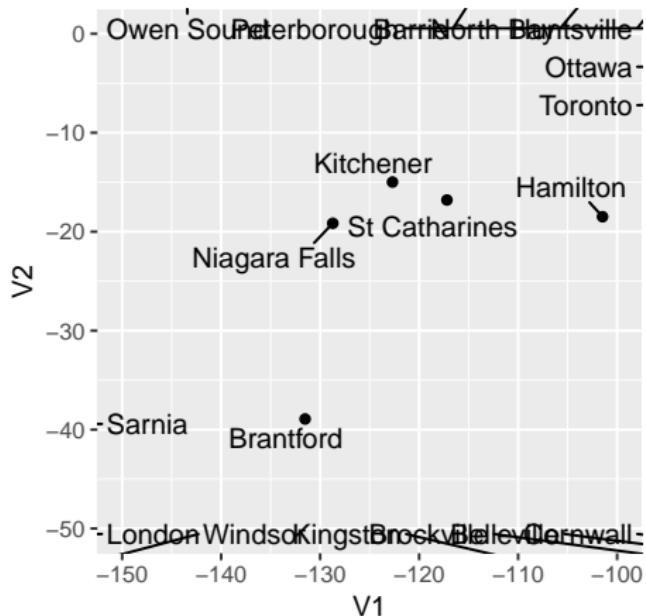
```
g2 = g + coord_fixed(xlim=c(-150,-100),ylim=c(-50,0))
```

Coordinate system already present. Adding new coordinate system, which will replace the existing one.

Zoomed-in plot

Ignore the arrows to points off the map:

g2



Does that make sense?

- ▶ Get a Google map of the area, with the points labelled.
- ▶ First geocode the cities of interest:

```
cities=c("Kitchener ON", "Hamilton ON", "Niagara Falls ON",
        "St Catharines ON", "Brantford ON")
latlong=geocode(cities)
latlong = bind_cols(city=cities,latlong) %>% print()
```

```
## # A tibble: 5 x 3
##   city           lon     lat
##   <chr>         <dbl>   <dbl>
## 1 Kitchener ON -80.5   43.5
## 2 Hamilton ON  -79.9   43.3
## 3 Niagara Falls ON -79.1   43.1
## 4 St Catharines ON -79.2   43.2
## 5 Brantford ON  -80.3   43.1
```

- ▶ Get a Google map of the area (experiment with zoom):

```
map=get_map("Hamilton ON", zoom=8)
```

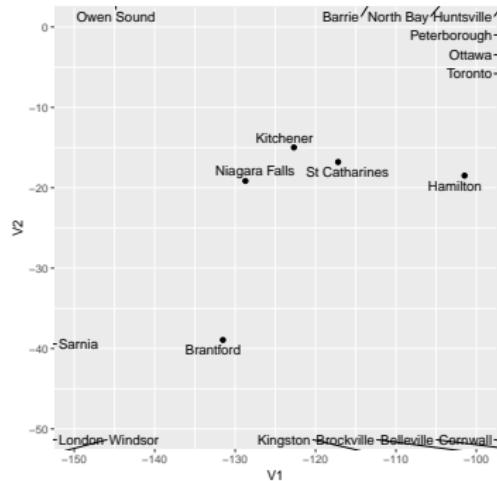
Making the Google map

Plot the map, plus the cities, plus labels for the cities:

```
gmap = ggmap(map)+  
    geom_point(data=latlong,  
                aes(x=lon,y=lat),  
                shape=3,colour="red") +  
    geom_text_repel(data=latlong,  
                    aes(label=city))
```

The mds map and Google map

g2



gmap



St Catharines and Niagara Falls should be the *other side* of Hamilton!

Quality of fit

- ▶ Read in “southern Ontario” data set from file:

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/southern-
ontario.csv"
ontario2=read_csv(my_url)
```

- ▶ Calling cmdscale with eig=T gives more info:

```
ontario2.2 = ontario2 %>% select_if(is.numeric) %>%
  cmdscale(eig=T)
names(ontario2.2)

## [1] "points" "eig"      "x"       "ac"       "GOF"

ontario2.2$GOF

## [1] 0.8381590 0.8914059

ontario2.3 = ontario2 %>% select_if(is.numeric) %>%
  cmdscale(3,eig=T)
ontario2.3$GOF

## [1] 0.8852559 0.9414948
```

Comments

- ▶ Coordinates now in points.
- ▶ GOF is R-squared-like measure saying how well map distances match real ones. Higher is better.
- ▶ For Ontario road distances, GOF better for 3 dimensions than 2, presumably to accommodate St Catharines and Niagara Falls?

3-dimensional coordinates. cities attached

```
ontario2.3$points %>% as_tibble() %>%
  mutate(city=ontario2$x)

## # A tibble: 19 x 4
##       V1      V2      V3 city
##     <dbl>    <dbl>    <dbl> <chr>
## 1 -38.7   122.     4.17 Barrie
## 2 146.    -82.8    1.53 Belleville
## 3 -132.   -38.9    14.1 Brantford
## 4 298.   -106.    -7.74 Brockville
## 5 397.   -104.   -22.0 Cornwall
## 6 -101.   -18.5    30.0 Hamilton
## 7  62.4   198.   -14.0 Huntsville
## 8 214.   -129.    10.8 Kingston
## 9 -123.   -15.0   -6.44 Kitchener
## 10 -208.  -51.6   -36.5 London
## 11 -129.  -19.1   155. Niagara Falls
## 12 146.   300.   -25.4 North Bay
## 13 368.   -4.30   -47.2 Ottawa
## 14 -145.  125.   -16.0 Owen Sound
## 15  82.5   0.551   -6.92 Peterborough
## 16 -299.  -39.4   -72.5 Sarnia
## 17 -117.  -16.8   123. St Catharines
## 18 -34.3   -4.75   15.8 Toronto
## 19 -388. -116.   -99.5 Windsor
```

RGL code for 3 dimensions

```
library(rgl)
plot3d(ontario.3)
text3d(ontario.3, text=d2$city)
```

Comparing MDS solution with “reality”: Procrustes rotation

- ▶ How to tell that an MDS map makes a good correspondence with “what should be”?
- ▶ Problem: MDS map might be rotated/scaled/reflected from reality.
- ▶ How to find rotation/scaling/reflection that best matches reality?
- ▶ Answer: **Procrustes rotation**.
- ▶ In R: procOPA in package shapes.

“True” coordinates

- ▶ Get latitudes and longitudes of cities by geocoding, as before. Glue “ON” onto city names to make sure we get right ones:

```
lookup=str_c(ontario2$x, " ON")
latlong=geocode(lookup)
latlong = bind_cols(city=ontario2$x,latlong) %>% print(n=4)

## # A tibble: 19 x 3
##   city      lon    lat
##   <chr>    <dbl> <dbl>
## 1 Barrie    -79.7  44.4
## 2 Belleville -77.4  44.2
## 3 Brantford -80.3  43.1
## 4 Brockville -75.7  44.6
## # ... with 15 more rows
```

- ▶ Not (x, y) coordinates: one degree of latitude is always 110.25 km, but one degree of longitude is only that at the equator (less than that as you move further north, down to 0 km at north pole).

“True” coordinates part 2

- ▶ Make coordinates by multiplying by cosine of “typical” latitude.
- ▶ Find mean latitude:

```
m=mean(latlong$lat); m  
## [1] 44.01851
```

- ▶ Turn into radians and find its cosine:

```
mult=cos(m*pi/180); mult  
## [1] 0.7191153
```

- ▶ Create “true” coords by multiplying the longitudes by that. This needs to be R matrix, not data frame:

```
truecoord=with(latlong, cbind(V1=lon*mult, V2=lat))
```

Using procOPA

- ▶ Feed 2 things into procOPA: first, “true” coordinates, second MDS coordinates.
- ▶ Get out:
 - ▶ (centred and scaled) first set of coordinates A_{hat}
 - ▶ (centred and scaled) second set of coordinates B_{hat}
 - ▶ sum of squared differences between two sets of coordinates OSS
 - ▶ Rotation matrix R
- ▶ A_{hat} and B_{hat} coordinates supposed to match as well as possible.

```
ontario.pro=procOPA(truecoord,
                      ontario2.2$points)
names(ontario.pro)

## [1] "R"      "s"      "Ahat"   "Bhat"   "OSS"    "rmsd"
```

Make data frames of output, glue together

- ▶ Two sets of coordinates, A_{hat} are actual, B_{hat} are from MDS.

```
A = ontario.pro$Ahat %>% as_tibble() %>%
    mutate(which="actual", city=ontario2$x)
B = ontario.pro$Bhat %>% as_tibble() %>%
    mutate(which="MDS", city=ontario2$x)
dp=bind_rows(A,B)
dp %>% sample_n(6)

## # A tibble: 6 x 4
##       V1     V2 which   city
##   <dbl>  <dbl> <chr>   <chr>
## 1 -2.84  -1.70 actual  Windsor
## 2 -0.848 -0.879 actual  Brantford
## 3 -1.33   0.551 actual  Owen Sound
## 4 -0.214  -0.365 actual  Toronto
## 5  3.05   0.715 MDS    Cornwall
## 6  2.50   1.28  MDS    Ottawa
```

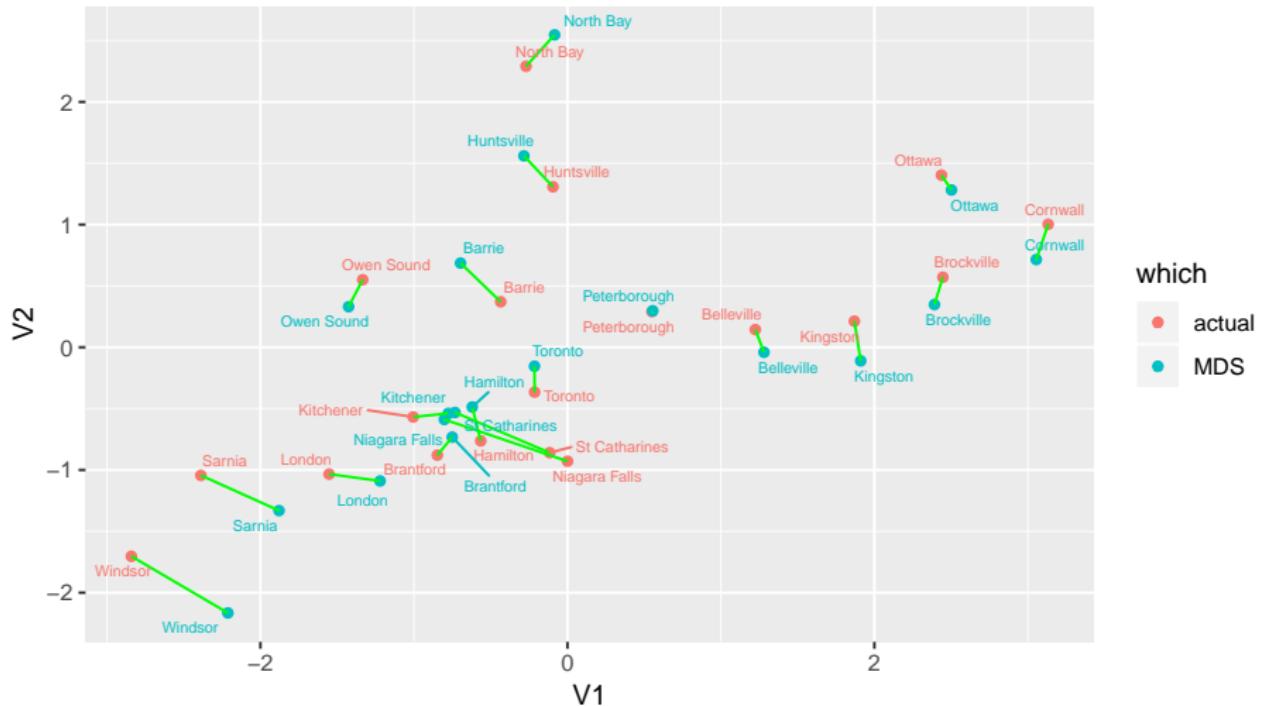
Procrustes rotation plot

- ▶ Strategy: plot all the locations, and colour them by whether they were the true location (red) or the MDS one (blue), which is in which. Label each location with the city name in the appropriate colour.
- ▶ I realized it was actually easy to join the two instances of a city by a line (in green, here, 3rd line) by setting group=city:

```
g_opa = ggplot(dp, aes(x=V1,y=V2,colour=which,  
label=city))+geom_point() +  
  geom_line(aes(group=city), colour="green") +  
  geom_text_repel(size=2)
```

- ▶ On plot, look to see whether points that are same city are joined by a short green line (good) or a long one (bad).

The maps



Comments

- ▶ True locations red, MDS locations blue
- ▶ Most things in roughly right place (esp. relative to other things)
- ▶ Extreme cities off by a bit, but OK relative to neighbours.
- ▶ St Catharines, Niagara Falls off by most.
- ▶ Sarnia, Windsor also off noticeably.
- ▶ These four cities had largest “third dimension” in 3D representation
`ontario2.3`.

Rotation matrix

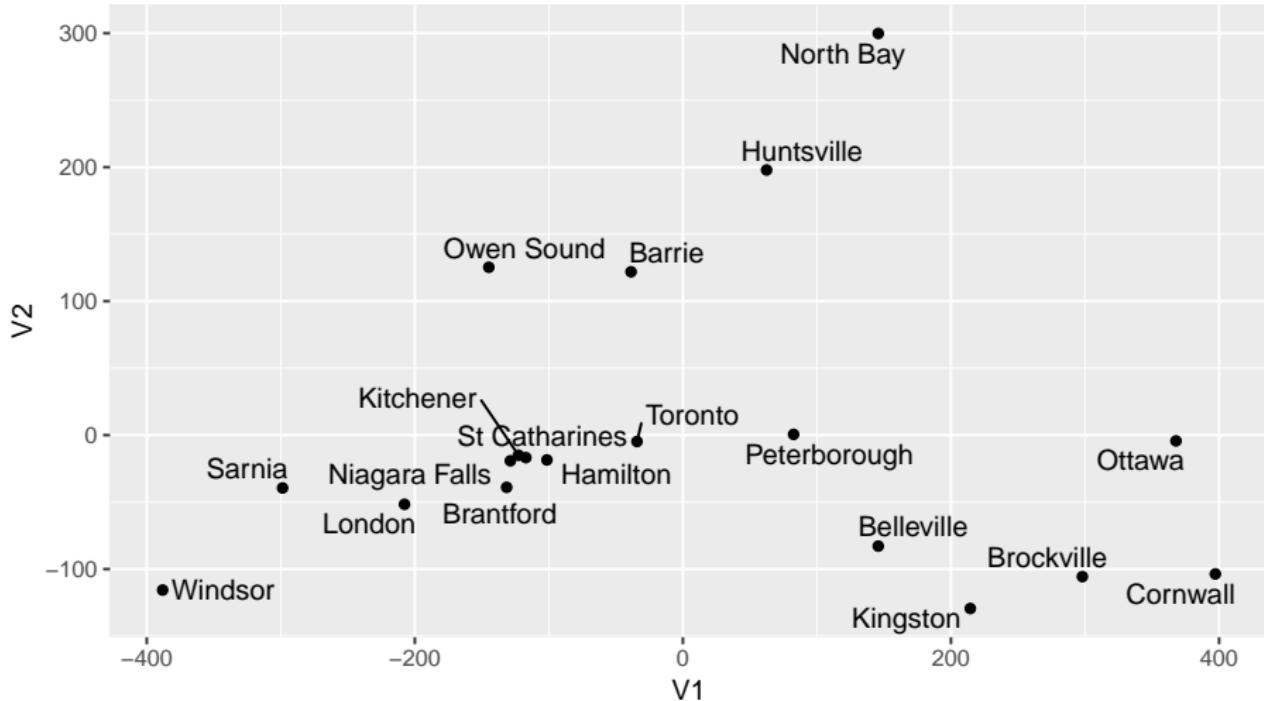
Shows how MDS map needs to be rotated to get best match with actual coordinates:

```
ontario.pro$R  
##           [,1]      [,2]  
## [1,]  0.8845749  0.4663981  
## [2,] -0.4663981  0.8845749
```

Rotation angle θ such that $\cos \theta = 0.885$, $\sin \theta = 0.466$: $\theta = 23$ degrees (counterclockwise).

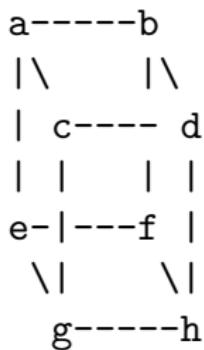
Is that right? Look at MDS map again

g



23 degrees counterclockwise seems about right.

A cube



Cube has side length 1, so distance across diagonal on same face is $\sqrt{2} \simeq 1.4$ and “long” diagonal of cube is $\sqrt{3} \simeq 1.7$.

Try MDS on this obviously 3-dimensional data.

Cube data as distances

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/cube.txt"
cube=read_delim(my_url, " ")

## Warning: Missing column names filled in: 'X2' [2], 'X3' [3], 'X5'
[5], 'X6' [6], 'X8' [8], 'X9' [9], 'X11' [11], 'X12' [12], 'X14' [14],
'X15' [15], 'X17' [17], 'X18' [18], 'X19' [19], 'X21' [21], 'X23' [23],
'X24' [24]

## Warning in rbind(names(probs), probs_f): number of columns of result
is not a multiple of vector length (arg 1)

## Warning: 8 parsing failures.

## row # A tibble: 5 x 5
#>   row col    row col  expected actual file
#>   <int> <chr> <chr>      <chr>    <chr>
#> 1     1 <NA>  25 columns 18 colu~ 'http://www.utsc.utoronto.ca/~
#> 2     2 <NA>  25 columns 19 colu~ 'http://www.utsc.utoronto.ca/~ row
#> 3     3 <NA>  25 columns 20 colu~ 'http://www.utsc.utoronto.ca/~ col 4
#> 4 <NA>  25 columns 19 colu~ 'http://www.utsc.utoronto.ca/~ expected 5
#> 5 <NA>  25 columns 16 colu~ 'http://www.utsc.utoronto.ca/~
#> ...
#> ....
#> ....
#> ....
#> ....
#> ....
```

Making dist object

```
cube.d=cube %>% select(-1) %>% as.dist()  
## Warning in as.dist.default(.): non-square matrix  
  
cube.d  
  
## Error in dimnames(df) <- if (is.null(labels))  
list(seq_len(size), seq_len(size)) else list(labels, :  
length of 'dimnames' [1] not equal to array extent
```

MDS and plotting commands

- ▶ By default in 2 dimensions; save the extra stuff for later:

```
cube.2 = cube.d %>% cmdscale(eig=T)  
  
## Error in cmdscale(., eig = T): NA values not allowed  
in 'd'
```

- ▶ Make data frame to plot, remembering the points to plot are in points now:

```
d = cube.2$points %>% as_tibble() %>%  
  mutate(corners=cube$x)  
  
## Error in eval(lhs, parent, parent): object 'cube.2'  
not found
```

- ▶ Plot points labelled by our names for the corners:

```
g=ggplot(d,aes(x=V1,y=V2,label=corners))+  
  geom_point()+geom_text_repel()
```

The “cube”

```
## Error in FUN(X[[i]], ...): object 'V1' not found
```

Not good.

2 and 3 dimensions

```
cube.3 = cube.d %>% cmdscale(3,eig=T)  
## Error in cmdscale(., 3, eig = T): NA values not allowed  
in 'd'  
  
cube.2$GOF  
## Error in eval(expr, envir, enclos): object 'cube.2' not  
found  
  
cube.3$GOF  
## Error in eval(expr, envir, enclos): object 'cube.3' not  
found
```

- ▶ Really need 3rd dimension to represent cube.

Non-metric scaling

- ▶ Sometimes distances not meaningful as *distances*
- ▶ Only order matters: closest should be closest, farthest farthest on map, but how much further doesn't matter.
- ▶ Non-metric scaling, aims to minimize **stress**, measure of lack of fit.
- ▶ Example: languages. Make map based on “similarity” of number names, without requiring that 1 is “eight times better” than 8.

The languages

- ▶ Recall language data (from cluster analysis): 1–10, measure dissimilarity between two languages by how many number names *differ* in first letter:

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/languages.txt"
number.d=read_table(my_url)
number.d

## # A tibble: 11 x 12
##   la     en     no     dk     nl     de     fr     es
##   <chr> <int> <int> <int> <int> <int> <int> <int>
## 1 en      0      2      2      7      6      6      6
## 2 no      2      0      1      5      4      6      6
## 3 dk      2      1      0      6      5      6      5
## 4 nl      7      5      6      0      5      9      9
## 5 de      6      4      5      5      0      7      7
## 6 fr      6      6      6      9      7      0      2
## 7 es      6      6      5      9      7      2      0
## 8 it      6      6      5      9      7      1      1
## 9 pl      7      7      6     10      8      5      3
## 10 hu     9      8      8      8      9     10     10
## 11 fi     9      9      9      9      9      9      9
## # ... with 4 more variables: it <int>, pl <int>,
## #   hu <int>, fi <int>
```

Non-metric scaling

- ▶ Turn language dissimilarities into `dist` object
- ▶ Run through `isoMDS` from MASS package; works like `cmdscale`.
- ▶ Map only reproduces *relative* closeness of languages.

```
d = number.d %>% select_if(is.numeric) %>%  
  as.dist()  
number.nm = d %>% isoMDS()  
  
## initial value 12.404671  
## iter 5 value 5.933653  
## iter 10 value 5.300747  
## final value 5.265236  
## converged  
  
names(number.nm)  
  
## [1] "points" "stress"
```

- ▶ `points` for plotting, `stress` measure of fit (lower better).

Results

- ▶ Stress is very low (5%, good):

```
number.nm$stress  
## [1] 5.265236
```

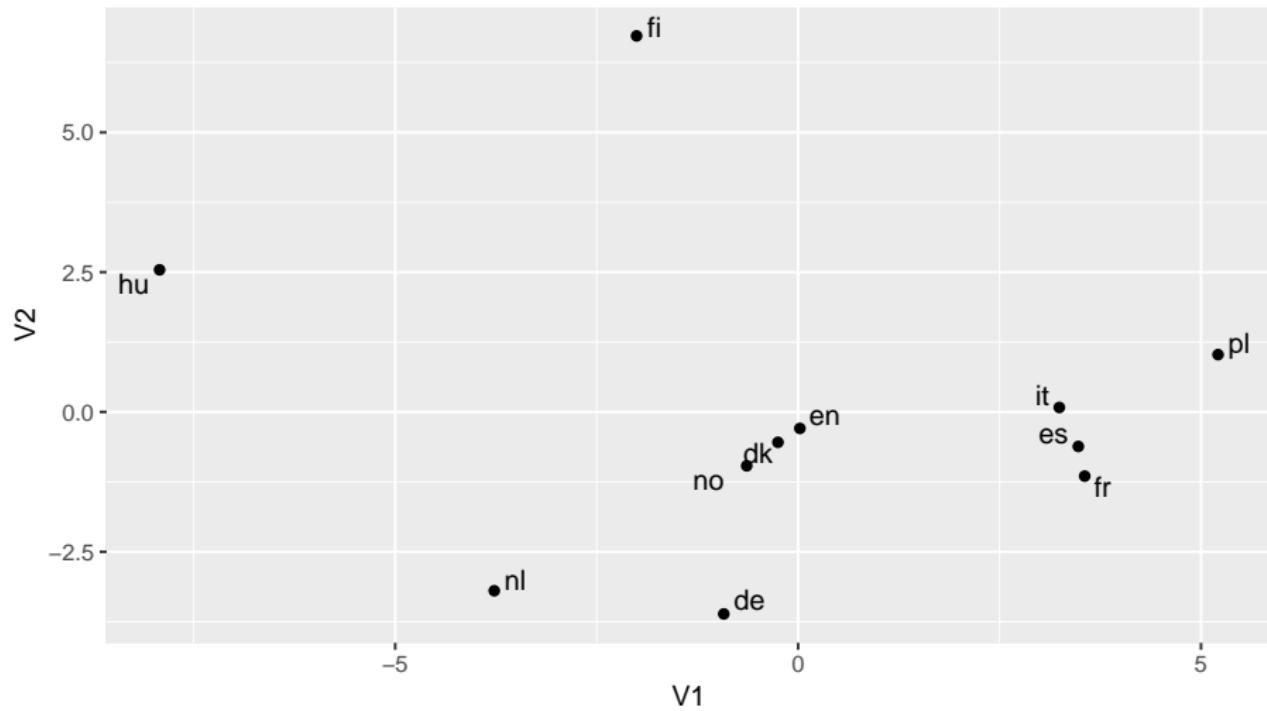
- ▶ Familiar process: make a data frame to plot. Use name dd for data frame this time since used d for distance object:

```
dd = number.nm$points %>% as_tibble() %>%  
  mutate(lang=number.d$la)
```

- ▶ Make plot:

```
g=ggplot(dd, aes(x=V1, y=V2, label=lang))+  
  geom_point()+geom_text_repel()
```

The languages map



Comments

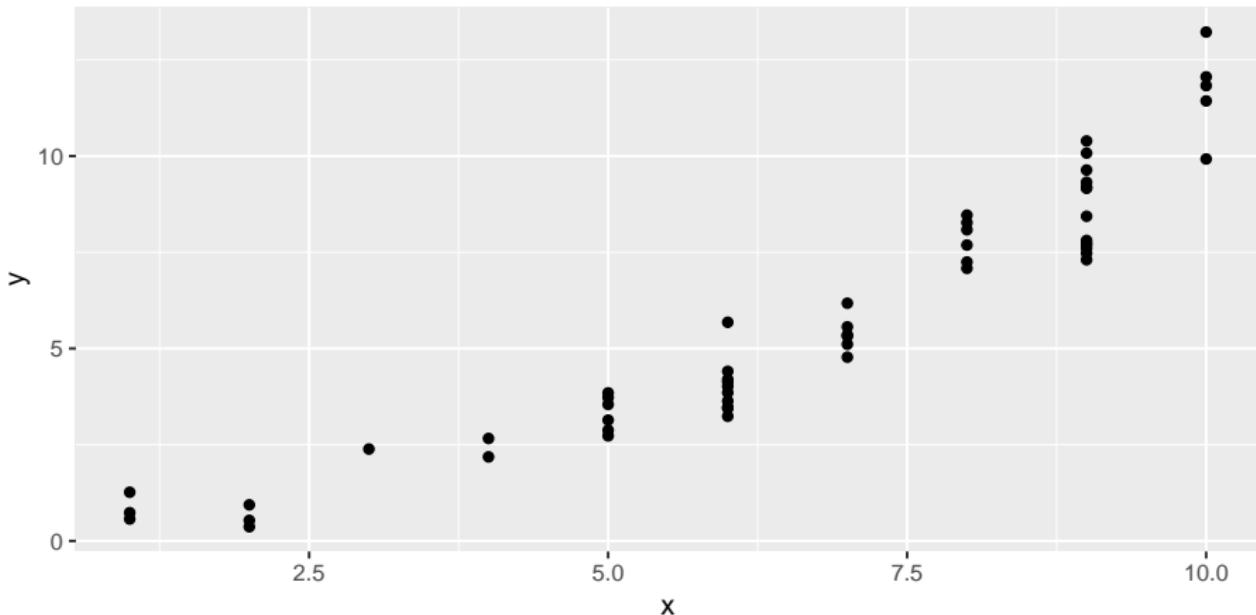
- ▶ Tight clusters: Italian-Spanish-French, English-Danish-Norwegian.
- ▶ Dutch and German close to English group.
- ▶ Polish close to French group.
- ▶ Hungarian, Finnish distant from everything else and each other!
- ▶ Similar conclusions as from the cluster analysis.

Shepard diagram

- ▶ Stress for languages data was 5.3%, very low.
- ▶ How do observed dissimilarities and map distances correspond?
- ▶ For low stress, expect larger dissimilarity to go with larger map distance, almost all the time.
- ▶ Not necessarily a linear trend since non-metric MDS works with *order* of values.
- ▶ Actual dissimilarity on x-axis; map distances on y-axis.

Shepard diagram for languages

```
Shepard(d, number.nm$points) %>% as_tibble() %>%  
  ggplot(aes(x=x, y=y)) + geom_point()
```



Actual dissimilarity x between higher: mapped distance y from MDS higher too. (MDS working well.)

Cube, revisited

```
cube.d = cube %>% select(-x) %>% as.dist(cube)

## Warning in as.dist.default(., cube): non-square matrix

cube.2=isoMDS(cube.d,trace=F) ; cube.2$stress

## Error in isoMDS(cube.d, trace = F): an initial configuration
must be supplied with NA/Infs in 'd'

## Error in eval(expr, envir, enclos): object 'cube.2' not found

cube.3=isoMDS(cube.d,k=3,trace=F) ; cube.3$stress

## Error in isoMDS(cube.d, k = 3, trace = F): an initial
configuration must be supplied with NA/Infs in 'd'

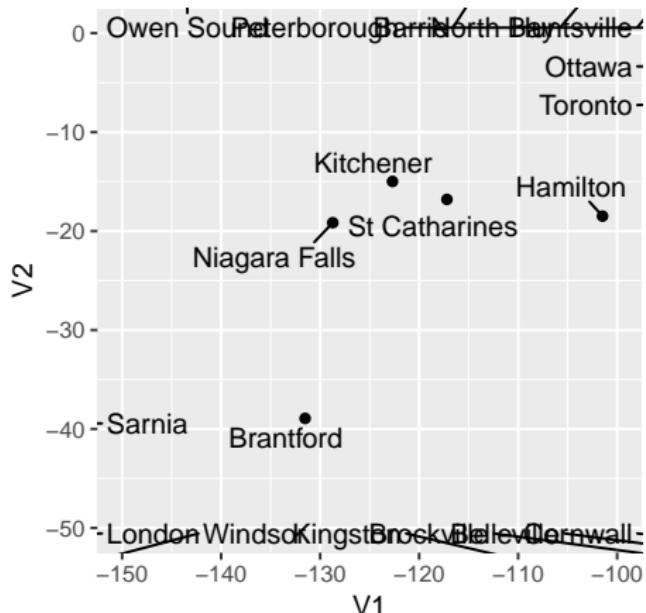
## Error in eval(expr, envir, enclos): object 'cube.3' not found
```

- ▶ Stress is 18% for 2 dimensions, basically 0% for 3.
- ▶ Three dimensions correct, two dimensions bad.
- ▶ Shepard diagrams for these:

```
cube2.sh=Shepard(cube.d,cube.2$points)
```

Shepard diagram for 2-dimensional cube

g2



Poor correspondence (not much trend).

Shepard diagram for 3-dimensional cube

```
g3
```

```
## Error in eval(expr, envir, enclos): object 'g3' not found
```

Almost perfect: all actual $x = 1$ go with smallest mapped distances; almost all $x = 1.7$ go with largest.

Guidelines for stress values, in %

Smaller is better:

Stress value	Interpretation
Less than 5	Excellent: no prospect of misinterpretation (rarely achieved)
5–10	Good: most distances reproduced well, small prospect of false inferences
10–20	Fair: usable, but some distances misleading.
More than 20	Poor: may be dangerous to interpret

- ▶ Languages: stress in “good” range.
- ▶ Cube:
 - ▶ 2 dimensions “fair”, almost “poor”;
 - ▶ 3 dimensions, “excellent”.

Section 12

Principal components

Principal Components

- ▶ Have measurements on (possibly large) number of variables on some individuals.
- ▶ Question: can we describe data using fewer variables (because original variables correlated in some way)?
- ▶ Look for direction (linear combination of original variables) in which values *most spread out*. This is *first principal component*.
- ▶ Second principal component then direction uncorrelated with this in which values then most spread out. And so on.

Principal components

- ▶ See whether small number of principal components captures most of variation in data.
- ▶ Might try to interpret principal components.
- ▶ If 2 components good, can make plot of data.
- ▶ (Like discriminant analysis, but no groups.)
- ▶ “What are important ways that these data vary?”

Packages

You might not have installed the first of these. See over for instructions.

```
library(ggbiplot) # see over
library(tidyverse)
library(ggrepel)
```

Installing ggbiplot

- ▶ ggbiplot not on CRAN, so usual `install.packages` will not work.
- ▶ Install package `devtools` first (once):

```
install.packages("devtools")
```

- ▶ Then install `ggbiplot` (once):

```
library(devtools)  
install_github("vqv/ggbiplot")
```

Small example: 2 test scores for 8 people

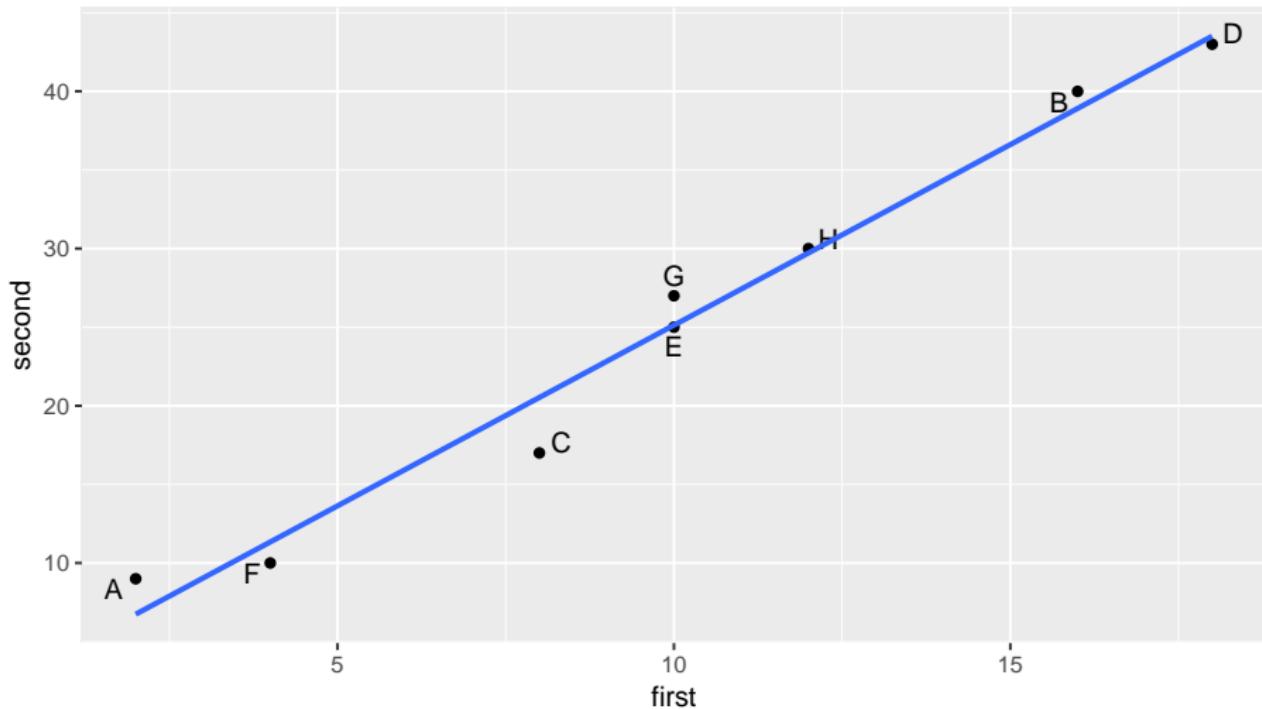
```
my_url="http://www.utsc.utoronto.ca/~butler/d29/test12.txt"
test12=read_table2(my_url)
test12

## # A tibble: 8 x 3
##   first second id
##   <int>  <int> <chr>
## 1     2      9 A
## 2    16     40 B
## 3     8     17 C
## 4    18     43 D
## 5    10     25 E
## 6     4     10 F
## 7    10     27 G
## 8    12     30 H
```

```
g=ggplot(test12,aes(x=first,y=second,label=id))+
  geom_point()+geom_text_repel()
```

The plot

```
g+geom_smooth(method="lm", se=F)
```



Principal component analysis

- ▶ Grab just the numeric columns:

```
test12_numbers = test12 %>% select_if(is.numeric)
```

- ▶ Strongly correlated, so data nearly 1-dimensional:

```
cor(test12_numbers)
```

```
##           first   second
## first  1.000000 0.989078
## second 0.989078 1.000000
```

- ▶ Make a score summarizing this one dimension. Like this:

```
test12.pc = test12_numbers %>% princomp(cor=T)
summary(test12.pc)
```

```
## Importance of components:
```

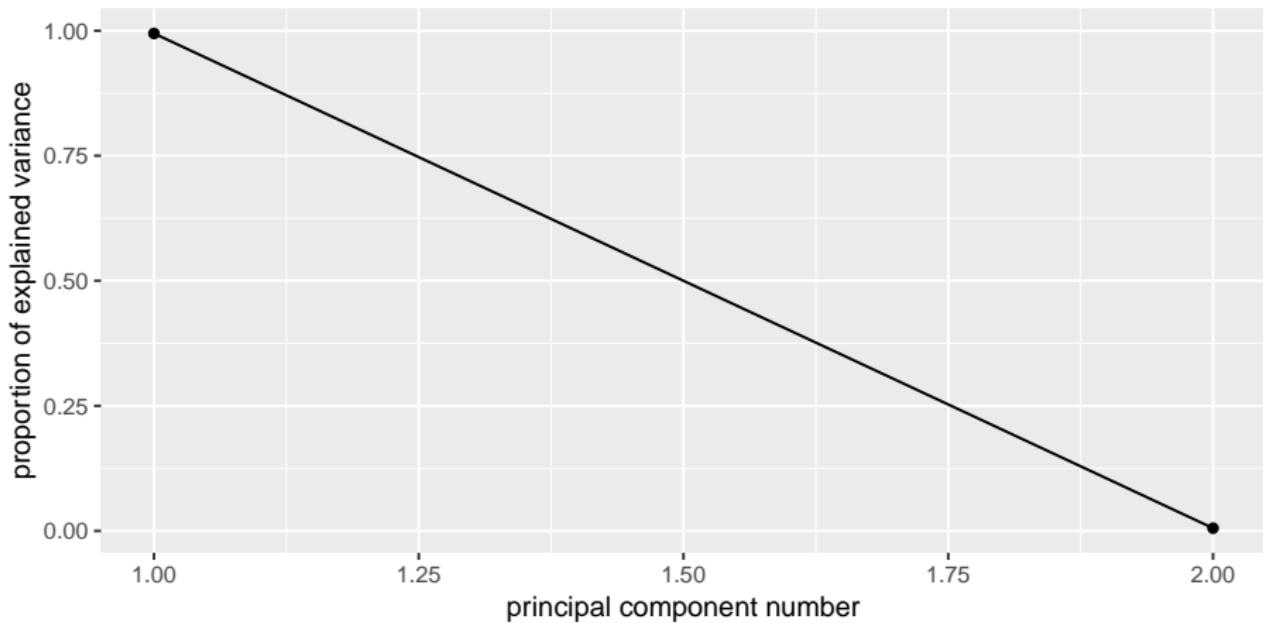
	Comp.1	Comp.2
## Standard deviation	1.410347	0.104508582
## Proportion of Variance	0.994539	0.005461022
## Cumulative Proportion	0.994539	1.0000000000

Comments

- ▶ “Standard deviation” shows relative importance of components (as for LDs in discriminant analysis)
- ▶ Here, first one explains almost all (99.4%) of variability.
- ▶ That is, look only at first component and ignore second.
- ▶ `cor=T` standardizes all variables first. Usually wanted, because variables measured on different scales. (Only omit if variables measured on same scale and expect similar variability.)

Scree plot

```
ggscreepplot(test12.pc)
```



Imagine scree plot continues at zero, so 2 components is a *big* elbow (take one component).

Component loadings

explain how each principal component depends on (standardized) original variables (test scores):

```
test12.pc$loadings  
  
##  
## Loadings:  
##          Comp.1  Comp.2  
## first    0.707   0.707  
## second   0.707  -0.707  
##  
##          Comp.1  Comp.2  
## SS loadings     1.0    1.0  
## Proportion Var  0.5    0.5  
## Cumulative Var 0.5    1.0
```

First component basically negative sum of (standardized) test scores. That is, person tends to score similarly on two tests, and a composite score would summarize performance.

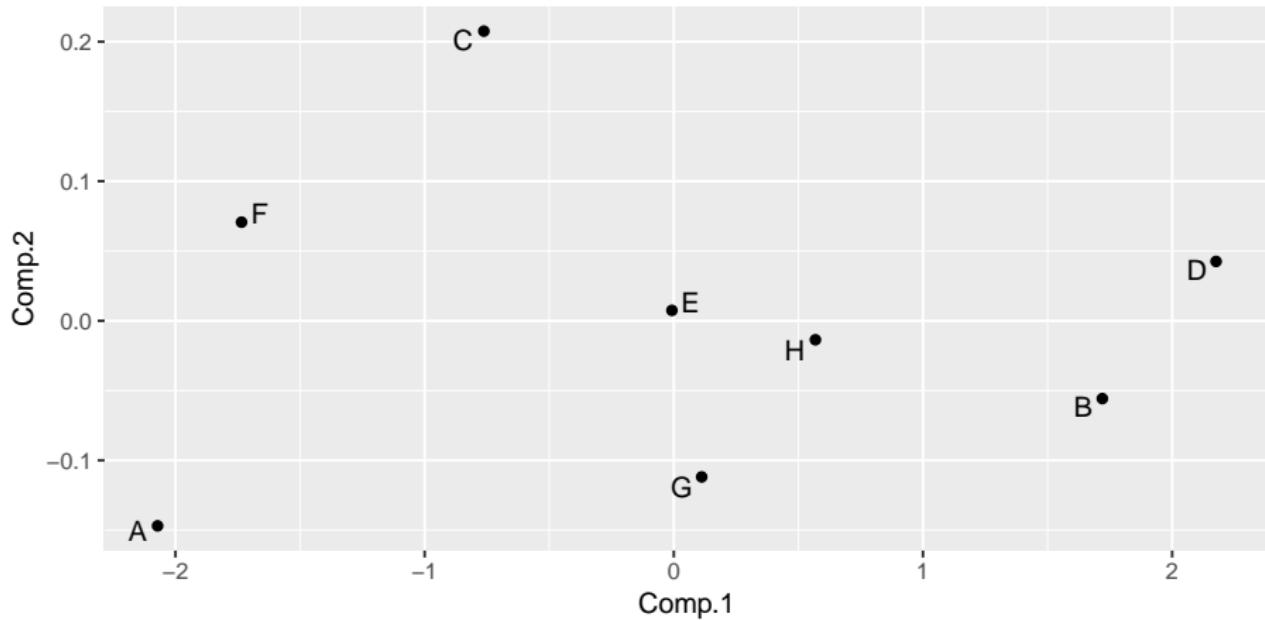
Component scores

```
d=data.frame(test12,test12.pc$scores) ; d  
##   first second id      Comp.1      Comp.2  
## 1     2       9  A -2.071819003 -0.146981782  
## 2    16      40  B  1.719862811 -0.055762223  
## 3     8      17  C -0.762289708  0.207589512  
## 4    18      43  D  2.176267535  0.042533250  
## 5    10      25  E -0.007460609  0.007460609  
## 6     4      10  F -1.734784030  0.070683441  
## 7    10      27  G  0.111909141 -0.111909141  
## 8    12      30  H  0.568313864 -0.013613668
```

- ▶ Person A is a low scorer, high positive comp. 1 score.
- ▶ Person D is high scorer, high negative comp. 1 score.
- ▶ Person E average scorer, near-zero comp. 1 score.
- ▶ comp. 2 says basically nothing.

Plot of scores

```
ggplot(d,aes(x=Comp.1,y=Comp.2,label=id))+  
  geom_point() + geom_text_repel()
```



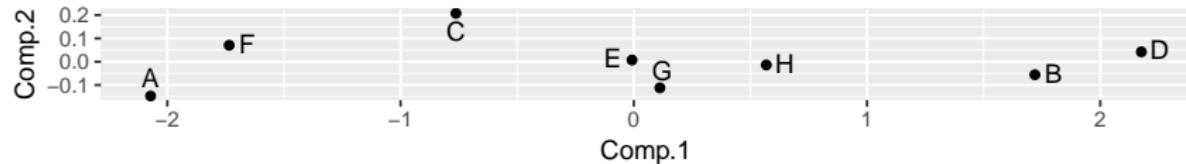
Comments

- ▶ Vertical scale exaggerates importance of comp. 2.
- ▶ Fix up to get axes on same scale:

```
g=ggplot(d,aes(x=Comp.1,y=Comp.2,label=id))+  
  geom_point() + geom_text_repel() +  
  coord_fixed()
```

- ▶ Shows how exam scores really spread out along one dimension:

```
g
```

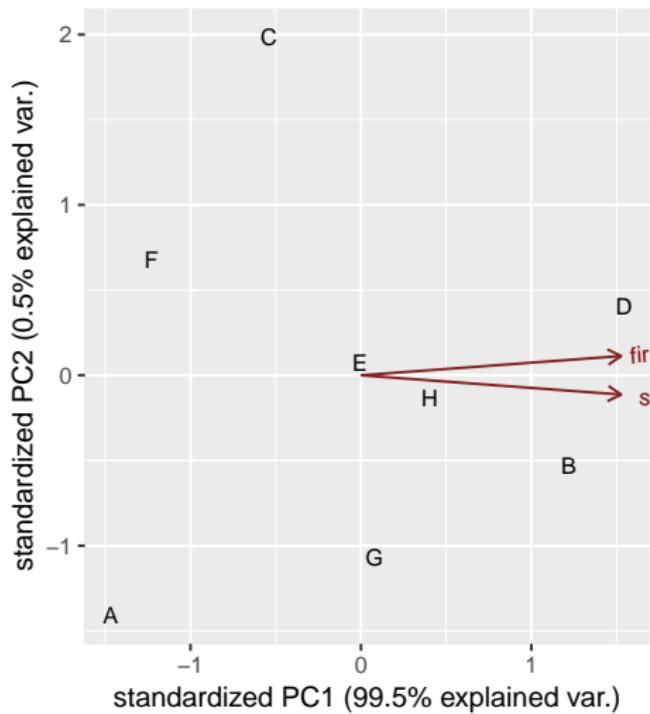


The biplot

- ▶ Plotting variables and individuals on one plot.
- ▶ Shows how components and original variables related.
- ▶ Shows how individuals score on each component, and therefore suggests how they score on each variable.
- ▶ Add `labels` option to identify individuals:

```
g=ggbiplot(test12.pc,labels=test12$id)
```

The biplot



Comments

- ▶ Variables point almost same direction (left). Thus very negative value on comp.1 goes with high scores on both tests, and test scores highly correlated.
- ▶ Position of individuals on plot according to scores on principal components, implies values on original variables. Eg.:
 - ▶ D very negative on comp.1, high scorer on both tests.
 - ▶ A and F very positive on comp.1, poor scorers on both tests.
 - ▶ C positive on comp.2, high score on first test relative to second.
 - ▶ A negative on comp.2, high score on second test relative to first.

Track running data

(1984) track running records for distances 100m to marathon, arranged by country. Countries labelled by (mostly) Internet domain names (ISO 2-letter codes):

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/men-track-field.txt"
track=read_table2(my_url)

## Error in open.connection(con, "rb"): HTTP error 404.

track %>% sample_n(12)

## Error in eval(lhs, parent, parent): object 'track' not found
```

Country names

Also read in a table to look country names up in later:

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/isocodes.txt"
iso=read_csv(my_url)

## Error in open.connection(con, "rb"): HTTP error 404.

iso

## Error in eval(expr, envir, enclos): object 'iso' not found
```

Data and aims

- ▶ Times in seconds 100m–400m, in minutes for rest (800m up).
- ▶ This taken care of by standardization.
- ▶ 8 variables; can we summarize by fewer and gain some insight?
- ▶ In particular, if 2 components tell most of story, what do we see in a plot?

Fit and examine principal components

```
track_num = track %>% select_if(is.numeric)

## Error in eval(lhs, parent, parent): object 'track' not found

track.pc=princomp(track_num,cor=T)

## Error in princomp(track_num, cor = T): object 'track_num' not found

summary(track.pc)

## Error in summary(track.pc): object 'track.pc' not found
```

Scree plot

```
ggscreepplot(track.pc)  
## Error in ggscreepplot(track.pc): object 'track.pc' not  
found
```

How many components?

- ▶ As for discriminant analysis, look for “elbow” in scree plot.
- ▶ See one here at 3 components; everything 3 and beyond is “scree”.
- ▶ So take 2 components.
- ▶ Note difference from discriminant analysis: want “large” rather than “small”, so go 1 step left of elbow.
- ▶ Another criterion: any component with eigenvalue bigger than about 1 is worth including. 2nd one here has eigenvalue just less than 1.
- ▶ Refer back to **summary**: cumulative proportion of variance explained for 2 components is 93.7%, pleasantly high. 2 components tell almost whole story.

How do components depend on original variables?

Loadings:

```
track.pc$loadings
```

```
## Error in eval(expr, envir, enclos): object 'track.pc' not found
```

Comments

- ▶ comp. 1 loads about equally (has equal weight) on times over all distances.
- ▶ comp. 2 has large positive loading for long distances, large negative for short ones.
- ▶ comp. 3: large negative for middle distance, large positive especially for short distances.
- ▶ Country overall good at running will have lower than average record times at all distances, so comp. 1 *large*. Conversely, for countries bad at running, comp. 1 very negative.
- ▶ Countries relatively better at sprinting (low times) will be *positive* on comp. 2; countries relatively better at distance running *negative* on comp. 2.

Commands for plots

- ▶ Principal component scores (first two). Also need country names.

```
d=data.frame(track.pc$scores,  
             country=track$country)  
  
## Error in data.frame(track.pc$scores, country =  
## track$country): object 'track.pc' not found  
  
names(d)  
  
## [1] "first"   "second"  "id"        "Comp.1"   "Comp.2"  
  
g1=ggplot(d,aes(x=Comp.1,y=Comp.2,  
                 label=country))+  
  geom_point() + geom_text_repel() +  
  coord_fixed()
```

- ▶ Biplot:

```
g2=ggbiplots(track.pc,labels=track$country)
```

```
## Error in ggbiplots(track.pc, labels = track$country);
```

Principal components plot

g1

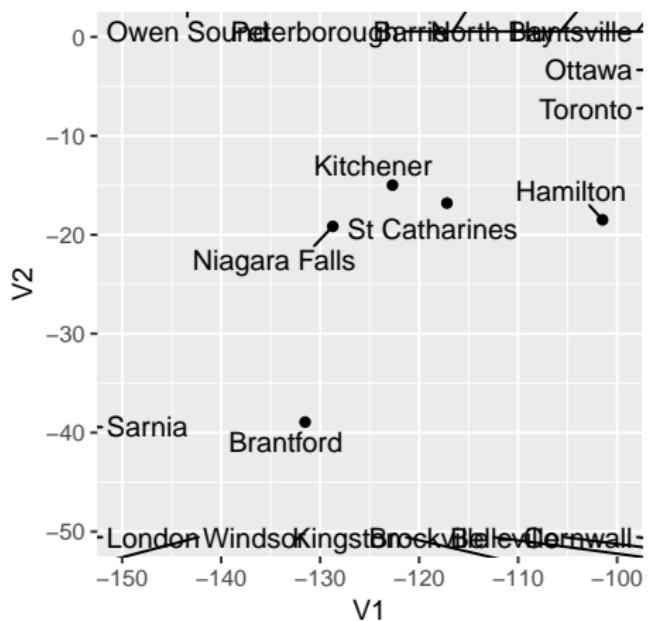
```
## Error in FUN(X[[i]], ...): object 'country' not found
```

Comments on principal components plot

- ▶ Good running countries at right of plot: US, UK, Italy, Russia, East and West Germany.
- ▶ Bad running countries at left: Western Samoa, Cook Islands.
- ▶ Better sprinting countries at top: US, Italy, Russia, Brazil, Greece. do is Dominican Republic, where sprinting records relatively good, distance records very bad.
- ▶ Better distance-running countries at bottom: Portugal, Norway, Turkey, Ireland, New Zealand, Mexico. ke is Kenya.

Biplot

g2



Comments on biplot

- ▶ Had to do some pre-work to interpret PC plot. Biplot more self-contained.
- ▶ All variable arrows point left; countries on left have large (bad) record times overall, countries on right good overall.
- ▶ Variable arrows extend negatively as well. Top left = bad at distance running, bottom right = good at distance running.
- ▶ Bottom left = bad at sprinting, top right = good at sprinting.
- ▶ Doesn't require so much pre-interpretation of components.

How do I know which country is which?

Need to look up two-letter abbreviations in ISO table, eg. for best 8 running countries:

```
d %>% arrange(desc(Comp.1)) %>%
  left_join(iso, by=c("country"="ISO2")) %>%
  select(Comp.1, country, Country) %>%
  slice(1:8)

## Error in tbl_vars(y): object 'iso' not found
```

Best 8 running countries

```
d %>% arrange(Comp.1) %>%
  left_join(iso, by=c("country"="ISO2")) %>%
  select(Comp.1, country, Country) %>%
  slice(1:8)

## Error in tbl_vars(y): object 'iso' not found
```

Worst 8 running countries

```
d %>% arrange(desc(Comp.1)) %>%
  left_join(iso, by=c("country"="ISO2")) %>%
  select(Comp.1, country, Country) %>%
  slice(1:8)

## Error in tbl_vars(y): object 'iso' not found
```

Better at distance running

```
d %>% arrange(desc(Comp.2)) %>%  
  left_join(iso,by=c("country"="ISO2")) %>%  
  select(Comp.2,country,Country) %>%  
  slice(1:8)  
  
## Error in tbl_vars(y): object 'iso' not found
```

Better at sprinting

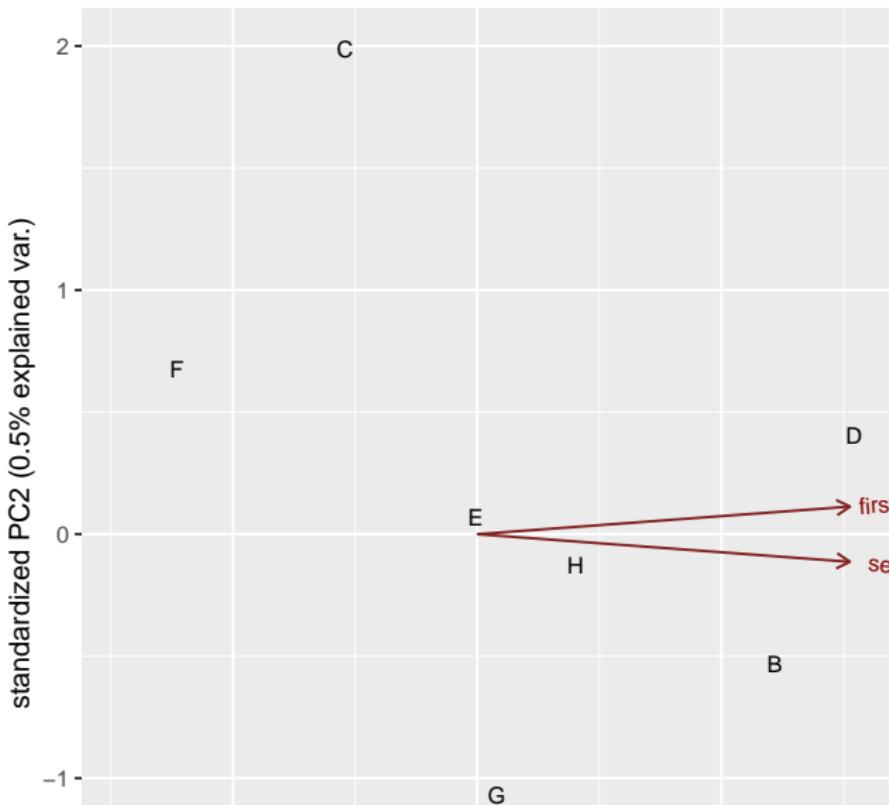
```
d %>% arrange(Comp.2) %>%  
  left_join(iso,by=c("country"="ISO2")) %>%  
  select(Comp.2,country,Country) %>%  
  slice(1:10)  
  
## Error in tbl_vars(y): object 'iso' not found
```

Plot with country names

```
g = d %>% left_join(iso,by=c("country"="ISO2")) %>%
  select(Comp.1,Comp.2,Country) %>%
  ggplot(aes(x=Comp.1,y=Comp.2,label=Country))+
  geom_point()+geom_text_repel(size=1)+
  coord_fixed()
## Error in tbl_vars(y): object 'iso' not found
```

The plot

gg



Principal components from correlation matrix

Create data file like this:

```
1          0.9705 -0.9600
0.9705    1        -0.9980
-0.9600   -0.9980  1
```

and read in like this:

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/cov.txt"
mat=read_table(my_url, col_names=F)
mat

## # A tibble: 3 x 3
##       X1      X2      X3
##     <dbl>  <dbl>  <dbl>
## 1  1.00  0.970 -0.960
## 2  0.970  1.00 -0.998
## 3 -0.960 -0.998  1.00
```

Pre-processing

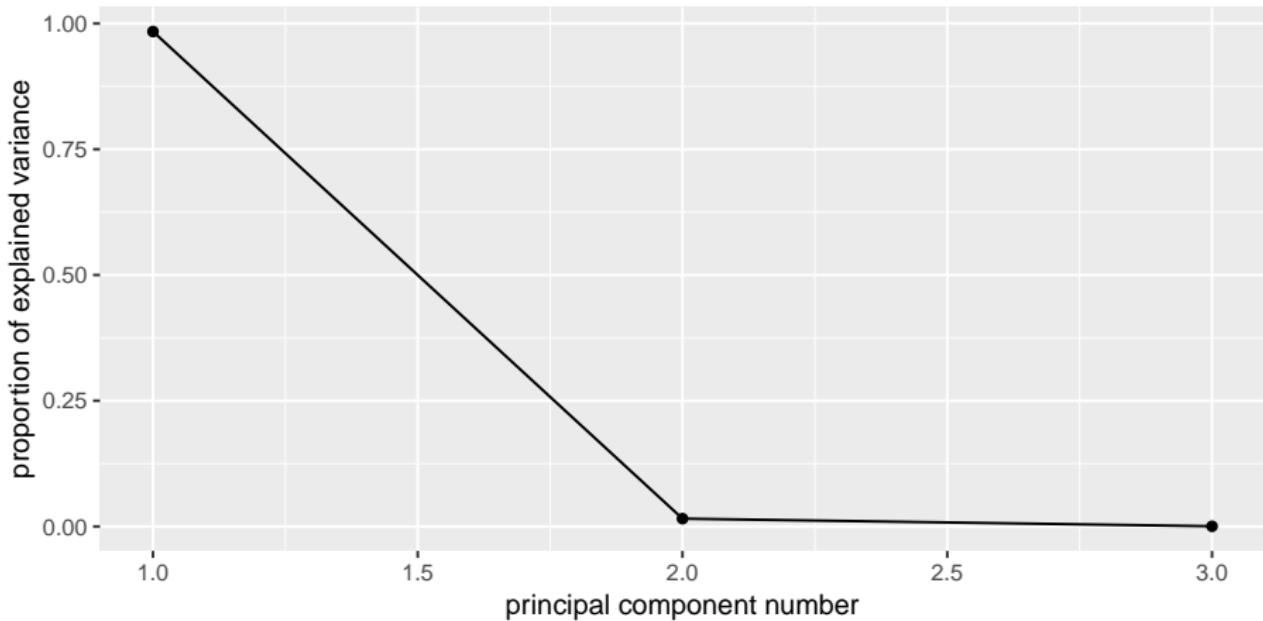
A little pre-processing required:

- ▶ Turn into matrix (from data frame)
- ▶ Feed into princomp as covmat=

```
mat.pc = mat %>% as.matrix() %>%  
  princomp(covmat=.)
```

Scree plot: one component fine

```
ggscreepplot(mat.pc)
```



Component loadings

Compare correlation matrix:

```
mat
## # A tibble: 3 x 3
##       X1      X2      X3
##   <dbl>  <dbl>  <dbl>
## 1  1     0.970 -0.96
## 2  0.970  1     -0.998
## 3 -0.96  -0.998  1
```

with component loadings

```
mat.pc$loadings
##
## Loadings:
##       Comp.1 Comp.2 Comp.3
## X1    0.573  0.812  0.112
## X2    0.581 -0.306 -0.755
## X3   -0.578  0.498 -0.646
##
##                               Comp.1 Comp.2 Comp.3
## SS loadings      1.000  1.000  1.000
## Proportion Var  0.333  0.333  0.333
## Cumulative Var  0.333  0.667  1.000
```

- ▶ When X1 large, X2 also large, X3 small.
- ▶ Then comp.1 *negative*.
- ▶ When X1 small, X2 small, X3 large.
- ▶ Then comp.1 *positive*.

No scores

- ▶ With correlation matrix rather than data, no component scores
- ▶ So no principal component plot
- ▶ and no biplot.

Section 13

Exploratory factor analysis

Principal components and factor analysis

- ▶ Principal components:
 - ▶ Purely mathematical.
 - ▶ Find eigenvalues, eigenvectors of correlation matrix.
 - ▶ No testing whether observed components reproducible, or even probability model behind it.
- ▶ Factor analysis:
 - ▶ some way towards fixing this (get test of appropriateness)
 - ▶ In factor analysis, each variable modelled as: “common factor” (eg. verbal ability) and “specific factor” (left over).
 - ▶ Choose the common factors to “best” reproduce pattern seen in correlation matrix.
 - ▶ Iterative procedure, different answer from principal components.

Packages

```
library(lavaan) # confirmatory factor analysis  
library(ggbiplot)  
library(tidyverse)
```

Example

- ▶ 145 children given 5 tests, called PARA, SENT, WORD, ADD and DOTS. 3 linguistic tasks (paragraph comprehension, sentence completion and word meaning), 2 mathematical ones (addition and counting dots).
- ▶ Correlation matrix of scores on the tests:

para	1	0.722	0.714	0.203	0.095
sent	0.722	1	0.685	0.246	0.181
word	0.714	0.685	1	0.170	0.113
add	0.203	0.246	0.170	1	0.585
dots	0.095	0.181	0.113	0.585	1

- ▶ Is there small number of underlying “constructs” (unobservable) that explains this pattern of correlations?

To start: principal components

Using correlation matrix:

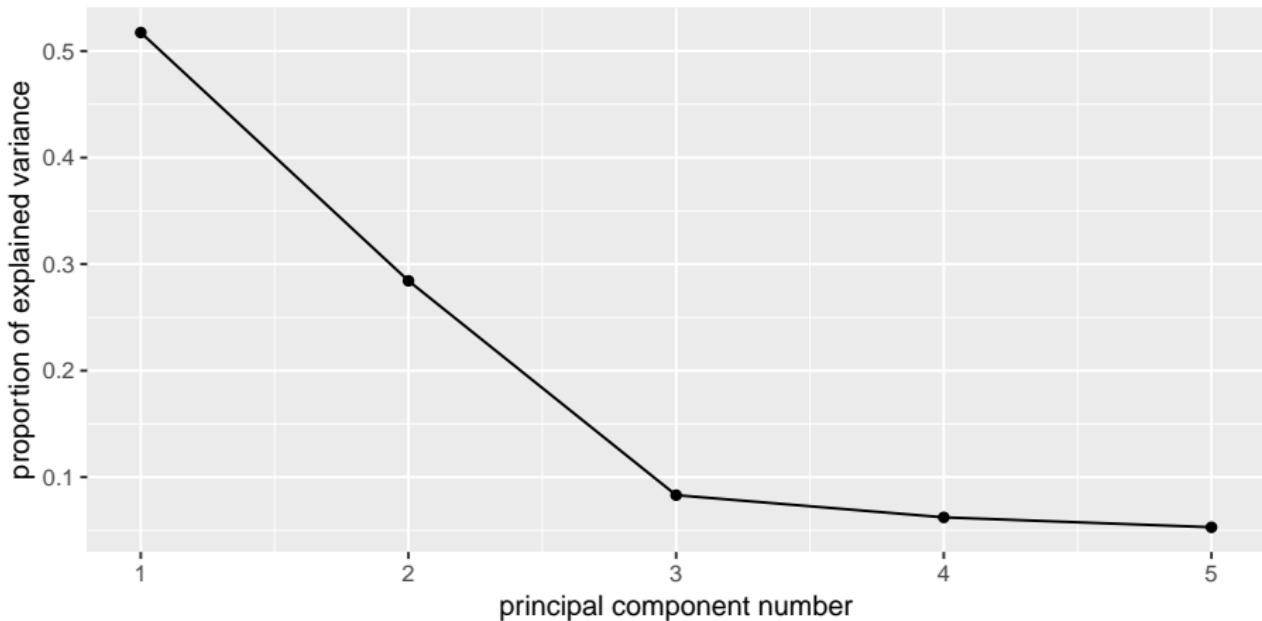
```
my_url="http://www.utsc.utoronto.ca/~butler/d29/rex2.txt"
kids = read_delim(my_url, " ")
kids

## # A tibble: 5 x 6
##   test    para   sent   word   add   dots
##   <chr>  <dbl>  <dbl>  <dbl>  <dbl>  <dbl>
## 1 para    1     0.722  0.714  0.203  0.095
## 2 sent    0.722  1     0.685  0.246  0.181
## 3 word    0.714  0.685  1      0.17   0.113
## 4 add     0.203  0.246  0.17   1      0.585
## 5 dots    0.095  0.181  0.113  0.585  1

kids.pc = kids %>%
  select_if(is.numeric) %>%
  as.matrix() %>%
  princomp(covmat=.)
```

Scree plot

```
ggscreepplot(kids.pc)
```



Principal component results

- ▶ Need 2 components. Loadings:

```
kids.pc$loadings  
##  
## Loadings:  
##          Comp.1 Comp.2 Comp.3 Comp.4 Comp.5  
## para    0.534  0.245  0.114      0.795  
## sent    0.542  0.164      0.660 -0.489  
## word    0.523  0.247 -0.144 -0.738 -0.316  
## add     0.297 -0.627  0.707  
## dots    0.241 -0.678 -0.680      0.143  
##  
##          Comp.1 Comp.2 Comp.3 Comp.4 Comp.5  
## SS loadings   1.0    1.0    1.0    1.0    1.0  
## Proportion Var 0.2    0.2    0.2    0.2    0.2  
## Cumulative Var 0.2    0.4    0.6    0.8    1.0
```

- ▶ First component has a bit of everything, though especially the first three tests.
- ▶ Second component rather more clearly add and dots.
- ▶ N

Factor analysis

- ▶ Specify number of factors first, get solution with exactly that many factors.
- ▶ Includes hypothesis test, need to specify how many children wrote the tests.
- ▶ Works from correlation matrix via `covmat` or actual data, like `princomp`.
- ▶ Introduces extra feature, *rotation*, to make interpretation of loadings (factor-variable relation) easier.

Factor analysis for the kids data

- ▶ Create “covariance list” to include number of children who wrote the tests.
- ▶ Feed this into factanal, specifying how many factors (2).

```
km = kids %>%
  select_if(is.numeric) %>%
  as.matrix()
km2=list(cov=km,n.obs=145)
kids.f2=factanal(factors=2,covmat=km2)
```

Uniquenesses

```
kids.f2$uniquenesses
```

```
##      para      sent      word      add      dots
## 0.2424457 0.2997349 0.3272312 0.5743568 0.1554076
```

- ▶ Uniquenesses say how “unique” a variable is (size of specific factor). Small uniqueness means that the variable is summarized by a factor (good).
- ▶ Mildly worried by how large add’s uniqueness is.
- ▶ Also see “communality” for this, where *large* is good.

Loadings

```
kids.f2$loadings
##
## Loadings:
##      Factor1 Factor2
## [1,] 0.867
## [2,] 0.820   0.166
## [3,] 0.816
## [4,] 0.167   0.631
## [5,]           0.918
##
##          Factor1 Factor2
## SS loadings    2.119   1.282
## Proportion Var 0.424   0.256
## Cumulative Var 0.424   0.680
```

- ▶ Loadings show how each factor depends on variables. Blanks indicate “small”, less than 0.1.
- ▶ Factor 1 clearly the “linguistic” tasks, factor 2 clearly the “mathematical” ones.
- ▶ Two factors together explain 68% of variability (like regression R-squared).

Are 2 factors enough?

```
kids.f2$STATISTIC
```

```
## objective  
## 0.5810578
```

```
kids.f2$dof
```

```
## [1] 1
```

```
kids.f2$PVAL
```

```
## objective  
## 0.445898
```

P-value not small, so 2 factors OK.

1 factor

```
kids.f1=factanal(factors=1,covmat=km2)
kids.f1$STATISTIC

## objective
## 58.16534

kids.f1$dof

## [1] 5

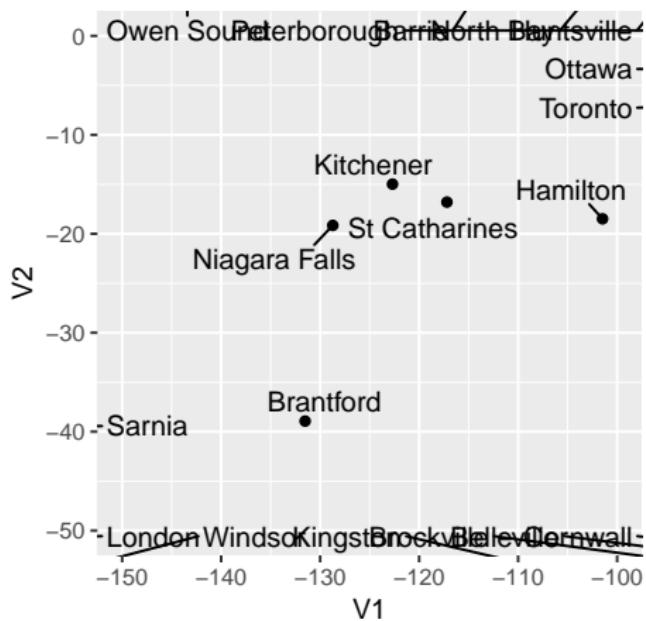
kids.f1$PVAL

##     objective
## 2.907856e-11
```

1 factor rejected (P-value small). Definitely need more than 1.

Track running records revisited

g2



Benefit of rotation

- ▶ 100m and marathon arrows almost perpendicular, but components don't match anything much:
 - ▶ sprinting: top left and bottom right
 - ▶ distance running: bottom left and top right.
- ▶ Can we arrange things so that components (factors) correspond to something meaningful?

Track records by factor analysis

Obtain factor scores (have actual data):

```
track

## Error in eval(expr, envir, enclos): object 'track' not found

track.f = track %>% select_if(is.numeric) %>%
  factanal(2,scores="r")

## Error in eval(lhs, parent, parent): object 'track' not found
```

Track data biplot

Not so nice-looking:

```
biplot(track.f$scores,track.f$loadings,  
xlab=xlab=track$country)  
  
## Error in biplot(track.f$scores, track.f$loadings, xlab  
# = track$country): object 'track.f' not found
```

Comments

- ▶ This time 100m “up” (factor 2), marathon “right” (factor 1).
- ▶ Countries most negative on factor 2 good at sprinting.
- ▶ Countries most negative on factor 1 good at distance running.

Rotated factor loadings

```
track.f$loadings  
## Error in eval(expr, envir, enclos): object 'track.f' not found
```

Which countries are good at sprinting or distance running?

Make a data frame with the countries and scores in:

```
scores=data.frame(country=track$country,  
                  track.f$scores)  
  
## Error in data.frame(country = track$country,  
track.f$scores): object 'track' not found  
  
scores %>% slice(1:6)  
  
## Error in eval(lhs, parent, parent): object 'scores' not  
found
```

The best sprinting countries

Most negative on factor 2:

```
scores %>% arrange(Factor2) %>%
  left_join(iso,by=c("country"="ISO2")) %>%
  select(Country,Factor1,Factor2) %>%
  slice(1:10)

## Error in eval(lhs, parent, parent): object 'scores' not found
```

The best distance-running countries

Most negative on factor 1:

```
scores %>% arrange(Factor1) %>%
  left_join(iso, by=c("country"="ISO2")) %>%
  select(Country,Factor1,Factor2) %>%
  slice(1:10)

## Error in eval(lhs, parent, parent): object 'scores' not found
```

A bigger example: BEM sex role inventory

- ▶ 369 women asked to rate themselves on 60 traits, like “self-reliant” or “shy”.
- ▶ Rating 1 “never or almost never true of me” to 7 “always or almost always true of me”.
- ▶ 60 personality traits is a lot. Can we find a smaller number of factors that capture aspects of personality?
- ▶ The whole BEM sex role inventory on next page.

The whole inventory

1. self reliant	21.reliable	41.warm
2. yielding	22.analytical	42.solemn
3. helpful	23.sympathetic	43.willing to take a stand
4. defends own beliefs	24.jealous	44.tender
5. cheerful	25.leadership ability	45.friendly
6. moody	26.sensitive to other's needs	46.aggressive
7. independent	27.truthful	47.gullible
8. shy	28.willing to take risks	48.inefficient
9. conscientious	29.understanding	49.acts as a leader
10.athletic	30.secretive	50.childlike
11.affectionate	31.makes decisions easily	51.adaptable
12.theatrical	32.compassionate	52.individualistic
13.assertive	33.sincere	53.does not use harsh language
14.flatterable	34.self-sufficient	54.unsystematic
15.happy	35.eager to soothe hurt feelings	55.competitive
16.strong personality	36.conceited	56.loves children
17.loyal	37.dominant	57.tactful
18.unpredictable	38.soft spoken	58.ambitious
19.forceful	39.likable	59.gentle
20.feminine	40.masculine	60.conventional

Some of the data

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/factor.txt"
bem=read_tsv(my_url)
bem

## # A tibble: 369 x 45
##   subno helpful reliant defbel yielding cheerful indpt
##   <int>    <int>    <int>    <int>    <int>    <int> <int>
## 1     1        7        7        5        5        7        7
## 2     2        5        6        6        6        2        3
## 3     3        7        6        4        4        5        5
## 4     4        6        6        7        4        6        6
## 5     5        6        6        7        4        7        7
## 6     7        5        6        7        4        6        6
## 7     8        6        4        6        6        6        3
## 8     9        7        6        7        5        6        7
## 9    10        7        6        6        4        4        5
## 10   11        7        4        7        4        7        5
## # ... with 359 more rows, and 38 more variables:
## #   athlet <int>, shy <int>, assert <int>, strpers <int>,
## #   forceful <int>, affect <int>, flatter <int>,
## #   loyal <int>, analyt <int>, feminine <int>,
## #   sympathy <int>, moody <int>, sensitiv <int>,
## #   undstand <int>, compass <int>, leaderab <int>,
## #   soothe <int>, risk <int>, decide <int>, selfsuff <int>,
## #   conscienc <int>, dominant <int>, masculin <int>,
## #   stand <int>, happy <int>, softspok <int>, warm <int>,
## #   truthful <int>, tender <int>, gullible <int>,
```

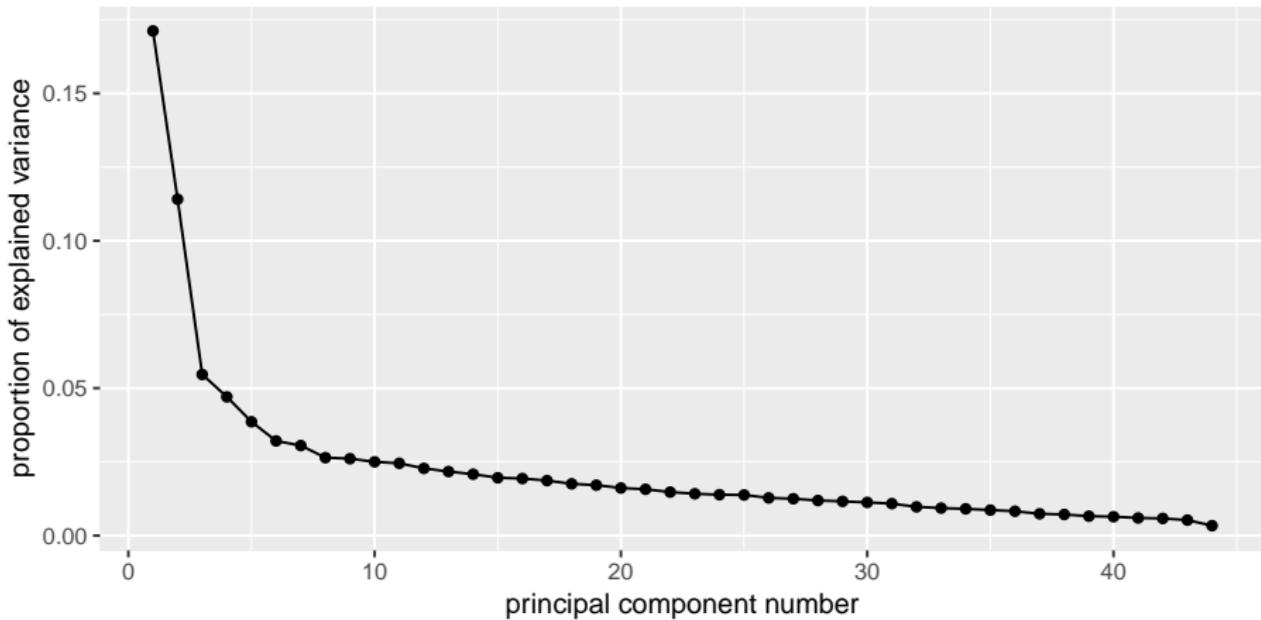
Principal components first

...to decide on number of factors:

```
bem.pc = bem %>% select(-subno) %>%  
  princomp(cor=T)
```

The scree plot

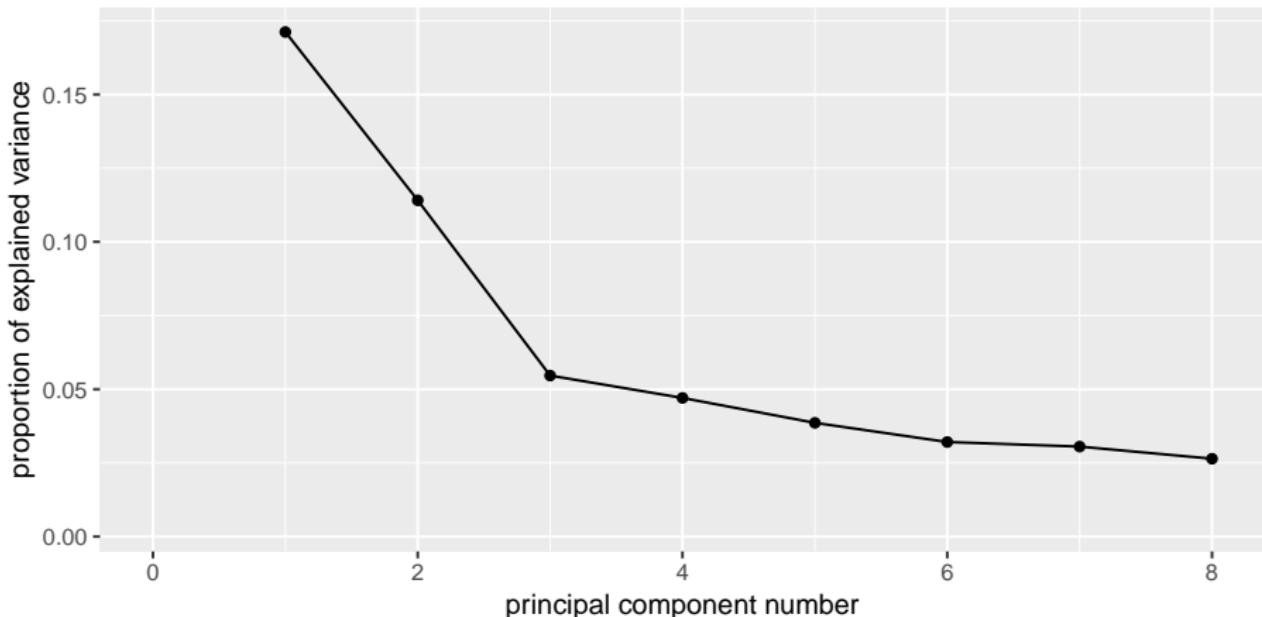
```
g=ggscreeplot(bem.pc) ; g
```



- ▶ No obvious elbow.

Zoom in to search for elbow

```
g+scale_x_continuous(limits=c(0,8))
```



Possible elbows at 3 (2 factors) and 6 (5).

but is 2 really good?

summary(bem.pc)

```
## Importance of components:
```

```
##          Comp.1    Comp.2    Comp.3
```

```
## Standard deviation 2.7444993 2.2405789 1.55049106
```

```
## Proportion of Variance 0.1711881 0.1140953 0.05463688
```

```
## Cumulative Proportion 0.1711881 0.2852834 0.33992029
```

```
##          Comp.4    Comp.5    Comp.6
```

```
## Standard deviation 1.43886350 1.30318840 1.18837867
```

```
## Proportion of Variance 0.04705291 0.03859773 0.03209645
```

```
## Cumulative Proportion 0.38697320 0.42557093 0.45766738
```

```
##          Comp.7    Comp.8    Comp.9
```

```
## Standard deviation 1.15919129 1.07838912 1.07120568
```

```
## Proportion of Variance 0.03053919 0.02643007 0.02607913
```

```
## Cumulative Proportion 0.48820657 0.51463664 0.54071577
```

```
##          Comp.10   Comp.11   Comp.12
```

```
## Standard deviation 1.04901318 1.03848656 1.00152287
```

```
## Proportion of Variance 0.02500974 0.02451033 0.02279655
```

```
## Cumulative Proportion 0.56572551 0.59023584 0.61303238
```

```
##          Comp.13   Comp.14   Comp.15
```

```
## Standard deviation 0.97753974 0.95697572 0.9287543
```

```
## Proportion of Variance 0.02171782 0.02081369 0.0196042
```

```
## Cumulative Proportion 0.63475020 0.65556390 0.6751681
```

```
##          Comp.16   Comp.17   Comp.18
```

```
## Standard deviation 0.92262649 0.90585705 0.8788668
```

```
## Proportion of Variance 0.01934636 0.01864948 0.0175547
```

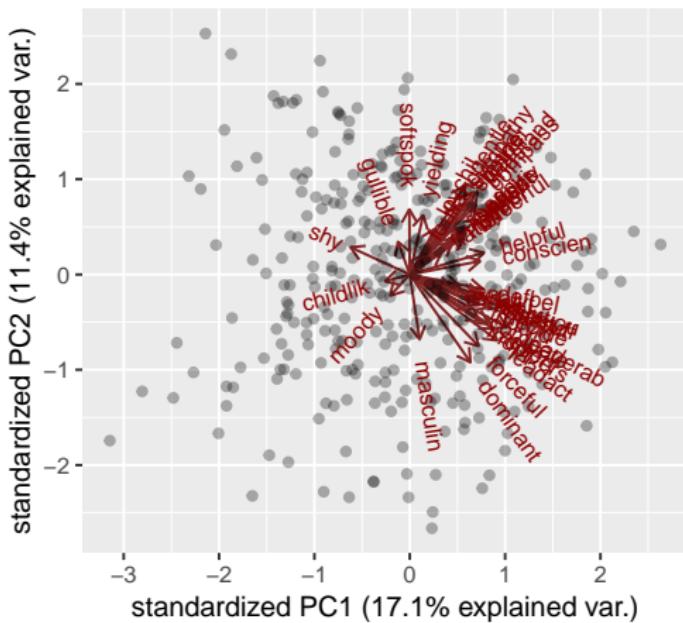
```
## Cumulative Proportion 0.69451445 0.71316392 0.7307186
```

Comments

- ▶ Want overall fraction of variance explained (“cumulative proportion”) to be reasonably high.
- ▶ 2 factors, 28.5%. Terrible!
- ▶ Even 56% (10 factors) not that good!
- ▶ Have to live with that.

Biplot

```
ggbiplot(bem.pc, alpha=0.3)
```



Comments

- ▶ Ignore individuals for now.
- ▶ Most variables point to 10 o'clock or 7 o'clock.
- ▶ Suggests factor analysis with rotation will get interpretable factors (rotate to 6 o'clock and 9 o'clock, for example).
- ▶ Try for 2-factor solution (rough interpretation, will be bad):

```
bem.2 = bem %>% select(-subno) %>%  
  factanal(factors=2)
```

- ▶ Show output in pieces (just print `bem.2` to see all of it).

Uniquenesses

bem.2\$uniquenesses

```
## helpful reliant defbel yielding cheerful indpt
## 0.7598223 0.7808058 0.7748448 0.8688473 0.8394916 0.7282742
## athlet shy assert strpers forceful affect
## 0.9229702 0.8239496 0.6329347 0.5679398 0.5631857 0.6616625
## flatter loyal analyt feminine sympathy moody
## 0.9409500 0.8035264 0.8968744 0.8829927 0.7231450 0.9730607
## sensitiv undstand compass leaderab soothe risk
## 0.8018851 0.6194392 0.5937073 0.4091894 0.6596103 0.7789761
## decide selfsuff conscienc dominant masculin stand
## 0.6938578 0.7210246 0.7974820 0.4942909 0.8453368 0.6024001
## happy softspok warm truthful tender gullible
## 0.8008966 0.8339058 0.4764762 0.8889983 0.4928919 0.9583435
## leadact childlik individ foulang lovchil compete
## 0.4166153 0.9800360 0.7941998 0.9821662 0.8924392 0.7942910
## ambitiou gentle
## 0.8101599 0.5064551
```

- ▶ Mostly high or very high (bad).
- ▶ Some smaller, eg.: Leadership ability (0.409), Acts like leader (0.417), Warm (0.476), Tender (0.493).
- ▶ Smaller uniquenesses captured by one of our two factors.

Factor loadings some

```
bem.2$loadings
```

```
##  
## Loadings:  
##          Factor1 Factor2  
## helpful    0.314   0.376  
## reliant    0.453   0.117  
## defbel     0.434   0.193  
## yielding   -0.131  0.338  
## cheerful   0.152   0.371  
## indpt      0.521  
## athlet     0.267  
## shy        -0.414  
## assert     0.605  
## strpers    0.657  
## forceful   0.649   -0.126  
## affect     0.178   0.554  
## flatter     0.223  
## loyal      0.151   0.417  
## analyt     0.295   0.127  
## feminine   0.113   0.323  
## sympathy    0.526  
## moody       -0.162  
## sensitiv   0.135   0.424  
## undstand    0.610  
## compass     0.114   0.627  
## leaderab   0.765
```

Making a data frame

There are too many to read easily, so make a data frame. This is a bit tricky:

```
loadings = as.data.frame(unclass(bem.2$loadings)) %>%
  mutate(trait=rownames(bem.2$loadings))
loadings %>% slice(1:10)

##          Factor1      Factor2     trait
## 1    0.3137466  0.376484908  helpful
## 2    0.4532904  0.117140647  reliant
## 3    0.4336574  0.192602996  defbel
## 4   -0.1309965  0.337629288 yielding
## 5    0.1523718  0.370530549 cheerful
## 6    0.5212403  0.005870336    indpt
## 7    0.2670788  0.075542858  athlet
## 8   -0.4144579 -0.065372760      shy
## 9    0.6049588  0.033004846   assert
## 10   0.6569855  0.020777649 strpers
```

Pick out the big ones on factor 1

Arbitrarily defining > 0.4 or < -0.4 as “big”:

```
loadings %>% filter(abs(Factor1)>0.4)
```

```
##          Factor1        Factor2      trait
## 1    0.4532904   0.117140647  reliant
## 2    0.4336574   0.192602996  defbel
## 3    0.5212403   0.005870336  indpt
## 4   -0.4144579  -0.065372760    shy
## 5    0.6049588   0.033004846  assert
## 6    0.6569855   0.020777649 strpers
## 7    0.6487190  -0.126405816 forceful
## 8    0.7654924   0.069513572 leaderab
## 9    0.4416176   0.161238425     risk
## 10   0.5416796   0.112807957  decide
## 11   0.5109964   0.133626767 selfsuff
## 12   0.6676490  -0.244855780 dominant
## 13   0.6066864   0.171848896    stand
## 14   0.7627129  -0.040667202 leadact
## 15   0.4448064   0.089146147  individ
## 16   0.4504188   0.053207281 compete
## 17   0.4136498   0.136869589 ambitiou
```

Factor 2, the big ones

```
loadings %>% filter(abs(Factor2)>0.4)

##          Factor1    Factor2      trait
## 1  0.17789112 0.5537994   affect
## 2  0.15121266 0.4166622   loyal
## 3  0.02301456 0.5256654 sympathy
## 4  0.13476970 0.4242037 sensitiv
## 5  0.09111299 0.6101294 undstand
## 6  0.11350643 0.6272223  compass
## 7  0.06061755 0.5802714   soothe
## 8  0.11893011 0.4300698   happy
## 9  0.07956978 0.7191610    warm
## 10 0.05113807 0.7102763   tender
## 11 -0.01873224 0.7022768   gentle
```

Plotting the two factors

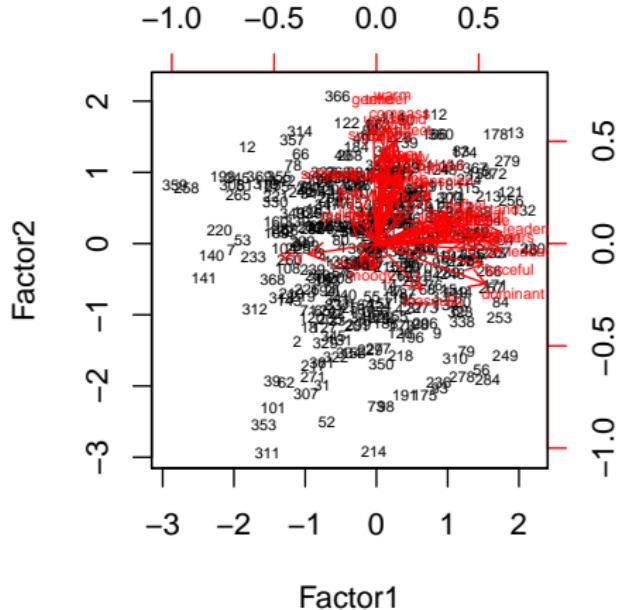
A bi-plot, this time with the variables reduced in size. Looking for unusual individuals.

Have to run factanal again to get factor scores for plotting.

```
bem.2a=factanal(bem[,-1],factors=2,scores="r")
biplot(bem.2a$scores,bem.2a$loadings,cex=c(0.5,0.5))
```

Numbers on plot are row numbers of bem data frame.

The (awful) biplot



Comments

- ▶ Variables mostly up ("feminine") and right ("masculine"), accomplished by rotation.
- ▶ Some unusual individuals: 311, 214 (low on factor 2), 366 (high on factor 2), 359, 258 (low on factor 1), 230 (high on factor 1).

Individual 366

```
bem %>% slice(366) %>% glimpse()

## # Observations: 1
## # Variables: 45
## # $ subno <int> 755
## # $ helpful <int> 7
## # $ reliant <int> 7
## # $ defbel <int> 5
## # $ yielding <int> 7
## # $ cheerful <int> 7
## # $ indpt <int> 7
## # $ athlet <int> 7
## # $ shy <int> 2
## # $ assert <int> 1
## # $ strpers <int> 3
## # $ forceful <int> 1
## # $ affect <int> 7
## # $ flatter <int> 9
## # $ loyal <int> 7
## # $ analyt <int> 7
## # $ feminine <int> 7
## # $ sympathy <int> 7
## # $ moody <int> 1
## # $ sensitiv <int> 7
## # $ undstand <int> 7
## # $ compass <int> 6
## # $ leaderab <int> 3
## # $ soothe <int> 7
## # $ risk <int> 7
## # $ decide <int> 7
## # $ selfsuff <int> 7
## # $ conscienc <int> 7
## # $ dominant <int> 1
## # $ masculin <int> 1
```

- ▶ High on factor 2, but hard to see which traits should have high scores (unless we remember).
- ▶ Idea: *tidy* original data frame to make easier to look things up.

Tidying original data

```
bem_tidy = bem %>% mutate(row=row_number()) %>%
  gather(trait,score,c(-subno,-row))
bem_tidy

## # A tibble: 16,236 x 4
##       subno     row trait   score
##       <int>   <int> <chr>   <int>
## 1       1       1 helpful     7
## 2       2       2 helpful     5
## 3       3       3 helpful     7
## 4       4       4 helpful     6
## 5       5       5 helpful     6
## 6       6       6 helpful     5
## 7       7       7 helpful     6
## 8       8       8 helpful     7
## 9      10      9 helpful     7
## 10     11     10 helpful     7
## # ... with 16,226 more rows
```

Recall data frame of loadings

```
loadings %>% slice(1:10)

##       Factor1      Factor2     trait
## 1   0.3137466  0.376484908  helpful
## 2   0.4532904  0.117140647  reliant
## 3   0.4336574  0.192602996  defbel
## 4  -0.1309965  0.337629288 yielding
## 5   0.1523718  0.370530549 cheerful
## 6   0.5212403  0.005870336    indpt
## 7   0.2670788  0.075542858  athlet
## 8  -0.4144579 -0.065372760      shy
## 9   0.6049588  0.033004846   assert
## 10  0.6569855  0.020777649 strpers
```

Want to add the factor scores for each trait to our tidy data frame `bem_tidy`. This is a left-join (over), matching on the column `trait` that is in both data frames (thus, the default):

Looking up loadings

```
bem_tidy = bem_tidy %>% left_join(loadings)

## Joining, by = "trait"

bem_tidy %>% sample_n(12)

## # A tibble: 12 x 6
##       subno     row trait      score  Factor1  Factor2
##       <int>   <int> <chr>    <int>    <dbl>    <dbl>
## 1     439     244 leadact     5  0.763  -0.0407
## 2     445     249 indpt      7  0.521   0.00587
## 3     687     352 sympathy    7  0.0230  0.526
## 4     291     169 risk       4  0.442   0.161
## 5     140      89 foullang    3 -0.00493  0.133
## 6     378     217 leaderab    4  0.765   0.0695
## 7     497     284 compass     5  0.114   0.627
## 8     247     141 gullible    5 -0.153   0.135
## 9     400     226 ambitiou    6  0.414   0.137
## 10    313     182 stand      5  0.607   0.172
## 11    469     266 loyal      6  0.151   0.417
## 12    306     177 softspok    6 -0.230   0.336
```

Individual 366, high on Factor 2

So now pick out the rows of the tidy data frame that belong to individual 366 (`row=366`) and for which the Factor2 score exceeds 0.4 in absolute value (our “big” from before):

```
bem_tidy %>% filter(row==366, abs(Factor2)>0.4)

## # A tibble: 11 x 6
##   subno   row trait    score Factor1 Factor2
##   <int> <int> <chr>    <int>    <dbl>    <dbl>
## 1 755    366 affect     7  0.178  0.554
## 2 755    366 loyal      7  0.151  0.417
## 3 755    366 sympathy   7  0.0230 0.526
## 4 755    366 sensitiv   7  0.135  0.424
## 5 755    366 undstand   7  0.0911 0.610
## 6 755    366 compass    6  0.114  0.627
## 7 755    366 soothe     7  0.0606 0.580
## 8 755    366 happy      7  0.119  0.430
## 9 755    366 warm       7  0.0796 0.719
## 10 755   366 tender     7  0.0511 0.710
## 11 755   366 gentle     7 -0.0187 0.702
```

As expected, high scorer on these.

Several individuals

Rows 311 and 214 were *low* on Factor 2, so their scores should be low.
Can we do them all at once?

```
bem_tidy %>% filter(row %in% c(366,311,214),  
                      abs(Factor2)>0.4)  
  
## # A tibble: 33 x 6  
##   subno   row trait    score Factor1 Factor2  
##   <int> <int> <chr>    <int>    <dbl>    <dbl>  
## 1 369    214 affect     1  0.178    0.554  
## 2 534    311 affect     5  0.178    0.554  
## 3 755    366 affect     7  0.178    0.554  
## 4 369    214 loyal      7  0.151    0.417  
## 5 534    311 loyal      4  0.151    0.417  
## 6 755    366 loyal      7  0.151    0.417  
## 7 369    214 sympathy   4  0.0230   0.526  
## 8 534    311 sympathy   4  0.0230   0.526  
## 9 755    366 sympathy   7  0.0230   0.526  
## 10 369   214 sensitiv   7  0.135    0.424  
## # ... with 23 more rows
```

Can we display each individual in own column?

Individual by column

Un-tidy, that is, spread:

```
bem_tidy %>% filter(row %in% c(366,311,214),  
                      abs(Factor2)>0.4) %>%  
  select(-subno,-Factor1,-Factor2) %>%  
  spread(row,score)  
  
## # A tibble: 11 x 4  
##   trait    `214` `311` `366`  
##   <chr>    <int> <int> <int>  
## 1 affect      1     5     7  
## 2 compass     5     4     6  
## 3 gentle      2     3     7  
## 4 happy       4     3     7  
## 5 loyal       7     4     7  
## 6 sensitiv    7     4     7  
## 7 soothe      3     4     7  
## 8 sympathy    4     4     7  
## 9 tender      3     4     7  
## 10 undstand   5     3     7  
## 11 warm       1     3     7
```

366 high, 311 middling, 214 (sometimes) low.

Individuals 230, 258, 359

These were high, low, low on factor 1. Adapt code:

```
bem_tidy %>% filter(row %in% c(359,258,230),abs(Factor1)>0.4) %>%  
  select(-subno,-Factor1,-Factor2) %>% spread(row,score)  
  
## # A tibble: 17 x 4  
##   trait    `230` `258` `359`  
##   <chr>    <int> <int> <int>  
## 1 ambitiou     7     2     4  
## 2 assert       7     3     1  
## 3 compete      6     2     1  
## 4 decide       7     1     2  
## 5 defbel       7     1     1  
## 6 dominant      7     1     1  
## 7 forceful      7     1     1  
## 8 individ       7     3     3  
## 9 indpt        7     7     1  
## 10 leadact      7     1     1  
## 11 leaderab     7     1     1  
## 12 reliant       7     4     1  
## 13 risk         7     5     7  
## 14 selfsuff     7     4     1  
## 15 shy          2     7     5  
## 16 stand        7     1     6  
## 17 strpers      7     1     3
```

Is 2 factors enough?

Suspect not:

```
bem.2$PVAL  
##      objective  
## 1.458183e-150
```

2 factors resoundingly rejected. Need more. Have to go all the way to 15 factors to not reject:

```
bem.15 = bem %>% select(-subno) %>%  
  factanal(factors=15)  
bem.15$PVAL  
## objective  
## 0.132617
```

Even then, only just over 50% of variability explained.

Let's have a look at the important things in those 15 factors.

Get factor loadings

into a data frame, as before:

```
loadings = as.data.frame(unclass(bem.15$loadings)) %>%
  mutate(trait=rownames(bem.15$loadings))
```

then show the highest few loadings on each factor.

Factor 1

```
loadings %>% arrange(desc(abs(Factor1))) %>%
  select(Factor1, trait) %>% slice(1:10)

##      Factor1    trait
## 1  0.8127595 compass
## 2  0.6756043 undstand
## 3  0.6611293 sympathy
## 4  0.6408327 sensitiv
## 5  0.5971006 soothe
## 6  0.3481290      warm
## 7  0.2797159      gentle
## 8  0.2788627      tender
## 9  0.2501505      helpful
## 10 0.2340594      conscienc
```

Compassionate, understanding, sympathetic, soothing: thoughtful of others.

Factor 2

```
loadings %>% arrange(desc(abs(Factor2))) %>%
  select(Factor2, trait) %>% slice(1:10)

##          Factor2      trait
## 1    0.7615492 strpers
## 2    0.7160312 forceful
## 3    0.6981500 assert
## 4    0.5041921 dominant
## 5    0.3929344 leaderab
## 6    0.3669560     stand
## 7    0.3507080 leadact
## 8   -0.3131682 softspok
## 9   -0.2866862       shy
## 10   0.2602525    analyt
```

Strong personality, forceful, assertive, dominant: getting ahead.

Factor 3

```
loadings %>% arrange(desc(abs(Factor3))) %>%
  select(Factor3, trait) %>% slice(1:10)

##          Factor3      trait
## 1    0.6697542   reliant
## 2    0.6475496 selfsuff
## 3    0.6204018   indpt
## 4    0.3899607   helpful
## 5   -0.3393605  gullible
## 6    0.3333813  individ
## 7    0.3319003  decide
## 8    0.3294806  conscienc
## 9    0.2877396 leaderab
## 10   0.2804170  defbel
```

Self-reliant, self-sufficient, independent: going it alone.

Factor 4

```
loadings %>% arrange(desc(abs(Factor4))) %>%
  select(Factor4, trait) %>% slice(1:10)

##      Factor4     trait
## 1  0.6956206   gentle
## 2  0.6920303   tender
## 3  0.5992467   warm
## 4  0.4465546   affect
## 5  0.3942568 softspok
## 6  0.2779793 lovchil
## 7  0.2444249 undstand
## 8  0.2442119   happy
## 9  0.2125905   loyal
## 10 0.2022861   soothe
```

Gentle, tender, warm (affectionate): caring for others.

Factor 5

```
loadings %>% arrange(desc(abs(Factor5))) %>%
  select(Factor5, trait) %>% slice(1:10)

##      Factor5     trait
## 1  0.6956846 compete
## 2  0.6743459 ambition
## 3  0.3453425      risk
## 4  0.3423456   individ
## 5  0.2808623    athlet
## 6  0.2695570 leaderab
## 7  0.2449656    decide
## 8  0.2064415 dominant
## 9  0.1928159 leadact
## 10 0.1854989 strpers
```

Ambitious, competitive (with a bit of risk-taking and individualism): Being the best.

Factor 6

```
loadings %>% arrange(desc(abs(Factor6))) %>%
  select(Factor6, trait) %>% slice(1:10)

##          Factor6      trait
## 1    0.8675651 leadact
## 2    0.6078869 leaderab
## 3    0.3378645 dominant
## 4    0.2014835 forceful
## 5   -0.1915632      shy
## 6    0.1789256      risk
## 7    0.1703440 masculin
## 8    0.1639190    decide
## 9    0.1594585   compete
## 10   0.1466037   athlet
```

Acts like a leader, leadership ability (with a bit of Dominant): Taking charge.

Factor 7

```
loadings %>% arrange(desc(abs(Factor7))) %>%
  select(Factor7, trait) %>% slice(1:10)

##          Factor7      trait
## 1    0.6698996    happy
## 2    0.6667105  cheerful
## 3   -0.5219125    moody
## 4    0.2191425   athlet
## 5    0.2126626     warm
## 6    0.1719953   gentle
## 7   -0.1640302  masculin
## 8    0.1601472  reliant
## 9    0.1472926  yielding
## 10   0.1410481 lovchil
```

Acts like a leader, leadership ability (with a bit of Dominant): Taking charge.

Factor 8

```
loadings %>% arrange(desc(abs(Factor8))) %>%
  select(Factor8, trait) %>% slice(1:10)

##          Factor8      trait
## 1    0.6296764   affect
## 2    0.5158355  flatter
## 3   -0.2512066 softspok
## 4    0.2214623     warm
## 5    0.1878549    tender
## 6    0.1846225 strpers
## 7   -0.1804838     shy
## 8    0.1801992  compete
## 9    0.1658105    loyal
## 10   0.1548617  helpful
```

Affectionate, flattering: Making others feel good.

Factor 9

```
loadings %>% arrange(desc(abs(Factor9))) %>%
  select(Factor9, trait) %>% slice(1:10)

##          Factor9      trait
## 1    0.8633171    stand
## 2    0.3403294   defbel
## 3    0.2446971  individ
## 4    0.1941110     risk
## 5   -0.1715481     shy
## 6    0.1710978  decide
## 7    0.1197126   assert
## 8    0.1157729  conscienc
## 9    0.1120308   analyt
## 10   -0.1115140  gullible
```

Taking a stand.

Factor 10

```
loadings %>% arrange(desc(abs(Factor10))) %>%
  select(Factor10, trait) %>% slice(1:10)

##          Factor10      trait
## 1    0.80751267 feminine
## 2   -0.26378513 masculin
## 3    0.24507184 softspok
## 4    0.23175597 conscien
## 5    0.20192035 selfsuff
## 6    0.17584233 yielding
## 7    0.14127067 gentle
## 8    0.11282028 flatter
## 9    0.10934531 decide
## 10   -0.09407978 lovchil
```

Feminine. (A little bit of not-masculine!)

Factor 11

```
loadings %>% arrange(desc(abs(Factor11))) %>%
  select(Factor11, trait) %>% slice(1:10)

##          Factor11      trait
## 1  0.91622589    loyal
## 2  0.18949077   affect
## 3  0.15883857 truthful
## 4  0.12464529  helpful
## 5  0.10440664  analyt
## 6  0.10076794   tender
## 7  0.09720457  lovchil
## 8  0.09635223 gullible
## 9  0.09350623  cheerful
## 10 0.08207596 conscient
```

Loyal.

Factor 12

```
loadings %>% arrange(desc(abs(Factor12))) %>%
  select(Factor12, trait) %>% slice(1:10)

##      Factor12    trait
## 1   0.6106933 childlik
## 2  -0.2845004 selfsuff
## 3  -0.2786751 conscien
## 4   0.2588843     moody
## 5   0.2013245       shy
## 6  -0.1669301    decide
## 7   0.1542031 masculin
## 8   0.1455526 dominant
## 9   0.1379163  compass
## 10 -0.1297408 leaderab
```

Childlike. (With a bit of moody, shy, not-self-sufficient, not-conscientious.)

Factor 13

```
loadings %>% arrange(desc(abs(Factor13))) %>%
  select(Factor13, trait) %>% slice(1:10)

##      Factor13    trait
## 1   0.5729242 truthful
## 2  -0.2776490 gullible
## 3   0.2631046    happy
## 4   0.1885152     warm
## 5  -0.1671924     shy
## 6   0.1646031    loyal
## 7  -0.1438127 yielding
## 8  -0.1302900    assert
## 9   0.1137074   defbel
## 10 -0.1105583  lovchil
```

Truthful. (With a bit of happy and not-gullible.)

Factor 14

```
loadings %>% arrange(desc(abs(Factor14))) %>%
  select(Factor14, trait) %>% slice(1:10)

##      Factor14    trait
## 1  0.4429926  decide
## 2  0.2369714 selfsuff
## 3  0.1945034 forceful
## 4 -0.1862756 softspok
## 5  0.1604175   risk
## 6 -0.1484606 strpers
## 7  0.1461972 dominant
## 8  0.1279456   happy
## 9  0.1154479 compass
## 10 0.1054078 masculin
```

Decisive. (With a bit of self-sufficient and not-soft-spoken.)

Factor 15

```
loadings %>% arrange(desc(abs(Factor15))) %>%
  select(Factor15, trait) %>% slice(1:10)

##          Factor15      trait
## 1   -0.3244092  compass
## 2    0.2471884  athlet
## 3    0.2292980 sensitiv
## 4    0.1986878     risk
## 5   -0.1638296  affect
## 6    0.1632164   moody
## 7   -0.1118135  individ
## 8    0.1100678     warm
## 9    0.1047347 cheerful
## 10   0.1012342  reliant
```

Not-compassionate, athletic, sensitive: A mixed bag. ("Cares about self"?)

Anything left out? Uniquenesses

```
data.frame(uniq=bem.15$uniquenesses) %>%
  rownames_to_column() %>%
  arrange(desc(uniq)) %>% slice(1:10)

##      rowname      uniq
## 1  foul lang 0.9136126
## 2   love child 0.8242992
## 3   analyt 0.8120934
## 4 yielding 0.7911748
## 5   masculin 0.7228739
## 6   athlet 0.7217327
## 7      shy 0.7033071
## 8  gullible 0.7000779
## 9   flatter 0.6625008
## 10  helpful 0.6516863
```

Uses foul language especially, also loves children and analytical. So could use even more factors.

Section 14

Confirmatory factor analysis

Confirmatory factor analysis

- ▶ Exploratory: what do data suggest as hidden underlying factors (in terms of variables observed)?
- ▶ Confirmatory: have *theory* about how underlying factors depend on observed variables; test whether theory supported by data:
 - ▶ does theory provide *some* explanation (better than nothing)
 - ▶ can we do better?
- ▶ Also can compare two theories about factors: is more complicated one significantly better than simpler one?

Children and tests again

- ▶ Previously had this correlation matrix of test scores (based on 145 children):

km

```
##      para   sent   word   add   dots
## [1,] 1.000 0.722 0.714 0.203 0.095
## [2,] 0.722 1.000 0.685 0.246 0.181
## [3,] 0.714 0.685 1.000 0.170 0.113
## [4,] 0.203 0.246 0.170 1.000 0.585
## [5,] 0.095 0.181 0.113 0.585 1.000
```

- ▶ Will use package `lavaan` for confirmatory analysis.
- ▶ Can use actual data or correlation matrix.
- ▶ Latter (a bit) more work, as we see.

Two or three steps

1. Make sure correlation matrix (if needed) is handy.
2. Specify factor model (from theory)
3. Fit factor model: does it fit acceptably?

Specifying a factor model

- ▶ Jargon: thing you cannot observe called **latent variable**.
- ▶ Thing you *can* observe called **manifest variable**.
- ▶ Model predicts latent variables from manifest variables.
- ▶ Model with one factor including all the tests:

```
test.model.1='ability=~para+sent+word+add+dots'
```

- ▶ and a model that we really believe, that there are two factors, a verbal and a mathematical:

```
test.model.2='  
    verbal=~para+sent+word  
    math=~add+dots'
```

- ▶ Note the format: really all one line between single quotes, but putting it on several lines makes the layout clearer.
- ▶ Also note special notation $=~$ for “this latent variable depends on these observed variables”.

Fitting a 1-factor model

- ▶ Need to specify model, correlation matrix, n like this:

```
fit1=cfa(test.model.1, sample.cov=km,  
         sample.nobs=145)
```

- ▶ Has `summary`, or briefer version like this:

```
fit1  
  
## lavaan 0.6-3 ended normally after 16 iterations  
##  
## Optimization method NLMINB  
## Number of free parameters 10  
##  
## Number of observations 145  
##  
## Estimator ML  
## Model Fit Test Statistic 59.886  
## Degrees of freedom 5  
## P-value (Chi-square) 0.000
```

- ▶ Test of fit: null “model fits” *rejected*. We can do better.

Two-factor model

```
fit2=cfa(test.model.2,sample.cov=km,sample.nobs=145)
fit2

## lavaan 0.6-3 ended normally after 25 iterations
##
## Optimization method                           NLMINB
## Number of free parameters                   11
##
## Number of observations                      145
##
## Estimator                                 ML
## Model Fit Test Statistic                  2.951
## Degrees of freedom                         4
## P-value (Chi-square)                      0.566
```

- ▶ This fits OK: 2-factor model supported by the data.
- ▶ 1-factor model did not fit. We really need 2 factors.
- ▶ Same conclusion as from factanal earlier.

Comparing models

- ▶ Use anova as if this were a regression:

```
anova(fit1,fit2)

## Chi Square Difference Test
##
##      Df     AIC     BIC   Chisq Chisq diff Df diff Pr(>Chisq)
## fit2  4 1776.7 1809.4  2.9509
## fit1  5 1831.6 1861.4 59.8862      56.935       1 4.504e-14
##
## fit2
## fit1 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- ▶ 2-factor model fits significantly better than 1-factor.
- ▶ No surprise!

Track and field data, yet again

- ▶ cfa works easier on actual data, such as the running records:

```
track %>% print(n=6)  
## Error in eval(lhs, parent, parent): object 'track' not found
```

- ▶ Specify factor model. Factors seemed to be “sprinting” (up to 800m) and “distance running” (beyond):

```
track.model='  
sprint=~m100+m200+m400+m800  
distance=~m1500+m5000+m10000+marathon'
```

Fit and examine the model

- ▶ Fit the model. The observed variables are on different scales, so we should standardize them first via std.ov:

```
track.1 = track %>% select(-country) %>%
  cfa(track.model, data=., std.ov=T)

## Error in eval(lhs, parent, parent): object 'track' not found

track.1

## Error in eval(expr, envir, enclos): object 'track.1' not found
```

- ▶ This fits badly. Can we do better?
- ▶ Idea: move middle distance races (800m, 1500m) into a third factor.

Factor model 2

- ▶ Define factor model:

```
track.model.2='  
sprint=~m100+m200+m400  
middle=~m800+m1500  
distance=~m5000+m10000+marathon'
```

- ▶ Fit and examine:

```
track.2 = track %>% select(-country) %>%  
  cfa(track.model.2,data=.,std.ov=T)  
  
## Error in eval(lhs, parent, parent): object 'track' not found  
  
track.2  
  
## Error in eval(expr, envir, enclos): object 'track.2' not found
```

- ▶ Fits marginally better, though still badly.

Comparing the two models

- ▶ Second model doesn't fit well, but is it better than first?

```
anova(track.1,track.2)
```

```
## Error in anova(track.1, track.2): object 'track.1' not found
```

- ▶ Oh yes, a lot better.

Section 15

Multiway frequency tables

Packages

```
library(tidyverse)

## -- Attaching packages ----- tidyverse 1.2.1 --
## v ggplot2 3.1.0     v purrrr  0.2.5
## v tibble  1.4.2     v dplyr    0.7.8
## v tidyrr   0.8.1    v stringr  1.3.1
## v readr    1.1.1    vforcats 0.3.0

## -- Conflicts ----- tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()   masks stats::lag()
```

Multi-way frequency analysis

- ▶ A study of gender and eyewear-wearing finds the following frequencies:

Gender	Contacts	Glasses	None
Female	121	32	129
Male	42	37	85

- ▶ Is there association between eyewear and gender?
- ▶ Normally answer this with chisquare test (based on observed and expected frequencies from null hypothesis of no association).
- ▶ Two categorical variables and a frequency.
- ▶ We assess in way that generalizes to more categorical variables.

The data file

gender	contacts	glasses	none
female	121	32	129
male	42	37	85

- ▶ This is *not tidy!*
- ▶ Two variables are gender and eyewear, and those numbers all frequencies.

```
my_url = "http://www.utsc.utoronto.ca/~butler/d29/eyewear.txt"
eyewear = read_delim(my_url, " ")
eyewear

## # A tibble: 2 x 4
##   gender contacts glasses none
##   <chr>     <int>    <int> <int>
## 1 female      121       32    129
## 2 male        42        37     85
```

Tidying the data

```
eyes = eyewear %>%
  gather(eyewear,frequency,contacts:none)
eyes

## # A tibble: 6 x 3
##   gender eyewear frequency
##   <chr>  <chr>      <int>
## 1 female contacts     121
## 2 male   contacts     42
## 3 female glasses      32
## 4 male   glasses      37
## 5 female none         129
## 6 male   none         85

xt=xtabs(frequency~gender+eyewear,data=eyes)
xt

##           eyewear
## gender   contacts glasses none
##   female      121      32    129
##   male        42       37     85
```

Modelling

- ▶ Last table on previous page is “reconstituted” contingency table, for checking.
- ▶ Predict frequency from other factors and combos. `glm` with `poisson` family.

```
eyes.1=glm(frequency~gender*eyewear,data=eyes,  
           family="poisson")
```

- ▶ Called **log-linear model**.

What can we get rid of?

```
drop1(eyes.1,test="Chisq")

## Single term deletions
##
## Model:
## frequency ~ gender * eyewear
##           Df Deviance    AIC      LRT  Pr(>Chi)
## <none>        0.000 47.958
## gender:eyewear 2   17.829 61.787 17.829 0.0001345 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Conclusions

- ▶ drop1 says what we can remove at this step. Significant = must stay.
- ▶ Cannot remove anything.
- ▶ Frequency depends on gender-wear *combination*, cannot be simplified further.
- ▶ Gender and eyewear are *associated*.
- ▶ Stop here.

prop.table

Original table:

```
xt  
  
##           eyewear  
## gender   contacts glasses none  
##   female      121       32   129  
##   male        42       37   85
```

Calculate eg. row proportions like this:

```
prop.table(xt,margin=1)  
  
##           eyewear  
## gender   contacts   glasses      none  
##   female 0.4290780 0.1134752 0.4574468  
##   male    0.2560976 0.2256098 0.5182927
```

- ▶ margin says what to make add to 1.
- ▶ More females wear contacts and more males wear glasses.

No association

- ▶ Suppose table had been as shown below:

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/eyewear2.txt"
eyewear2=read_table(my_url)
eyes2 = eyewear2 %>% gather(eyewear,frequency,contacts:none)
xt2=xtabs(frequency~gender+eyewear,data=eyes2)
xt2

##           eyewear
## gender   contacts glasses none
##   female      150       30    120
##   male        75       16     62

prop.table(xt2,margin=1)

##           eyewear
## gender   contacts   glasses      none
##   female 0.5000000 0.1000000 0.4000000
##   male   0.4901961 0.1045752 0.4052288
```

- ▶ Females and males wear contacts and glasses in same proportions (though more females and more contact-wearers). No *association* between gender and eyewear.

Analysis for revised data

```
eyes.2=glm(frequency~gender*eyewear,data=eyes2,
  family="poisson")
drop1(eyes.2,test="Chisq")

## Single term deletions
##
## Model:
## frequency ~ gender * eyewear
##           Df Deviance    AIC      LRT Pr(>Chi)
## <none>        0.000000 47.467
## gender:eyewear 2  0.047323 43.515 0.047323  0.9766
```

No longer any association. Take out interaction.

No interaction

```
eyes.3=update(eyes.2,.~.-gender:eyewear)
drop1(eyes.3,test="Chisq")

## Single term deletions
##
## Model:
## frequency ~ gender + eyewear
##          Df Deviance      AIC      LRT  Pr(>Chi)
## <none>     0.047  43.515
## gender    1   48.624  90.091  48.577 3.176e-12 ***
## eyewear   2  138.130 177.598 138.083 < 2.2e-16 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- ▶ More females (gender effect)
- ▶ more contact-wearers (eyewear effect)
- ▶ no association (no interaction).

Chest pain, being overweight and being a smoker

- ▶ In a hospital emergency department, 176 subjects who attended for acute chest pain took part in a study.
- ▶ Each subject had a normal or abnormal electrocardiogram reading (ECG), were overweight (as judged by BMI) or not, and were a smoker or not.
- ▶ How are these three variables related, or not?

The data

In modelling-friendly format:

ecg	bmi	smoke	count
abnormal	overweight	yes	47
abnormal	overweight	no	10
abnormal	normalweight	yes	8
abnormal	normalweight	no	6
normal	overweight	yes	25
normal	overweight	no	15
normal	normalweight	yes	35
normal	normalweight	no	30

First step

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/ecg.txt"
chest=read_delim(my_url, " ")
chest.1=glm(count~ecg*bmi*smoke,data=chest,
            family="poisson")
drop1(chest.1,test="Chisq")

## Single term deletions
##
## Model:
## count ~ ecg * bmi * smoke
##           Df Deviance     AIC      LRT Pr(>Chi)
## <none>        0.0000 53.707
## ecg:bmi:smoke 1    1.3885 53.096 1.3885   0.2387
```

That 3-way interaction comes out.

Removing the 3-way interaction

```
chest.2=update(chest.1,.~.-ecg:bmi:smoke)
drop1(chest.2,test="Chisq")

## Single term deletions
##
## Model:
## count ~ ecg + bmi + smoke + ecg:bmi + ecg:smoke + bmi:smoke
##          Df Deviance    AIC      LRT  Pr(>Chi)
## <none>     1.3885 53.096
## ecg:bmi   1  29.0195 78.727 27.6310 1.468e-07 ***
## ecg:smoke 1   4.8935 54.601  3.5050  0.06119 .
## bmi:smoke 1   4.4689 54.176  3.0803  0.07924 .
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

At $\alpha = 0.05$, `bmi:smoke` comes out.

Removing bmi:smoke

```
chest.3=update(chest.2,.~.-bmi:smoke)
drop1(chest.3,test="Chisq")

## Single term deletions
##
## Model:
## count ~ ecg + bmi + smoke + ecg:bmi + ecg:smoke
##           Df Deviance    AIC      LRT  Pr(>Chi)
## <none>        4.469 54.176
## ecg:bmi     1   36.562 84.270 32.094 1.469e-08 ***
## ecg:smoke   1   12.436 60.144  7.968  0.004762 **
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

ecg:smoke has become significant. So we have to stop.

Understanding the final model

- ▶ Thinking of ecg as “response” that might depend on anything else.
- ▶ What is associated with ecg? Both `bmi` on its own and `smoke` on its own, but *not* the combination of both.
- ▶ `ecg:bmi` table:

```
xtabs(count~ecg+bmi,data=chest)
```

		bmi	
		normal	weight
##	ecg	normal	overweight
	abnormal	14	57
##	normal	65	40

- ▶ Most normal weight people have a normal ECG, but a majority of overweight people have an *abnormal* ECG. That is, knowing about BMI says something about likely ECG.

ecg:smoke

- ▶ `ecg:smoke` table:

```
xtabs(count~ecg+smoke, data=chest)
```

```
##           smoke
##   ecg      no  yes
##   abnormal 16  55
##   normal    45  60
```

- ▶ Most nonsmokers have a normal ECG, but smokers are about 50–50 normal and abnormal ECG.
- ▶ Don't look at `smoke:bmi` table since not significant.

Simpson's paradox: the airlines example

Airport	Alaska Airlines		America West	
	On time	Delayed	On time	Delayed
Los Angeles	497	62	694	117
Phoenix	221	12	4840	415
San Diego	212	20	383	65
San Francisco	503	102	320	129
Seattle	1841	305	201	61
Total	3274	501	6438	787

Use status as variable name for “on time/delayed”.

- ▶ Alaska: 13.3% flights delayed ($501/(3274 + 501)$).
- ▶ America West: 10.9% ($787/(6438 + 787)$).
- ▶ America West more punctual, right?

Arranging the data

- ▶ Can only have single thing in columns, so we have to construct column names like this:

airport	aa_ontime	aa_delayed	aw_ontime	aw_delayed
LosAngeles	497	62	694	117
Phoenix	221	12	4840	415
SanDiego	212	20	383	65
SanFrancisco	503	102	320	129
Seattle	1841	305	201	61

- ▶ Some tidying gets us the right layout, with frequencies all in one column and the airline and delayed/on time status separated out:

```
my_url = "http://www.utsc.utoronto.ca/~butler/d29/airlines.txt"
airlines = read_table2(my_url)
punctual = airlines %>%
  gather(line.status, freq, contains("_")) %>%
  separate(line.status, c("airline", "status"))
```

The data frame punctual

```
## # A tibble: 20 x 4
##   airport      airline status   freq
##   <chr>        <chr>   <chr>   <int>
## 1 LosAngeles    aa     ontime    497
## 2 Phoenix       aa     ontime    221
## 3 SanDiego      aa     ontime    212
## 4 SanFrancisco  aa     ontime    503
## 5 Seattle        aa     ontime   1841
## 6 LosAngeles    aa     delayed   62
## 7 Phoenix        aa     delayed   12
## 8 SanDiego       aa     delayed   20
## 9 SanFrancisco  aa     delayed  102
## 10 Seattle       aa     delayed  305
## 11 LosAngeles   aw     ontime   694
## 12 Phoenix       aw     ontime  4840
## 13 SanDiego      aw     ontime   383
## 14 SanFrancisco aw     ontime   320
## 15 Seattle        aw     ontime   201
## 16 LosAngeles   aw     delayed  117
## 17 Phoenix        aw     delayed  415
## 18 SanDiego       aw     delayed  65
## 19 SanFrancisco aw     delayed 129
## 20 Seattle        aw     delayed  61
```

Proportions delayed by airline

- ▶ Two-step process: get appropriate subtable:

```
xt=xtabs(freq~airline+status, data=punctual)
xt

##           status
## airline delayed onttime
##      aa      501    3274
##      aw      787    6438
```

- ▶ and then calculate appropriate proportions:

```
prop.table(xt, margin=1)

##           status
## airline   delayed   onttime
##      aa 0.1327152 0.8672848
##      aw 0.1089273 0.8910727
```

- ▶ More of Alaska Airlines' flights delayed (13.3% vs. 10.9%).

Proportion delayed by airport, for each airline

```
xt=xtabs(freq~airline+status+airport,data=punctual)
xp=prop.table(xt,margin=c(1,3))
ftable(xp,row.vars=c("airport","airline"),
col.vars="status")

##                                     status      delayed      ontime
## airport      airline
## LosAngeles   aa          0.11091234 0.88908766
##                  aw          0.14426634 0.85573366
## Phoenix       aa          0.05150215 0.94849785
##                  aw          0.07897241 0.92102759
## SanDiego     aa          0.08620690 0.91379310
##                  aw          0.14508929 0.85491071
## SanFrancisco aa          0.16859504 0.83140496
##                  aw          0.28730512 0.71269488
## Seattle       aa          0.14212488 0.85787512
##                  aw          0.23282443 0.76717557
```

Simpson's Paradox

Airport	Alaska	America West
Los Angeles	11.4	14.4
Phoenix	5.2	7.9
San Diego	8.6	14.5
San Francisco	16.9	28.7
Seattle	14.2	23.2
Total	13.3	10.9

- ▶ America West more punctual overall,
- ▶ but worse at *every single* airport!
- ▶ How is that possible?
- ▶ Log-linear analysis sheds some light.

Model 1 and output

```
punctual.1=glm(freq~airport*airline*status,
  data=punctual,family="poisson")
drop1(punctual.1,test="Chisq")

## Single term deletions
##
## Model:
## freq ~ airport * airline * status
##                               Df Deviance     AIC      LRT Pr(>Chi)
## <none>                      0.0000 183.44
## airport:airline:status    4   3.2166 178.65 3.2166  0.5223
```

Remove 3-way interaction

```
punctual.2=update(punctual.1,~-airport:airline:status)
drop1(punctual.2,test="Chisq")

## Single term deletions
##
## Model:
## freq ~ airport + airline + status + airport:airline + airport:status +
##      airline:status
##          Df Deviance     AIC      LRT  Pr(>Chi)
## <none>           3.2  178.7
## airport:airline  4   6432.5 6599.9 6429.2 < 2.2e-16 ***
## airport:status   4    240.1  407.5  236.9 < 2.2e-16 ***
## airline:status   1     45.5   218.9   42.2 8.038e-11 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Stop here.

Understanding the significance

- ▶ airline:status:

```
xt=xtabs(freq~airline+status,data=punctual)
prop.table(xt, margin=1)

##          status
## airline    delayed      ontime
##       aa 0.1327152 0.8672848
##       aw 0.1089273 0.8910727
```

- ▶ More of Alaska Airlines' flights delayed overall.
- ▶ Saw this before.

Understanding the significance (2)

- ▶ airport:status:

```
xt=xtabs(freq~airport+status, data=punctual)
prop.table(xt, margin=1)

##           status
## airport      delayed      ontime
##   LosAngeles 0.13065693 0.86934307
##   Phoenix     0.07780612 0.92219388
##   SanDiego    0.12500000 0.87500000
##   SanFrancisco 0.21916509 0.78083491
##   Seattle     0.15199336 0.84800664
```

- ▶ Flights into San Francisco (and maybe Seattle) are often late, and flights into Phoenix are usually on time.
- ▶ Considerable variation among airports.

Understanding the significance (3)

- ▶ airport:airline:

```
xt=xtabs(freq~airport+airline,data=punctual)
prop.table(xt,margin=2)

##           airline
## airport          aa        aw
## LosAngeles  0.14807947 0.11224913
## Phoenix     0.06172185 0.72733564
## SanDiego    0.06145695 0.06200692
## SanFrancisco 0.16026490 0.06214533
## Seattle     0.56847682 0.03626298
```

- ▶ What fraction of each airline's flights are to each airport.
- ▶ Most of Alaska Airlines' flights to Seattle and San Francisco.
- ▶ Most of America West's flights to Phoenix.

The resolution

- ▶ Most of America West's flights to Phoenix, where it is easy to be on time.
- ▶ Most of Alaska Airlines' flights to San Francisco and Seattle, where it is difficult to be on time.
- ▶ Overall comparison looks bad for Alaska because of this.
- ▶ But, *comparing like with like*, if you compare each airline's performance *to the same airport*, Alaska does better.
- ▶ Aggregating over the very different airports was a (big) mistake: that was the cause of the Simpson's paradox.
- ▶ Alaska Airlines is *more punctual* when you do the proper comparison.

Ovarian cancer: a four-way table

- ▶ Retrospective study of ovarian cancer done in 1973.
- ▶ Information about 299 women operated on for ovarian cancer 10 years previously.
- ▶ Recorded:
 - ▶ stage of cancer (early or advanced)
 - ▶ type of operation (radical or limited)
 - ▶ X-ray treatment received (yes or no)
 - ▶ 10-year survival (yes or no)
- ▶ Survival looks like response (suggests logistic regression).
- ▶ Log-linear model finds any associations at all.

The data

after tidying:

stage	operation	xray	survival	freq
early	radical	no	no	10
early	radical	no	yes	41
early	radical	yes	no	17
early	radical	yes	yes	64
early	limited	no	no	1
early	limited	no	yes	13
early	limited	yes	no	3
early	limited	yes	yes	9
advanced	radical	no	no	38
advanced	radical	no	yes	6
advanced	radical	yes	no	64
advanced	radical	yes	yes	11
advanced	limited	no	no	3
advanced	limited	no	yes	1
advanced	limited	yes	no	13
advanced	limited	yes	yes	5

Stage 1

hopefully looking familiar by now:

```
my_url="http://www.utsc.utoronto.ca/~butler/d29/cancer.txt"
cancer=read_delim(my_url, " ")
cancer %>% print(n=6)

## # A tibble: 16 x 5
##   stage operation xray survival freq
##   <chr>    <chr>   <chr>   <chr>   <int>
## 1 early    radical  no     no      10
## 2 early    radical  no     yes     41
## 3 early    radical  yes    no      17
## 4 early    radical  yes    yes     64
## 5 early    limited  no     no      1
## 6 early    limited  no     yes     13
## # ... with 10 more rows

cancer.1=glm(freq~stage*operation*xray*survival,
             data=cancer,family="poisson")
```

Output 1

See what we can remove:

```
drop1(cancer.1,test="Chisq")

## Single term deletions
##
## Model:
## freq ~ stage * operation * xray * survival
##                               Df Deviance     AIC      LRT
## <none>                      0.00000 98.130
## stage:operation:xray:survival 1  0.60266 96.732 0.60266
##                               Pr(>Chi)
## <none>
## stage:operation:xray:survival 0.4376
```

Non-significant interaction can come out.

Stage 2

```
cancer.2=update(cancer.1,~.  
-stage:operation:xray:survival)  
drop1(cancer.2,test="Chisq")  
  
## Single term deletions  
##  
## Model:  
## freq ~ stage + operation + xray + survival + stage:operation +  
##     stage:xray + operation:xray + stage:survival + operation:survival +  
##     xray:survival + stage:operation:xray + stage:operation:survival +  
##     stage:xray:survival + operation:xray:survival  
##  
##             Df Deviance      AIC      LRT  
## <none>          0.60266 96.732  
## stage:operation:xray    1  2.35759 96.487 1.75493  
## stage:operation:survival 1  1.17730 95.307 0.57465  
## stage:xray:survival     1  0.95577 95.085 0.35311  
## operation:xray:survival 1  1.23378 95.363 0.63113  
##  
##             Pr(>Chi)  
## <none>  
## stage:operation:xray      0.1853  
## stage:operation:survival   0.4484  
## stage:xray:survival       0.5524  
## operation:xray:survival   0.4269
```

Least significant term is stage:xray:survival: remove.

Take out stage:xray:survival

```
cancer.3=update(cancer.2, .~.-stage:xray:survival)
drop1(cancer.3,test="Chisq")

## Single term deletions
##
## Model:
## freq ~ stage + operation + xray + survival + stage:operation +
##      stage:xray + operation:xray + stage:survival + operation:survival +
##      xray:survival + stage:operation:xray + stage:operation:survival +
##      operation:xray:survival
##                               Df Deviance     AIC      LRT
## <none>                  0.95577 95.085
## stage:operation:xray     1  3.08666 95.216  2.13089
## stage:operation:survival 1  1.56605 93.696  0.61029
## operation:xray:survival 1  1.55124 93.681  0.59547
##                               Pr(>Chi)
## <none>
## stage:operation:xray     0.1444
## stage:operation:survival 0.4347
## operation:xray:survival 0.4403
```

operation:xray:survival comes out next.

Remove operation:xray:survival

```
cancer.4=update(cancer.3,.~.-operation:xray:survival)
drop1(cancer.4,test="Chisq")

## Single term deletions
##
## Model:
## freq ~ stage + operation + xray + survival + stage:operation +
##       stage:xray + operation:xray + stage:survival + operation:survival +
##       xray:survival + stage:operation:xray + stage:operation:survival
##                                     Df Deviance    AIC    LRT Pr(>Chi)
## <none>                      1.5512 93.681
## xray:survival      1  1.6977 91.827 0.1464  0.70196
## stage:operation:xray 1  6.8420 96.972 5.2907  0.02144
## stage:operation:survival 1  1.9311 92.061 0.3799  0.53768
##
## <none>
## xray:survival
## stage:operation:xray      *
## stage:operation:survival
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Comments

- ▶ stage:operation:xray has now become significant, so won't remove that.
- ▶ Shows value of removing terms one at a time.
- ▶ There are no higher-order interactions containing both xray and survival, so now we get to test (and remove) xray:survival.

Remove xray:survival

```
cancer.5=update(cancer.4,.~.-xray:survival)
drop1(cancer.5,test="Chisq")

## Single term deletions
##
## Model:
## freq ~ stage + operation + xray + survival + stage:operation +
##       stage:xray + operation:xray + stage:survival + operation:survival +
##       stage:operation:xray + stage:operation:survival
##                                     Df Deviance    AIC      LRT Pr(>Chi)
## <none>                  1.6977 91.827
## stage:operation:xray     1   6.9277 95.057 5.2300   0.0222
## stage:operation:survival 1   2.0242 90.154 0.3265   0.5677
##
## <none>
## stage:operation:xray    *
## stage:operation:survival
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Remove stage:operation:survival

```
cancer.6=update(cancer.5,.~.-stage:operation:survival)
drop1(cancer.6,test="Chisq")

## Single term deletions
##
## Model:
## freq ~ stage + operation + xray + survival + stage:operation +
##      stage:xray + operation:xray + stage:survival + operation:survival +
##      stage:operation:xray
##                               Df Deviance      AIC      LRT Pr(>Chi)
## <none>                  2.024  90.154
## stage:survival          1   135.198 221.327 133.173  <2e-16
## operation:survival      1     4.116  90.245   2.092   0.1481
## stage:operation:xray    1     7.254  93.384   5.230   0.0222
##
## <none>
## stage:survival      ***
## operation:survival
## stage:operation:xray *
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Last step?

Remove operation:survival.

```
cancer.7=update(cancer.6,.~.-operation:survival)
drop1(cancer.7,test="Chisq")

## Single term deletions
##
## Model:
## freq ~ stage + operation + xray + survival + stage:operation +
##       stage:xray + operation:xray + stage:survival + stage:operation:xray
##                               Df Deviance      AIC      LRT Pr(>Chi)
## <none>                  4.116  90.245
## stage:survival          1   136.729 220.859 132.61    <2e-16
## stage:operation:xray   1     9.346  93.475   5.23    0.0222
##
## <none>
## stage:survival        ***
## stage:operation:xray  *
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Finally done!

Conclusions

- ▶ What matters is things associated with survival (survival is “response”).
- ▶ Only significant such term is `stage:survival`:

```
xt=xtabs(freq~stage+survival, data=cancer)
prop.table(xt, margin=1)

##                 survival
## stage             no      yes
##   advanced  0.8368794 0.1631206
##   early     0.1962025 0.8037975
```

- ▶ Most people in early stage of cancer survived, and most people in advanced stage did not survive.
- ▶ This true *regardless* of type of operation or whether or not X-ray treatment was received. These things have no impact on survival.

What about that other interaction?

```
xt=xtabs(freq~operation+xray+stage,data=cancer)
ftable(prop.table(xt,margin=3))

##                                stage    advanced      early
## operation xray
## limited    no            0.02836879 0.08860759
##                  yes          0.12765957 0.07594937
## radical    no            0.31205674 0.32278481
##                  yes          0.53191489 0.51265823
```

- ▶ Out of the people at each stage of cancer (since `margin=3` and `stage` was listed 3rd).
- ▶ The association is between `stage` and `xray` *only for those who had the limited operation.*
- ▶ For those who had the radical operation, there was no association between `stage` and `xray`.
- ▶ This is of less interest than associations with `survival`.

General procedure

- ▶ Start with “complete model” including all possible interactions.
- ▶ `drop1` gives highest-order interaction(s) remaining, remove least non-significant.
- ▶ Repeat as necessary until everything significant.
- ▶ Look at subtables of significant interactions.
- ▶ Main effects not usually very interesting.
- ▶ Interactions with “response” usually of most interest: show association with response.