Chapter 4

The Continuous-Time Fourier Transform

—Frequency Domain Analysis of LTI System

- **Representation of Aperiodic Signals**

- **LTI System's Response to Aperiodic Signals**

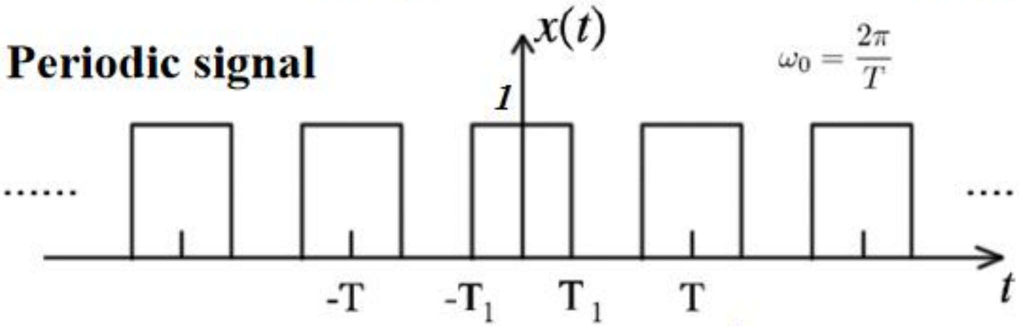
4.0 Introduction

<Example 3.5>

$$x_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \Rightarrow T x_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0} = \left.\underbrace{\frac{2\sin(\omega T_1)}{\omega}}_{X(j\omega)}\right|_{\omega=k\omega_0}$$

Periodic signal

$\omega_0 = \frac{2\pi}{T}$

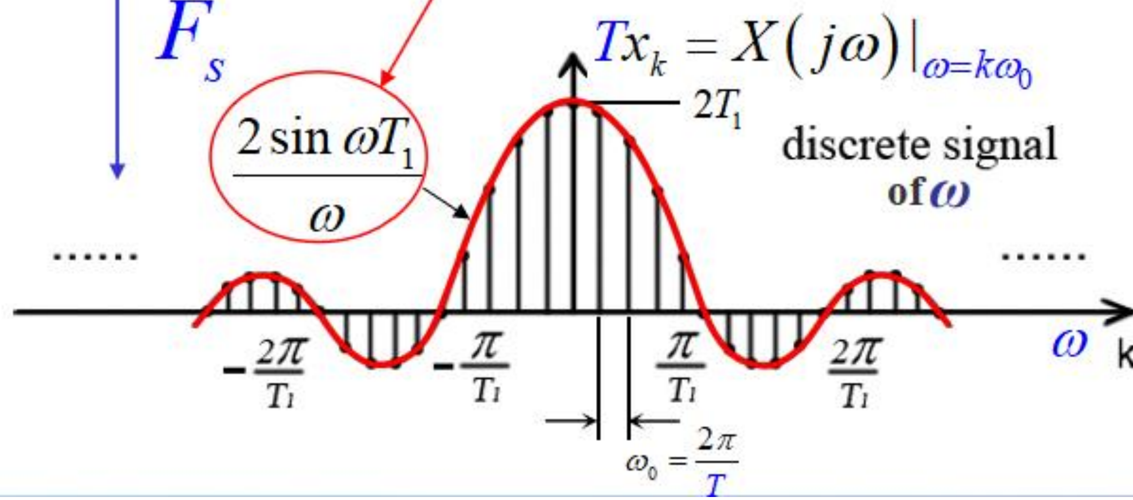


$\Updownarrow F_s$

$\Downarrow$ $T \to \infty$

$\omega_0 = \frac{2\pi}{T} \to 0$

$T x_k = X(j\omega)|_{\omega=k\omega_0}$



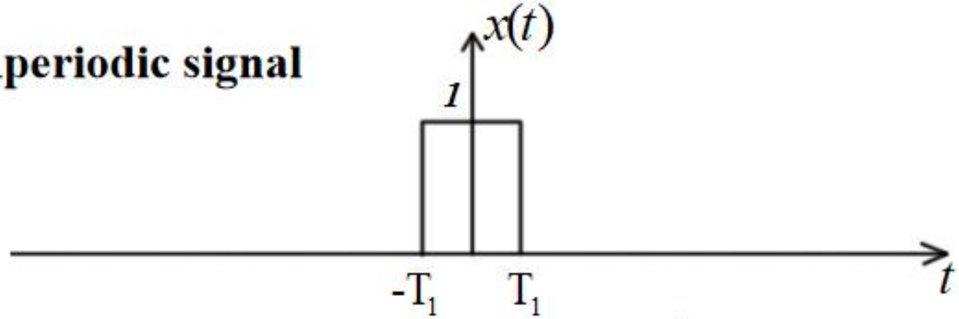
$T \to \infty$

$$\omega_0 = \frac{2\pi}{T} \to 0$$

$X(j\omega)$

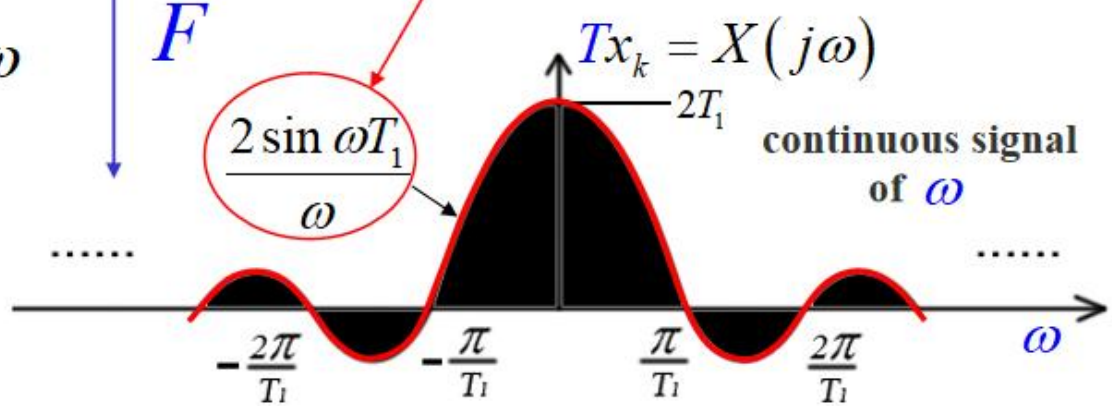
$$= \left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\omega=k\omega_0}$$

Aperiodic signal



F

$$\begin{cases} x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega \\ X(j\omega) = \displaystyle\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,d\omega \end{cases}$$



$$x(t) \xleftrightarrow{F} X(j\omega)$$

Fourier Transform

<General>

Periodic signal $\omega_0 = 2\pi / T$

$$
\begin{cases}
x(t) = \sum_{k=-\infty}^{+\infty} x_k e^{jk\omega_0 t} \\
x_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt
\end{cases}
$$

$\Downarrow$ ***reformation***

$$
\frac{2\pi}{T}
$$

$$
X(j\omega)\Big|_{\omega = k\omega_0}
$$

$$
\begin{cases}
x(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} T x_k e^{\overset{\omega}{\overbrace{jk\omega_0}} t} \cdot \omega_0 \\
T x_k = \int_T x(t) e^{-jk\omega_0 t} dt
\end{cases}
\xrightarrow[\omega_0 \to 0]{T \to \infty}
$$

$$
X(jk\omega_0) = X(j\omega)\Big|_{\omega = k\omega_0}
$$

Aperiodic signal

$$
\begin{cases}
x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \\
X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} d\omega
\end{cases}
$$

4.1.2 Convergence of Fourier Transform

1. $x(t)$ *must be absolutely integrable :*

$$\int_{-\infty}^{+\infty} |x(t)|\, dt < +\infty$$

2. $x(t)$ *have finite number of maxima and minima within any finite interval.*

3. $x(t)$ *have a finite number of discontinuities within any finite interval, and each of these discontinuities must be finite.*

任意信号x(t)的傅里叶变换求解思路

- 简单信号的傅里叶变换（要求都会）
- 结合傅里叶变换性质

4.1.3 Examples of Fourier Transform

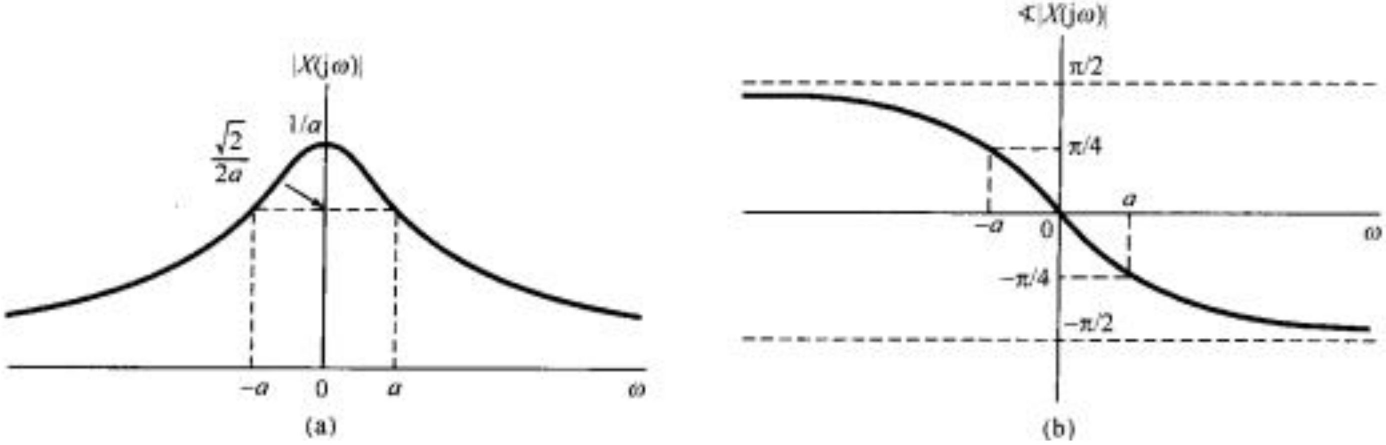
(Basic Fourier Transform Pairs： used later)

<4.1>

$$x(t)=e^{-at}u(t)\overset{F}{\longleftrightarrow}X(j\omega)=\frac{1}{j\omega+a}\quad(a>0)$$

②

$$\left|X(j\omega)\right|=\frac{1}{\sqrt{\omega^2+a^2}};\quad \angle X(j\omega)=-\tan^{-1}(\frac{\omega}{a})$$

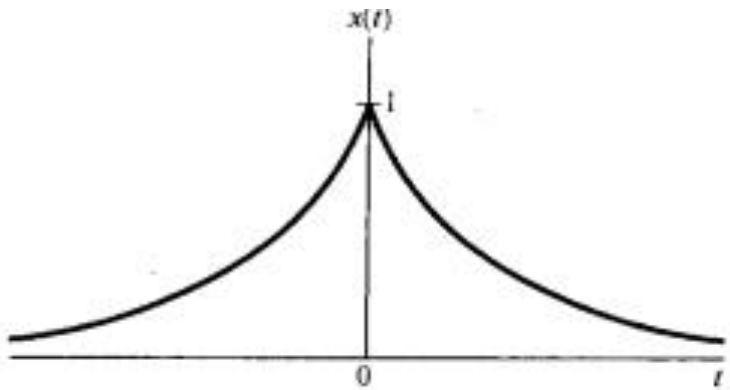


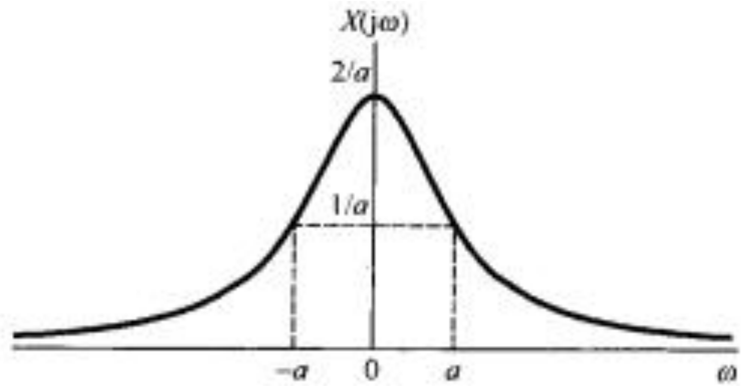
<4.2>

$$x(t) = e^{-a|t|} \underset{2}{\overset{F}{\longleftrightarrow}} X(j\omega) = \frac{2a}{\omega^2 + a^2} \qquad (a > 0)$$

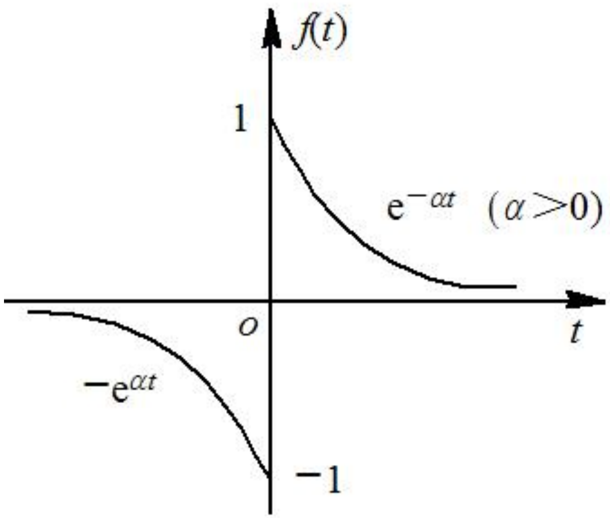
$$X(jw) = \frac{2a}{w^2 + a^2} \qquad \angle X(j\omega) = 0$$

(Figure sho

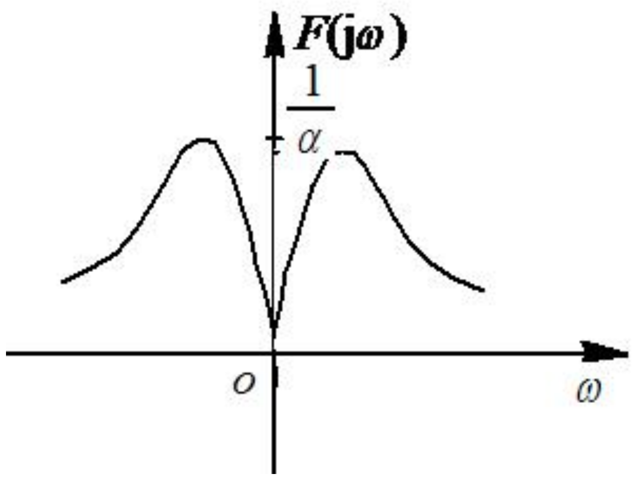




求下图所示信号$f(t)$的频谱函数。



$$f(t)=\begin{cases} e^{-at} & t>0 \\ -e^{-at} & t<0 \end{cases} \qquad (a>0)$$



$$X(j\omega)=\int_{-\infty}^{0}-e^{at}e^{-j\omega t}dt+\int_{0}^{\infty}e^{-at}e^{-j\omega t}dt$$

$$=-\frac{1}{a-j\omega}+\frac{1}{a+j\omega}=j\frac{-2\omega}{a^2+\omega^2}$$

考考你

已知信号 $f(t)=e^{-2|t|}$ ，则其频谱函数为(　　)

（A） $\dfrac{1}{jw+2}$

（B） $\dfrac{2}{w^2+2}$

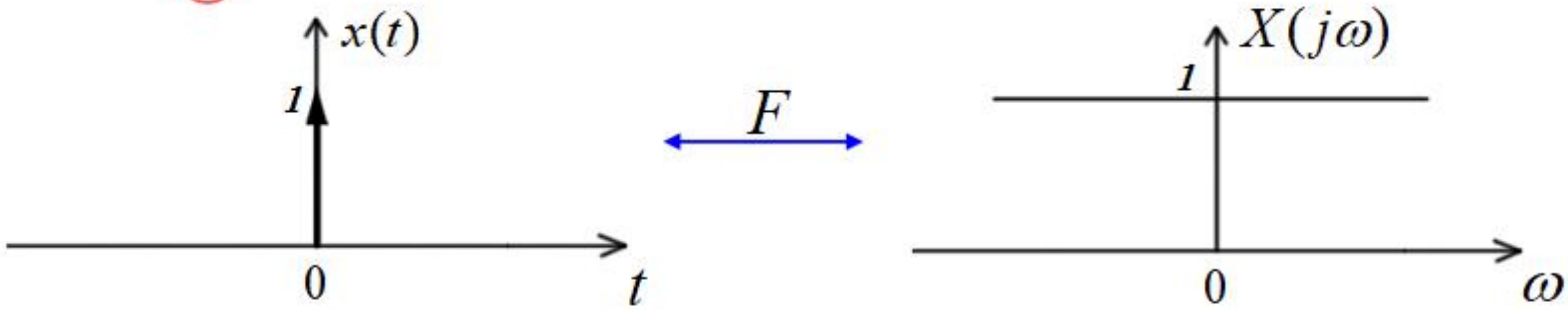
（C） $\dfrac{4}{w^2+2}$

（D） $\dfrac{4}{w^2+4}$

<4.3>

$$x(t)=\delta(t) \xleftrightarrow[\longrightarrow]{F} X(j\omega)=1$$

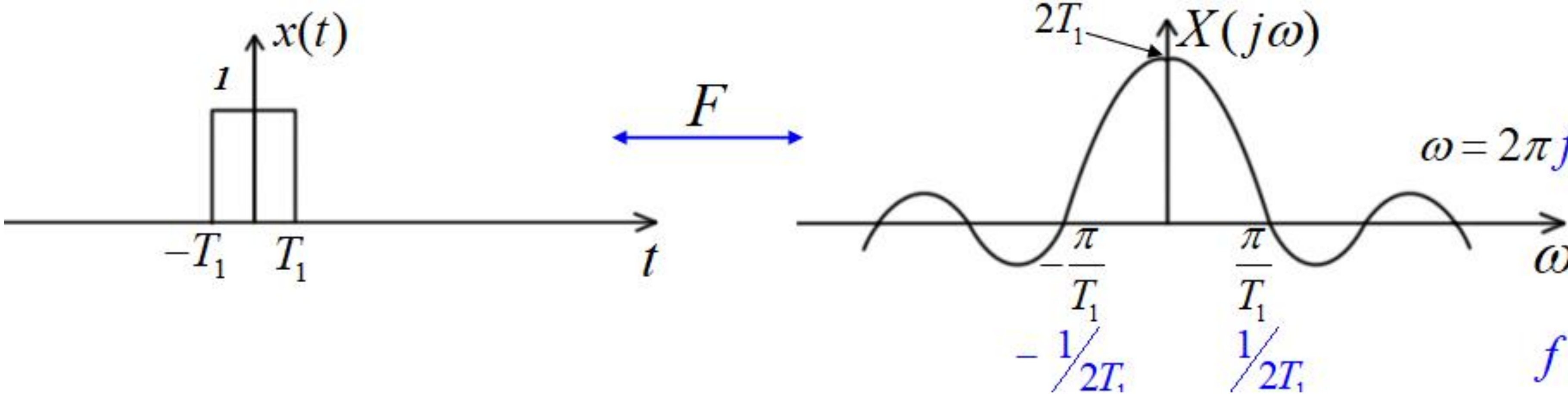
②



$$\delta'(t) \longleftrightarrow ?$$

<4.4>

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \xleftrightarrow[\rightarrow \; 2]{F} X(j\omega) = 2\frac{\sin \omega T_1}{\omega}$$



4.5 常数1

有一些函数不满足绝对可积这一充分条件，如1，ε(t)等，但傅里叶变换却存在。直接用定义式不好求解。

可构造一函数序列$\{f_n(t)\}$逼近f (t)，即

$$f(t)=\lim_{n\to\infty} f_n(t)$$

而$f_n(t)$满足绝对可积条件，并且$\{f_n(t)\}$的傅里叶变换所形成的序列$\{F_n(j\omega)\}$是极限收敛的。则可定义f(t)的傅里叶变换F (jω)为

$$X(j\omega)=\lim_{k\to\infty} a_k(j\omega)$$

这样定义的傅里叶变换也称为广义傅里叶变换。

构造 $x_\alpha(t)=e^{-\alpha|t|}$， $\alpha>0 \longleftrightarrow X_\alpha(j\omega)=\dfrac{2\alpha}{\alpha^2+\omega^2}$

$$x(t)=1=\lim_{\alpha\to 0} x_\alpha(t)$$

所以 $X(j\omega)=\lim\limits_{\alpha\to 0} X_\alpha(j\omega)=\lim\limits_{\alpha\to 0}\dfrac{2\alpha}{\alpha^2+\omega^2}=\begin{cases}0, & \omega\neq 0\\ \infty, & \omega=0\end{cases}$

又

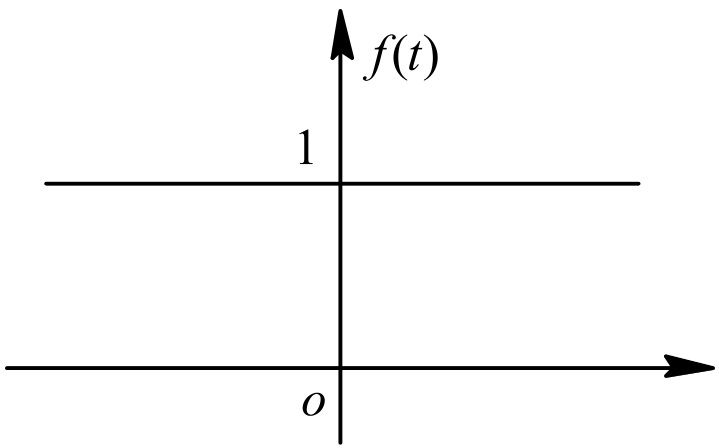
$$\lim_{\alpha\to 0}\int_{-\infty}^{\infty}\frac{2\alpha}{\alpha^2+\omega^2}d\omega=\lim_{\alpha\to 0}\int_{-\infty}^{\infty}\frac{2}{1+\left(\frac{\omega}{\alpha}\right)^2}d\frac{\omega}{\alpha}=\lim_{\alpha\to 0}2\arctan\frac{\omega}{\alpha}\Big|_{-\infty}^{\infty}=2\pi$$

因此， $1\longleftrightarrow 2\pi\delta(\omega)$

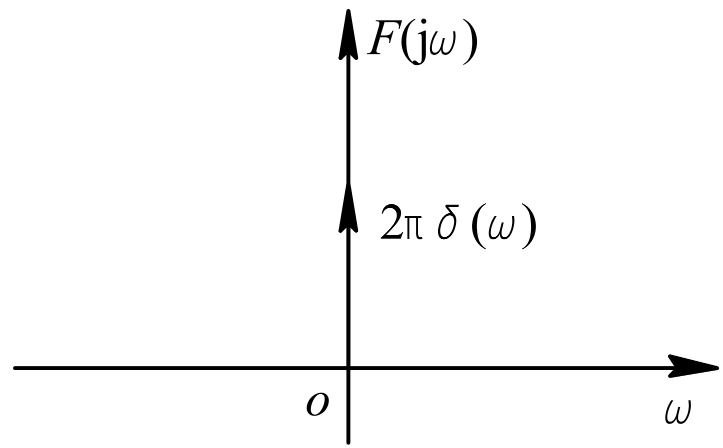
另一种求法： $\delta(t)\longleftrightarrow 1$代入反变换定义式

$$1\longleftrightarrow\int_{-\infty}^{\infty}e^{-j\omega t}dt=2\pi\delta(-\omega)=2\pi\delta(\omega)$$

$1 \longleftrightarrow 2\pi\delta(\omega)$



(a)



(b)

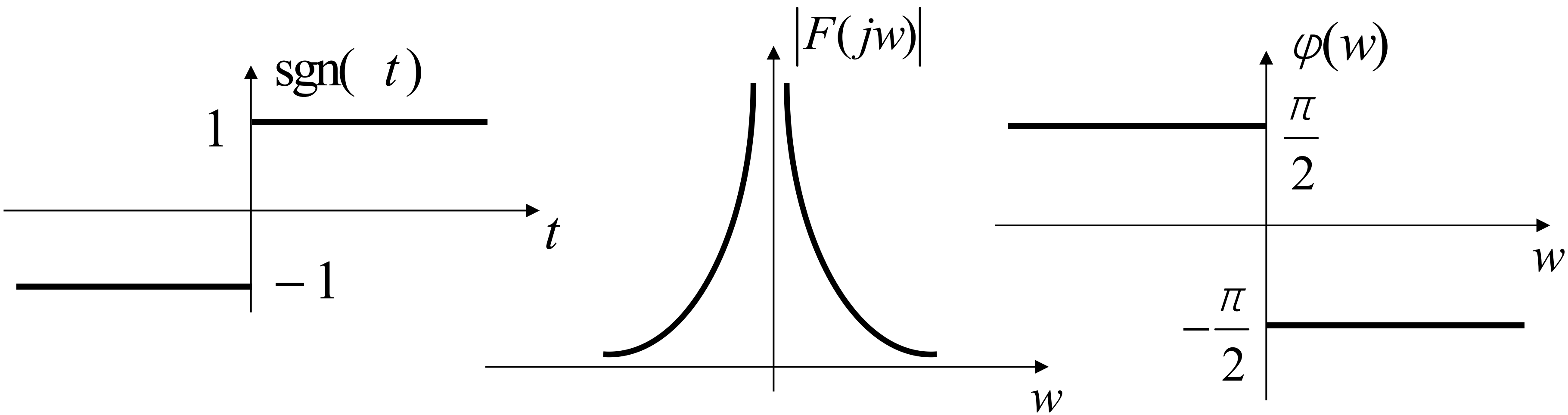
直流信号的频谱集中在w=0处，时域没有变化。

4.6 符号函数

$$\mathrm{sgn}(t)=\begin{cases}-1, & t<0\\ 1, & t>0\end{cases}\qquad f_a(t)=\begin{cases}-\mathrm{e}^{at}, & t<0\\ \mathrm{e}^{-at}, & t>0\end{cases}\qquad a>0$$

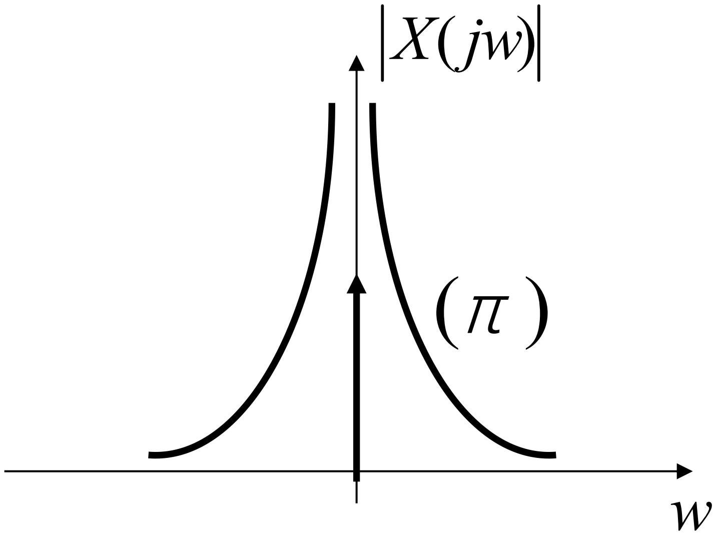
$$\mathrm{sgn}(t)=\lim_{a\to 0} f_a(t),\quad f_a(t)\longleftrightarrow F_a(j\omega)=\frac{1}{a+j\omega}-\frac{1}{a-j\omega}=-\frac{j2\omega}{a^2+\omega^2}$$

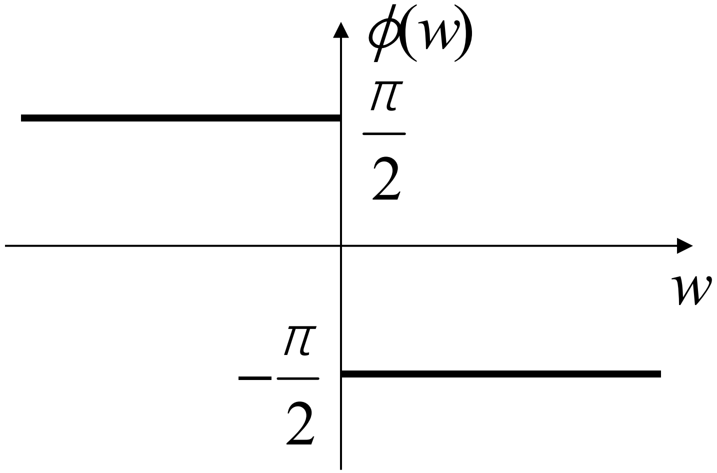
$$\mathrm{sgn}(t)\longleftrightarrow \lim_{a\to 0} F_a(j\omega)=\lim_{a\to 0}\left(-\frac{j2\omega}{a^2+\omega^2}\right)=\frac{2}{j\omega}$$



4.7 阶跃函数u(t)

$$u(t)=\frac{1}{2}+\frac{1}{2}\operatorname{sgn}(t)\longleftrightarrow \pi\delta(\omega)+\frac{1}{j\omega}$$

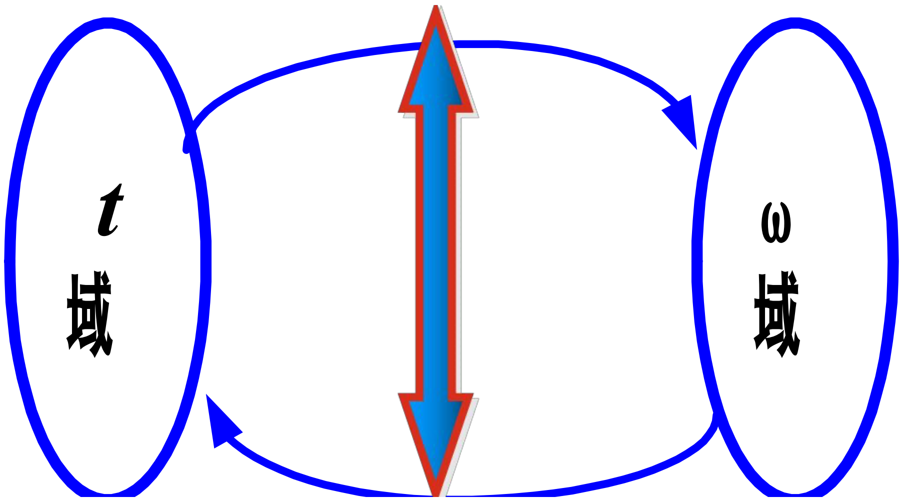




归纳记忆：

1. F 变换对

$$X(j\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}d\omega$$



$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}d\omega$$

2. 常用函数 F 变换对：

$$\delta(t)\longleftrightarrow 1$$

$$1\longleftrightarrow 2\pi\delta(\omega)$$

$$u(t)\longleftrightarrow \pi\delta(\omega)+\frac{1}{j\omega}$$

$$e^{-\alpha t}u(t)\longleftrightarrow \frac{1}{j\omega+a}$$

$$g_\tau(t)\longleftrightarrow \tau Sa\left(\frac{\omega\tau}{2}\right)$$

$$\mathrm{sgn}(t)\longleftrightarrow \frac{2}{j\omega}$$

$$e^{-\alpha|t|}\longleftrightarrow \frac{2a}{a^2+\omega^2}$$

$$\frac{1}{2} \leftrightarrow$$

$$\frac{1}{2}u(t) \leftrightarrow$$

$$e^{-t}u(t) \leftrightarrow$$

$$g_2(t) \leftrightarrow$$

$$e^{-|t|}u(t) \leftrightarrow$$

$$2\delta(t) \leftrightarrow$$

$$2sgn(t) \leftrightarrow$$

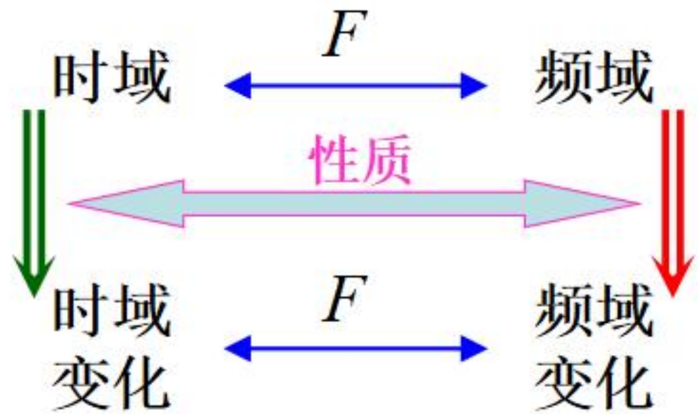
4.3 Properties of the Continuous-Time Fourier Transform

$$\begin{cases} x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega \\ X(j\omega) = \displaystyle\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt \end{cases} \quad \updownarrow F$$

① $x(t) = F^{-1}[X(j\omega)]$ $\quad e^{-at}u(t) = F^{-1}[\dfrac{1}{j\omega + a}]$

② $X(j\omega) = F\ [x(t)]$ $\quad \dfrac{1}{j\omega + a} = F\ [e^{-at}u(t)]$

Property:



4.3.1 Linearity

$$\sum_i a_i x_i \xleftrightarrow{F} \sum_i a_i X_i$$

$$\begin{cases} x(t) \xleftrightarrow{F} X(j\omega) \\ y(t) \xleftrightarrow{F} Y(j\omega) \end{cases}$$

$$\Downarrow$$

$$ax(t) + by(t) \xleftrightarrow{F} aX(j\omega) + bY(j\omega)$$

$$F[ax(t) + by(t)] \xleftrightarrow{F} aF[x(t)] + bF[y(t)]$$

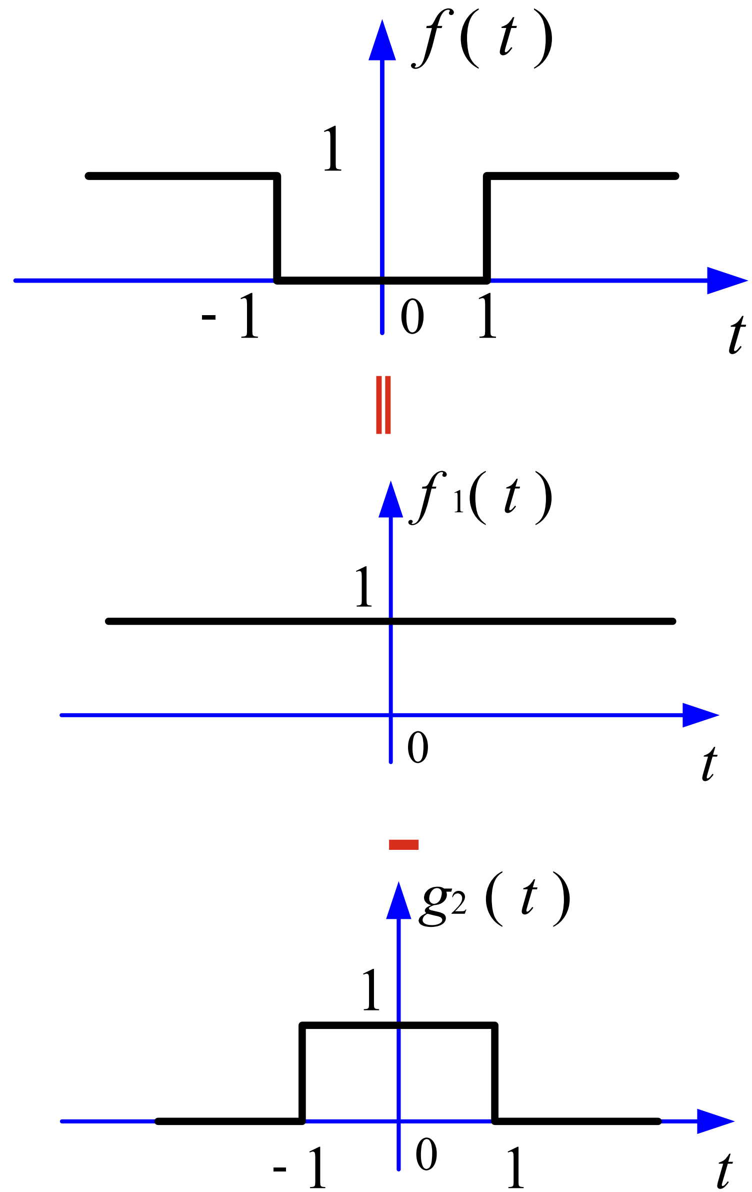
For example $F(j\omega) = ?$

Ans: $f(t) = f_1(t) - g_2(t)$

$$f_1(t) = 1 \longleftrightarrow 2\pi\delta(\omega)$$

$$g_2(t) \longleftrightarrow 2Sa(\omega)$$

$$\therefore\ F(j\omega) = 2\pi\delta(\omega) - 2Sa(\omega)$$



4.3.2 Time shifting

$$shift\ \ t_0 \xleftrightarrow{F} \cdot e^{-j\omega t_0}$$

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$\Downarrow$$

$$x(t-t_0) \xleftrightarrow[\longrightarrow]{F} e^{-j\omega t_0} X(j\omega)$$

1 *Proof*

For example $F(j\omega) = ?$

Ans: $f_1(t) = g_6(t-5)$,

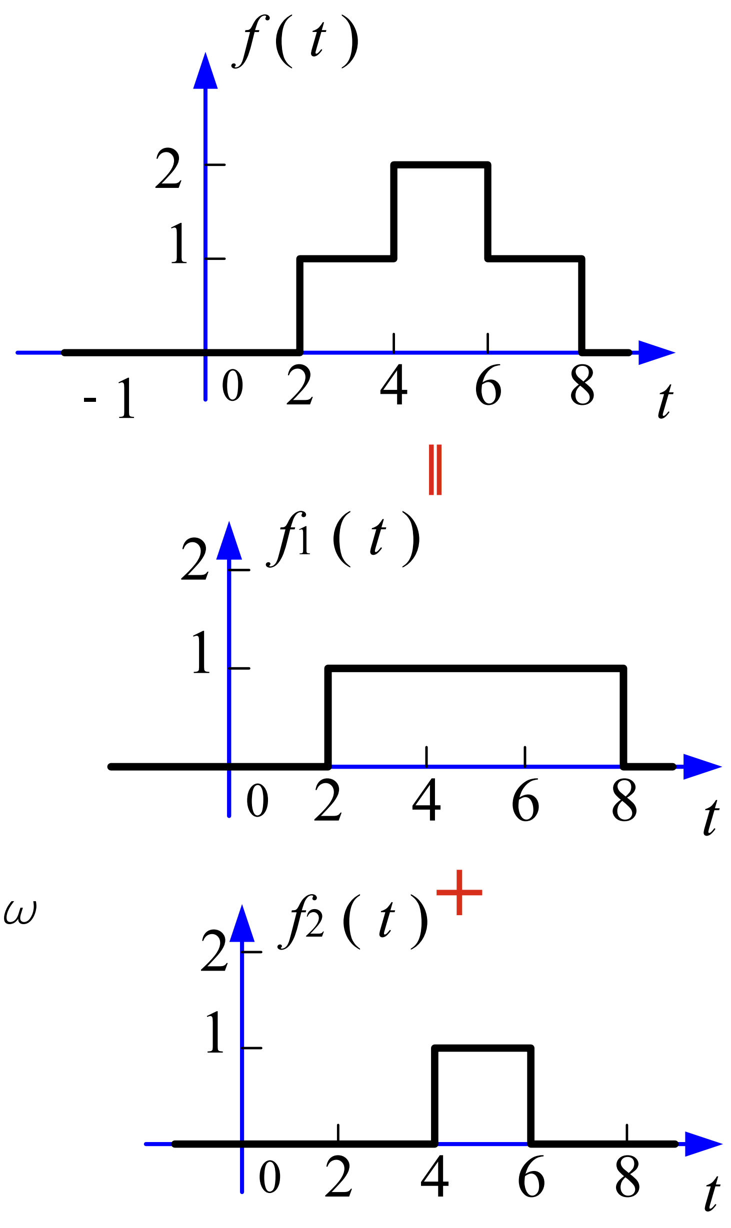
$f_2(t) = g_2(t-5)$

$$g_6(t-5) \longleftrightarrow 6\,\mathrm{Sa}(3\omega)\,\mathrm{e}^{-j5\omega}$$

$$g_2(t-5) \longleftrightarrow 2\,\mathrm{Sa}(\omega)\,\mathrm{e}^{-j5\omega}$$

$\therefore F(j\omega) =$

$$[6\,\mathrm{Sa}(3\omega) + 2\,\mathrm{Sa}(\omega)]\,\mathrm{e}^{-j5\omega}$$



4.3.2' Frequency Shifting

$\cdot e^{j\omega_0 t} \xleftrightarrow{F} shift \quad \omega_0$

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$② \Downarrow$$

$$e^{j\omega_0 t}x(t) \xleftrightarrow{F} X[j(\omega-\omega_0)]$$

For example 1

$$f(t) = e^{j3t} \longleftrightarrow F(j\omega) = ?$$

For example 2

$$f(t) = \cos\omega_0 t \longleftrightarrow F(j\omega) = ?$$

Ans: $$f(t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

$$F(j\omega) = \pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$$

$$f(t) = \sin\omega_0 t \longleftrightarrow F(j\omega) = ?$$

For example 3

Given that $f(t) \longleftrightarrow F(j\omega)$

The modulated signal $f(t)\cos\omega_0 t \longleftrightarrow ?$

$$F\left[f(t)\cos\omega_0 t\right] = \frac{1}{2}\left\{F\left[j(\omega+\omega_0)\right] + F\left[j(\omega-\omega_0)\right]\right\}$$

例： 求矩形脉冲调幅信号的频谱，已知 $f(t)=g_{\tau}(t)\cos w_0 t$，其中$g_{\tau}(t)$ 为矩形脉冲，脉幅为1, 脉宽为τ。

4.3.2'' Time Reversal

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$\Downarrow$$

$$x(-t) \xleftrightarrow{F} X(-j\omega)$$

① ② *Proof*

4.3.3 Conjugate Symmetry

$$* \xleftrightarrow{F} *(-)$$

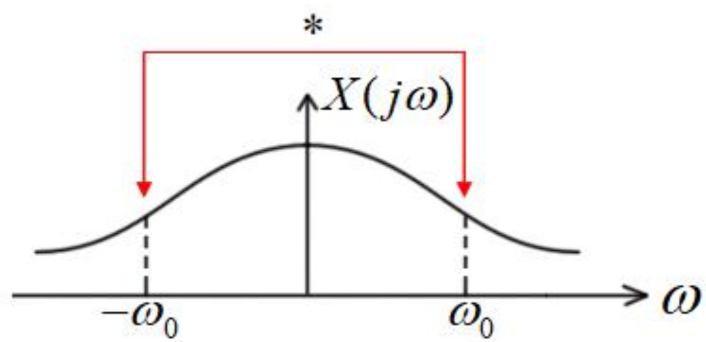
$$x(t) \xleftrightarrow{F} X(j\omega)$$

① $\Downarrow$

$$x^*(t) \xleftrightarrow{F} X^*(-j\omega)$$

<Especially for real $x(t)$ >

(**1**) *if* $x(t)$ *is real,* $X(j\omega)$ *is conjugate symmetry*



real

$$
\begin{tikzcd}
x(t) \arrow[r, leftrightarrow, "F"] \arrow[d, "\text{共轭}"] & X(j\omega) \arrow[d, "\text{共轭}"] \\
x^*(t) \arrow[r, leftrightarrow, "F"] & X^*(-j\omega)
\end{tikzcd}
$$

$X(j\omega) = X^*(-j\omega)$ **or** $X^*(j\omega) = X(-j\omega)$

Conjugate Symmetry

$x(t)$ *is real* :

$$X(j\omega) = \underbrace{|X(j\omega)|}_{\text{even}}\, e^{j\underbrace{\angle X(j\omega)}_{\text{odd}}} = \underbrace{R(\omega)}_{\text{even}} + j\underbrace{I(\omega)}_{\text{odd}}$$

<Proof>

$$X(j\omega) = |X(j\omega)|\, e^{j\angle X(j\omega)} = R(\omega) + jI(\omega)$$

$\Downarrow *$

$\omega \Rightarrow -\omega$

$$X^*(j\omega) = |X(j\omega)|\, e^{-j\angle X(j\omega)} = R(\omega) - jI(\omega)$$

Conjugate Symmetry $\|$

even | odd | even | odd

$$X(-j\omega) = |X(-j\omega)|\, e^{j\angle X(-j\omega)} = R(-\omega) + jI(-\omega)$$

(2) ***x(t)*** *is real even,* $X(j\omega)$ *is a real even*

$$
\begin{tikzcd}
x^*(t) \arrow[r, leftrightarrow, "F"] & X^*(-j\omega) \\
x(t) \arrow[r, leftrightarrow, "F"] \arrow[u, "\text{共轭}"'] \arrow[d, "\text{反转}"'] & X(j\omega) \arrow[u, "\text{共轭}"'] \arrow[d, "\text{反转}"'] \\
x(-t) \arrow[r, leftrightarrow, "F"] & X(-j\omega)
\end{tikzcd}
$$

real ‖   ‖ real

even ‖   ‖ even

Real Even $\xleftrightarrow{F}$ ***Real Even***

(3) ***x(t)*** *is real odd,* $X(j\omega)$ *is a pure imaginary odd*

$$
\begin{tikzcd}
x^*(t) \arrow[r, leftrightarrow, "F"] & X^*(-j\omega) \\
x(t) \arrow[r, leftrightarrow, "F"] \arrow[u, "\text{共轭}"'] \arrow[d, "\text{反转}"'] & X(j\omega) \arrow[u, "\text{共轭}"'] \arrow[d, "\text{反转}"'] \\
-x(-t) \arrow[r, leftrightarrow, "F"] & -X(-j\omega)
\end{tikzcd}
$$

real ‖   ‖ imaginary

odd ‖   ‖ odd

Real Odd $\xleftrightarrow{F}$ ***Imaginary Odd***

(4) Even and odd part of real x(t)
~ Real and imaginary part of $X(j\omega)$

$$
\begin{array}{ccccc}
\text{real} & & \text{real even} & & \text{real odd} \\
x(t) & = & x_e(t) & + & x_o(t) \\
F\updownarrow & & F\updownarrow & & F\updownarrow \\
X(j\omega) & = & R(\omega) & + & jI(\omega) \\
\text{conjugate symmetry} & & \text{real even} & & \text{imaginary odd}
\end{array}
$$

Chapter 1
$$
\begin{cases}
\text{even} \quad x_e(t) = \dfrac{x(t)+x(-t)}{2} \\
\text{odd} \quad x_o(t) = \dfrac{x(t)-x(-t)}{2}
\end{cases}
$$

$\therefore$ For real signal $x(t)$:

偶部的F = F的实部 ***Transform of Even Part = Real Part of Transform***

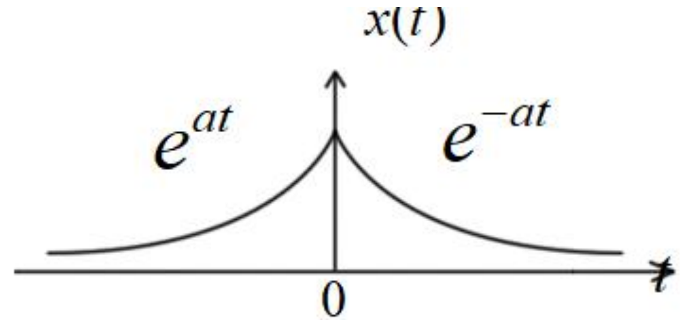
奇部的F = F的虚部 ***Transform of Odd Part = Imaginary Part of Transform***

<4.10>

$$x(t) = e^{-a|t|} \xleftrightarrow[\;?\;]{F} \frac{2a}{a^2+\omega^2} = X(j\omega)$$

real even **real even**

Solution:



$$x_1(t) = e^{-at}u(t) \xleftrightarrow[\surd]{F} \frac{1}{a+j\omega} = X_1(j\omega)$$

$$x(t) = e^{-at}u(t) + e^{at}u(-t) = 2\cdot\frac{x_1(t)+x_1(-t)}{2} = 2x_{1e}(t)$$

$$\Big\downarrow F \qquad\qquad F$$

$$X(j\omega) = 2\,\mathrm{Re}[X_1(j\omega)] = \frac{2a}{a^2+\omega^2}$$

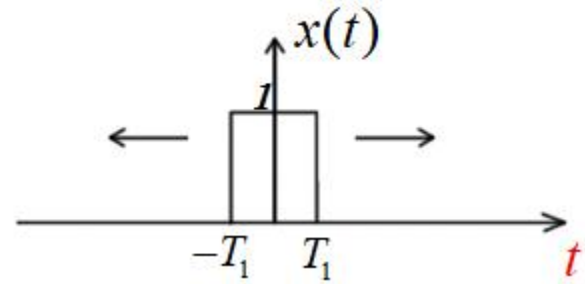
偶部的F = F的实部 ***Transform of Even Part = Real Part of Transform***

4.3.5 Time and Frequency Scaling

$$x(t) \xleftrightarrow{F} X(j\omega)$$

1 $\Downarrow$

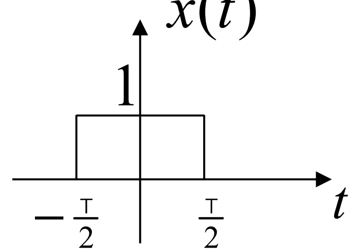
$$x(at) \xleftrightarrow{F} \frac{1}{|a|} X(j\frac{\omega}{a})$$

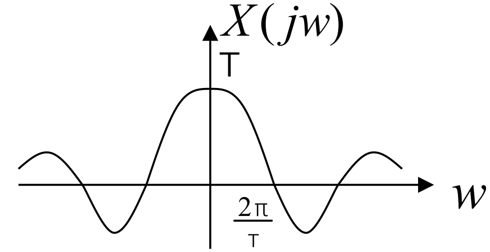


若 $F\left[f(t)\right] = F(\mathrm{j}\omega)$，则 $F\left[f(at)\right] = \frac{1}{|a|}F\left(\mathrm{j}\frac{\omega}{a}\right)$

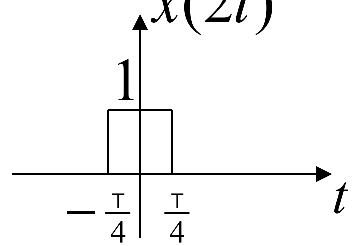
$$x(t) \xleftrightarrow{FT} X(jw)$$

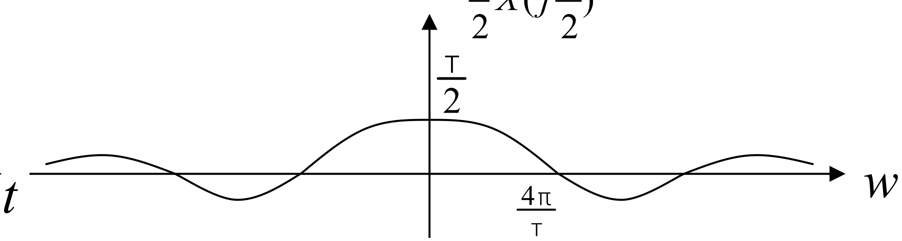
$$X(jw) = \tau Sa\left(w\frac{\tau}{2}\right)$$



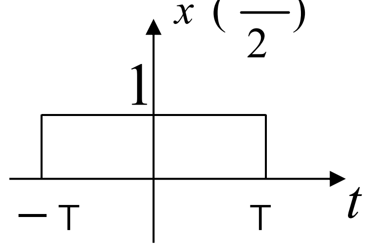


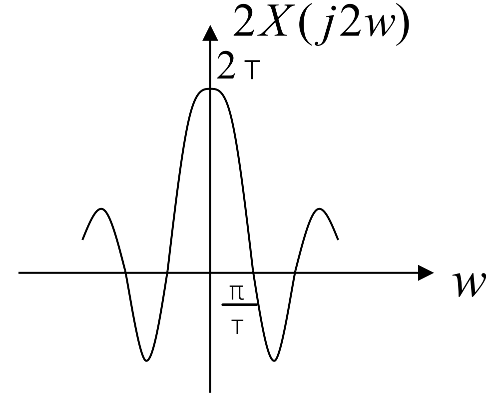
$$x(2t) \xleftrightarrow{FT} \frac{\tau}{2}Sa\left(w\frac{\tau}{4}\right)$$





$$x\left(\frac{t}{2}\right) \xleftrightarrow{FT} 2\tau Sa(w\tau)$$





- 尺度变换性质表明，信号的持续时间与其频带宽度成反比。
- 在通信系统中，为了快速传输信号，即加快信息传递速度，对信号进行时域压缩，将以扩展频带为代价；
- 若压缩信号的频带宽度，则需要增加信号的持续时间为代价；
- 这是通信中时长与带宽的矛盾，也是通信速度与信道容量的矛盾，故在实际应用中要权衡考虑。

例1： 已知$f(t) \longleftrightarrow F(j\omega)$，求 $f(at-b) \longleftrightarrow ?$

解： $f(t-b) \longleftrightarrow e^{-j\omega b} F(j\omega)$

$$f(at-b) \longleftrightarrow \frac{1}{|a|} e^{-j\frac{\omega}{a}b} F\left(j\frac{\omega}{a}\right)$$

或

$$f(at) \longleftrightarrow \frac{1}{|a|} F\left(j\frac{\omega}{a}\right)$$

$$f(at-b) = f\left[a(t-\frac{b}{a})\right] \longleftrightarrow \frac{1}{|a|} e^{-j\frac{\omega}{a}b} F\left(j\frac{\omega}{a}\right)$$

例2： $f(t) = \dfrac{1}{-jt+1} \longleftrightarrow F(j\omega) = ?$

解： $e^{-t}\varepsilon(t) \longleftrightarrow \dfrac{1}{j\omega+1}$

对称性：$\dfrac{1}{jt+1} \longleftrightarrow 2\pi e^{\omega}\varepsilon(-\omega)$

尺度变换：**a = -1**

$$\therefore \quad \frac{1}{-jt+1} \longleftrightarrow 2\pi e^{-\omega}\varepsilon(\omega)$$

考考你

若$F_1(jw)=F[f_1(t)]$，则$F_2(jw)=F[f_1(4-2t)]$ =(   )

(A) $\frac{1}{2}F_1(jw)e^{-j4w}$

(B) $\frac{1}{2}F_1(-j\frac{w}{2})e^{-j4w}$

(C) $F_1(-jw)e^{-jw}$

(D) $\frac{1}{2}F_1(-j\frac{w}{2})e^{-j2w}$

4.3.4 Differentiation and Integration in Time Domain

A. Differentiation

$$^{(k)} \xleftrightarrow{F} \cdot (j\omega)^k$$

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$① \Downarrow$$

$$x'(t) \xleftrightarrow{F} j\omega X(j\omega)$$

B. Integration $\int_{-\infty}^{t} \xleftrightarrow[\int_{-\infty}^{+\infty}=0]{F} \cancel{\;}\,/\,j\omega$

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$\int_{-\infty}^{t} \Bigg\Downarrow$$

（不常用） $\int_{-\infty}^{+\infty} x(t)e^{-j0t}dt = \int_{-\infty}^{+\infty} x(\tau)d\tau$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

or

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$$\int_{-\infty}^{t} \Bigg\Downarrow \quad \int_{-\infty}^{+\infty} x(\tau)d\tau = 0$$

（时域面积为0, 无直流分量） （常用）

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{1}{j\omega}X(j\omega)$$

C. Examples

<4.11>

$$u(t) \xleftrightarrow{F} \frac{1}{j\omega} + \pi\delta(\omega)$$

例1:

$$f(t)=1/t^2 \leftrightarrow ?$$

解:

$$\mathrm{sgn}(t) \leftrightarrow \frac{2}{j\omega}$$

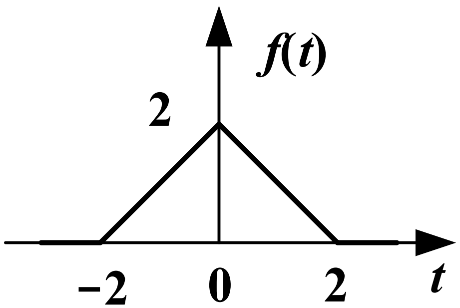
$$\frac{2}{jt} \leftrightarrow 2\pi\,\mathrm{sgn}(-\omega)$$

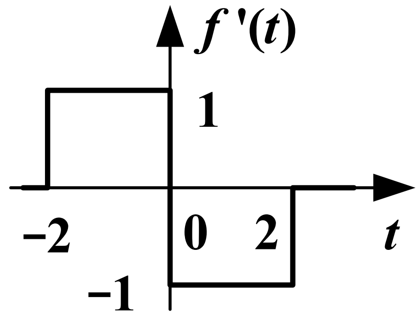
$$\frac{1}{t} \leftrightarrow -j\pi\,\mathrm{sgn}(\omega)$$

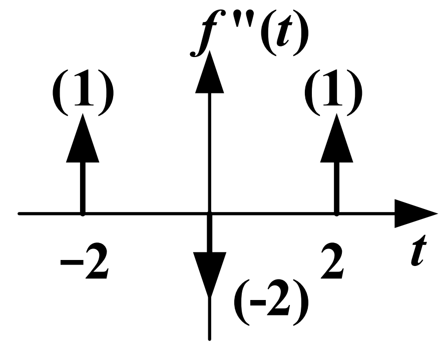
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{t}\right) \leftrightarrow -(j\omega)j\pi\,\mathrm{sgn}(\omega)=\pi\ \omega\,\mathrm{sgn}(\omega)$$

$$\frac{1}{t^2} \leftrightarrow -\pi\ \omega\,\mathrm{sgn}(\omega)=-\pi\,|\,\omega\,|$$

例2:







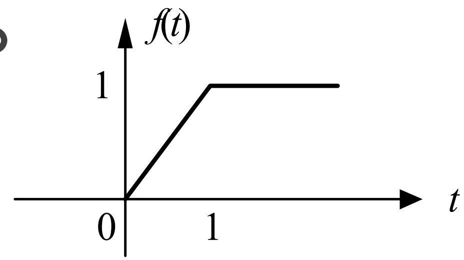
试确定 $f\,(\mathrm{t}) \leftrightarrow F\,(j\omega)$。

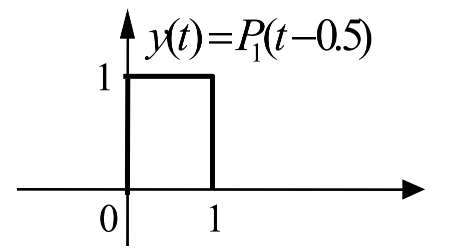
解: $f''(t) = \delta(t+2) - 2\,\delta(t) + \delta(t-2)$

$$F_2(j\omega) = \mathcal{F}\,[f''(t)] = \mathrm{e}^{j2\omega} - 2 + \mathrm{e}^{-j2\omega} = 2\cos(2\omega) - 2$$

$$F\,(j\omega) = \frac{F_2(j\omega)}{(j\omega)^2} = \frac{2 - 2\cos(2\omega)}{\omega^2}$$

例3：试利用积分特性求图示信号 $f(t)$ 的频谱函数。





解：先求 $f'(t)$ 的频谱。

$f'(t)$ 微分后如 $y(t)$ 所示，对应的频谱 $Y(j\omega)$ 易求：

$$p_1(t-0.5) \xleftrightarrow{FT} Y(j\omega)=\mathrm{Sa}(0.5\omega)e^{-j0.5\omega}$$

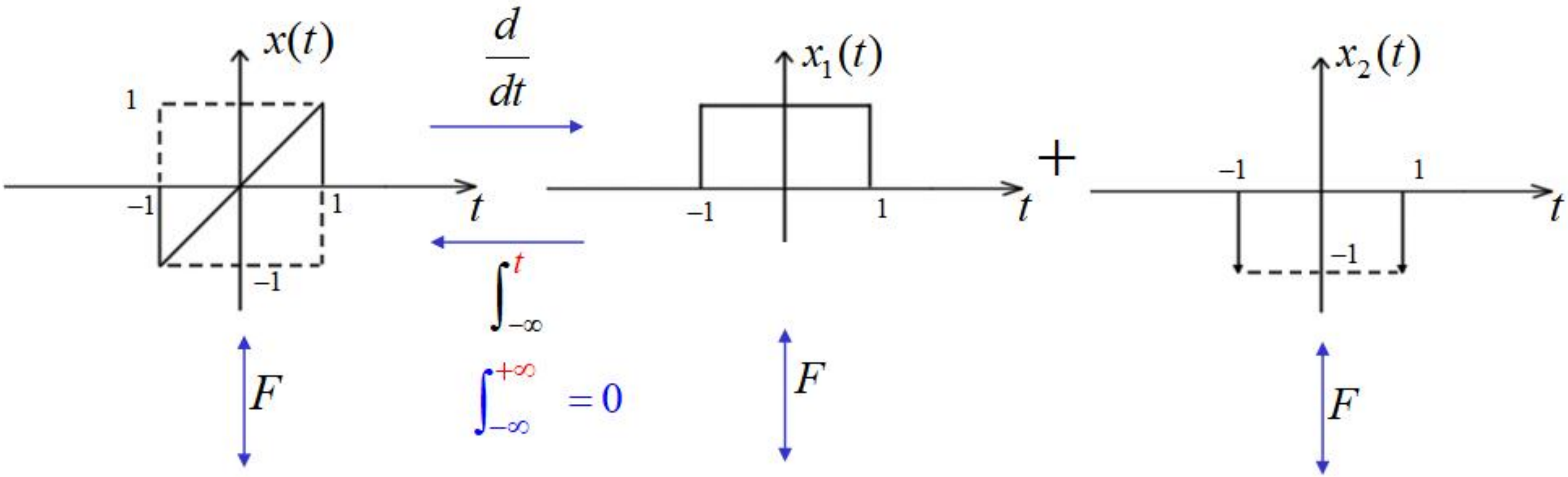
利用时域积分特性，可得

$$F(j\omega)=\frac{1}{j\omega}Y(j\omega)+\pi Y(0)\delta(\omega)$$

$$=\frac{1}{j\omega}\mathrm{Sa}(0.5\omega)e^{-j0.5\omega}+\pi\delta(\omega)$$

因为 $f(t)$ 中有直流分量,不能用微分性质。

<4.12> $\overset{\surd}{x(t)} \xleftrightarrow{F} X(j\omega)^{?}$ ② ?



$$\frac{1}{j\omega}\left(\frac{2\sin\omega}{\omega}-2\cos\omega\right)=\frac{2\sin\omega}{j\omega^2}-\frac{2\cos\omega}{j\omega}$$

B. Integration

$$/(-jt) \xleftrightarrow[\int_{-\infty}^{+\infty}=0]{F} \int_{-\infty}^{\omega}$$

$$\begin{array}{ccc} x(t) & \xleftrightarrow{F} & X(j\omega) \\ \Big\Downarrow \text{Can be proved} & & \Big\Downarrow \int_{-\infty}^{\omega} \\ \dfrac{1}{-jt}\cdot x(t)+\pi x(0)\delta(t) & \xleftrightarrow{F} & \int_{-\infty}^{\omega} X(j\eta)d\eta \end{array}$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\cdot e^{j\omega 0}d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\eta)d\eta$$

or

$$\begin{array}{ccc} x(t) & \xleftrightarrow{F} & X(j\omega) \\ /(-jt)\Big\Downarrow & & \int_{-\infty}^{+\infty}=0 \Big\Downarrow \int_{-\infty}^{\omega} \\ \dfrac{x(t)}{-jt} & \xleftrightarrow{F} & \int_{-\infty}^{\omega} X(j\eta)d\eta \end{array}$$

4.3.4' Differentiation and Integration in Frequency Domain

A. Differentiation

$$\bullet(-jt)^k \overset{F}{\longleftrightarrow} {}^{(k)}$$

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega)$$

② $\Downarrow$

$$-jt\cdot x(t) \overset{F}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}$$

例1： 试确定 $f(t) = tu(t) \leftrightarrow F(j\omega)=?$

解： $u(t) \leftrightarrow \pi\delta(\omega) + \dfrac{1}{j\omega}$ $\qquad -jtu(t) \leftrightarrow \dfrac{\mathrm{d}}{\mathrm{d}\omega}\left[\pi\delta(\omega) + \dfrac{1}{j\omega}\right]$

$$tu(t) \leftrightarrow j\pi\delta'(\omega) - \frac{1}{\omega^2}$$

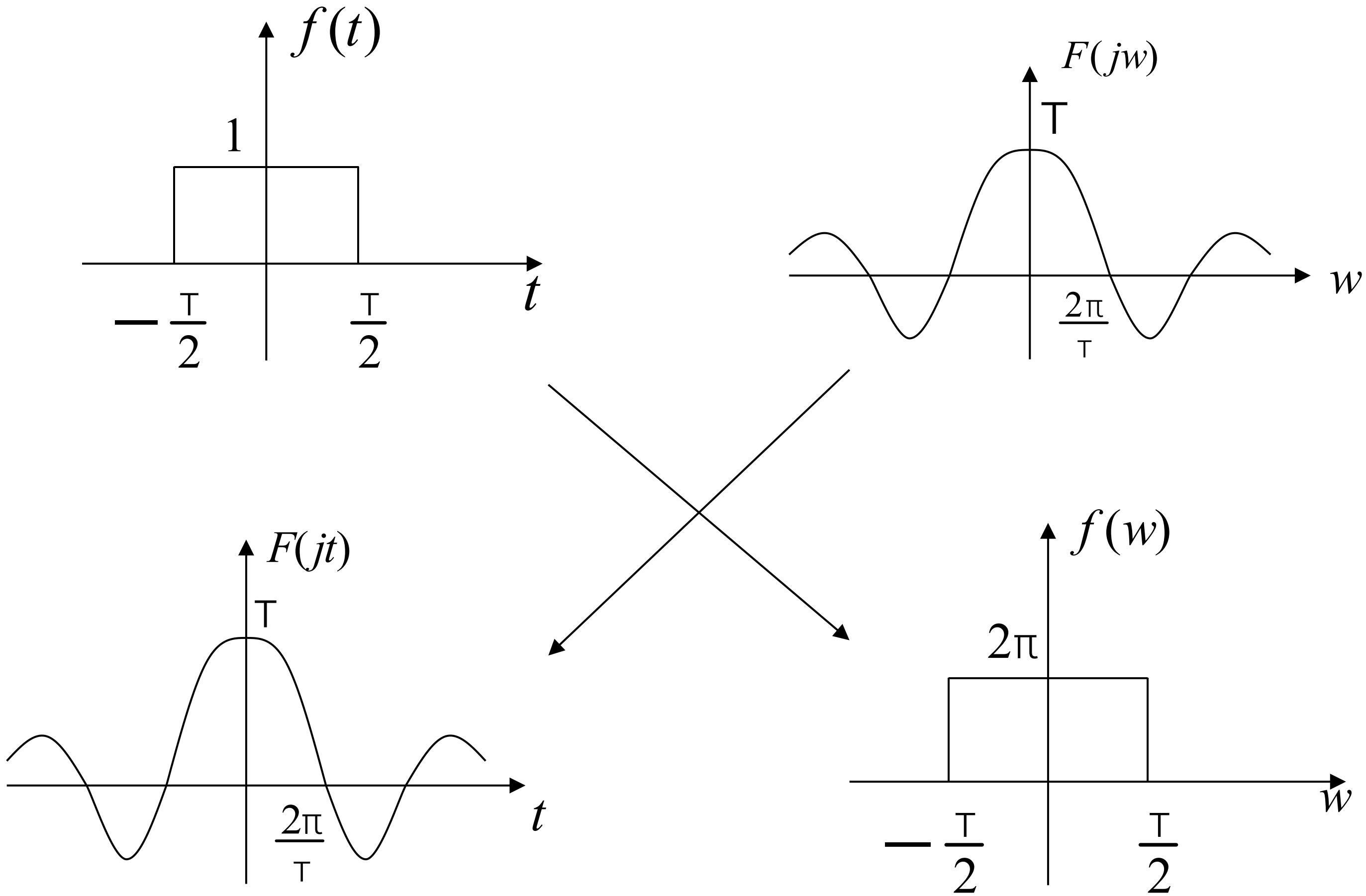
4.3.6 Duality (对偶性)

$$\bigcirc \xleftrightarrow{F} \bigcirc$$

$$\begin{tikzcd} x(t) \arrow[r, leftrightarrow, "F"] \arrow[d, Rightarrow] \arrow[dr] & X(j\omega) \arrow[dl] \\ X(jt) \arrow[r, leftrightarrow, "F"] & 2\pi x(-\omega) \end{tikzcd}$$

Can be proved

对称性质 $\boxed{F(jt) \longleftrightarrow 2\pi f(-\omega)}$



例： 求 $F\left[\frac{1}{t}\right]$

$$\boxed{F(jt) \longleftrightarrow 2\pi f(-\omega)}$$

解： 因为 $F\,[\mathrm{sgn}(t)] = \dfrac{2}{j\omega}$，

所以 $F\left[\dfrac{2}{jt}\right] = 2\pi\,\mathrm{sgn}(-\omega) = -2\pi\,\mathrm{sgn}(\omega)$

这样 $F\left[\dfrac{1}{t}\right] = -j\pi\,\mathrm{sgn}(\omega)$

For example

$$f(t)=\frac{1}{1+t^2}\longleftrightarrow F(j\omega)=?$$

Ans: $$\mathrm{e}^{-\alpha|t|}\longleftrightarrow\frac{2\alpha}{\alpha^2+\omega^2}$$

if α=1, $$\mathrm{e}^{-|t|}\longleftrightarrow\frac{2}{1+\omega^2}$$

$$\therefore\quad \frac{2}{1+t^2}\longleftrightarrow 2\pi\,\mathrm{e}^{-|\omega|}\qquad \frac{1}{1+t^2}\longleftrightarrow \pi\,\mathrm{e}^{-|\omega|}$$

*** if** $$\boxed{f(t)=\frac{t^2-2t+3}{t^2-2t+2}}\qquad\boxed{F(j\omega)=?}$$