# Response to reviewer 3

June 6, 2025

**Paper entitled "Averaged equations for disperse two-phase flow with interfacial transport. IJMF"**

*This paper derives the continuum transport equations that govern two phase systems where one phase is of the form of dispersed, but not necessarily rigid, particles. The method of derivation, despite slight differences from previous approaches, appears to be quite standard. The main extensions that I can detect are that transport are derived for properties associated with the interfaces between the dispersed and continuous phases. The paper is nicely introduced, the analysis is clear and appears to be correct - at least the standard two phase flow equations for bulk properties are recovered. Overall, I like the clarity of exposition and the connections the authors make with previous papers on the subject. However, I do have some concerns, which are listed below. I believe most of these concerns can be addressed if the authors put their minds to it, but it will require some rewriting, clearer explanation and possibly a better discussion of the main advances.*

We thank the Referee for his /her positive feedback on our manuscript. Before providing a point-by-point response to the Referee's comments, we just wish to mention the main changes introduced in the paper to address the concerns raised by the other referees, some of which overlap with those specified by the current referee. We have added two new sections (Sections V and VI) to the revised version of the article.

Section V presents a set of averaged equations, including higher-order moments, to describe the dynamics of droplets immersed in a Newtonian fluid. Particular emphasis is placed on the choice of the stress decomposition and on the equation governing the second-order mass moment and first moment of momentum equation, which plays a key role in describing the deformation of the dispersed phase.

In Section VI, we derive the closure terms in the dilute, viscous-dominated regime. Specifically, we quantify the influence of surface tension gradient on the forces and moments acting on the droplets. Additionally, we discuss several covariance closure terms that emerge in the averaged equations. Finally we demonstrate how the leading order deformation of the droplets can be obtained thanks to the second-order mass moment and first moment of momentum equation.

We have also rewritten the conclusion. In light of these modifications, the title has been changed in "Averaged equations for disperse two-phase flow with interfacial properties and their closures for dilute suspension of droplets".

1. It is not clear what the purpose of the paper is, and in what substantive way it departs or extends previous studies, many of which the authors have already cited. Arriving at known results by another route might be a useful and rewarding exercise for the person doing it, but to qualify as a paper it must make an original contribution. What are the original contributions of the paper? To make this clear, I suggest the authors replace or expand the last paragraph of the introductory section to outline the new contributions in the paper and state in what way the analysis presented departs from previous papers.

   We agree with the referee. We have expanded the last paragraph of the introduction to outline the new contributions of the present paper. Although this problem has been addressed by several research groups, including Lhuillier (1992) and Zhang et Prosperetti (1994), our contribution lies in presenting a unified and simple theoretical framework for deriving the hybrid set of governing equations. First the Lagrangian balance equations are derived without any simplifying assumptions, thus allowing for: (1) the incorporation of interfacial properties while retaining a Lagrangian framework, and (2) the representation of internal gradients within droplets through the use of distribution moments. Moreover we believe that even if the link between Lagrangian and Eulerian averaging approaches is well understood, it is often overlooked and has never been fully demonstrated in such general scenarios that the present one. A central feature of our approach is the emphasis on moment equations for the dispersed phase, as previously evidenced by Lhuillier & Nadim (2009) in the context of solid particles. We argue that, regardless of the specific problem under consideration - including those involving fluid particles of complex shape - the hybrid formulation offers a more physically grounded alternative to the traditional two-fluid model for describing dispersed flows. To illustrate our methodology, we provide closure laws in the dilute, viscous-dominated regime, with special attention to the role of surface tension gradients and the closure of higher-order moment equations.

2. One original contribution that I see is the transport equations of interfacial properties. This is a reasonable extension, but if it is the only one, the paper does not have to be this long and does not have to repeat the derivation of transport equations for bulk properties of the dispersed and continuous phases. Additionally, what do the authors foresee the interfacial transport equations being useful for? For example, for a scalar property such as surfactant concentration, transport equations for the zeroth moment is quite simple - one can even write them down intuitively. I can see that transport equations for higher moments might be useful, for example in computing the mean Marangoni drift of bubbles in a liquid. There isn't much discussion of these aspects, which makes one wonder what the motivation is for deriving the equations. We thank the Referee for this excellent remark which motivated us to address the influence of surface tension gradients on the governing equations and closure relations. These gradients give rise to the well-known Marangoni drift but also contribute to the effective stress in the suspension. Those effects are now discussed in detail in Section VI of the revised manuscript.

   In response to the reviewer's comment that " the paper does not have to be this long and does not have to repeat the derivation of transport equations for bulk properties of the dispersed and continuous phases." , we have chosen to retain the derivations for both the bulk and interfacial transport equations. We believe that including both contributes to a more pedagogical and coherent presentation of the overall derivation.

3. A related point is the utility of the transport equations for the moments of particle properties. The utility of the first moment of the momentum of the momentum balance is immediately obvious - its anti-symmetric part is the balance of angular momentum. Where would one need balances for higher moments? Moreover, the main difficulty with these equations, as the authors have already found, is that one needs closures for covariances, for example between concentration and velocity fluctuations.

   In general, if the internal motion or shape of the droplets exhibits $n$ degrees of freedom, then $n$ moment equations are required to fully describe their dynamics. However, we believe that in most practical applications, only the first few moments are needed to accurately capture the motion of both the dispersed and continuous phases. In this work, we demonstrate that the leading-order deformation of the droplets can be effectively described using the second-order mass moment equation.

   Motivated by the referee's comments, we also include a discussion of several covariance closure terms that arise in the averaged equations. In particular, for viscous-dominated flows (i.e., in the Stokes regime), the relevant quantities are uncorrelated, leading to vanishing covariance terms. However, this assumption no longer holds at finite inertia, where these covariance terms become non-negligible. We place special emphasis on the covariance of velocity, which is non-zero at finite Reynolds numbers. Although we do not derive an explicit closed-form expression for the velocity covariance—owing to the well-known divergence paradox associated with this term in dilute flows—we provide a functional tensorial closure form that can be used in practical modeling.

4. The discussion of the Lagrangian versus Eulerian description of the dispersed phases is misleading - only the equations describing individual particle properties are Lagrangian. Once they are volume (or ensemble) averaged, the continuum equations become Eulerian. The many references to Lagrangian versus Eulerian descriptions makes it confusing for the reader. If I have misunderstood the argument the authors are making, it is likely that an average reader would too. Indeed we have replaced the term "Lagrangian" by "Lagrangian-based" model

5. A lot of discussion in the paper is about the equivalence between the 'phase averaged equations' and 'particle averaged' transport equations. I see that they do cite the previous studies that have shown this equivalence, but I'm wondering what the point to showing the equivalence if it has already been done. In previous studies, the equivalence between the 'phase averaged equations' and 'particle averaged' has been demonstrated only in the context of the momentum equation, without accounting for interfacial properties or mass transfer (although Lhuillier 2001 considered specifically the interfacial terms in a different context). In this work, we extend the demonstration to the more general case, including these effects. Beyond the technical contribution, our aim is also pedagogical: we seek to clearly illustrate how Eulerian-based and Lagrangian-based formulations are interconnected.

Minor points:

- In the Abstract and elsewhere, it is stated 'Notably, the non-convective flux inside the inclusion does not appear in the conservation law using this formulation.' Isn't this obvious? The non-convective flux will simply move properties from one place to another within a particle. Yes it is obvious however it remains unclear for the interfacial convective fluxes, therefore it seemed important for us to point that out, at least in the body of the text (it has been removed from the abstract), especially Regarding the surface forces

- Abstract and elsewhere: what is the 'distributional form' of the interfacial transport equation. Equation 2.10 represents the distributional form of the interfacial transport equation. While we acknowledge that this terminology may not be entirely satisfactory, it follows the convention used by Teigen et al. (2009), and we have therefore chosen to retain it.
- p. 5: 'We also recognize a term related to mass transfer proportional to $(u_\Gamma - u_k)$.' Isn't this is only true if $f = \rho$? Indeed this is not accurate the comments have been removed
- p. 6: 'The ensemble average quantities are assumed to satisfy the following properties ...'. Is it an assumption or is it exact? It is an assumption (Drew, 1983)
- After (2.15): Is it probability or probability density? This point was also raised by another reviewer. The term "probability..." has been replaced with "interface area per unit volume"
- p. 8: 'indexed, $\alpha$,' - delete the commas corrected
- p. 11: 'pioneered by (Lhuillier, 1992)' - should be pioneered by Lhuillier (1992) corrected