Physical interpretation of the two fluid model

Nicolas Fintzi[1,2] and Jean-Lou Pierson[1]

[1] *IFP Energies Nouvelles, Rond-point de l'echangeur de Solaize, 69360 Solaize*
[2] *Sorbonne Université, Institut Jean le Rond ∂'Alembert, 4 place Jussieu, 75252 PARIS CEDEX 05, France*

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Abstract

1 Singularity solution for capillary motion of a droplet

At the leading order in droplet volume fraction, the closure problem is equivalent to that of an isolated droplet in an infinite medium **?**. Hence, we consider the problem of an isolated droplet, immersed in an arbitrary quadratic flow without the presence of fluid inertia. The disturbances pressure and velocity field are noted $\mathbf{u}_{out}$, $\mathbf{u}_{in}$, p_{out} and p_{in}, for the velocity outside the droplet, the velocity inside the droplet, the pressure outside the droplet and the pressure inside the droplet, respectively. The corresponding "undisturbed field" are noted $\mathbf{u}$ and p.

In dimensionless form $(\mathbf{u}_{out}, \mathbf{u}_{in})$ are governed by the Stokes equations, namely,

$$\nabla \cdot \mathbf{u}_{in} = 0 \qquad\qquad \nabla \cdot \mathbf{u}_{out} = 0 \tag{1.1}$$

$$-\nabla p_{in} + \nabla^2 \mathbf{u}_{in} = 0 \qquad\qquad -\nabla p_{out} + \nabla^2 \mathbf{u}_{out} = 0 \tag{1.2}$$

As these field are disturbance field they follow,

$$\lim_{r \to \infty} (\mathbf{u}_{out}, p_{out}) = 0 \tag{1.3}$$

far from the droplet centered at the origin ($\mathbf{x} = 0$). At the surface of the droplet ($r = 1$) the continuity of velocity and the far field condition impose,

$$\mathbf{u}_{in} = \mathbf{u}_{out} \qquad\qquad \mathbf{u}_{in} \cdot \mathbf{n} = -\mathbf{u} \cdot \mathbf{n} \tag{1.4}$$

where it must be noted that both $\mathbf{u}_{in}$ and $\mathbf{u}$ are evaluated at the points on the surface of the droplet, hence $\mathbf{u}$ may not always be a constant vector.

Let us introduce the dimensionless stress tensor $\boldsymbol{\sigma}_{in/out} = -p_{in/out}\boldsymbol{\delta} + \nabla\mathbf{u}_{in/out} +^\dagger \nabla\mathbf{u}_{in/out}$ and the dimensionless shear stress tensor, $\boldsymbol{\tau}_{in/out} = \nabla\mathbf{u}_{in/out} +^\dagger \nabla\mathbf{u}_{in/out}$. The undisturbed shear stress reads $\boldsymbol{\tau} = \nabla\mathbf{u} +^\dagger \nabla\mathbf{u}$ With these notations the tangential shear stress jump reads at $r = 1$ reads as,

$$\mathbf{n} \cdot (\boldsymbol{\tau}_{out} - \lambda\boldsymbol{\tau}_{in} + \boldsymbol{\tau} - \lambda\boldsymbol{\tau}) \cdot (\boldsymbol{\delta} - \mathbf{nn}) = Ma(\boldsymbol{\delta} - \mathbf{nn}) \cdot \nabla\gamma \tag{1.5}$$

where Ma is the Marangoni number defined as (**?**) which represent the effect of non-uniform surface tension.

In the most general case senario $\nabla\gamma$ as well as $\mathbf{u}$ are not constant vectors. However, in the disturbance field problem it is assumed that these quantities varies very slowly compared to the droplet radius, on which we evaluate these fields. Hence, we may introduce the following relationship,

$$\mathbf{u}(\mathbf{n}) = \mathbf{u}|_{\mathbf{x}=0} + \mathbf{r} \cdot \nabla\mathbf{u}|_{\mathbf{x}=0} + \frac{1}{2}\mathbf{rr} \cdot \nabla\nabla\mathbf{u}|_{\mathbf{x}=0} + \dots$$

$$\boldsymbol{\tau}(\mathbf{n}) = \boldsymbol{\tau}|_{\mathbf{x}=0} + \mathbf{r} \cdot \nabla\boldsymbol{\tau}|_{\mathbf{x}=0} + \frac{1}{2}\mathbf{rr} \cdot \nabla\nabla\boldsymbol{\tau}|_{\mathbf{x}=0} + \dots$$

$$\nabla\gamma(\mathbf{n}) = \nabla\gamma|_{\mathbf{x}=0} + \mathbf{r} \cdot \nabla\nabla\gamma|_{\mathbf{x}=0} + \frac{1}{2}\mathbf{rr} \cdot \nabla\nabla\nabla\gamma|_{\mathbf{x}=0}.$$

Each gradient posses their own dimensionless number We deduce that the disturbance fields are linear with the constant tensor $\mathbf{u}|_{\mathbf{x}=0}$, $\nabla\gamma|_{\mathbf{x}=0}$ and their derivatives through the boundary conditions Eq. (1.5) and Eq. (1.4). Hence, we deduce that $\mathbf{u}_{in/out}$ and $p_{in/out}$ must be linear combination of spherical harmonics function proportional to $\mathbf{u}|_{\mathbf{x}=0}$, $\nabla\gamma|_{\mathbf{x}=0}$, and their derivatives.

Using the standard procedure of spherical harmonics we can stipulate that (**??**),

$$p_{out} = \{\mathcal{P}_{u\text{-}out}^{(1)} + \mathcal{P}_{u\text{-}out}^{(2)} \cdot \nabla + \mathcal{P}_{u\text{-}out}^{(3)} : \nabla\nabla\} \cdot \mathbf{u} + \{\mathcal{P}_{\gamma\text{-}out}^{(1)} + \mathcal{P}_{\gamma\text{-}out}^{(2)} \cdot \nabla + \mathcal{P}_{\gamma\text{-}out}^{(3)} : \nabla\nabla\} \cdot \nabla\gamma$$

$$(\mathbf{u}_{out})_i = \{\mathcal{U}_{u\text{-}out}^{(1)} + \mathcal{U}_{u\text{-}out}^{(2)} \cdot \nabla + \mathcal{U}_{u\text{-}out}^{(3)} : \nabla\nabla\} \cdot \mathbf{u} + \{\mathcal{U}_{\gamma\text{-}out}^{(1)} + \mathcal{U}_{\gamma\text{-}out}^{(2)} \cdot \nabla + \mathcal{U}_{\gamma\text{-}out}^{(3)} : \nabla\nabla\} \cdot \nabla\gamma$$

where $\mathcal{P}_{out}^{(n)}, \mathcal{U}_{out}^{(n)}$ are tensor of order n and $n+1$ respectively that are function of $\mathbf{r}$ and the viscosity ratio λ. Because the pressure is an harmonics function the tensor $\mathcal{P}_{(.)}^{(n)}$ take the form,

$$\mathcal{P}_{(.)}^{(2)} = \left(\frac{1}{r}\right)\boldsymbol{\delta} + \nabla\nabla\left(\frac{1}{r}\right) \qquad \mathcal{P}_{(.)}^{(3)} = \left(\frac{1}{r}\right)\boldsymbol{\delta} + \nabla\nabla\left(\frac{1}{r}\right) \tag{1.6}$$

Similar consideration can be applied to the interior velocity fields and pressure, yielding,

$$\begin{pmatrix}\mathbf{u}_{out} \\ p_{out} \\ \mathbf{u}_{in} \\ p_{in}\end{pmatrix} = \begin{pmatrix} \mathcal{U}_{u\text{-}out}^{(1)} + \mathcal{U}_{u\text{-}out}^{(2)} \cdot \nabla + \mathcal{U}_{u\text{-}out}^{(3)} : \nabla\nabla & \mathcal{U}_{\gamma\text{-}out}^{(1)} + \mathcal{U}_{\gamma\text{-}out}^{(2)} \cdot \nabla + \mathcal{U}_{\gamma\text{-}out}^{(3)} : \nabla\nabla \\ \mathcal{P}_{u\text{-}out}^{(1)} + \mathcal{P}_{u\text{-}out}^{(2)} \cdot \nabla + \mathcal{P}_{u\text{-}out}^{(3)} : \nabla\nabla & \mathcal{P}_{\gamma\text{-}out}^{(1)} + \mathcal{P}_{\gamma\text{-}out}^{(2)} \cdot \nabla + \mathcal{P}_{\gamma\text{-}out}^{(3)} : \nabla\nabla \\ \mathcal{U}_{u\text{-}in}^{(1)} + \mathcal{U}_{u\text{-}in}^{(2)} \cdot \nabla + \mathcal{U}_{u\text{-}in}^{(3)} : \nabla\nabla & \mathcal{U}_{\gamma\text{-}in}^{(1)} + \mathcal{U}_{\gamma\text{-}in}^{(2)} \cdot \nabla + \mathcal{U}_{\gamma\text{-}in}^{(3)} : \nabla\nabla \\ \mathcal{P}_{u\text{-}in}^{(1)} + \mathcal{P}_{u\text{-}in}^{(2)} \cdot \nabla + \mathcal{P}_{u\text{-}in}^{(3)} : \nabla\nabla & \mathcal{P}_{\gamma\text{-}in}^{(1)} + \mathcal{P}_{\gamma\text{-}in}^{(2)} \cdot \nabla + \mathcal{P}_{\gamma\text{-}in}^{(3)} : \nabla\nabla \end{pmatrix} \cdot \begin{pmatrix}\mathbf{u} \\ \nabla\gamma\end{pmatrix}$$

The tensor $\mathcal{U}_{u\text{-}out}^{(1)}, \mathcal{U}_{u\text{-}in}^{(1)}, \mathcal{P}_{u\text{-}out}^{(1)}$ and $\mathcal{P}_{u\text{-}in}^{(1)}$ represent the effect of uniform relative translation on the disturbance fields, hence these expression to correspond to the Hadamard-Ribczynski solution (**??**). The dependency with the mean gradient velocity $\nabla\mathbf{u}$ is defined through the tensor $\mathcal{U}_{u\text{-}out}^{(2)}, \mathcal{U}_{u\text{-}in}^{(2)}, \mathcal{P}_{u\text{-}out}^{(2)}$ and $\mathcal{P}_{u\text{-}in}^{(2)}$. The solution of a drop in pure linear flow have been derived in **???**. Finally the disturbance field of a droplet in quadratic flow may be defined through the tensor $\mathcal{U}_{u\text{-}out}^{(3)}, \mathcal{U}_{u\text{-}in}^{(3)}, \mathcal{P}_{u\text{-}out}^{(3)}$ and $\mathcal{P}_{u\text{-}in}^{(3)}$. The definition of the fomer tensor may be found in **?**.

Regarding the effect of non-uniform surface tension on the disturbance fields, solution are less commun. For instance one might refer to **?** we they compute the disturbance fields due to a constant surface tension gradient, hence defining $\mathcal{U}_{\gamma\text{-}out}^{(1)}, \mathcal{U}_{\gamma\text{-}in}^{(1)}, \mathcal{P}_{\gamma\text{-}out}^{(1)}$ and $\mathcal{P}_{\gamma\text{-}in}^{(1)}$. Then no-study could be found that express the disturbance fields, in term of $\nabla\nabla\gamma$ or higher order derivatives[1] .

Note that only the surface force at $r = 1$ is used to compute the closure terms, hence we provide they explicite expression here,

$$(\boldsymbol{\sigma}_{out} \cdot \mathbf{n}|_{r=1})_i = \{\mathcal{S}_{u\text{-}out}^{(1)} + \mathcal{S}_{u\text{-}out}^{(2)} \cdot \nabla + \mathcal{S}_{u\text{-}out}^{(3)} : \nabla\nabla\} \cdot \mathbf{u} + \{\mathcal{S}_{\gamma\text{-}out}^{(1)} + \mathcal{S}_{\gamma\text{-}out}^{(2)} \cdot \nabla + \mathcal{S}_{\gamma\text{-}out}^{(3)} : \nabla\nabla\} \cdot \nabla\gamma$$

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[1]In **?**, Appendix C They provide a solution which could correspond to $\mathcal{U}_{\gamma\text{-}out}^{(1)}, \mathcal{U}_{\gamma\text{-}in}^{(1)}, \mathcal{P}_{\gamma\text{-}out}^{(1)}$ and $\mathcal{P}_{\gamma\text{-}in}^{(1)}$ but only in the case $\nabla \cdot \nabla\gamma = 0$ since their tensor $\mathbf{B}$ is assumed traceless.

with,

$$\mathcal{S}^{(1)}_{u\text{-}out} = 3\frac{1}{\lambda+1}x^i x^j + \left(\frac{3}{2}\right)\frac{\lambda}{\lambda+1}\delta^{ij}$$

$$\mathcal{S}^{(2)}_{u\text{-}out} = 8\frac{1}{\lambda+1}x^i x^j x^k + \left(\frac{1}{2}\right)\frac{1}{\lambda+1}(3\lambda-2)(\delta^{ij}x^k + \delta^{ik}x^j)$$

$$\mathcal{S}^{(3)}_{u\text{-}out} = -\left(\frac{1}{4}\right)\frac{\lambda^2+15\lambda+4}{\lambda^2+5\lambda+4}\delta^{jk}x^i x^l - \left(\frac{1}{8}\right)\frac{\lambda(3\lambda-8)}{\lambda^2+5\lambda+4}\delta^{il}\delta^{jk}$$
$$+\left(\frac{1}{8}\right)\frac{(17\lambda^2+20\lambda-32)}{\lambda^2+5\lambda+4}\delta^{il}x^j x^k + \left(\frac{1}{8}\right)\frac{(\lambda^2+20\lambda-16)}{\lambda^2+5\lambda+4}(\delta^{ij}x^k x^l + \delta^{ik}x^j x^l)$$
$$+\left(\frac{25}{4}\right)\frac{1}{\lambda+1}x^i x^j x^k x^l$$

$$\mathcal{S}^{(1)}_{\gamma\text{-}out} = -\frac{2}{\lambda+1}x^i x^j + \frac{1}{\lambda+1}\delta^{ij}$$

$$\mathcal{S}^{(2)}_{\gamma\text{-}out} = -\left(\frac{8}{5}\right)\frac{1}{\lambda+1}x^i x^j x^k + \left(\frac{1}{2}\right)\frac{1}{\lambda+1}(\delta^{ij}x^k + \delta^{ik}x^j) + \left(\frac{1}{5}\right)\frac{1}{\lambda+1}\delta^{jk}x^i$$

$$\mathcal{S}^{(3)}_{\gamma\text{-}out} = -\left(\frac{5}{7}\right)\frac{1}{\lambda+1}x^i x^j x^k x^l + \left(\frac{1}{105}\right)\frac{1}{\lambda+1}(\delta^{jk}x^i x^l + \delta^{jl}x^i x^k + \delta^{kl}x^i x^j)$$
$$+\left(\frac{1}{6}\right)\frac{1}{\lambda+1}(\delta^{ij}x^k x^l + \delta^{ik}x^j x^l + \delta^{il}x^j x^k)$$

It will also be of interest to compute the surface velocity,

$$\mathcal{U}^{(1)}_u = -\frac{1}{\lambda+1}\left(\lambda+\frac{1}{2}\right)\delta^{ij} - \left(\frac{1}{2}\right)\frac{1}{\lambda+1}x^i x^j$$

$$\mathcal{U}^{(2)}_u = -\frac{1}{\lambda+1}x^i x^j x^k - \left(\frac{1}{2}\right)\lambda\frac{1}{\lambda+1}(\delta^{ij}x^k + \delta^{ik}x^j)$$

$$\mathcal{U}^{(3)}_u = -\left(\frac{5}{8}\right)\frac{1}{\lambda+1}x^i x^j x^k x^l - \left(\frac{1}{8}\right)\frac{1}{\lambda^2+5\lambda+4}(4\lambda^2+5\lambda-4)\delta^{il}x^j x^k$$
$$-\left(\frac{5}{8}\right)\lambda\frac{1}{\lambda^2+5\lambda+4}(\delta^{ij}x^k x^l + \delta^{ik}x^j x^l + \delta^{il}\delta^{jk} - \delta^{jk}x^i x^l)$$

$$\mathcal{U}^{(1)}_\gamma = -\left(\frac{1}{3}\right)\frac{1}{\lambda+1}\delta^{ij} + \left(\frac{1}{3}\right)\frac{1}{\lambda+1}x^i x^j$$

$$\mathcal{U}^{(2)}_\gamma = -\left(\frac{1}{10}\right)\frac{1}{\lambda+1}(\delta^{ij}x^k + \delta^{ik}x^j) + \left(\frac{1}{5}\right)\frac{1}{\lambda+1}x^i x^j x^k$$

$$\mathcal{U}^{(3)}_\gamma = -\left(\frac{1}{42}\right)\frac{1}{\lambda+1}(\delta^{ij}x^k x^l + \delta^{ik}x^j x^l + \delta^{il}x^j x^k) + \left(\frac{1}{14}\right)\frac{1}{\lambda+1}x^i x^j x^k x^l$$
$$+\left(\frac{2}{315}\right)\frac{1}{\lambda+1}(-\delta^{ij}\delta^{kl} - \delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk} + \delta^{jk}x^i x^l + \delta^{jl}x^i x^k + \delta^{kl}x^i x^j)$$

1.1   Compute the force traction terms

In the main text one needs to integrate the force traction of the disturbance field $\boldsymbol{\sigma}_{out}$ over the surface of the droplet to obtain the closure. Particularily we need to compute, these three integrals,

$$\oint_{\Gamma_\alpha}(\boldsymbol{\sigma}_{out}\cdot\mathbf{n})_i d\Gamma \qquad\qquad \oint_{\Gamma_\alpha}x_m(\boldsymbol{\sigma}_{out}\cdot\mathbf{n})_i d\Gamma \qquad\qquad \oint_{\Gamma_\alpha}x_n x_m(\boldsymbol{\sigma}_{out}\cdot\mathbf{n})_i d\Gamma$$

which are the three first hydrodynamic moments of the force traction.

To do so one needs to notice that by integrating over the unit sphere we obtain,

$$\frac{1}{4\pi}\oint_{\Gamma_\alpha}\delta_{ij}d\Gamma = 1$$

$$\frac{3}{4\pi}\oint_{\Gamma_\alpha}x_ix_jd\Gamma = 1$$

$$\frac{15}{4\pi}\oint_{\Gamma_\alpha}x_ix_jx_kx_ld\Gamma = (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})$$

$$\frac{105}{4\pi}\oint_{\Gamma_\alpha}x_ix_jx_kx_lx_mx_nd\Gamma = \delta_{nm}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})$$
$$+ \delta_{nl}(\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{kj})$$
$$+ \delta_{nk}(\delta_{ij}\delta_{ml} + \delta_{im}\delta_{jl} + \delta_{il}\delta_{mj})$$
$$+ \delta_{nj}(\delta_{im}\delta_{kl} + \delta_{ik}\delta_{ml} + \delta_{il}\delta_{km})$$
$$+ \delta_{ni}(\delta_{mj}\delta_{kl} + \delta_{mk}\delta_{jl} + \delta_{ml}\delta_{kj})$$

[maybe the 105 is wrong]Additionally all odd order tesnor x_i or $x_ix_jx_k$, integrated over the surface of the sphere goes to zero.

$$\oint_{\Gamma_\alpha}\boldsymbol{\sigma}_{out}\cdot\mathbf{n}d\Gamma = 2\pi\frac{2+3\lambda}{1+\lambda}\mathbf{u}_r + \pi\frac{\lambda}{\lambda+1}\nabla^2\mathbf{u} + \frac{4\pi}{3}\frac{1}{\lambda+1}\nabla\gamma + \frac{2\pi}{15(\lambda+1)}\nabla^2(\nabla\gamma)$$

$$\oint_{\Gamma_\alpha}\mathbf{x}\boldsymbol{\sigma}_{out}\cdot\mathbf{n}d\Gamma = \frac{2\pi(5\lambda+2)}{5(\lambda+1)}[\nabla\mathbf{u} + (\nabla\mathbf{u})^\dagger] + \frac{12\pi}{25(\lambda+1)}\nabla\nabla\gamma - \frac{4\pi}{25(\lambda+1)}\boldsymbol{\delta}(\nabla\cdot\nabla)\gamma$$

$$\oint_{\Gamma_\alpha}\mathbf{xx}\boldsymbol{\sigma}_{out}\cdot\mathbf{n}d\Gamma = \frac{4\pi}{5(\lambda+1)}(\mathbf{u}_r\boldsymbol{\delta} + \boldsymbol{\delta}\mathbf{u}_r) + \frac{2\pi(5\lambda+2)}{5(\lambda+1)}\boldsymbol{\delta}\mathbf{u}_r - \frac{8\pi}{15(\lambda+1)}(\nabla\gamma\boldsymbol{\delta} + \boldsymbol{\delta}\nabla\gamma) + \frac{4\pi}{5(\lambda+1)}\boldsymbol{\delta}\nabla\gamma$$

It will also be useful to compute the droplet internal shear for the closure problem. Hence one needs the velocity at the droplet interface.

$$\int_{\Omega_\alpha}\boldsymbol{\tau}_{in}d\Omega = \int_{\Omega_\alpha}[\nabla\mathbf{u}_{in} + (\nabla\mathbf{u}_{in})^\dagger]d\Omega = \oint_{\Gamma_\alpha}(\mathbf{n}\mathbf{u}_{in} + \mathbf{u}_{in}\mathbf{n})d\Gamma \tag{1.7}$$

$$\int_{\Omega_\alpha}\boldsymbol{\tau}_{in}d\Omega = \int_{\Omega_\alpha}[\nabla\mathbf{u}_{in} + (\nabla\mathbf{u}_{in})^\dagger]d\Omega = \oint_{\Gamma_\alpha}(\mathbf{n}\mathbf{u}_{in} + \mathbf{u}_{in}\mathbf{n})d\Gamma \tag{1.8}$$

Thus, we may compute that gives,

$$\oint_{\Gamma_\alpha}(\mathbf{n}\mathbf{u}_{in} + \mathbf{u}_{in}\mathbf{n})d\Gamma = -\frac{4\pi}{15}\frac{5\lambda+2}{\lambda+1}[\nabla\mathbf{u} + (\nabla\mathbf{u})^\dagger] - \frac{8\pi}{25}\frac{1}{\lambda+1}\nabla\nabla\gamma + \frac{8\pi}{75}\frac{1}{\lambda+1}\boldsymbol{\delta}(\nabla\cdot\nabla)\gamma$$

$$\oint_{\Gamma_\alpha}\mathbf{r}(\mathbf{n}\mathbf{u}_{in} + \mathbf{u}_{in}\mathbf{n})d\Gamma = -\frac{2\pi}{15}\frac{10\lambda+7}{\lambda+1}(\boldsymbol{\delta}\mathbf{u}_r + \mathbf{u}_r\boldsymbol{\delta}) - \frac{4\pi}{15}\frac{1}{\lambda+1}\boldsymbol{\delta}\mathbf{u}_r - \frac{4\pi}{15}\frac{1}{\lambda+1}(\boldsymbol{\delta}\nabla\gamma + \nabla\gamma\boldsymbol{\delta})$$
$$+ \frac{8\pi}{45}\frac{1}{\lambda+1}\boldsymbol{\delta}\nabla\gamma$$

$$\oint_{\Gamma_\alpha}2\mathbf{e}_{in}d\Gamma = -\frac{4\pi}{15}\frac{5\lambda+2}{\lambda+1}[\nabla\mathbf{u} + (\nabla\mathbf{u})^\dagger] - \frac{8\pi}{25}\frac{1}{\lambda+1}\nabla\nabla\gamma + \frac{8\pi}{75}\frac{1}{\lambda+1}\boldsymbol{\delta}(\nabla\cdot\nabla)\gamma$$

$$\oint_{\Gamma_\alpha}2\mathbf{r}\mathbf{e}_{in}d\Gamma = -\frac{2\pi}{15}\frac{20\lambda+17}{\lambda+1}(\boldsymbol{\delta}\mathbf{u}_r + \mathbf{u}_r\boldsymbol{\delta}) - \frac{4\pi}{15}\frac{1}{\lambda+1}\boldsymbol{\delta}\mathbf{u}_r - \frac{4\pi}{15}\frac{1}{\lambda+1}(\boldsymbol{\delta}\nabla\gamma + \nabla\gamma\boldsymbol{\delta})$$
$$+ \frac{8\pi}{45}\frac{1}{\lambda+1}\boldsymbol{\delta}\nabla\gamma$$

Deformation of the droplets

The usual procedure to determine the droplet deformation is to use the normal stress balance. However to point out the meaning of the first- and second-moment of momentum here we introduce an alternative approaches based on Lagrangian balance laws.

$$\oint_{\Gamma_\alpha} (\boldsymbol{\sigma}_\Gamma^0)_{ik} d\Gamma \quad + \quad \int_{\Omega_\alpha} (\boldsymbol{\sigma}_d^0)_{ik} d\Omega \quad = \quad +\frac{1}{2} \oint_{\Gamma_\alpha} \left[(\boldsymbol{\sigma}_f^0 \cdot \mathbf{n})_i r_k + (\boldsymbol{\sigma}_f^0 \cdot \mathbf{n})_k r_i \right] d\Gamma \quad (1.9)$$

$$\int_{\Omega_\alpha} r_j(\boldsymbol{\sigma}_d^0)_{ik} + r_k(\boldsymbol{\sigma}_d^0)_{ji} d\Omega + \oint_{\Gamma_\alpha} r_j(\boldsymbol{\sigma}_\Gamma^0)_{ik} + r_k(\boldsymbol{\sigma}_\Gamma^0)_{ji} d\Gamma = + \oint_{\Gamma_\alpha} r_k r_j(\boldsymbol{\sigma}_f^0 \cdot \mathbf{n})_i d\Gamma + \int_{\Omega_\alpha} r_k r_j \rho_d(\mathbf{b}_d^0)_i d\Omega$$
$$(1.10)$$

the mean part of the stress then read

$$\ldots = \int_{\Omega_\alpha} r_k r_j (\nabla \cdot \boldsymbol{\Sigma})_i d\Omega + (1-\lambda) \oint_{\Gamma_\alpha} (\boldsymbol{\Sigma})_{ik} r_j d\Gamma + (1-\lambda) \oint_{\Gamma_\alpha} (\boldsymbol{\Sigma})_{ij} r_k d\Gamma \quad (1.11)$$

Using the trik of **?** we may re-write the second moment of moment with the symmetry relation $B_{ijk} + B_{jik} - B_{kij}$ which yiel, Indeed, consider the

$$\int_{\Omega_\alpha} 2r_k(\boldsymbol{\sigma}_d^0)_{ij} d\Omega + \oint_{\Gamma_\alpha} 2r_k(\boldsymbol{\sigma}_\Gamma^0)_{ij} d\Gamma = + \oint_{\Gamma_\alpha} r_k r_j(\boldsymbol{\sigma}_f^0 \cdot \mathbf{n})_i + r_k r_i(\boldsymbol{\sigma}_f^0 \cdot \mathbf{n})_j - r_j r_i(\boldsymbol{\sigma}_f^0 \cdot \mathbf{n})_k d\Gamma \quad (1.12)$$

$$+ \int_{\Omega_\alpha} r_k r_j \rho_d(\mathbf{b}_d^0)_i + r_k r_i \rho_d(\mathbf{b}_d^0)_j - r_j r_i \rho_d(\mathbf{b}_d^0)_k d\Omega \quad (1.13)$$

The traceless part (on the index ij) of this equation yields,

$$\int_{\Omega_\alpha} 2r_k(\boldsymbol{\sigma}_d^0)_{ij}^{dev} d\Omega + \oint_{\Gamma_\alpha} 2r_k(\boldsymbol{\sigma}_\Gamma^0)_{ij}^{dev} d\Gamma = + \oint_{\Gamma_\alpha} r_k r_j(\boldsymbol{\sigma}_f^0 \cdot \mathbf{n})_i + r_k r_i(\boldsymbol{\sigma}_f^0 \cdot \mathbf{n})_j - r_j r_i(\boldsymbol{\sigma}_f^0 \cdot \mathbf{n})_k d\Gamma \quad (1.14)$$

$$- \frac{1}{3}\delta_{ij} \oint_{\Gamma_\alpha} r_k r_l(\boldsymbol{\sigma}_f^0 \cdot \mathbf{n})_l + r_k r_l(\boldsymbol{\sigma}_f^0 \cdot \mathbf{n})_l - r_l r_l(\boldsymbol{\sigma}_f^0 \cdot \mathbf{n})_k d\Gamma \quad (1.15)$$

$$+ \int_{\Omega_\alpha} r_k r_j(\nabla \cdot \boldsymbol{\Sigma})_i + r_k r_i(\nabla \cdot \boldsymbol{\Sigma})_j - r_j r_i(\nabla \cdot \boldsymbol{\Sigma})_k d\Omega \quad (1.16)$$

$$- \frac{1}{3}\delta_{ij} \int_{\Omega_\alpha} r_k r_l(\nabla \cdot \boldsymbol{\Sigma})_l + r_k r_l(\nabla \cdot \boldsymbol{\Sigma})_l - r_l r_l(\nabla \cdot \boldsymbol{\Sigma})_k d\Omega \quad (1.17)$$

$$+ \mu_f(1-\lambda) \oint_{\Gamma_\alpha} 4\mathbf{E}_{ij} r_k d\Gamma \quad (1.18)$$

$$+ \rho_d \int_{\Omega_\alpha} r_k r_j(\mathbf{b}_d^0)_i + r_k r_i(\mathbf{b}_d^0)_j - r_j r_i(\mathbf{b}_d^0)_k d\Omega \quad (1.19)$$

$$- \rho_d \frac{1}{3}\delta_{ij} \int_{\Omega_\alpha} r_k r_l(\mathbf{b}_d^0)_l + r_k r_l(\mathbf{b}_d^0)_l - r_l r_l(\mathbf{b}_d^0)_k d\Omega \quad (1.20)$$

Because $\nabla \cdot \boldsymbol{\Sigma} = -\rho_f \mathbf{g}$ one recover teh buoyancy forces for the remaining terms, but these cancel with the dependency of the secoond mom with the $\mathbf{u}_r$ and thus contribute to nothing to the deformation.

So in the end we are left with

$$\int_{\Omega_\alpha} 2r_k(\boldsymbol{\sigma}')_{ij}^{dev} d\Omega + \oint_{\Gamma_\alpha} 2r_k(\boldsymbol{\sigma}_\Gamma^0)_{ij}^{dev} d\Gamma = + \oint_{\Gamma_\alpha} r_k r_j(\boldsymbol{\sigma}_f' \cdot \mathbf{n})_i + r_k r_i(\boldsymbol{\sigma}_f' \cdot \mathbf{n})_j - r_j r_i(\boldsymbol{\sigma}_f' \cdot \mathbf{n})_k d\Gamma \tag{1.21}$$

$$- \frac{1}{3}\delta_{ij} \oint_{\Gamma_\alpha} r_k r_l(\boldsymbol{\sigma}_f' \cdot \mathbf{n})_l + r_k r_l(\boldsymbol{\sigma}_f' \cdot \mathbf{n})_l - r_l r_l(\boldsymbol{\sigma}_f' \cdot \mathbf{n})_k d\Gamma \tag{1.22}$$

$$+ \mu_f(1-\lambda) \oint_{\Gamma_\alpha} 4\mathbf{E}_{ij} r_k d\Gamma \tag{1.23}$$

$$+ (\rho_d - \rho_f) \int_{\Omega_\alpha} r_k r_j \mathbf{g}_i + r_k r_i \mathbf{g}_j - r_j r_i \mathbf{g}_k d\Omega \tag{1.24}$$

$$- (\rho_d - \rho_f)\frac{1}{3}\delta_{ij} \int_{\Omega_\alpha} r_k r_l \mathbf{g}_l + r_k r_l \mathbf{g}_l - r_l r_l \mathbf{g}_k d\Omega \tag{1.25}$$

1.2 Volume jacobian

let $\mathbf{M}$ be a point in the deformable particle then,

$$\mathbf{M} = r(1 + f(\varphi, \theta))\mathbf{e}_r = r(1 + f(\varphi, \theta)) \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix} \tag{1.26}$$

let,

$$\mathbf{e}_\theta = \frac{\partial \mathbf{e}_r}{\partial \theta} = \begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ -\sin\theta \end{pmatrix} \qquad \mathbf{e}_\varphi = \frac{\partial \mathbf{e}_r}{\partial \varphi} = \begin{pmatrix} -\sin\varphi\sin\theta \\ \cos\varphi\sin\theta \\ 0 \end{pmatrix} \tag{1.27}$$

$$\tag{1.28}$$

Hence,

$$\frac{\partial \mathbf{M}}{\partial r} = (1 + f)\mathbf{e}_r \tag{1.29}$$

$$\frac{\partial \mathbf{M}}{\partial \varphi} = r(f_\varphi \mathbf{e}_r + (1 + f)\mathbf{e}_\varphi) \tag{1.30}$$

$$\frac{\partial \mathbf{M}}{\partial \theta} = r(f_\theta \mathbf{e}_r + (1 + f)\mathbf{e}_\theta) \tag{1.31}$$

The jacobian

$$\left|\frac{\partial \mathbf{M}}{\partial(r, \varphi, \theta)}\right| =$$

$$\begin{vmatrix} (1+f)\cos\varphi\sin\theta & r(f_\varphi\cos\varphi\sin\theta - (1+f)\sin\varphi\sin\theta) & r(f_\theta\cos\varphi\sin\theta + (1+f)\cos\varphi\cos\theta) \\ (1+f)\sin\varphi\sin\theta & r(f_\varphi\sin\varphi\sin\theta + (1+f)\cos\varphi\sin\theta) & r(f_\theta\sin\varphi\sin\theta + (1+f)\sin\varphi\cos\theta) \\ (1+f)\cos\theta & rf_\varphi\cos\theta & r(f_\theta\cos\theta - (1+f)\sin\theta) \end{vmatrix}$$

But also,

$$\left|\frac{\partial \mathbf{M}}{\partial(r, \varphi, \theta)}\right| = \left|\frac{\partial \mathbf{M}}{\partial r} \quad \frac{\partial \mathbf{M}}{\partial \varphi} \quad \frac{\partial \mathbf{M}}{\partial \theta}\right| = \frac{\partial \mathbf{M}}{\partial r} \cdot \frac{\partial \mathbf{M}}{\partial \varphi} \times \frac{\partial \mathbf{M}}{\partial \theta}$$

$$= r^2(1+f)\mathbf{e}_r \cdot (f_\varphi \mathbf{e}_r + (1+f)\mathbf{e}_\varphi) \times (f_\theta \mathbf{e}_r + (1+f)\mathbf{e}_\theta)$$

$$= r^2(1+f)^2 \mathbf{e}_r \cdot (f_\varphi \mathbf{e}_r \times \mathbf{e}_\theta + f_\theta \mathbf{e}_\varphi \times \mathbf{e}_r + (1+f)\mathbf{e}_\varphi \times \mathbf{e}_\theta)$$

$$= r^2(1+f)^3 \sin\theta$$

Because,

$$\mathbf{e}_r \cdot (\mathbf{e}_\varphi \times \mathbf{e}_\theta) = \sin\theta \tag{1.32}$$

So that the final jacobian reads,

$$|\frac{\partial\mathbf{M}}{\partial(r,\varphi,\theta)}| = (1+f)^3 r^2(\sin\theta) \tag{1.33}$$

1.3 Surface jacobian

Let use the general parametrsiaition,

$$\mathbf{M}(\theta,\varphi) = (1+f(\theta,\varphi))\mathbf{e}_r = (1+f(\theta,\varphi)) \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix} \tag{1.34}$$

The vector in the basis reads,

$$\frac{\partial\mathbf{M}}{\partial\varphi} = f_\varphi\mathbf{e}_r + (1+f)\mathbf{e}_\varphi \tag{1.35}$$

$$\frac{\partial\mathbf{M}}{\partial\theta} = f_\theta\mathbf{e}_r + (1+f)\mathbf{e}_\theta \tag{1.36}$$

with,

$$\mathbf{e}_\theta = \frac{\partial\mathbf{e}_r}{\partial\theta} = \begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ -\sin\theta \end{pmatrix} \qquad\qquad \mathbf{e}_\varphi = \frac{\partial\mathbf{e}_r}{\partial\varphi} = \begin{pmatrix} -\sin\varphi\sin\theta \\ \cos\varphi\sin\theta \\ 0 \end{pmatrix} \tag{1.37}$$

$$\tag{1.38}$$

Hence the general formulation of the Jacobi determinant is the norm of the vector,

$$\mathbf{J} = \frac{\partial\mathbf{M}}{\partial\varphi} \times \frac{\partial\mathbf{M}}{\partial\theta} = (1+f)[f_\varphi\mathbf{e}_r \times \mathbf{e}_\theta + f_\theta\mathbf{e}_\varphi \times \mathbf{e}_r + (1+f)\mathbf{e}_\varphi \times \mathbf{e}_\theta] \tag{1.39}$$

$$\tag{1.40}$$

$$\mathbf{e}_r \times \mathbf{e}_\theta = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \qquad \mathbf{e}_\varphi \times \mathbf{e}_r = \begin{pmatrix} \sin\theta\cos\theta\cos\varphi \\ \sin\theta\cos\theta\sin\varphi \\ -\sin^2\theta \end{pmatrix} \qquad \mathbf{e}_\varphi \times \mathbf{e}_\theta = \begin{pmatrix} -\sin^2\theta\cos\varphi \\ -\sin^2\theta\sin\varphi \\ -\cos\theta\sin\theta \end{pmatrix} \tag{1.41}$$

The absolute value of this tensor is the sqareroot of the scalar product

$$\begin{aligned} \mathbf{J} \cdot \mathbf{J} &= (1+f)^2[f_\varphi\mathbf{e}_r \times \mathbf{e}_\theta + f_\theta\mathbf{e}_\varphi \times \mathbf{e}_r + (1+f)\mathbf{e}_\varphi \times \mathbf{e}_\theta] \cdot [f_\varphi\mathbf{e}_r \times \mathbf{e}_\theta + f_\theta\mathbf{e}_\varphi \times \mathbf{e}_r + (1+f)\mathbf{e}_\varphi \times \mathbf{e}_\theta] \\ &= (1+f)^2[f_\varphi^2(\mathbf{e}_r \times \mathbf{e}_\theta) \cdot (\mathbf{e}_r \times \mathbf{e}_\theta) + f_\varphi f_\theta(\mathbf{e}_r \times \mathbf{e}_\theta) \cdot (\mathbf{e}_\varphi \times \mathbf{e}_r) + f_\varphi(1+f)(\mathbf{e}_r \times \mathbf{e}_\theta) \cdot (\mathbf{e}_\varphi \times \mathbf{e}_\theta) \\ &\quad + f_\theta f_\varphi(\mathbf{e}_\varphi \times \mathbf{e}_r) \cdot (\mathbf{e}_r \times \mathbf{e}_\theta) + f_\theta^2(\mathbf{e}_\varphi \times \mathbf{e}_r) \cdot (\mathbf{e}_\varphi \times \mathbf{e}_r) + f_\theta(1+f)(\mathbf{e}_\varphi \times \mathbf{e}_r) \cdot (\mathbf{e}_\varphi \times \mathbf{e}_\theta) \\ &\quad + (1+f)f_\varphi(\mathbf{e}_\varphi \times \mathbf{e}_\theta) \cdot (\mathbf{e}_r \times \mathbf{e}_\theta) + (1+f)f_\theta(\mathbf{e}_\varphi \times \mathbf{e}_\theta) \cdot (\mathbf{e}_\varphi \times \mathbf{e}_r) + (1+f)^2(\mathbf{e}_\varphi \times \mathbf{e}_\theta) \cdot (\mathbf{e}_\varphi \times \mathbf{e}_\theta)] \end{aligned} \tag{1.42}$$

Because, for any vector $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ we have,

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \tag{1.43}$$

and that,

$$\mathbf{e}_r \cdot \mathbf{e}_r = 1 \qquad\qquad \mathbf{e}_\theta \cdot \mathbf{e}_\theta = 1 \qquad\qquad \mathbf{e}_\varphi \cdot \mathbf{e}_\varphi = \sin^2\theta \tag{1.44}$$

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the last equaiton becomes,

$$\mathbf{J} \cdot \mathbf{J} = (1+f)^2[f_\varphi^2 + f_\theta^2 \sin^2\theta + (1+f)^2 \sin^2\theta] \tag{1.45}$$

And so the surface eara element becomes,

$$dS = |\mathbf{J}|d\theta d\varphi = (1+f)[f_\varphi^2 + \sin^2\theta f_\theta^2 + \sin^2\theta(1+f)^2]^{1/2}d\theta d\varphi \tag{1.46}$$

VERYY WROUNG OR MAYBE I JUST FORGET THE $\sin$ tNow let consider that $f(\theta,\varphi) \to Caf(\theta,\varphi)$. The above resuts reads

$$dS = (1 + 2Caf)d\theta d\varphi + O(Ca^2 d\theta\varphi) \tag{1.47}$$

So that,

$$\int dS = \int_0^{2\pi}\int_0^\pi (1+2Caf)d\theta d\varphi + O(Ca^2) \tag{1.48}$$

According to the parametrsiaition the distance function of the surface reads,

$$\mathscr{F} = r - (1 + Caf) = 0 \tag{1.49}$$

hence,

$$\nabla\mathscr{F} = \mathbf{e}_r - Ca\nabla f \tag{1.50}$$

while

$$\frac{1}{|\nabla\mathscr{F}|} = (1 - 2Ca(\mathbf{e}_r \cdot \nabla)f)^{-1/2} \approx (1 + Ca(\mathbf{e}_r \cdot \nabla)f) + O(Ca^2) \tag{1.51}$$

Thus,

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \mathbf{e}_r - Ca(\boldsymbol{\delta} - \mathbf{e}_r\mathbf{e}_r) \cdot \nabla f = \mathbf{e}_r - Ca\nabla_{||}f \tag{1.52}$$

According to that results we may write,

$$\mathbf{nn} = (\mathbf{e}_r - Ca\nabla_{||}f)(\mathbf{e}_r - Ca\nabla_{||}f)$$
$$= \mathbf{e}_r\mathbf{e}_r - Ca(\mathbf{e}_r\nabla_{||}f + \nabla_{||}f\mathbf{e}_r)$$

Then it is clear that,

$$(\frac{1}{3}\boldsymbol{\delta} - \mathbf{nn})dS = [\frac{1}{3}\boldsymbol{\delta} - \mathbf{e}_r\mathbf{e}_r + Ca(\mathbf{e}_r\nabla_{||}f + \nabla_{||}f\mathbf{e}_r)](1 + 2Caf)d\theta d\varphi \tag{1.53}$$

$$= \left[\frac{1}{3}\boldsymbol{\delta} - \mathbf{e}_r\mathbf{e}_r + Ca[2f(\frac{1}{3}\boldsymbol{\delta} - \mathbf{e}_r\mathbf{e}_r) + \mathbf{e}_r\nabla_{||}f + \nabla_{||}f\mathbf{e}_r]\right]d\theta d\varphi \tag{1.54}$$

$$\tag{1.55}$$

Assuming

$$f = H_{kl}e_k e_l \tag{1.56}$$

$$\int (\delta_{ij} - n_i n_j)dS = \frac{4\pi}{3}\frac{8}{5}H_{ij} - \frac{32\pi}{45}\delta_{ij}H_{kk} \tag{1.57}$$

The volume of the particle is given by,

$$\int_{\Omega_\alpha} d\Omega = \int_{0<r<1}\int_{\theta=0}^\pi\int_{\varphi=0}^{2\pi}(1 + 3Ca\mathbf{H}:\mathbf{e}_r\mathbf{e}_r)r^2\sin\theta dr d\theta d\varphi = \frac{4\pi}{3}(1 + Ca\mathbf{H}:\boldsymbol{\delta}) \tag{1.58}$$

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From the conservation of volume we deduce that

$$\mathbf{H} : \boldsymbol{\delta} = 0 \tag{1.59}$$

$$\mathbf{M}_p = \int_{\Omega_\alpha} \mathbf{r}\mathbf{r} d\Omega = \int r^4 (1 + Caf)^5 \sin\theta \mathbf{e}_r \mathbf{e}_r = \frac{4\pi}{15}\boldsymbol{\delta} + \mathbf{H} : \int r^4 (5Ca\mathbf{e}_r\mathbf{e}_r)\mathbf{e}_r\mathbf{e}_r \sin\theta dr d\theta d\varphi \tag{1.60}$$

$$= \frac{4\pi}{15}[\boldsymbol{\delta} + \mathbf{H}_{kl} : (\boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{kl} + \boldsymbol{\delta}_{ik}\boldsymbol{\delta}_{jl} + \boldsymbol{\delta}_{il}\boldsymbol{\delta}_{jk})] \tag{1.61}$$

$$= \frac{4\pi}{15}[\boldsymbol{\delta} + 2Ca\mathbf{H}] \tag{1.62}$$

$$\mathbf{n} = \mathbf{e}_r - Ca(\boldsymbol{\delta} - \mathbf{e}_r\mathbf{e}_r) \cdot \nabla f = \mathbf{e}_r - Ca\nabla_{||}f \tag{1.63}$$

$$\mathbf{r}^{(n)} = \mathbf{e}_r^{(n)} r^n (1 + Caf)^n = \mathbf{e}_r^{(n)} r^n (1 + nCaf) + O(r^n Ca) \tag{1.64}$$

$$\boldsymbol{\delta}/3 - \mathbf{n}\mathbf{n} = \boldsymbol{\delta}/3 - (\mathbf{e}_r - Ca\nabla_{||}f)(\mathbf{e}_r - Ca\nabla_{||}f) = \boldsymbol{\delta}/3 - \mathbf{e}_r\mathbf{e}_r + Ca[\mathbf{e}_r\nabla_{||}f +^\dagger \mathbf{e}_r\nabla_{||}f] \tag{1.65}$$

$$\mathbf{r}^{(n)}(\boldsymbol{\delta}/3 - \mathbf{n}\mathbf{n}) = \mathbf{e}_r^{(n)}(\boldsymbol{\delta}/3 - \mathbf{e}_r\mathbf{e}_r) + Ca\mathbf{e}_r^{(n)} nf(\boldsymbol{\delta}/3 - \mathbf{e}_r\mathbf{e}_r) + Ca\mathbf{e}_r^{(n)}[\mathbf{e}_r\nabla_{||}f +^\dagger \mathbf{e}_r\nabla_{||}f] \tag{1.66}$$

So that

$$\mathbf{r}^{(n)}(\boldsymbol{\delta}/3 - \mathbf{n}\mathbf{n})dS = \mathbf{r}^{(n)}(\boldsymbol{\delta}/3 - \mathbf{n}\mathbf{n})dS^0 + 2Caf\mathbf{e}_r^{(n)}(\boldsymbol{\delta}/3 - \mathbf{e}_r\mathbf{e}_r)dS^0 \tag{1.67}$$

$$= \mathbf{e}_r^{(n)}(\boldsymbol{\delta}/3 - \mathbf{e}_r\mathbf{e}_r) + Ca\mathbf{e}_r^{(n)}(n+2)f(\boldsymbol{\delta}/3 - \mathbf{e}_r\mathbf{e}_r) + Ca\mathbf{e}_r^{(n)}[\mathbf{e}_r\nabla_{||}f +^\dagger \mathbf{e}_r\nabla_{||}f]dS^0 \tag{1.68}$$

$$= \mathbf{e}_r^{(n)}(\boldsymbol{\delta}/3 - \mathbf{e}_r\mathbf{e}_r) + Ca\mathbf{e}_r^{(n)}[(n+2)f(\boldsymbol{\delta}/3 - \mathbf{e}_r\mathbf{e}_r) + \mathbf{e}_r\nabla_{||}f +^\dagger \mathbf{e}_r\nabla_{||}f]dS^0 \tag{1.69}$$

$$\tag{1.70}$$

So that,

$$\oint_{\Gamma_\alpha} \gamma(\boldsymbol{\delta} - \mathbf{n}\mathbf{n})d\Gamma = \gamma_0 \oint_{\Gamma_\alpha} (\boldsymbol{\delta} - \mathbf{n}\mathbf{n})d\Gamma + \nabla\gamma \cdot \oint_{\Gamma_\alpha} \mathbf{r}(\boldsymbol{\delta} - \mathbf{n}\mathbf{n})d\Gamma + \nabla\nabla\gamma : \oint_{\Gamma_\alpha} \mathbf{r}\mathbf{r}(\boldsymbol{\delta} - \mathbf{n}\mathbf{n})d\Gamma \tag{1.71}$$

where,

$$\oint_{\Gamma_\alpha} (\boldsymbol{\delta}/3 - \mathbf{n}\mathbf{n})_{ij}d\Gamma = \pi\frac{32}{15}(H_{ij} - H_{qq}\delta_{ij}/3)$$

$$\oint_{\Gamma_\alpha} (\boldsymbol{\delta}/3 - \mathbf{n}\mathbf{n})_{ij}r_k d\Gamma = \pi\frac{8}{7}(H_{ijk} - H_{iqq}\delta_{jk}/5 - H_{qjq}\delta_{ik}/5 - H_{qqk}\delta_{ij}/5)$$

$$\oint_{\Gamma_\alpha} (\boldsymbol{\delta}/3 - \mathbf{n}\mathbf{n})_{ij}r_k r_l d\Gamma = \pi\frac{64}{105}H_{ijkl}^{dev} + \pi\frac{16}{35}H_{ij}\delta_{kl} + \pi\frac{32}{315}H_{kl}\delta_{ij} + \pi\frac{16}{315}H_{qq}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$- \pi\frac{8}{15}(H_{ik}\delta_{jl} + H_{jl}\delta_{ik} + H_{il}\delta_{jk} + H_{jk}\delta_{il}) - \pi\frac{176}{945}H_{qq}\delta_{ij}\delta_{kl}$$

$$= \pi\frac{64}{105}H_{ijkl}^{dev} + \pi\frac{16}{35}H_{ij}\delta_{kl} + \pi\frac{32}{315}H_{kl}\delta_{ij} + \pi\frac{16}{315}H_{qq}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$- \pi\frac{8}{15}(H_{ik}\delta_{jl} + H_{jl}\delta_{ik} + H_{il}\delta_{jk} + H_{jk}\delta_{il}) - \pi\frac{176}{945}H_{qq}\delta_{ij}\delta_{kl}$$

$$= \pi\frac{64}{105}H_{ijkl}^{dev} + \pi\frac{16}{35}\{H_{ij}\delta_{kl} + \frac{2}{9}H_{kl}\delta_{ij} + \frac{1}{6}H_{qq}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$- \frac{1}{6}(H_{ik}\delta_{jl} + H_{jl}\delta_{ik} + H_{il}\delta_{jk} + H_{jk}\delta_{il}) - \pi\frac{11}{27}H_{qq}\delta_{ij}\delta_{kl}\}$$

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Or

$$(\nabla\nabla\gamma)_{kl}\oint_{\Gamma_\alpha}(\boldsymbol{\delta}/3-\mathbf{nn})_{ij}r_kr_ld\Gamma=\pi\frac{64}{105}H_{ijkl}^{dev}(\nabla\nabla\gamma)_{kl}$$

$$-\frac{\pi16}{105}[(\nabla\nabla\gamma)_{ik}(\mathbf{H}_p^{(2)})_{jk}+(\nabla\nabla\gamma)_{jk}(\mathbf{H}_p^{(2)})_{ik}-\frac{1}{3}\nabla^2\gamma(\mathbf{H}_p^{(2)})_{ij}-\frac{2}{3}(\nabla\nabla\gamma)_{kl}(\mathbf{H}_p^{(2)})_{kl}\delta_{ij}]$$

Also,

$$b_l\oint_{\Gamma_\alpha}r_kr_l(\boldsymbol{\delta}/3-\mathbf{nn})_{ij}d\Gamma=\pi\frac{64}{105}b_lH_{ijkl}-\frac{176}{945}b_k\delta_{ij}H_{kk}^2-\frac{8}{105}b_l(\delta_{jk}H_{il}+\delta_{ik}H_{jl}-\frac{4}{3}\delta_{ik}H_{jl})$$

$$-\frac{8}{105}(b_iH_{jk}+b_j\delta_{ik})+\frac{16}{315}(b_iH_{ll}\delta_{jk}+b_jH_{ll}\delta_{ik})+\frac{16}{35}(b_kH_{ij})\quad(1.72)$$

1.3.1 Singularity solution and faxen laws

Any flow produced by the motion of a body may be expressed in terms of free-space green function and its derivatives,

$$\mathcal{G}_{ij}=r^{-3}(\delta_{ij}r^2+x_ix_j)=(\delta_{ij}-x_i\partial_j)r^{-1}\quad(1.73)$$
$$\mathcal{G}_{ij,k}=\partial_k(\delta_{ij}-x_i\partial_j)r^{-1}=(\delta_{ij}\partial_k-\delta_{ik}\partial_j-x_i\partial_{jk})r^{-1}\quad(1.74)$$
$$\mathcal{G}_{ij,kl}=(\delta_{ij}\partial_{kl}-\delta_{ik}\partial_{jl}-\delta_{il}\partial_{jk}-x_i\partial_{jkl})r^{-1}\quad(1.75)$$
$$\mathcal{G}_{ij,klm}=(\delta_{ij}\partial_{klm}-\delta_{ik}\partial_{jlm}-\delta_{il}\partial_{jkm}-\delta_{im}\partial_{jkl}-x_i\partial_{jklm})r^{-1}\quad(1.76)$$
$$\mathcal{G}_{ij,klmn}=(\delta_{ij}\partial_{klmn}-\delta_{ik}\partial_{jlmn}-\delta_{il}\partial_{jkmn}-\delta_{im}\partial_{jkln}-\delta_{in}\partial_{jklm}-x_i\partial_{jklmn})r^{-1}\quad(1.77)$$
$$(1.78)$$

Note that the laplacien of teh green func is,

$$\mathcal{G}_{ij,kl}=(-\partial_{ji}-\partial_{ji})r^{-1}=-2\partial_{ji}r^{-1}\quad(1.79)$$
$$\mathcal{G}_{ij,kll}=(\delta_{ij}\partial_{kll}-\delta_{ik}\partial_{jll}-\delta_{il}\partial_{ijk}-\delta_{im}\partial_{ijk}-x_i\partial_{jkll})r^{-1}\quad(1.80)$$
$$\mathcal{G}_{ij,klmm}=(-\delta_{im}\partial_{ijkl}-\delta_{in}\partial_{jkl})r^{-1}=-2\partial_{ijkl}r^{-1}\quad(1.81)$$

Flow field of a droplet in quadratic flow may be expressed in term of spherical harmonics as,

$$\mathcal{P}_{jkl}=(C_0\delta_{kl}\partial_j+C_1\delta_{jl}\partial_k+C_2\delta_{jk}\partial_l+C_3\partial_{jkl})r^{-1}\quad(1.82)$$
$$r\mathcal{U}_{ijkl}=x_i(C_0\delta_{kl}\partial_j+C_1\delta_{jl}\partial_k+C_2\delta_{jk}\partial_l+C_3\partial_{jkl})+C_4\delta_{ij}\delta_{kl}+C_5\delta_{ik}\delta_{jl}+C_6\delta_{il}\delta_{kj}\quad(1.83)$$
$$+C_7\delta_{ij}\partial_{kl}+C_8\delta_{ik}\partial_{jl}+C_9\delta_{il}\partial_{kj}+C_{10}\delta_{kl}\partial_{ij}+C_{11}\delta_{jl}\partial_{ik}+C_{12}\delta_{kj}\partial_{il}+C_{13}\partial_{ijkl}\quad(1.84)$$

If one need to find a flow in a $\nabla\nabla\nabla\mathbf{u}=\partial_{ijk}u_l$ field he must shear for the constant of the form of,

$$\mathcal{P}_{jklm}=(C_0\delta_{kl}\partial_j+C_1\delta_{jl}\partial_k+C_2\delta_{jk}\partial_l+C_3\partial_{jkl})r^{-1}\quad(1.85)$$
$$r\mathcal{U}_{ijkl}=x_i(C_0\delta_{kl}\partial_j+C_1\delta_{jl}\partial_k+C_2\delta_{jk}\partial_l+C_3\partial_{jkl})+C_4\delta_{ij}\delta_{kl}+C_5\delta_{ik}\delta_{jl}+C_6\delta_{il}\delta_{kj}\quad(1.86)$$
$$+C_7\delta_{ij}\partial_{kl}+C_8\delta_{ik}\partial_{jl}+C_9\delta_{il}\partial_{kj}+C_{10}\delta_{kl}\partial_{ij}+C_{11}\delta_{jl}\partial_{ik}+C_{12}\delta_{kj}\partial_{il}+C_{13}\partial_{ijkl}\quad(1.87)$$

2 deformable droplet singularity solution

2.1 Physical problem

The points $\mathbf{r}_\Gamma(t)$ lying on the droplet surface or volume may be computed as,

$$\mathbf{r}_\Gamma=r[1+f(\theta,\varphi)]\mathbf{e}_r\quad(2.1)$$

In dimensionless form $(\mathbf{u}_{out}, \mathbf{u}_{in})$ are governed by the Stokes equations, namely,

$$\nabla \cdot \mathbf{u}_{in} = 0 \qquad\qquad\qquad\qquad \nabla \cdot \mathbf{u}_{out} = 0 \tag{2.2}$$

$$-\nabla p_{in} + \mu_{in}\nabla^2\mathbf{u}_{in} = 0 \qquad\qquad\qquad -\nabla p_{out} + \mu_{out}\nabla^2\mathbf{u}_{out} = 0 \tag{2.3}$$

As these field are disturbance field they follow,

$$\lim_{r\to\infty}(\mathbf{u}_{out}, p_{out}) = 0 \tag{2.4}$$

far from the droplet centered at the origin $(\mathbf{x} = 0)$. At the surface of the droplet $(r = 1 + f(\theta, \varphi))$ the continuity of velocity and the far field condition impose,

$$\mathbf{u}_{in} = \mathbf{u}_{out} \qquad (\mathbf{u}_{in} + \mathbf{u} - \mathbf{u}_\Gamma)\cdot\mathbf{n} = (\mathbf{u}_{in} + \mathbf{u}_r)\cdot\mathbf{n} = 0 \qquad \text{at } r = 1 + f(\theta, \varphi) \tag{2.5}$$

where it must be noted that both $\mathbf{u}_{in}$ and $\mathbf{u}$ are evaluated at the points on the surface of the droplet, hence $\mathbf{u}$ may not always be a constant vector. $\mathbf{u}_\Gamma$ is the interface of teh droplet, which may be replaced by the droplet center of mass for undeformable droplets, otherwise it is related to the moment of momentum and mass.

Let us introduce the dimensionless stress tensor $\boldsymbol{\sigma}_{in/out} = -p_{in/out}\boldsymbol{\delta} + \mu_{in/out}[\nabla\mathbf{u}_{in/out} +^\dagger \nabla\mathbf{u}_{in/out}]$ and the dimensionless shear stress tensor, $2\mathbf{e}_{in/out} = \nabla\mathbf{u}_{in/out} +^\dagger \nabla\mathbf{u}_{in/out}$. The undisturbed shear stress reads $2\mathbf{e} = \nabla\mathbf{u} +^\dagger \nabla\mathbf{u}$ With these notations the tangential shear stress jump reads at $r = 1$ reads as,

$$\mathbf{n}\cdot(\mathbf{e}_{out} - \lambda\mathbf{e}_{in} + \mathbf{e} - \lambda\mathbf{e})\cdot(\boldsymbol{\delta} - \mathbf{nn}) = (\boldsymbol{\delta} - \mathbf{nn})\cdot\mathbf{b} \qquad \text{at } r = 1 + f(\theta, \varphi) \tag{2.6}$$

with,

$$\mathbf{b} = \frac{\nabla\gamma}{2\mu_f} \tag{2.7}$$

The velocity $\mathbf{u}$ is imposed, the velocity $\mathbf{u}_\Gamma$ is just the velocity of the interface which may be obtained by differetiating the point lying on the interface, namely,

$$\mathbf{u}_\Gamma = \mathbf{u}_\alpha + \partial_t\mathbf{r}_\Gamma = \mathbf{u}_\alpha + \mathbf{e}_r\partial_t f(\theta, \varphi, t) \tag{2.8}$$

Dimensionless problem

The relative velocity at the interfaces as well as the mean stresss may be expanded as a taylor series such that,

$$\mathbf{u}_r(\mathbf{n}) = \mathbf{u}_r|_{\mathbf{x}=0} + \mathbf{r}\cdot\nabla\mathbf{u}_r|_{\mathbf{x}=0} + \ldots + \frac{1}{n!}\mathbf{r}^{(n)}\odot\nabla^{(n)}\mathbf{u}_r|_{\mathbf{x}=0}$$

$$\mathbf{e}(\mathbf{n}) = \mathbf{e}|_{\mathbf{x}=0} + \mathbf{r}\cdot\nabla\mathbf{e}|_{\mathbf{x}=0} + \ldots + \frac{1}{n!}\mathbf{r}^{(n)}\odot\nabla^{(n)}\mathbf{e}|_{\mathbf{x}=0}$$

Note that if the droplet is rotating or deforming it means that $\mathbf{u}_\Gamma = \mathbf{u}_\alpha + \mathbf{r}\cdot\mathbf{P} + \ldots$ hence these boundaries includes the fact that droplets deform.

Because of these relations one may be able to re-formulate the above problem in terms of generalized tensor such that,

$$\mathbf{u}_{out/in} = \mathbf{U}_{out/in}^{(n+2)}\odot\nabla^{(n)}\mathbf{u}_r \tag{2.9}$$

$$p_{out/in} = \mu_{out/in}\mathbf{P}_{out/in}^{(n+1)}\odot\nabla^{(n)}\mathbf{u}_r \tag{2.10}$$

$$\boldsymbol{\sigma}_{out/in} = \mu_{out/in}\boldsymbol{\Sigma}_{out/in}^{(n+3)}\odot\nabla^{(n)}\mathbf{u}_r = \mu_{out/in}[\boldsymbol{\delta}\mathbf{P}_{out/in}^{(n+1)} + \nabla\mathbf{U}_{out/in}^{(n+2)} +^\dagger \nabla\mathbf{U}_{out/in}^{(n+2)}]\odot\nabla^{(n)}\mathbf{u}_r \tag{2.11}$$

$$\mathbf{e}_{out/in} = 2\mathbf{E}_{out/in}^{(n+3)}\odot\nabla^{(n)}\mathbf{u}_r = [\nabla\mathbf{U}_{in}^{(n+2)} +^\dagger \nabla\mathbf{U}_{in}^{(n+2)}]\odot\nabla^{(n)}\mathbf{u}_r \tag{2.12}$$

$$\mathbf{b} = \mathbf{B}^{(n+1)}\odot\nabla^{(n)}\mathbf{u} \tag{2.13}$$

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where $\odot$ represent the maximum contraction operators. Note that $\mathbf{B} = \boldsymbol{\delta}$ means that $\mathbf{b}$ is aligned with $\mathbf{u}_r$. Hence the generalized tensor $\mathbf{B}$ indicated that one might choose a vector $\mathbf{b}$ such that it is in an other direction than $\mathbf{u}_r$ which is physically unlikely. We follow the convention of **?** for the transpose operator[†]. The mean averaged stress can then be written,

$$\frac{2}{(n-1)!}\mathbf{r}^{(n\text{-}1)} \odot \nabla^{(n\text{-}1)}\mathbf{e} = \frac{1}{(n-1)!}\mathbf{r}^{(n\text{-}1)} \odot \nabla^{(n\text{-}1)}[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\dagger}]$$

$$= \frac{1}{(n-1)!}\mathbf{r}^{(n\text{-}1)} \odot [\nabla^{(n)}\mathbf{u} + \nabla^{(n)}\mathbf{u}^{\dagger}] = \frac{1}{(n-1)!}\mathbf{r}^{(n\text{-}1)}[\boldsymbol{\delta}\boldsymbol{\delta} + (\boldsymbol{\delta}\boldsymbol{\delta})^{\dagger}] \odot \nabla^{(n)}\mathbf{u} \quad (2.14)$$

In index notation the mean stress could be noted,

$$e_{ij} = \frac{1}{(n-1)!}(\mathbf{r}^{(n\text{-}1)})_{k_1 k_2 \ldots k_{n-1}}[(\nabla^{(n)}\mathbf{u})_{k_1 k_2 \ldots k_{n-1}ij} + (\nabla^{(n)}\mathbf{u})_{k_1 k_2 \ldots k_{n-1}ji}]$$

$$= \frac{1}{(n-1)!}(\mathbf{r}^{(n\text{-}1)})_{k_1 k_2 \ldots k_{n-1}}[\delta_{ik_n}\delta_{jk_{n+1}} + \delta_{jk_n}\delta_{ik_{n+1}}](\nabla^{(n)}\mathbf{u})_{k_1 k_2 \ldots k_{n-1}k_n k_{n+1}} \quad (2.15)$$

The interface is assumed already deformed, following the form,

$$f(\varphi, \theta, t) = \sum_{n=2}^{\infty} \mathbf{S}^{(n)} \odot \mathbf{H}^{(n)} \tag{2.16}$$

$$\partial_t f(\varphi, \theta, t) = \sum_{n=2}^{\infty} \mathbf{S}^{(n)} \odot \dot{\mathbf{H}}^{(n)} \tag{2.17}$$

So that in general the equation to solve becomes,

$$\nabla \cdot \mathbf{U}_{in/out}^{(n+2)} = 0$$

$$\nabla^2 \mathbf{U}_{in/out}^{(n+2)} = \nabla\mathbf{P}_{in/out}^{(n+1)}$$

with the boundary condition,

$$\mathbf{U}_{in}^{(n+2)} - \mathbf{U}_{out}^{(n+2)} = 0 \quad \forall r = 1 + f(\theta, \varphi) \tag{2.18}$$

$$\mathbf{n} \cdot [\mathbf{U}_{in}^{(n+2)} + \frac{1}{n!}\boldsymbol{\delta}\mathbf{r}^{(n)}] = 0 \quad \forall r = 1 + f(\theta, \varphi) \tag{2.19}$$

$$\mathbf{n} \cdot \{\mathbf{E}_{out}^{(n+3)} - \lambda\mathbf{E}_{in}^{(n+3)} + \frac{(1-\lambda)}{(n-1)!}\mathbf{r}^{(n-1)}[\boldsymbol{\delta}\boldsymbol{\delta} + (\boldsymbol{\delta}\boldsymbol{\delta})^{\dagger}] - \mathbf{n}\mathbf{B}^{(n+2)}\} \cdot (\boldsymbol{\delta} - \mathbf{n}\mathbf{n}) = 0 \quad \forall r = 1 + f(\theta, \varphi) \tag{2.20}$$

note that we could factor by teh whole $\mathbf{u}_r$ hence the velocity of deformation doesn't comes into account at this stage.

Droplet deformation

Because we cannot apply these boundary condition directly on the interface $r = 1 + f$ we assume $f \to \epsilon f$ where ϵ is small and f of $O(1)$ and express any of the above funciton as a taylor expansion in terms of ϵ such that, any tensor $\mathbf{F}$ can be expressed as,

$$\mathbf{F} = {}_0\mathbf{F} + \epsilon\, {}_1\mathbf{F} + O(\epsilon^2) \tag{2.21}$$

Then, any function to be evaluated on the droplet interface might be written as,

$$\mathbf{F}(r = 1 + \epsilon f) = \mathbf{F}|_{r=1} + \epsilon f \frac{\partial \mathbf{F}}{\partial r}\Big|_{r=1} = {}_0\mathbf{F}|_{r=1} + \epsilon \left[{}_1\mathbf{F} + f\frac{\partial\, {}_0\mathbf{F}}{\partial r}\right]_{r=1} + O(\epsilon) \tag{2.22}$$

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the position vector in perturbed coordinates may be approximated by,

$$\mathbf{r}^{(n)} = \mathbf{e}_r^{(n)} r^n (1 + \epsilon f)^n = \mathbf{e}_r^{(n)} r^n (1 + n\epsilon f) + O(r^n \epsilon) \tag{2.23}$$

where $\mathbf{e}_r = {}_0\mathbf{n}$ is the unit radial vector going outward the interface. Not ethat the approximation is true only for small r however.

Likewise,

$$\mathbf{n} = \mathbf{e}_r - \epsilon(\boldsymbol{\delta} - \mathbf{e}_r \mathbf{e}_r) \cdot \nabla f = \mathbf{e}_r - \epsilon \nabla_{||} f \tag{2.24}$$

$$\boldsymbol{\delta} - \mathbf{nn} = \boldsymbol{\delta} - (\mathbf{e}_r - \epsilon \nabla_{||} f)(\mathbf{e}_r - \epsilon \nabla_{||} f) = \boldsymbol{\delta} - \mathbf{e}_r \mathbf{e}_r + \epsilon[\mathbf{e}_r \nabla_{||} f +^\dagger \mathbf{e}_r \nabla_{||} f] \tag{2.25}$$

$$\mathbf{n}(\boldsymbol{\delta} - \mathbf{nn}) = \mathbf{e}_r(\boldsymbol{\delta} - \mathbf{e}_r \mathbf{e}_r) - \epsilon \nabla_{||} f(\boldsymbol{\delta} - \mathbf{e}_r \mathbf{e}_r) + \epsilon \mathbf{e}_r[\mathbf{e}_r \nabla_{||} f +^\dagger \mathbf{e}_r \nabla_{||} f] \tag{2.26}$$

Expanding any of tehse terms and isolating any power of ϵ yields, at $O(\epsilon^0)$, the following system of equations,

$$\nabla \cdot {}_0\mathbf{U}_{in/out}^{(n+2)} = 0$$

$$\nabla^2 {}_0\mathbf{U}_{in/out}^{(n+2)} = \nabla {}_0\mathbf{P}_{in/out}^{(n+1)}$$

with the boundary condition,

$${}_0\mathbf{U}_{in}^{(n+2)} - {}_0\mathbf{U}_{out}^{(n+2)} = 0 \quad \forall r = 1 \tag{2.27}$$

$$\mathbf{e}_r \cdot [{}_0\mathbf{U}_{in}^{(n+2)} + \frac{r^n}{n!} \boldsymbol{\delta} \mathbf{e}_r^{(n)}] = 0 \quad \forall r = 1 \tag{2.28}$$

$$\mathbf{e}_r \cdot \{{}_0\mathbf{E}_{out}^{(n+3)} - \lambda {}_0\mathbf{E}_{in}^{(n+3)} + \frac{(1-\lambda)r^{n-1}}{(n-1)!} \mathbf{e}_r^{(n-1)}[\boldsymbol{\delta}\boldsymbol{\delta} + (\boldsymbol{\delta}\boldsymbol{\delta})^\dagger] - \mathbf{e}_r {}_0\mathbf{B}^{(n+2)}\} \cdot (\boldsymbol{\delta} - \mathbf{e}_r \mathbf{e}_r) = 0 \quad \forall r = 1 \tag{2.29}$$

At the next order in $O(Ca)$ we obtain the following system,

$$\nabla \cdot {}_1\mathbf{U}_{in/out}^{(n+2)} = 0$$

$$\nabla^2 {}_1\mathbf{U}_{in/out}^{(n+2)} = \nabla {}_1\mathbf{P}_{in/out}^{(n+1)}$$

with the boundary condition,

$${}_1\mathbf{U}_{in}^{(n+2)} - {}_1\mathbf{U}_{out}^{(n+2)} + f\frac{\partial}{\partial r}[{}_0\mathbf{U}_{in}^{(n+2)} - {}_0\mathbf{U}_{out}^{(n+2)}] = 0 \quad \forall r = 1 \tag{2.30}$$

$$\mathbf{e}_r \cdot [{}_1\mathbf{U}_{in}^{(n+2)} + f\frac{\partial}{\partial r} {}_0\mathbf{U}_{in}^{(n+2)} + \frac{r^n}{(n-1)!} \boldsymbol{\delta} \mathbf{e}_r^{(n)} f] - \nabla_{||} f \cdot [{}_0\mathbf{U}_{in}^{(n+2)} + \frac{r^n}{n!} \boldsymbol{\delta} \mathbf{e}_r^{(n)}] = 0 \quad \forall r = 1 \tag{2.31}$$

the last BC is even more involving it reads,

$$\mathbf{e}_r \cdot \{{}_1\mathbf{E}_{out}^{(n+3)} - \lambda {}_1\mathbf{E}_{in}^{(n+3)} + f\frac{\partial}{\partial r}({}_0\mathbf{E}_{out}^{(n+3)} - \lambda {}_0\mathbf{E}_{in}^{(n+3)})$$

$$+ \frac{(1-\lambda)r^{n-1}}{(n-2)!} f\mathbf{e}_r^{(n-1)}[\boldsymbol{\delta}\boldsymbol{\delta} + (\boldsymbol{\delta}\boldsymbol{\delta})^\dagger] - {}_1\mathbf{B}^{(n+2)} - f\frac{\partial}{\partial r}\mathbf{B}^{(n+2)}\} \cdot (\boldsymbol{\delta} - \mathbf{e}_r \mathbf{e}_r)$$

$$+ [\mathbf{e}_r(\mathbf{e}_r \nabla_{||} f +^\dagger \mathbf{e}_r \nabla_{||} f) - \nabla_{||} f(\boldsymbol{\delta} - \mathbf{e}_r \mathbf{e}_r)]$$

$$: \{{}_0\mathbf{E}_{out}^{(n+3)} - \lambda {}_0\mathbf{E}_{in}^{(n+3)} + \frac{(1-\lambda)r^{n-1}}{(n-1)!} \mathbf{e}_r^{(n-1)}[\boldsymbol{\delta}\boldsymbol{\delta} + (\boldsymbol{\delta}\boldsymbol{\delta})^\dagger] - {}_0\mathbf{B}^{(n+2)}\} = 0 \tag{2.32}$$

solving this problem yields the final sol.

One can then factor out by any order of deformation and find

$$\frac{1}{a\pi\mu_f}\oint_{\Gamma_\alpha}\boldsymbol{\sigma}\cdot\mathbf{n}d\Gamma = 2\frac{3\lambda+2}{\lambda+1}\mathbf{u}_r + Ca\frac{(9\lambda^2+11\lambda+20)}{5(\lambda+1)^2}\left[\mathbf{H}_p+\mathbf{H}_p^\dagger-\frac{2}{3}(\mathbf{H}_p:\boldsymbol{\delta})\boldsymbol{\delta}\right]\cdot\mathbf{u}_r + O(Ca^2) \quad (2.33)$$

since

$$\mathbf{H}_p = \frac{19\lambda+16}{8(\lambda+1)}\left(\frac{a\mu_f}{\gamma Ca}\right)\mathbf{E}_f = \frac{19\lambda+16}{8(\lambda+1)}\left(\frac{a}{U}\right)\mathbf{E}_f = \frac{19\lambda+16}{8(\lambda+1)}\mathbf{E}_f^* \quad (2.34)$$

hence $\mathbf{H}$ is proportional to the dimentionless shear not the shear it self (which make sense because it has no unit).

$$\frac{1}{a\pi\mu_f}\oint_{\Gamma_\alpha}\boldsymbol{\sigma}\cdot\mathbf{n}d\Gamma = 2\frac{3\lambda+2}{\lambda+1}\mathbf{u}_r + Ca\frac{(9\lambda^2+11\lambda+20)(19\lambda+16)}{40(\lambda+1)^3}\left[2\mathbf{E}_f^*-\frac{2}{3}(\mathbf{E}_f^*:\boldsymbol{\delta})\boldsymbol{\delta}\right]\cdot\mathbf{u}_r \quad (2.35)$$

One may introduce the scales $a\pi\mu_f\mathbf{u}_r \to a\pi\mu_f U\mathbf{u}_r^*$ hence;

$$\frac{1}{a\pi\mu_f U}\oint_{\Gamma_\alpha}\boldsymbol{\sigma}\cdot\mathbf{n}d\Gamma = 2\frac{3\lambda+2}{\lambda+1}\mathbf{u}_r + Ca\frac{(9\lambda^2+11\lambda+20)(19\lambda+16)}{40(\lambda+1)^3}\left[2\mathbf{E}_f^*-\frac{2}{3}(\mathbf{E}_f^*:\boldsymbol{\delta})\boldsymbol{\delta}\right]\cdot\mathbf{u}_r^* \quad (2.36)$$

Hence the coupling terms in the dimensioneless drag force (left-hand side) are of order Ca !

2.1.1 Perturbed solution of the Hybrid-model

Remark that at this order in accuracy $\mathcal{O}(Ca^0)$ the linear momentum equations are not coupled with the dispersed phase moments ($\mathbf{S}_p$, $\boldsymbol{\mu}_p$, and $\mathbf{M}_p$). Consequently, in this first approach ?????? are not needed. Hence, one can simply inject ?? to ??, into ?? and ?? to obtain a closed form of the hybrid model, namely,

$$\phi_f + \phi = 1 \tag{2.37}$$

$$\mathbf{u}_f = \mathbf{u} + \phi\mathbf{u}_r/\phi_f \approx \mathbf{u} + \phi\mathbf{u}_r \tag{2.38}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{2.39}$$

$$(\partial_t + \mathbf{u}\cdot\nabla)\phi = \nabla\cdot(\phi\mathbf{u}_r) \tag{2.40}$$

$$\rho_d\phi(\partial_t + \mathbf{u}_p\cdot\nabla)\mathbf{u}_p = \phi(\nabla\cdot\boldsymbol{\Sigma}+\rho_d\mathbf{g}) + \nabla\cdot\boldsymbol{\sigma}_p^{\text{eff}} + \mathbf{F} \tag{2.41}$$

$$\phi_f\rho_f(\partial_t + \mathbf{u}_f\cdot\nabla)\mathbf{u}_f = \phi_f(\nabla\cdot\boldsymbol{\Sigma}+\rho_f\mathbf{g}) + \nabla\cdot\boldsymbol{\sigma}^{\text{eff}} - \mathbf{F} \tag{2.42}$$

$$\tag{2.43}$$

$$\mathbf{F} = \phi\frac{\mu_f}{a^2}\frac{3(2+3\lambda)}{2(1+\lambda)}\mathbf{u}_r + \phi\mu_f\frac{3\lambda}{4(\lambda+1)}\nabla^2\mathbf{u} \tag{2.44}$$

$$\boldsymbol{\sigma}_p^{\text{eff}} = -\rho_d C_{up}^1(\phi,\lambda)\mathbf{u}_r\mathbf{u}_r - \rho_d C_{up}^2(\phi,\lambda)(\mathbf{u}_r\cdot\mathbf{u}_r)\boldsymbol{\delta} \tag{2.45}$$

$$\boldsymbol{\sigma}_f^{\text{eff}} = \mu_f\phi\frac{5\lambda+2}{\lambda+1}\mathbf{E} + \mu_f\frac{3\lambda}{4(\lambda+1)}[\nabla(\phi\mathbf{u}_r)+\nabla(\phi\mathbf{u}_r)^\dagger] - \mu_f\frac{3\lambda-2}{4(\lambda+1)}\nabla\cdot(\phi\mathbf{u}_r)\boldsymbol{\delta}$$
$$ -\rho_f C_{uu}^1(\phi,\lambda)\mathbf{u}_r\mathbf{u}_r - \rho_f C_{uu}^2(\phi,\lambda)(\mathbf{u}_r\cdot\mathbf{u}_r)\boldsymbol{\delta} \tag{2.46}$$

2.1.2 We neglect the $O(Re)$ terms

Or in an other form the total momentum ea,

$$\nabla \cdot \mathbf{u} = 0 \tag{2.47}$$

$$(\partial_t + \mathbf{u} \cdot \nabla)\phi = \nabla \cdot (\mathbf{w}) \tag{2.48}$$

$$0 = -\nabla p_f + \mu_f \nabla^2 \mathbf{u} + (\rho_f + \phi\rho)\mathbf{g} + \nabla \cdot (\boldsymbol{\sigma}^{\text{eff}}) \tag{2.49}$$

$$\boldsymbol{\sigma}^{\text{eff}} = \mu_f \phi \frac{5\lambda + 2}{\lambda + 1}(\nabla\mathbf{u} +^{\dagger} \nabla\mathbf{u}) + \mu_f \frac{3\lambda}{4(\lambda + 1)}\nabla\mathbf{w} + \mu_f \frac{2}{4(\lambda + 1)}(\nabla \cdot \mathbf{w})\boldsymbol{\delta} \tag{2.50}$$

$$\mathbf{w} = -\phi\rho\frac{a^2 2(\lambda + 1)}{\mu_f 3(3\lambda + 2)}\mathbf{g} - \frac{a^2 \lambda}{2(3\lambda + 2)}\phi\nabla^2\mathbf{u} \tag{2.51}$$

$$\tag{2.52}$$

where $\rho = \rho_d - \rho_f$ and ,

$$-\phi\nabla^2 p_f = O(\phi^2) \tag{2.53}$$

$$\phi(-\nabla p_f + \mu_f \nabla^2 \mathbf{u}) = -\phi\mathbf{g}\rho_f + O(\phi^2) \tag{2.54}$$

$$\phi(-\nabla\nabla p_f + \mu_f \nabla^2 \nabla\mathbf{u}) = O(\phi^2) \tag{2.55}$$

$$\phi\nabla^4 \mathbf{u} = O(\phi^2) \tag{2.56}$$

Let assume a small perturbed state aroud ($\phi_0 = cst$ and $\mathbf{u}_0 = 0$) such that,

$$\phi = \phi_0 + \varphi$$

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}$$

$$p = p_0 + q$$

$$\mathbf{w} = \mathbf{w}_0 + \mathbf{w}_1$$

$$\boldsymbol{\sigma}^{\text{eff}} = \boldsymbol{\sigma}_0^{\text{eff}} + \boldsymbol{\sigma}_1^{\text{eff}}$$

Hence the equilibrium varibale follows,

$$\mathbf{u}_0 \cdot \nabla\phi_0 = \nabla \cdot \mathbf{w}_0$$

$$0 = -\nabla p_0 + (\rho_f + \phi_0\rho)\mathbf{g} + \nabla \cdot (\boldsymbol{\sigma}_0^{\text{eff}})$$

$$0 = -\nabla^2 p_0 + \nabla\nabla : \boldsymbol{\sigma}_0^{\text{eff}}$$

$$\boldsymbol{\sigma}_0^{\text{eff}} = +\mu_f \frac{3\lambda}{4(\lambda + 1)}\nabla\mathbf{w}_0 + \mu_f \frac{2}{4(\lambda + 1)}(\nabla \cdot \mathbf{w}_0)\boldsymbol{\delta}$$

$$\mathbf{w}_0 = -\phi_0\rho\frac{a^2 2(\lambda + 1)}{\mu_f 3(3\lambda + 2)}\mathbf{g}$$

Including the perturbed sol neglecting the non-linear termes and removing the stable eq gives

$$\partial_t \varphi = \nabla \cdot \mathbf{w}_1$$

$$0 = -\nabla q + \mu_f \nabla^2 \mathbf{v} + \varphi\rho\mathbf{g} + \nabla \cdot \boldsymbol{\sigma}_1^{\text{eff}}$$

$$0 = -\nabla^2 q + \rho\mathbf{g} \cdot \nabla\varphi + \nabla\nabla : \boldsymbol{\sigma}_1^{\text{eff}}$$

$$\boldsymbol{\sigma}_1^{\text{eff}} = \mu_f \phi_0 \frac{5\lambda + 2}{\lambda + 1}(\nabla\mathbf{v} +^{\dagger} \nabla\mathbf{v}) + \mu_f \frac{3\lambda}{4(\lambda + 1)}\nabla\mathbf{w}_1 + \mu_f \frac{2}{4(\lambda + 1)}\nabla \cdot \mathbf{w}_1\boldsymbol{\delta}$$

$$\mathbf{w}_1 = -\varphi\rho\frac{a^2 2(\lambda + 1)}{\mu_f 3(3\lambda + 2)}\mathbf{g} - \frac{a^2 \lambda}{2(3\lambda + 2)}\phi_0\nabla^2\mathbf{v}$$

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of course we recall that $\nabla \cdot \mathbf{v} = 0$. Hence the last two equation may be injected in the first three one to get three eq. for $\varphi, q, \mathbf{v}$ namely,

$$\partial_t \varphi = -\rho \frac{a^2 2(\lambda+1)}{\mu_f 3(3\lambda+2)} \mathbf{g} \cdot \nabla \varphi$$

$$0 = -\nabla q + \mu_f(1 + \phi_0 \frac{5\lambda+2}{\lambda+1})\nabla^2 \mathbf{v} + \varphi \rho \mathbf{g} - \frac{a^2}{6(3\lambda+2)}\rho \mathbf{g} \cdot (+3\lambda \boldsymbol{\delta}\nabla^2 + 2\nabla\nabla)\varphi$$

$$0 = -\nabla^2 q + \rho \mathbf{g} \cdot (1 - \frac{a^2}{6}\nabla^2)\nabla\varphi$$

$$\boldsymbol{\sigma}_1^{\text{eff}} = \mu_f \phi_0 \frac{5\lambda+2}{\lambda+1}(\nabla\mathbf{v} +^\dagger \nabla\mathbf{v})$$

$$+ \mu_f \frac{3\lambda}{4(\lambda+1)}(-\rho \frac{a^2 2(\lambda+1)}{\mu_f 3(3\lambda+2)}\mathbf{g}\nabla\varphi - \frac{a^2\lambda}{2(3\lambda+2)}\phi_0\nabla^2\nabla\mathbf{v})$$

$$+ \mu_f \frac{2}{4(\lambda+1)}(-\rho \frac{a^2 2(\lambda+1)}{\mu_f 3(3\lambda+2)}\mathbf{g} \cdot \nabla\varphi)\boldsymbol{\delta}$$

$$\mathbf{w}_1 = -\varphi\rho \frac{a^2 2(\lambda+1)}{\mu_f 3(3\lambda+2)}\mathbf{g} - \frac{a^2\lambda}{2(3\lambda+2)}\phi_0\nabla^2\mathbf{v}$$

Using fourier transform   Let now consider the fourier transform,

$$\mathbf{A}(\mathbf{k}, \omega) = \iiiint \mathbf{A}(\mathbf{x}, t)e^{-i(\mathbf{k}\cdot\mathbf{x}+\omega t)}d\mathbf{k}d\omega \tag{2.57}$$

So the the above system of equaiton reads,

$$\omega\varphi = -\rho \frac{a^2 2(\lambda+1)}{\mu_f 3(3\lambda+2)}\mathbf{g} \cdot \mathbf{k}\varphi \tag{2.58}$$

$$0 = i\mathbf{k}q - \mu_f(1 + \phi_0 \frac{5\lambda+2}{\lambda+1})k^2\mathbf{v} + \varphi\rho\mathbf{g} - \frac{a^2}{6(3\lambda+2)}\rho\mathbf{g} \cdot (-3\lambda\boldsymbol{\delta}k^2 - 2\mathbf{k}\mathbf{k})\varphi \tag{2.59}$$

$$0 = +k^2 q - \rho\mathbf{g} \cdot (1 + \frac{a^2}{6}k^2)i\mathbf{k}\varphi \tag{2.60}$$

Or in matrix form,

$$\omega \begin{pmatrix} \varphi \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\rho\frac{a^2 2(\lambda+1)}{\mu_f 3(3\lambda+2)}\mathbf{g} \cdot \mathbf{k} & 0 & 0 \\ \rho\mathbf{g} + \frac{a^2}{6(3\lambda+2)}\rho\mathbf{g} \cdot (3\lambda\boldsymbol{\delta}k^2 + 2\mathbf{k}\mathbf{k}) & -\mu_f(1 + \phi_0\frac{5\lambda+2}{\lambda+1})k^2 & i\mathbf{k} \\ -\rho\mathbf{g} \cdot (1 + \frac{a^2}{6}k^2)i\mathbf{k} & 0 & k^2 \end{pmatrix} \cdot \begin{pmatrix} \varphi \\ \mathbf{v} \\ q \end{pmatrix} \tag{2.61}$$

3   Perturbed solution of the Hybrid-model

Remark that at this order in accuracy $\mathcal{O}(Ca^0)$ the linear momentum equations are not coupled with the dispersed phase moments ($\mathbf{S}_p, \boldsymbol{\mu}_p$, and $\mathbf{M}_p$). Consequently, in this first approach ?????? are not needed.

Hence, one can simply inject **??** to **??**, into **??** and **??** to obtain a closed form of the hybrid model, namely,

$$\phi_f + \phi = 1 \tag{3.1}$$

$$\mathbf{u}_f = \mathbf{u} + \phi\mathbf{u}_r/\phi_f \approx \mathbf{u} + \phi\mathbf{u}_r \tag{3.2}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{3.3}$$

$$(\partial_t + \mathbf{u} \cdot \nabla)\phi = \nabla \cdot (\phi\mathbf{u}_r) \tag{3.4}$$

$$\rho_d\phi(\partial_t + \mathbf{u}_p \cdot \nabla)\mathbf{u}_p = \phi(\nabla \cdot \boldsymbol{\Sigma} + \rho_d\mathbf{g}) + \nabla \cdot \boldsymbol{\sigma}_p^{\text{eff}} + \mathbf{F} \tag{3.5}$$

$$\phi_f\rho_f(\partial_t + \mathbf{u}_f \cdot \nabla)\mathbf{u}_f = \phi_f(\nabla \cdot \boldsymbol{\Sigma} + \rho_f\mathbf{g}) + \nabla \cdot \boldsymbol{\sigma}^{\text{eff}} - \mathbf{F} \tag{3.6}$$

$$\tag{3.7}$$

$$\mathbf{F} = \phi\frac{\mu_f}{a^2}\frac{3(2 + 3\lambda)}{2(1 + \lambda)}\mathbf{u}_r + \phi\mu_f\frac{3\lambda}{4(\lambda + 1)}\nabla^2\mathbf{u} \tag{3.8}$$

$$\boldsymbol{\sigma}_p^{\text{eff}} = -\rho_d C_{up}^1(\phi, \lambda)\mathbf{u}_r\mathbf{u}_r - \rho_d C_{up}^2(\phi, \lambda)(\mathbf{u}_r \cdot \mathbf{u}_r)\boldsymbol{\delta} \tag{3.9}$$

$$\boldsymbol{\sigma}_f^{\text{eff}} = \mu_f\phi\frac{5\lambda + 2}{\lambda + 1}\mathbf{E} + \mu_f\frac{3\lambda}{4(\lambda + 1)}[\nabla(\phi\mathbf{u}_r) + \nabla(\phi\mathbf{u}_r)^\dagger] - \mu_f\frac{3\lambda - 2}{4(\lambda + 1)}\nabla \cdot (\phi\mathbf{u}_r)\boldsymbol{\delta}$$
$$\quad - \rho_f C_{uu}^1(\phi, \lambda)\mathbf{u}_r\mathbf{u}_r - \rho_f C_{uu}^2(\phi, \lambda)(\mathbf{u}_r \cdot \mathbf{u}_r)\boldsymbol{\delta} \tag{3.10}$$

3.0.1   We neglect the $O(Re)$ terms

Or in an other form the total momentum ea,

$$\nabla \cdot \mathbf{u} = 0 \tag{3.11}$$

$$(\partial_t + \mathbf{u} \cdot \nabla)\phi = \nabla \cdot (\mathbf{w}) \tag{3.12}$$

$$0 = -\nabla p_f + \mu_f\nabla^2\mathbf{u} + (\rho_f + \phi\rho)\mathbf{g} + \nabla \cdot (\boldsymbol{\sigma}^{\text{eff}}) \tag{3.13}$$

$$\boldsymbol{\sigma}^{\text{eff}} = \mu_f\phi\frac{5\lambda + 2}{\lambda + 1}(\nabla\mathbf{u} +^\dagger \nabla\mathbf{u}) + \mu_f\frac{3\lambda}{4(\lambda + 1)}\nabla\mathbf{w} + \mu_f\frac{2}{4(\lambda + 1)}(\nabla \cdot \mathbf{w})\boldsymbol{\delta} \tag{3.14}$$

$$\mathbf{w} = -\phi\rho\frac{a^2 2(\lambda + 1)}{\mu_f 3(3\lambda + 2)}\mathbf{g} - \frac{a^2\lambda}{2(3\lambda + 2)}\phi\nabla^2\mathbf{u} \tag{3.15}$$

$$\tag{3.16}$$

where $\rho = \rho_d - \rho_f$ and ,

$$-\phi\nabla^2 p_f = O(\phi^2) \tag{3.17}$$

$$\phi(-\nabla p_f + \mu_f\nabla^2\mathbf{u}) = -\phi\mathbf{g}\rho_f + O(\phi^2) \tag{3.18}$$

$$\phi(-\nabla\nabla p_f + \mu_f\nabla^2\nabla\mathbf{u}) = O(\phi^2) \tag{3.19}$$

$$\phi\nabla^4\mathbf{u} = O(\phi^2) \tag{3.20}$$

Let assume a small perturbed state aroud ($\phi_0 = cst$ and $\mathbf{u}_0 = 0$) such that,

$$\phi = \phi_0 + \varphi$$

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}$$

$$p = p_0 + q$$

$$\mathbf{w} = \mathbf{w}_0 + \mathbf{w}_1$$

$$\boldsymbol{\sigma}^{\text{eff}} = \boldsymbol{\sigma}_0^{\text{eff}} + \boldsymbol{\sigma}_1^{\text{eff}}$$

Hence the equilibrium varibale follows,

$$\mathbf{u}_0 \cdot \nabla \phi_0 = \nabla \cdot \mathbf{w}_0$$
$$0 = -\nabla p_0 + (\rho_f + \phi_0 \rho)\mathbf{g} + \nabla \cdot (\boldsymbol{\sigma}_0^{\text{eff}})$$
$$0 = -\nabla^2 p_0 + \nabla\nabla : \boldsymbol{\sigma}_0^{\text{eff}}$$
$$\boldsymbol{\sigma}_0^{\text{eff}} = +\mu_f \frac{3\lambda}{4(\lambda+1)}\nabla\mathbf{w}_0 + \mu_f \frac{2}{4(\lambda+1)}(\nabla \cdot \mathbf{w}_0)\boldsymbol{\delta}$$
$$\mathbf{w}_0 = -\phi_0 \rho \frac{a^2 2(\lambda+1)}{\mu_f 3(3\lambda+2)}\mathbf{g}$$

Including the perturbed sol neglecting the non-linear termes and removing the stable eq gives

$$\partial_t \varphi = \nabla \cdot \mathbf{w}_1$$
$$0 = -\nabla q + \mu_f \nabla^2 \mathbf{v} + \varphi \rho \mathbf{g} + \nabla \cdot \boldsymbol{\sigma}_1^{\text{eff}}$$
$$0 = -\nabla^2 q + \rho \mathbf{g} \cdot \nabla \varphi + \nabla\nabla : \boldsymbol{\sigma}_1^{\text{eff}}$$
$$\boldsymbol{\sigma}_1^{\text{eff}} = \mu_f \phi_0 \frac{5\lambda+2}{\lambda+1}(\nabla\mathbf{v} +^\dagger \nabla\mathbf{v}) + \mu_f \frac{3\lambda}{4(\lambda+1)}\nabla\mathbf{w}_1 + \mu_f \frac{2}{4(\lambda+1)}\nabla \cdot \mathbf{w}_1 \boldsymbol{\delta}$$
$$\mathbf{w}_1 = -\varphi \rho \frac{a^2 2(\lambda+1)}{\mu_f 3(3\lambda+2)}\mathbf{g} - \frac{a^2 \lambda}{2(3\lambda+2)}\phi_0 \nabla^2 \mathbf{v}$$

of course we recall that $\nabla \cdot \mathbf{v} = 0$. Hence the last two equation may be injected in the first three one to get three eq, for $\varphi, q, \mathbf{v}$ namely,

$$\partial_t \varphi = -\rho \frac{a^2 2(\lambda+1)}{\mu_f 3(3\lambda+2)}\mathbf{g} \cdot \nabla \varphi$$
$$0 = -\nabla q + \mu_f(1 + \phi_0 \frac{5\lambda+2}{\lambda+1})\nabla^2 \mathbf{v} + \varphi \rho \mathbf{g} - \frac{a^2}{6(3\lambda+2)}\rho \mathbf{g} \cdot (+3\lambda\boldsymbol{\delta}\nabla^2 + 2\nabla\nabla)\varphi$$
$$0 = -\nabla^2 q + \rho \mathbf{g} \cdot (1 - \frac{a^2}{6}\nabla^2)\nabla \varphi$$
$$\boldsymbol{\sigma}_1^{\text{eff}} = \mu_f \phi_0 \frac{5\lambda+2}{\lambda+1}(\nabla\mathbf{v} +^\dagger \nabla\mathbf{v})$$
$$+ \mu_f \frac{3\lambda}{4(\lambda+1)}(-\rho\frac{a^2 2(\lambda+1)}{\mu_f 3(3\lambda+2)}\mathbf{g}\nabla\varphi - \frac{a^2\lambda}{2(3\lambda+2)}\phi_0\nabla^2\nabla\mathbf{v})$$
$$+ \mu_f \frac{2}{4(\lambda+1)}(-\rho\frac{a^2 2(\lambda+1)}{\mu_f 3(3\lambda+2)}\mathbf{g}\cdot\nabla\varphi)\boldsymbol{\delta}$$
$$\mathbf{w}_1 = -\varphi\rho\frac{a^2 2(\lambda+1)}{\mu_f 3(3\lambda+2)}\mathbf{g} - \frac{a^2\lambda}{2(3\lambda+2)}\phi_0\nabla^2\mathbf{v}$$

Using fourier transform Let now consider the fourier transform,

$$\mathbf{A}(\mathbf{k}, \omega) = \iiiint \mathbf{A}(\mathbf{x}, t)e^{-i(\mathbf{k}\cdot\mathbf{x}+\omega t)}d\mathbf{k}d\omega \tag{3.21}$$

So the the above system of equaiton reads,

$$\omega\varphi = -\rho\frac{a^2 2(\lambda+1)}{\mu_f 3(3\lambda+2)}\mathbf{g}\cdot\mathbf{k}\varphi \tag{3.22}$$

$$0 = i\mathbf{k}q - \mu_f(1+\phi_0\frac{5\lambda+2}{\lambda+1})k^2\mathbf{v} + \varphi\rho\mathbf{g} - \frac{a^2}{6(3\lambda+2)}\rho\mathbf{g}\cdot(-3\lambda\boldsymbol{\delta}k^2 - 2\mathbf{kk})\varphi \tag{3.23}$$

$$0 = +k^2 q - \rho\mathbf{g}\cdot(1+\frac{a^2}{6}k^2)i\mathbf{k}\varphi \tag{3.24}$$

Or in matrix form,

$$\omega\begin{pmatrix}\varphi\\0\\0\end{pmatrix} = \begin{pmatrix} -\rho\frac{a^2 2(\lambda+1)}{\mu_f 3(3\lambda+2)}\mathbf{g}\cdot\mathbf{k} & 0 & 0\\ \rho\mathbf{g}+\frac{a^2}{6(3\lambda+2)}\rho\mathbf{g}\cdot(3\lambda\boldsymbol{\delta}k^2+2\mathbf{kk}) & -\mu_f(1+\phi_0\frac{5\lambda+2}{\lambda+1})k^2 & i\mathbf{k}\\ -\rho\mathbf{g}\cdot(1+\frac{a^2}{6}k^2)i\mathbf{k} & 0 & k^2 \end{pmatrix}\cdot\begin{pmatrix}\varphi\\\mathbf{v}\\q\end{pmatrix} \tag{3.25}$$

4 A simple example : 1D sedimentation

We consider the 1D sedimentation of spherical droplet in a fluid at rest although or model can be generalized to cases with non-zero bulk velocity. For now we neglect the effect of bulk and micro scale inertia. Although in 1D the bulk inertia vanishes (but the unsteady terms remain), while we may incorporate the incluence of finite inertia on the effective stress (TO DO). The above equation of motion reduces to,

$$\phi_f + \phi = 1 \tag{4.1}$$

$$\phi_f u_f = -\phi u_p = \phi u_r \quad \text{or} \quad u = 0 \tag{4.2}$$

$$\partial_t\phi + \partial_z(u_p\phi) = 0 \tag{4.3}$$

$$0 = \phi\partial_z\Sigma_{zz} - \phi\rho_d g - \phi\frac{\mu_f}{a^2}\frac{3(2+3\lambda)}{2(1+\lambda)}u_p \tag{4.4}$$

$$0 = \phi_f\partial_z\Sigma_{zz} - \phi_f\rho_f g + \mu_f\frac{3\lambda-2}{4(\lambda+1)}\partial_{zz}(\phi u_p) + \phi\frac{\mu_f}{a^2}\frac{3(2+3\lambda)}{2(1+\lambda)}u_p \tag{4.5}$$

From which we immediately evidenced the effect of the second order force moment : spatial diffusion of the volume fraction and of the relative velocity. However if we inject this term in mass conservation equation we obtain a dispersion term ! Moreover in constrast to the classical Richardson-Zaki scaling (or in our dilute configuration the Hadamard force), the relative velocity is not uniform in space. The previous system of equaitions lead to,

$$\partial_t\phi + \partial_z(u_p\phi) = 0 \tag{4.6}$$

$$0 = (1-\phi)g(\rho_d - \rho_f) + \mu_f\frac{3\lambda-2}{4(\lambda+1)}\partial_{zz}(\phi u_p) + \frac{\mu_f}{a^2}\frac{3(2+3\lambda)}{2(1+\lambda)}u_p \tag{4.7}$$

which leads to a more complex equations than the traditional continuity wave equation.

5 TO DO : perform the stability analysis *a la Jackson* considering inertia

Do the dilute fluidisation of droplet is stable ?