Theoretical calculation of the droplet velocity varience in mono disperse buoyant emulsions for low inertia and dilute regime.

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Abstract

In this study we derive an analytical expression for the particle phase velocity variance tensor. This derivation is restricted to the dilute regime where the dispersed phase volume fraction labeled ϕ is vanishingly small and for particle Reynolds number $Re \ll 1$. In this context we derive an analytical formula for the particle velocity variance generated by translating bubbles or droplets inside a Newtonian fluid under the effect of gravity. To by-pass the integral convergence difficulties generated due to the $\mathcal{O}(r^{-1})$ decay of the disturbance velocity field we make use of the Nearest-Particle-Statistical (NPS) as it is introduced in Zhang et al. (2023).

1 The original problem with classic pair stats

we should have a factor of $1/N$ or something in front of the int

2 Strategy

As in the previous chapter we reformulate teh ensemble average of the partcile phase velocity variance,

$$\langle \delta_p \mathbf{u}'_\alpha \mathbf{u}'_\alpha \rangle \tag{2.1}$$

in terms of the nearest particle statistics conditioned quantities. We use (2.15) of Zhang (2021), namely,

$$\int_{\mathbb{R}^3} \sum_{j \neq i} \delta(\mathbf{x}_j[\mathscr{F}, t] - \mathbf{y}) h_{ij}[t, \mathscr{F}] d\mathbf{y} = 1 \tag{2.2}$$

And introduce the relation,

$$\langle \delta_p \mathbf{u}'_\alpha \mathbf{u}'_\alpha \rangle = n_p[\mathbf{x}, t] \int_{\mathbb{R}^3} (\mathbf{v}_p^{\text{nst}} \mathbf{v}_p^{\text{nst}})[\mathbf{x}, \mathbf{y}, t] P_{\text{nst}}[\mathbf{y}|\mathbf{x}, t] d\mathbf{y} + \int_{\mathbb{R}^3} \left\langle \delta_p \sum_{j \neq i} h_{ij} (\mathbf{u}''_\alpha - \mathbf{u}_p^{\text{nst}})(\mathbf{u}''_\alpha - \mathbf{u}_p^{\text{nst}}) \right\rangle d\mathbf{y} \tag{2.3}$$

where we recall that P_{nst} is the probability of finding the nearest neighbor at the position of $\mathbf{r}$ knowing that the particle is present at $\mathbf{x}$ at time t. $\mathbf{v}_p^{\text{nst}} = \mathbf{u}_p^{\text{nst}} - \mathbf{u}_p$ where $\mathbf{u}_p^{\text{nst}}$ si the mean particle velocities with nearest neighbor at $\mathbf{r}$. And $\mathbf{u}''_\alpha = \mathbf{u}_\alpha - \mathbf{u}_p^{\text{nst}}$ is the particle velocity fluctuation around the conditional mean $\mathbf{u}_p^{\text{nst}}$.

3 How to derive of the conditional particle velocity

Bring the problem to the two particle conditional field. This is done by making a balence between the velocity and the buoyancy.

4 derivation of the conditional average with the method of reflexion

According to Kim and Karrila (2013); Zhang (2021) teh particle phase velcoity fluctuation can be obtained directly with the faxen laws. Following the notation of Kim and Karrila (2013) we note $\mathbf{v}_1$ and $\mathbf{v}_2$ the velocity fields generated by the particle at $\mathbf{x}$ and at $\mathbf{y}$ respectively. The external body forces inducing the motion of the particles are noted $\mathbf{b}$ and $\mathbf{b}_2$. For instance, we keep them arbitrary however note that in the DNS $\mathbf{b}_1 = \mathbf{b}_2 = m\mathbf{g} = \mathbf{b}$.

The substantial difference between this analysis and Kim and Karrila (2013) analysis is that $\mathbf{v}_2$ is the nearest particle averaged. Meaning that, $\mathbf{v}_2$ is given by,

$$\mathbf{v}_2 = \mathbf{b} \cdot \left[1 + \frac{\lambda}{2(3\lambda+2)}\nabla^2\right] \frac{\mathcal{G}(\mathbf{z},\mathbf{y})}{8a\pi\mu_f} - \phi\frac{\mathbf{b}}{4\pi\mu_f a}|\mathbf{z}-\mathbf{y}|^2 \tag{4.1}$$

where all the distance have been made dimenisonless by a. and,

$$\mathbf{b} = 2\pi\mu_f a\left(\frac{2+3\lambda}{1+\lambda}\right)\mathbf{U}_1^{(0)} \tag{4.2}$$

$$\mathbf{b} = 2\pi\mu_f a\left(\frac{2+3\lambda}{1+\lambda}\right)\mathbf{U}_2^{(0)} \tag{4.3}$$

with $\mathbf{k}_2 = \mathbf{b}_2/U$. We recall that,

$$\mathcal{G}(\mathbf{z},\mathbf{y}) = \frac{\boldsymbol{\delta}}{r} + \frac{\mathbf{rr}}{r^3} \tag{4.4}$$

with $\mathbf{r} = |\mathbf{z}-\mathbf{x}|$. Notice and it will be useful for the following analysis that the gradient of $\mathcal{G}$ and

r over its first variable reads as,

$$\partial_k \mathcal{G}_{ij} = \frac{(-\delta_{ij} r_k + \delta_{ki} r_j + \delta_{kj} r_i)}{r^3} - 3\frac{r_i r_j r_k}{r^5} \tag{4.5}$$

$$\partial_{kl} \mathcal{G}_{ij} = \frac{(-\delta_{ij}\delta_{kl} + \delta_{ki}\delta_{jl} + \delta_{kj}\delta_{il})}{r^3} - 3\frac{(-\delta_{ij} r_k r_l + \delta_{ki} r_j r_l + \delta_{kj} r_i r_l)}{r^5} \tag{4.6}$$

$$-3\frac{(\delta_{il} r_j r_k + r_i r_k \delta_{jl} + r_i r_j \delta_{kl})}{r^5} + 15\frac{r_i r_j r_k r_l}{r^7} \tag{4.7}$$

$$\nabla^2 \mathcal{G}_{ij} = \frac{2\boldsymbol{\delta}}{r^3} - 6\frac{\mathbf{rr}}{r^5} \tag{4.8}$$

$$\partial_k \nabla^2 \mathcal{G}_{ij} = -6\frac{(\delta_{ij} r_k + \delta_{ki} r_j + r_i \delta_{jk})}{r^5} + 30\frac{r_i r_j r_k}{r^7} \tag{4.9}$$

$$\nabla^4 \mathcal{G}_{ij} = 0 \tag{4.10}$$

$$\partial_k r^2 = 2 r_k \tag{4.11}$$

$$\partial_{kl} r^2 = 2\delta_{kl} \tag{4.12}$$

Regarding, $\mathbf{v}_1$ it can be considered at the lowest order as the one of an isolated particle.

$$\mathbf{v}_1 = \mathbf{b} \cdot \left[1 + \frac{\lambda}{2(3\lambda+2)}\nabla^2\right] \frac{\mathcal{G}(\mathbf{z},\mathbf{x})}{8\pi\mu_f}. \tag{4.13}$$

At the first order in reflextion the velocity of the dorplet at $\mathbf{x}$ knowing its nearest neighbor is at $\mathbf{y}$, can be computed according to faxen law as,

$$2\pi\mu_f a \mathbf{U}_1^{(1)} = \left(1 + \frac{\lambda}{2(3\lambda+2)}\nabla^2\right) \mathbf{v}_2|_{\mathbf{z}=\mathbf{x}} \tag{4.14}$$

where $\mathbf{U}^{(1)}$ is the first reflexion. From the expression of $\mathbf{G}$ and its derivative evaluated at $\mathbf{x}$ we may write,

$$2\pi\mu_f a \mathbf{U}_1^{(1)} = \frac{\mathbf{b}}{4} \cdot \left(1 + \frac{\lambda}{2(3\lambda+2)}\nabla^2\right) \left\{ \left(1 + \frac{\lambda}{2(3\lambda+2)}\nabla^2\right) \mathcal{G}|_{\mathbf{z}=\mathbf{x}} - 2\phi r^2 \right\} \tag{4.15}$$

$$= \frac{\mathbf{b}}{4} \cdot \left(1 + \frac{\lambda}{3\lambda+2}\nabla^2 + \frac{\lambda^2}{4(3\lambda+2)^2}\nabla^4\right) \mathcal{G}|_{\mathbf{z}=\mathbf{x}} - \phi \mathbf{b}(\frac{r^2}{2} + 1) \tag{4.16}$$

$$= \frac{\mathbf{b}}{4} \cdot \left\{ \frac{\boldsymbol{\delta}}{r} + \frac{\mathbf{rr}}{r^3} + \frac{2\lambda}{3\lambda+2}\left(\frac{\boldsymbol{\delta}}{r^3} - 3\frac{\mathbf{rr}}{r^5}\right) - \phi\boldsymbol{\delta}(\frac{r^2}{2} + 1) \right\} \tag{4.17}$$

In other word,

$$\mathbf{U}_1^{(1)} = \left(\frac{2+3\lambda}{\lambda+1}\right) \frac{\mathbf{U}_1^{(0)}}{4} \cdot \left\{ \frac{\boldsymbol{\delta}}{r} + \frac{\mathbf{rr}}{r^3} + \frac{2\lambda}{3\lambda+2}\left(\frac{\boldsymbol{\delta}}{r^3} - 3\frac{\mathbf{rr}}{r^5}\right) - \phi\boldsymbol{\delta}(\frac{r^2}{2} + 1) \right\} \tag{4.18}$$

The particle phase pdf is given in dimensionless form by,

$$P_{\text{nst}}[\mathbf{y}|\mathbf{x}] = n_p e^{-\phi(r^3 - 8)}$$

Integrating the first int we have,

$$\langle \delta_p \mathbf{u}'_\alpha \mathbf{u}'_\alpha \rangle = n_p[\mathbf{x}, t] \int_{\mathbb{R}^3} (\mathbf{v}_p^{\text{nst}} \mathbf{v}_p^{\text{nst}})[\mathbf{x}, \mathbf{y}, t] P_{\text{nst}}[\mathbf{y}|\mathbf{x}, t] d\mathbf{y}$$

which gives,

$$\langle \delta_p \mathbf{u}'_\alpha \mathbf{u}'_\alpha \rangle / n_p =$$

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