Validation of the Unsteady Heat Diffusion Solver

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Analytical solution of steady or transient diffusion problem is obtained via different techniques such as seperation of variables or laplace transform. In general the resulting analytical equations are in the form of an infinite series. In some certain configurations these analytical solutions can be simplified a bit. Here I will try to supply such simpler configurations so that you don't need to deal with complicated mathematical formulas.

Steady Configuration

As you know the code you are developing will be an unsteady code. To get steady result from your unsteady code, you should run the simulation in time until the solution is not changing any more. So you will just wait long enough for the physical system to reach a steady-state. To be sure that the solution is not changing, you should define some sort of a residual and check whether this value is dropping below a certain predefined value. This resdual can be defined in many ways such as

- the maximum value of the change of temperature between two consecutive iterations or preferably
- the RMS value of the change of temperature between two consecutive iterations.

These should also be normalized with the value at the initial time step. So you will have a residuial of unity at the first time step and it will decrease in time until it reaches a user defined value (such as 1.0e-6). This way you can understand whether your solver reached a steady solution. To better unerstand how these residuals are defined, please google "L1 norm" and "L2 norm".

For the steady validation you can use the validation case on page 52-53 of the PhD thesis of Gerstenberger [1]. You can find the pdf file (Gerstenberger-XFEM.pdf) in L2P and you can reach the necessary mesh files in the SVN repository. In Figure 1, you can see the face group numbers for each boundary of the concentric cylinders.

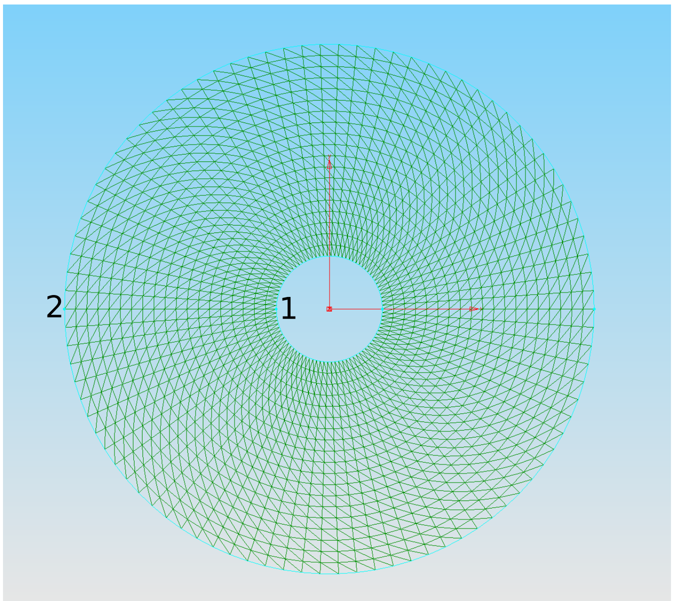

Figure 1: Face groups for the cylinder mesh

The inner and outer surfaces are set to constant temperatures T_i and T_a respectively and the analytical steady-state heat distribution in the cylinder is given as

$$T_{exact}(r) = T_i - (ln(r) - ln(r_i))\frac{T_i - T_a}{ln(r_a) - ln(r_i)}$$

To compare your results with the analytical solution, generating colorful images is not enough :) Please create some graphs such that the temperature on a line from the inner wall to outer wall is compared with the analytical result.

Unsteady Configuration

For the unsteady validation you can use the Example 6.4.2 on page 159 of the the book "Fundamentals of the Finite Element Method for Heat and Fluid Flow" by Lewis et al. [2]. Again the file (LewisNithiarasuSeetharamu.pdf) is uploaded to L2P and the necessary mesh files can be found in the repository. You can see the face groups for each boundary of the solution domain in Figure 2.

Here the left side of the domain is subjected to a constant heat flux and all other walls are insulated (heat flux is zero) and the corresponding exact temperature distribution is given as a function of time and x coordinate as

$$T_{exact}(x,t) = 2(t/\pi)^{1/2}\left[exp(-x^2/4t) - (1/2)x\sqrt{\frac{\pi}{t}}erfc\left(\frac{x}{2\sqrt{t}}\right)\right]$$

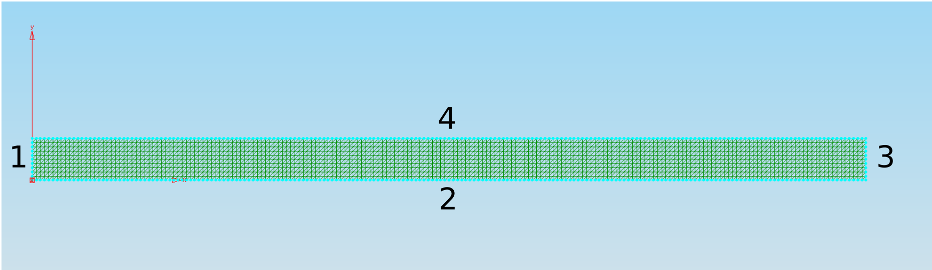

Figure 2: Face groups for the rod mesh

You can make a comparison with your code at different time instants and evaluate the correctness of your Neumann boundary conditions as well as the accuracy of your solution. Again to make a scientific comparison, create not only colorful images but also charts showing the temperature distribution along the rod. Don't forget to mention the time instant for that solution.

Derivation of Analytical Solutions

If you are interested in the derivation of the analytical solutions, you can have a look at the lecture notes given in the below links. It is possible to test more detailed configurations based on the more general solutions given in these notes. You can also check an advanced heat transfer book or advanced engineering mathematics book for the seperation of variables method.

- http://www.ewp.rpi.edu/hartford/~ernesto/C_Su2003/MMHCD/Notes/Notes_pdf/
- http://www.ewp.rpi.edu/hartford/~wallj2/CHT/Notes/

References

[1] Axel Gerstenberger. *An XFEM based fixed-grid approach to fluid-structure interaction.* PhD thesis, Ph. D. Thesis, 2010.

[2] Roland Wynne Lewis, Perumal Nithiarasu, and Kankanhalli Seetharamu. *Fundamentals of the finite element method for heat and fluid flow.* John Wiley & Sons, 2004.