CSD018U2M : Discrete Mathematical Structures
QuestionSet - 01: Sets, Functions and Relations

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1. List the members of the following sets:
   (a) $\{x \mid x \text{ is a real number such that } x^2 = 1\}$
   (b) $\{x \mid x \text{ is a positive integer less than } 12\}$
   (c) $\{x \mid x \text{ is the square of an integer and } x < 100\}$
   (d) $\{x \mid x \text{ is an integer such that } x^2 = 2\}$

2. Use set builder notation to give a description of each of the following sets.
   (a) $\{0, 3, 6, 9, 12\}$
   (b) $\{-3, -2, -1, 0, 1, 2, 3\}$
   (c) $\{m, n, o, p\}$

3. Determine whether each of the following statements is true or false:
   (a) $x \in \{x\}$
   (b) $\{x\} \subseteq \{x\}$
   (c) $\{x\} \in \{x\}$
   (d) $\{x\} \in \{\{x\}\}$
   (e) $\Phi \subseteq \{x\}$
   (f) $\Phi \in \{x\}$

4. Russell's paradox: Show that the set K, such that $K = \{S \mid S \text{ is a set and } S \notin S\}$ does not exist.

5. Prove or disprove: The set $A = \{S \mid S \text{ is a set}\}$ does not exist.

6. Prove that the empty set Φ is unique.

7. Prove or disprove: The complement of
   $$(\bar{A} \cap B) \cap (A \cup \bar{B}) \cap (A \cup C)$$
   is
   $$(A \cup \bar{B}) \cup (\bar{A} \cap (B \cap \bar{C})).$$

8. Determine which of the following statements are true for all sets A, B, C and D. If a double implication fails, determine whether one or the other of all possible implication holds. If an equality fails, determine whether the statement becomes true if the "equals" symbol is replaced by one or the other of the inclusion symbols $\subseteq$ or $\supseteq$.
   (a) $A \subseteq B$ and $A \subseteq C \leftrightarrow A \subseteq (B \cup C)$.

(b) $A \subseteq B$ or $A \subseteq C \leftrightarrow A \subseteq (B \cup C)$.

(c) $A \subseteq B$ and $A \subseteq C \leftrightarrow A \subseteq (B \cap C)$.

(d) $A \subseteq B$ or $A \subseteq C \leftrightarrow A \subseteq (B \cap C)$.

(e) $A - (A - B) = B$.

(f) $A - (B - A) = A - B$.

(g) $A \cap (B - C) = (A \cap B) - (A \cap C)$.

(h) $A \cup (B - C) = (AUB) - (A \cup C)$.

(i) $(A \cap B) \cup (A - B) = A$.

(j) $A \subseteq C$ and $B \subseteq D \rightarrow (A \times B) \subseteq (C \times D)$.

(k) The converse of (j).

(l) The converse of (j), assuming that A and B are nonempty.

(m) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.

(n) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

(o) $A \times (B - C) = (A \times B) - (A \times C)$.

(p) $(A - B)X(C - D) = ((A \times C) - (B \times C)) - (A \times D)$.

(q) $(A \times B) - (C \times D) = (A - C) \times (B - D)$.

9. Prove that two equivalence classes E and E' are either disjoint or equal.

10. A **partition** of a set A is a collection of disjoint nonempty subsets of A whose union is all of A.

    Show that given any partition $\mathcal{D}$ of A, there is exactly one equivalence relation on A from which it is derived.

11. Define P_1RP_2 for all $P_1, P_2 \in \mathbb{R}^2$ if points P_1 and P_2 lie at the same distance from the origin. Is R an equivalence relation? Justify your answer. If your answer is yes, what are the equivalence classes?

12. Define P_1RP_2 for all $P_1, P_2 \in \mathbb{R}^2$ if points P_1 and P_2 have the same y-coordinate. Is R an equivalence relation? Justify your answer. If your answer is yes, what are the equivalence classes?

13. Let $\mathcal{L}'$ be the collection of all straight lines in the plane. Is $\mathbb{L}'$ a partition of the plane? Justify your answer.

14. Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be any two points in $\mathbb{R}^2$. Define P_1RP_2 for all $P_1, P_2 \in \mathbb{R}^2$ if $y_1 - x_1^2 = y_2 - x_2^2$. Is R an equivalence relation? Justify your answer. If your answer is yes, what are the equivalence classes?

15. Let $A = \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$. Define $(a, b)R(c, d)$ for all $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ if $ad = bc$. Is R an equivalence relation? Justify your answer. If your answer is yes, what are the equivalence classes?

16. Let $f : A \rightarrow B$ be a surjective function. Let us define a relation R on A by defining a_0Ra_1 if

    $$f(a_0) = f(a_1).$$

    Is R an equivalence relation? Justify your answer.

17. Let S and S' be the following subsets of the plane $\mathbb{R} \times \mathbb{R}$:

    $$\begin{aligned} S &= \{(x, y) \mid y = x + 1 \text{ and } 0 < x < 2\}, \\ S' &= \{(x, y) \mid y - x \text{ is an integer}\}. \end{aligned}$$

    (a) Are S and S' equivalence relations on $\mathbb{R}$? Justify your answer.

    (b) Find the equivalence class(es) of S or S' (or both) in case of being equivalence relation(s).

18. Define a relation on the plane $\mathbb{R}^2$ by setting

$$(x_0, y_0)R(x_1, y_1)$$

if either $y_0 - x_0^2 < y_1 - x_1^2$, or $y_0 - x_0^2 = y_1 - x_1^2$ and $x_0 < x_1$. What kind of relation R is - an equivalence relation, linear order relation or partial order relation? Justify your answer.

19. A set with an order relation $<$ is said to be **well-ordered** if every nonempty subset of A has a smallest element.

    Find an order relation on $\mathbb{Z}$ (the set of integers) which is well-ordering.

20. Refer lecture slides on Cardinality - in Exercise-3, show that f and g are bijections.

21. Use Schrӧder-Bernstein theorem to show that $|(0, 1)| = |(0, 1]|$.

    Theorem 1. *(Schrӧder-Bernstein theorem) If there is a one-to-one function f from A to B and a one-to-one function g from B to A, then there is a bijection between A and B.*

22. Show that the set of rational numbers $\mathbb{Q}$ is countable.

23. Let $\mathbb{R}$ be the set of real numbers. Prove or disprove - $|\mathbb{R}| = |\mathbb{R} \times \mathbb{R}|$. Using this result, can we conclude whether $|\mathbb{R}| = |\mathbb{C}|$ or not where $\mathbb{C}$ is the set of complex numbers? Justify your answer.

24. Assume that someone proves that the set of rational numbers in the interval $(0, 1]$ is uncountable by exactly imitating the proof for $(0, 1]$ is uncountable. Where is the flaw?