Chapter 1

An In-Depth Guide to Mathematical Symbols, Logic, and Foundational Concepts

A Collaboration with Gemini

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Contents

Chapter 1

A Brief History of Mathematical Notation

1.1 Why History Matters

Mathematical notation is the language of mathematics. Like any language, it didn't appear overnight; it evolved over millennia through the contributions of countless cultures and individuals. Early mathematics was largely **rhetorical**, meaning problems and solutions were written out in full sentences. The compact, powerful symbolism we use today is a relatively recent invention. Understanding its history helps us appreciate the elegance of modern notation and grasp why certain symbols were chosen and what they truly represent.

1.2 Ancient Beginnings: Words and Glyphs

- **Egypt (c. 2000 BCE):** The Egyptians used a hieroglyphic system for numbers, but their mathematics, famously recorded in documents like the **Rhind Papyrus**, was practical and descriptive. They had a sophisticated understanding of fractions but primarily worked with **unit fractions** (of the form $\frac{1}{n}$). A problem might be stated as, "A quantity and its seventh part together make 19. What is the quantity?" There was no "$x + \frac{x}{7} = 19$".

- **Babylon (c. 1800 BCE):** The Babylonians developed a **sexagesimal (base-60) positional system** using cuneiform script. This was a major advance, as the value of a symbol depended on its position, just like our base-10 system. This was powerful for astronomical calculations. However, they lacked a true concept of zero as a number and placeholder, which could lead to ambiguity. For example, the symbols for 61 and 2 could look identical without context.

- **Greece (c. 600 BCE - 400 CE):** Greek mathematics, epitomized by **Euclid's *Elements***, was based on rigorous logic and geometry. However, it was almost entirely rhetorical. Proofs were written as carefully constructed paragraphs of prose, accompanied by diagrams. The concept of an equation was expressed as a geometric relationship. For instance, the algebraic identity $(a + b)^2 = a^2 + 2ab + b^2$ was understood as a statement about the areas of squares and rectangles.

1.3 The Birth of Modern Symbols

The transition from rhetorical to symbolic mathematics was gradual, with key contributions from India, the Islamic world, and Renaissance Europe.

- **India (c. 500 CE):** Indian mathematicians, notably **Brahmagupta** and **Aryabhata**, made monumental contributions. They developed the Hindu-Arabic numeral system (0, 1, 2, ...), which included a symbol and a working concept for **zero**. This was a revolutionary idea that simplified calculation and paved the way for modern algebra.

- **The Islamic Golden Age (c. 800-1200 CE):** Scholars like **Al-Khwarizmi** in Baghdad synthesized and expanded upon Greek and Indian knowledge. His book, *Al-Jabr wa'l-Muqabala*, provided systematic methods for solving linear and quadratic equations. The word "**algebra**" comes from *al-jabr*, meaning "restoring" or "completion" (referring to moving a negative term to the other side of an equation). His work was still largely rhetorical, but it laid the procedural groundwork for the field.

- **Renaissance Europe (c. 1400-1600):** The introduction of Hindu-Arabic numerals to Europe (notably by Fibonacci) set the stage. The invention of the printing press helped standardize symbols. Key figures include:

  - **Robert Recorde (c. 1557):** Introduced the equals sign (=) in *The Whetstone of Witte*, stating, "...to avoid the tedious repetition of these words: 'is equal to': I will set a pair of parallel lines of one length, thus: =====, because no two things can be more equal."
  - **François Viète (c. 1591):** Made the crucial leap of using letters to represent variables and constants (vowels for unknowns, consonants for knowns), allowing for the statement of general formulas rather than just solving specific problems.
  - **René Descartes (1637):** In his work *La Géométrie*, he popularized the convention of using letters from the beginning of the alphabet (a, b, c) for known quantities and letters from the end (x, y, z) for unknown quantities. He also introduced modern exponential notation (x^3 instead of xxx).

- **The Age of Calculus and Analysis (17th-18th Centuries): Gottfried Wilhelm Leibniz** was a brilliant notation designer, giving us the integral symbol $\int$ (from a long 'S' for *summa*) and the $\frac{dy}{dx}$ notation for derivatives. **Leonhard Euler**, arguably the most prolific mathematician in history, introduced or popularized an incredible number of symbols we use today, including e for the base of the natural logarithm, i for the imaginary unit, π for the circle constant, Σ for summation, and the $f(x)$ notation for functions.

1.4 Logic and Set Theory: The Final Frontier

While algebra and calculus had developed sophisticated notation by the 18th century, logic and set theory were still described in words. This changed in the 19th and early 20th centuries.

- **George Boole (1854):** In *The Laws of Thought*, Boole developed an algebraic system for logic, treating propositions as variables that could be manipulated with operators like AND, OR, and NOT. This is the foundation of modern computer science.

- **Georg Cantor (1874):** Created modern set theory, introducing concepts of infinity and defining sets as collections of objects. His work was initially controversial but is now a cornerstone of mathematics.

- **Gottlob Frege, Giuseppe Peano, Bertrand Russell (late 19th - early 20th century):** These logicians developed the notation for formal logic, including the quantifiers "for all" ($\forall$) and "there exists" ($\exists$), solidifying the language used in rigorous proofs today.

This historical journey shows that the symbols we often take for granted are the refined product of centuries of thought, designed for clarity, precision, and power.

Chapter 2

Symbols of Arithmetic, Relations, and Sets

2.1 Arithmetic Operators: The Building Blocks

At the most basic level, mathematics manipulates quantities using operations. An **operator** is a symbol that represents an action or process. The ones below are **binary operators** because they act on two objects (operands).

2.1.1 Addition (+) and Subtraction (−)

Definition 2.1.1. *__Addition__ is a binary operation that combines two quantities (addends) to produce their total, called the __sum__. Its inverse is __Subtraction__, which finds the __difference__ between two quantities.*

Example 2.1.1. $7 + 5 = 12$. Here, 7 and 5 are addends, and 12 is the sum. $12 - 5 = 7$. This shows the inverse relationship. Subtracting 5 from the sum returns the original number, 7. In algebra, we generalize this: $a + b$ is the sum of variables a and b.

2.1.2 Multiplication (×, ·, juxtaposition) and Division (÷, /)

Definition 2.1.2. *__Multiplication__ can be thought of as repeated addition. It combines two quantities (factors) to produce their __product__. Its inverse is __Division__, which determines how many times one number (the divisor) is contained within another (the dividend), yielding a __quotient__.*

Remark 2.1.1. In algebra, the $\times$ symbol is often avoided because it can be confused with the variable x. Instead, a dot ($\cdot$) or, most commonly, **juxtaposition** (placing symbols next to each other) is used.

Example 2.1.2. $4 \times 3 = 4 + 4 + 4 = 12$. Algebraic forms: ab is preferred over $a \times b$ or $a \cdot b$. $12 \div 3 = 4$, or using the fraction bar, $\frac{12}{3} = 4$.

Exercise 2.1.1.

1. Compute $15 + (-8)$, $9 - 17$, 6×7, and $\frac{42}{7}$.
2. Express "five times the difference of x and y, all divided by z" in algebraic form.

2.2 Relations: Comparing Mathematical Objects

Relation symbols don't perform an operation; they state a relationship between two expressions. The result is a statement that is either true or false.

2.2.1 Equality (=) and Inequality (≠)

Definition 2.2.1. *The **equals sign** (=) asserts that the expressions on both sides represent the exact same mathematical object or value. It is an **equivalence relation**, meaning it is:*

- ***Reflexive:*** $a = a$
- ***Symmetric:*** *If* $a = b$, *then* $b = a$
- ***Transitive:*** *If* $a = b$ *and* $b = c$, *then* $a = c$

*The **not equal sign** (≠) asserts that the expressions have different values.*

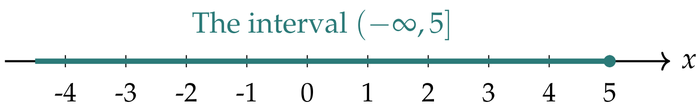
Example 2.2.1. $3^2 + 4^2 = 9 + 16 = 25$, while $5^2 = 25$. Therefore, $3^2 + 4^2 = 5^2$. $2 + 2 \neq 5$. This is a true statement.

2.2.2 Order Relations: $<, >, \leq, \geq$

Definition 2.2.2. *These symbols define an **order** on sets of numbers like the real numbers.*

- $a < b$: *"a is **strictly less than** b."*
- $a > b$: *"a is **strictly greater than** b."*
- $a \leq b$: *"a is **less than or equal to** b."*
- $a \geq b$: *"a is **greater than or equal to** b."*

Example 2.2.2. $4 < 9$ is true. $9 > 4$ is also true. The statement $x \leq 5$ means that x could be 5 or any number less than 5. On a number line, this is represented by a closed interval ending at 5.



Exercise 2.2.1.

1. Solve for y: $2y - 5 = 11$.
2. Is the statement $\frac{1}{3} = 0.33$ true or false? Why?
3. Describe the set of real numbers $\{x \in \mathbb{R} : x \geq -1\}$ in words and draw it on a number line.

2.3 Set Theory: The Language of Collections

Set theory is the foundation upon which much of modern mathematics is built. Its notation allows us to reason about collections of objects with precision.

Definition 2.3.1. *A **set** is a well-defined collection of distinct objects, considered as an object in its own right. The objects within a set are called its **elements** or **members**.*

- ***Roster Notation:*** *List the elements, e.g.,* $A = \{1, 2, 3\}$.
- ***Set-Builder Notation:*** *Describe the elements, e.g.,* $E = \{x \mid x \text{ is an even integer}\}$.

2.3.1 Membership ($\in, \notin$)

Definition 2.3.2. *The symbol $\in$ denotes set membership. $a \in A$ is read "a is an element of set A." The symbol $\notin$ means "is not an element of."*

Example 2.3.1. If $V = \{a, e, i, o, u\}$, then $a \in V$ but $b \notin V$.

2.3.2 Subsets ($\subseteq, \subset$)

Definition 2.3.3. *$A \subseteq B$ means A is a **subset** of B. This is true if every element of A is also an element of B. The symbol $\subset$ denotes a **proper subset**, meaning $A \subseteq B$ but $A \neq B$.*

Example 2.3.2. Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$. Then $A \subseteq B$ and $A \subset B$. Let $C = \{1, 2, 3\}$. Then $B \subseteq C$ but $B \not\subset C$ (since $B = C$).

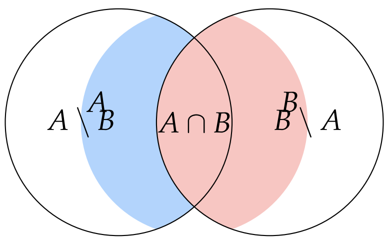
2.3.3 Special Sets and Set Operations ($\varnothing, \cup, \cap, \setminus$)

Definition 2.3.4.

- ***Empty Set** ($\varnothing$ **or** $\{\}$)**:** The unique set containing no elements.*
- ***Union** ($\cup$)**:** $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. It is the set of elements in either A, or B, or both.*
- ***Intersection** ($\cap$)**:** $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. It is the set of elements common to both A and B.*
- ***Set Difference** ($\setminus$)**:** $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$. It is the set of elements in A but not in B.*

Example 2.3.3. Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$.

- $A \cup B = \{1, 2, 3, 4, 5, 6\}$
- $A \cap B = \{3, 4\}$
- $A \setminus B = \{1, 2\}$
- $B \setminus A = \{5, 6\}$



Exercise 2.3.1.

1. Let $P = \{x \mid x \text{ is a prime number less than } 10\}$ and $E = \{x \mid x \text{ is an even number less tha}$
   (a) Write P and E in roster form.
   (b) Find $P \cup E$, $P \cap E$, and $P \setminus E$.
2. Is it true that for any set A, $\varnothing \subseteq A$? Why or why not? (Hint: Think about the definition of a subset.)

Chapter 3

Symbols of Logic and Truth Tables

3.1 Propositions: The Atoms of Logic

Definition 3.1.1. *A **proposition** (or statement) is a declarative sentence that has a definite **truth value**: it is either true (T) or false (F), but not both.*

Example 3.1.1.

- "The Earth is the third planet from the Sun." (A true proposition)
- "$2 + 2 = 5$" (A false proposition)
- "What time is it?" (Not a proposition; it's a question)
- "$x > 5$" (Not a proposition until x is defined; this is a **predicate**)

We use capital letters like P, Q, R to represent propositions.

3.2 Logical Connectives: Building Compound Statements

We can combine simple propositions to form compound ones using logical connectives. The meaning of each connective is defined precisely by a **truth table**, which shows the truth value of the compound statement for all possible combinations of truth values of its components.

3.2.1 Negation (¬): NOT

This is a unary operator; it acts on a single proposition. $\neg P$ is read "not P."

P	$\neg P$
T	F
F	T

Interpretation: $\neg P$ is true if and only if P is false.

3.2.2 Conjunction (∧): AND

This operator connects two propositions. $P \wedge Q$ is read "P and Q."

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Interpretation: The conjunction $P \wedge Q$ is true only when both P and Q are true.

3.2.3 Disjunction ($\vee$): OR

This operator also connects two propositions. $P \vee Q$ is read "P or Q."

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Interpretation: This is the **inclusive OR**. The disjunction $P \vee Q$ is true if at least one of P or Q is true. It is only false when both are false.

3.2.4 Implication ($\Rightarrow$ or $\rightarrow$): IF...THEN

The implication $P \Rightarrow Q$ is read "if P, then Q." P is called the **hypothesis** (or antecedent) and Q is the **conclusion** (or consequent). This is often the trickiest connective.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Interpretation: Think of an implication as a promise. If I say, "If it rains (P), then I will bring an umbrella (Q)," the only way I have broken my promise is if it rains (P is true) and I do not bring an umbrella (Q is false). In all other cases, my promise holds.

- If it doesn't rain (P is false), it doesn't matter whether I bring an umbrella or not. I haven't broken my promise. This is called being **vacuously true**.

3.2.5 Biconditional ($\Leftrightarrow$ or $\leftrightarrow$): IF AND ONLY IF

The biconditional $P \Leftrightarrow Q$ is read "P if and only if Q." It is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Interpretation: The biconditional is true if and only if P and Q have the same truth value.

Definition 3.2.1. *Two statements are* ***logically equivalent*** *if they have the same truth table. We denote this with* $\equiv$.

Example 3.2.1 (De Morgan's Laws). Let's show that $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$. We build a truth table for both sides.

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	T	**F**	F	F	**F**
T	F	F	**T**	F	T	**T**
F	T	F	**T**	T	F	**T**
F	F	F	**T**	T	T	**T**

Since the columns for $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ are identical, the statements are logically equivalent.

Exercise 3.2.1.

1. Construct a truth table for the statement $(P \Rightarrow Q) \wedge P$. What do you notice about this statement in relation to Q? This is the logical foundation of a famous rule of inference called *Modus Ponens*.

2. Give a real-life example of an implication that is vacuously true.

3. Use a truth table to show that $P \Rightarrow Q$ is logically equivalent to its **contrapositive**, $\neg Q \Rightarrow \neg P$.

Chapter 4

Quantifiers and Standard Number Sets

4.1 Quantifiers: Expressing Generality and Existence

Quantifiers allow us to express properties of entire sets of objects. They turn predicates (like "$x > 5$") into propositions with a definite truth value.

4.1.1 The Universal Quantifier ($\forall$): "For All"

Definition 4.1.1. *The statement $\forall x \in S, P(x)$ is read "For all x in the set S, the proposition $P(x)$ is true." To prove such a statement is true, you must show it holds for every single element of S. To prove it is false, you only need to find one* ***counterexample****.*

Example 4.1.1.

- $\forall x \in \mathbb{R}, x^2 \geq 0$. (True. The square of any real number is non-negative.)
- $\forall n \in \mathbb{N}, n$ is an odd number. (False. A counterexample is $n = 2$, which is in $\mathbb{N}$ but is not odd.)

4.1.2 The Existential Quantifier ($\exists$): "There Exists"

Definition 4.1.2. *The statement $\exists x \in S, P(x)$ is read "There exists at least one element x in the set S such that $P(x)$ is true." To prove this is true, you only need to find one such element. To prove it is false, you must show that for all elements of S, $P(x)$ is false.*

Example 4.1.2.

- $\exists p \in \{\text{Prime numbers}\}, p > 100$. (True. For example, 101 is a prime number.)
- $\exists x \in \mathbb{R}, x^2 = -1$. (False. The square of every real number is non-negative.)

4.1.3 Negating Quantifiers

Negating quantified statements is a crucial skill in proofs. The rules are:

$$\neg(\forall x, P(x)) \equiv \exists x, \neg P(x)$$

$$\neg(\exists x, P(x)) \equiv \forall x, \neg P(x)$$

Intuition: To deny that "all dogs are brown" is to claim that "there exists at least one dog that is not brown." To deny that "there exists a purple dog" is to claim that "all dogs are not purple."

Exercise 4.1.1.

1. Translate into symbolic logic: "Every integer is either even or odd." (Let $E(n)$ be "n is even" and $O(n)$ be "n is odd".)

2. Translate and state the truth value: $\exists n \in \mathbb{Z}, n^2 = 2$.

3. Write the negation of $\forall x \in \mathbb{R}, x > 0 \Rightarrow x^2 > 0$ both symbolically and in English.

4.2 The Hierarchy of Standard Number Sets

These sets are fundamental across all of mathematics. Each set is a subset of the next.

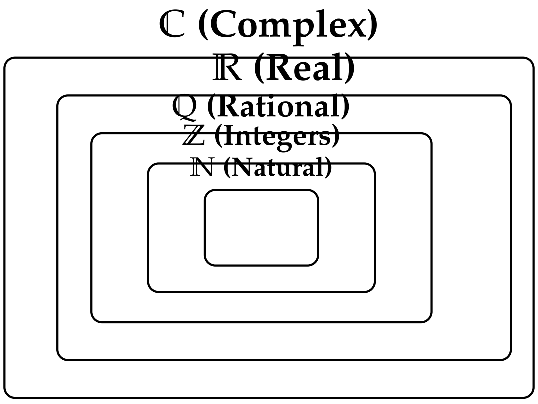


Figure 4.1: The relationship between number sets: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

- **Natural Numbers ($\mathbb{N}$):** The counting numbers. $\mathbb{N} = \{1, 2, 3, \ldots\}$.

  Remark 4.2.1. Some fields (especially computer science) include 0 in $\mathbb{N}$, writing $\mathbb{N} = \{0, 1, 2, \ldots\}$. When it matters, it is crucial to state your convention. We will use $\mathbb{N} = \{1, 2, 3, ...\}$.

- **Integers ($\mathbb{Z}$):** The whole numbers, both positive, negative, and zero. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The 'Z' comes from the German word *Zahlen* (numbers).

- **Rational Numbers ($\mathbb{Q}$):** Numbers that can be expressed as a ratio (quotient) of two integers, $\frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$. Examples: $\frac{1}{2}, -7, 0.25$.

- **Real Numbers ($\mathbb{R}$):** All numbers on the number line. This includes all rational numbers and also **irrational numbers**—numbers that cannot be expressed as a ratio of integers, like $\sqrt{2}$, π, and e. Irrational numbers have non-terminating, non-repeating decimal expansions.

- **Complex Numbers ($\mathbb{C}$):** Numbers of the form $a + bi$, where $a, b \in \mathbb{R}$ and i is the imaginary unit, defined by $i^2 = -1$. They are essential for solving equations like $x^2 + 1 = 0$.

Exercise 4.2.1. 1. To which of the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ does the number -4 belong?

2. Give an example of a number that is in $\mathbb{R}$ but not in $\mathbb{Q}$.

3. Is the set of integers closed under division? (Meaning, if you divide any integer by any non-zero integer, is the result always an integer?) Explain.

Chapter 5

A Glossary of Common Greek Letters

Greek letters are ubiquitous in mathematics and science. While their usage can vary, many have conventional meanings.

5.1 Common Lowercase Letters

α, β, γ **(alpha, beta, gamma)** Often used for angles in geometry, coefficients in equations (e.g., $\alpha x^2 + \beta x + \gamma$), and error types in statistics (Type I and Type II errors).

δ, ϵ **(delta, epsilon)** The language of calculus and analysis. ϵ typically represents an arbitrarily small positive number, while δ is another small number that depends on ϵ. The famous $\epsilon - \delta$ definition of a limit is a cornerstone of analysis. δ can also represent the Dirac delta function.

θ, ϕ, ψ **(theta, phi, psi)** Commonly used for angles (especially in polar/spherical coordinates) and parameters in statistical models. In quantum mechanics, ψ represents a wave function.

λ, μ **(lambda, mu)** Central to statistics and linear algebra. μ almost always denotes the **mean** (average) of a population. λ is used for **eigenvalues** in linear algebra, and as the rate parameter in Poisson distributions.

π **(pi)** The constant ratio of a circle's circumference to its diameter in Euclidean geometry, approximately 3.14159.

ρ **(rho)** Often used for density in physics and for the correlation coefficient in statistics.

σ **(sigma)** Denotes the **standard deviation** of a population in statistics. Its square, σ^2, is the variance. In number theory, it can also represent a sum-of-divisors function.

ω **(omega)** Typically used for angular frequency in physics and engineering.

5.2 Common Uppercase Letters

Δ **(Delta)** Represents a "change in" a variable, e.g., $\Delta x = x_f - x_i$. It is also used for the discriminant of a quadratic equation ($b^2 - 4ac$) and for the Laplace operator.

Σ **(Sigma)** The **summation operator**. $\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$.

Π **(Pi)** The **product operator**. $\prod_{k=1}^{n} a_k = a_1 \cdot a_2 \cdot \cdots \cdot a_n$.

Ω **(Omega)** In probability theory, it represents the entire **sample space** (the set of all possible outcomes). In computer science, it is used for Big-Ω notation, an asymptotic lower bound on function growth.

Φ **(Phi)** Often represents the cumulative distribution function (CDF) of the standard normal distribution in statistics.

Exercise 5.2.1.

1. Write the sum of the first 5 perfect squares ($1^2 + 2^2 + 3^2 + 4^2 + 5^2$) using Σ notation.
2. Write the factorial of a number n, defined as $n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$, using Π notation.
3. The probability density function of a normal distribution is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$. What do μ and σ represent in this context?

Chapter 6

Translating Between English and Mathematics

The ultimate goal of this notation is to translate complex ideas from natural language into a precise, unambiguous symbolic form.

6.1 English to Symbolic Form

Example 6.1.1 (Primes). "All prime numbers greater than 2 are odd."

- **Identify the universe:** Prime numbers, $\mathbb{P}$.
- **Identify the quantifier:** "All" $\rightarrow \forall$.
- **Identify the structure:** "If P then Q" $\rightarrow P \Rightarrow Q$.
- **Symbolic form:** $\forall p \in \mathbb{P}, (p > 2 \Rightarrow p \text{ is odd})$.

Example 6.1.2 (Calculus). "The limit of the function $f(x)$ as x approaches c is L." This is a very complex statement. Its formal definition is a triumph of symbolic precision.

- **Meaning:** We can make $f(x)$ arbitrarily close to L by making x sufficiently close to c.
- **Symbolic form:** $\forall \epsilon > 0, \exists \delta > 0$ such that $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$.
- **Breakdown:**
  - $\forall \epsilon > 0$: For any small positive distance (tolerance) from L...
  - $\exists \delta > 0$: ...there exists a small positive distance from c...
  - $0 < |x - c| < \delta$: ...such that if x is within that distance of c (but not equal to c)...
  - $|f(x) - L| < \epsilon$: ...then the function value $f(x)$ is within the tolerance ϵ of L.

6.2 Symbolic Form to English

Example 6.2.1. $\exists x \in \mathbb{Z}$ such that x is prime $\wedge\, x$ is even.

- **$\exists x \in \mathbb{Z}$:** "There exists an integer..."
- **x is prime $\wedge\, x$ is even:** "...that is both prime and even."
- **Full sentence:** "There exists an integer that is both prime and even." (This is true, the integer is 2).

Exercise 6.2.1.

1. Translate: "If an integer is divisible by 6, then it is divisible by both 2 and 3."
2. Translate: "The square of any real number is non-negative."
3. Translate the following symbolic statement into a clear English sentence: $\forall y \in \mathbb{R}^+, \exists x \in \mathbb{R}, x^2 = y$. (Note: $\mathbb{R}^+$ means the set of positive real numbers).

Chapter Review Exercises

1. **History and Concepts:** Briefly explain the difference between rhetorical, syncopated, and symbolic algebra, giving an example of a key figure or development in the transition to symbolic algebra.

2. **Sets:** Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be the universal set. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{x \in U \mid x \text{ is even}\}$.

    (a) List the elements of set B.

    (b) Find $A \cap B$.

    (c) Find $A \cup B$.

    (d) Find $A \setminus B$.

    (e) Find the complement of A, denoted $A^c = U \setminus A$.

3. **Logic:** Construct a full truth table for the statement $(P \vee Q) \Rightarrow (P \wedge Q)$. Under what conditions is this statement true?

4. **Quantifiers:** Consider the statement: "Every person has a mother."

    (a) Let P be the set of all people. Let $M(x, y)$ be the predicate "x is the mother of y". Translate the statement into symbolic form using quantifiers. (Be careful with the order!)

    (b) Write the negation of this statement in symbolic form and then translate the negation back into a natural English sentence.

5. **Translation and Notation:**

    (a) Use Σ notation to express the sum of all odd integers from 1 to 99.

    (b) A function $f : \mathbb{R} \to \mathbb{R}$ is called *injective* (or one-to-one) if for every two distinct inputs, their outputs are also distinct. Translate this definition into a formal symbolic statement using quantifiers.

Appendix A

Solutions to Exercises

Solutions for Chapter 2

Exercises 2.1

1. $15 + (-8) = 7, \quad 9 - 17 = -8, \quad 6 \times 7 = 42, \quad \frac{42}{7} = 6.$

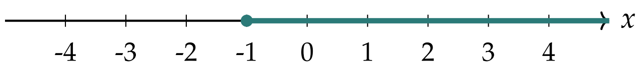
2. The difference of x and y is $(x - y)$. Five times this is $5(x - y)$. All divided by z is $\frac{5(x-y)}{z}$.

Exercises 2.2

1. $2y - 5 = 11 \implies 2y = 16 \implies y = 8.$

2. False. $\frac{1}{3}$ is the repeating decimal $0.333\ldots$, which is not equal to the terminating decimal 0.33. $0.33 = \frac{33}{100}$.

3. The set $\{x \in \mathbb{R} : x \geq -1\}$ is "the set of all real numbers greater than or equal to -1."



Exercises 2.3

1. (a) $P = \{2, 3, 5, 7\}, \quad E = \{2, 4, 6, 8\}.$
   (b) $P \cup E = \{2, 3, 4, 5, 6, 7, 8\}.$
   (c) $P \cap E = \{2\}.$
   (d) $P \setminus E = \{3, 5, 7\}.$

2. Yes, $\varnothing \subseteq A$ for any set A. The definition of subset is: "$X \subseteq Y$ if every element of X is also in Y." To show $\varnothing \not\subseteq A$, we would need to find an element in $\varnothing$ that is not in A. Since $\varnothing$ has no elements, this is impossible. The statement is vacuously true.

Solutions for Chapter 3

Exercises 3.2

1. Truth table for $(P \Rightarrow Q) \wedge P$:

P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \wedge P$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

Notice that the final column is true if and only if both P and Q are true. It is logically equivalent to $P \wedge Q$. However, in *Modus Ponens*, we assert that P is true and $P \Rightarrow Q$ is true, forcing us to conclude Q must be true (only the first row applies).

2. "If my dog is a unicorn, then my dog can fly." Since the hypothesis "my dog is a unicorn" is false, the entire implication is vacuously true, regardless of whether my dog can fly.

3. Truth table for $P \Rightarrow Q$ vs. $\neg Q \Rightarrow \neg P$:

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$
T	T	**T**	F	F	**T**
T	F	**F**	T	F	**F**
F	T	**T**	F	T	**T**
F	F	**T**	T	T	**T**

Since the columns for $P \Rightarrow Q$ and its contrapositive are identical, they are logically equivalent.

Solutions for Chapter 4

Exercises 4.1

1. $\forall n \in \mathbb{Z}, E(n) \vee O(n)$.

2. "There exists an integer n whose square is 2." This is false. $\sqrt{2}$ is an irrational number, not an integer.

3. The original statement is $\forall x \in \mathbb{R}, x > 0 \Rightarrow x^2 > 0$.

   - **Symbolic Negation:** $\neg(\forall x \in \mathbb{R}, x > 0 \Rightarrow x^2 > 0) \equiv \exists x \in \mathbb{R}, \neg(x > 0 \Rightarrow x^2 > 0)$. Using the fact that $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$, this becomes: $\exists x \in \mathbb{R}, (x > 0 \wedge x^2 \leq 0)$.
   - **English Negation:** "There exists a real number that is positive and whose square is less than or equal to zero." (This is false).

Exercises 4.2

1. The number -4 belongs to $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$. It does not belong to $\mathbb{N}$.

2. Examples include π, $\sqrt{2}$, e. These are real numbers but cannot be written as a fraction of two integers.

3. No. For example, $3 \in \mathbb{Z}$ and $4 \in \mathbb{Z}$, but $3 \div 4 = \frac{3}{4}$, which is in $\mathbb{Q}$ but not in $\mathbb{Z}$.

Solutions for Chapter 5

Exercises 5.2

1. $\sum_{k=1}^{5} k^2$.
2. $\prod_{k=1}^{n} k = n!$.
3. μ represents the mean of the distribution (the center of the bell curve) and σ represents the standard deviation (a measure of the spread or width of the bell curve).

Solutions for Chapter 6

Exercises 6.2

1. Let $D_k(n)$ be "n is divisible by k." The statement is: $\forall n \in \mathbb{Z}, D_6(n) \Rightarrow (D_2(n) \wedge D_3(n))$.
2. $\forall x \in \mathbb{R}, x^2 \geq 0$.
3. "For every positive real number y, there exists a real number x whose square is y." In other words, "Every positive real number has a real square root."

Solutions to Chapter Review Exercises

1. **Rhetorical algebra** uses full sentences (e.g., Al-Khwarizmi). **Syncopated algebra** uses some abbreviations but is still mainly prose. **Symbolic algebra** uses abstract symbols for variables and operations, allowing for general formulas. A key development was **François Viète's** use of letters to represent variables and constants, which generalized algebra from solving specific problems to exploring general relationships.
2. Let $U = \{1, ..., 10\}$, $A = \{1, 2, 3, 4, 5\}$.
   (a) $B = \{2, 4, 6, 8, 10\}$.
   (b) $A \cap B = \{2, 4\}$.
   (c) $A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}$.
   (d) $A \setminus B = \{1, 3, 5\}$.
   (e) $A^c = U \setminus A = \{6, 7, 8, 9, 10\}$.
3. Truth table for $(P \vee Q) \Rightarrow (P \wedge Q)$:

P	Q	$P \vee Q$	$P \wedge Q$	$(P \vee Q) \Rightarrow (P \wedge Q)$
T	T	T	T	**T**
T	F	T	F	**F**
F	T	T	F	**F**
F	F	F	T	**T**

   The statement is true if and only if P and Q have the same truth value. (Notice this is logically equivalent to the biconditional $P \Leftrightarrow Q$).

4. (a) The statement means that for any person y, there is a person x who is their mother. So: $\forall y \in P, \exists x \in P, M(x, y)$. The order is crucial: $\exists x, \forall y, M(x, y)$ would mean a single person x is the mother of everyone!
   (b) **Symbolic Negation:** $\neg(\forall y \in P, \exists x \in P, M(x, y)) \equiv \exists y \in P, \forall x \in P, \neg M(x, y)$.

(c) **English Negation:** "There exists a person who has no mother."

5. (a) The odd integers are of the form $2k-1$. For $k=1$, we get 1. For $k=50$, we get 99. So the sum is $\sum_{k=1}^{50}(2k-1)$.

   (b) "For every two elements x_1, x_2 in the domain, if they are not equal, then their function values are not equal." Symbolically: $\forall x_1, x_2 \in \mathbb{R}, (x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2))$. An equivalent form using the contrapositive is often more useful: $\forall x_1, x_2 \in \mathbb{R}, (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$.