Response to reviewer 1

June 6, 2025

Paper entitled "Averaged equations for disperse two-phase flow with interfacial transport. IJMF"

This manuscript derives the averaged equations for disperse multiphase flows. Authors are certainly aware that this subject has been studied many times before by the cited literature. This manuscript does have its uniqueness, especially in its treatments of the interface terms and the moments. However, the authors do not provide an example to show how their method is more advantageous or reveals something new. As currently written, the derivations are too abstract. To publish this manuscript, I suggest the author to come up an example in which the closures can be easily obtained by their methods. For instance, authors can take advantage of their treatment of the interfaces study thermal capillary motion of bubbles to show the advantage of the method, or study rotating or deformable particles to show some unique or non-Newtonian behavior..

We thank the Referee for his /her positive feedback on our manuscript. Before providing a point-by-point response to the Referee's comments, we just wish to mention the main changes introduced in the paper to address the concerns raised by the other referees. In particular we have added new paragraphs in the introduction to outline the new contributions of the present paper and rewritten the conclusion.

Following the Referee's suggestion, we have added two new sections (Sections V and VI) to the revised version of the article.

Section V presents a set of averaged equations, including higher-order moments, to describe the dynamics of droplets immersed in a Newtonian fluid. Particular emphasis is placed on the choice of the stress decomposition and on the equation governing the second-order mass moment and first moment of momentum equation, which plays a key role in describing the deformation of the dispersed phase.

In Section VI, we derive the closure terms in the dilute, viscous-dominated regime. Specifically, we quantify the influence of surface tension gradient on the forces and moments acting on the droplets. Additionally, we discuss several covariance closure terms that emerge in the averaged equations. Finally we demonstrate how the leading order deformation of the droplets can be obtained thanks to the second-order mass moment and first moment of momentum equation.

In light of these modifications, the title has been changed in "Averaged equations for disperse two-phase flow with interfacial properties and their closures for dilute suspension of droplets".

Following are some technical suggestions.

1. Angular brackets are used for both inner product and averaged quantities. My suggestion is to use round brackets for the inner product to make the manuscript easier to read. Done.

2. In the first paragraph of sec. 2.2: use "I" to denote the identity tensor, instead of delta to clearly distinguish it from the Dirac delta function. We chose not to use **I** for the identity tensor because it is already used to denote the inertia tensor of the droplets in Section V. Additionally, the Dirac delta function is not typeset in bold, whereas the identity tensor is, which helps maintain a clear distinction between the two.

3. After eq. (2.15) "... with $\phi_I =< \delta_\Gamma >$ the probability of finding the interface at the point x at time t". The probability of finding an interface at a given location is always zero. This quantity is not the probability. It is the interface area per unit volume, also called the specific area.
   We thank the referee for his/her suggestion. This has been corrected

4. Please note that although $\nabla_{||} n$ is well defined but not ∇n; therefore, should not be used
   Although ∇n is not meaningful in the sense of function-since $\partial n / \partial n$ is not defined-we argue that it can still be interpreted in the sense of distributions, as shown in Appendix A and in the work of Orlando et al. (2023). Moreover, the use of ∇n is common in the literature (see Nadim, 1996; Lhuillier, 2003; Orlando et al., 2023). Since we believe this notation does not cause significant confusion, we have decided to retain it.