CSD018U2M : Discrete Mathematical Structures
QuestionSet - 02: Number Theory-I

September, 2023

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1. Find all positive integers n such that $n^2 + 1$ is divisible by $n + 1$.
2. Let a, b, c, d are integers such that $a \neq c$. Suppose $a - c \mid ab + cd$. Is it true that $a - c \mid ad + bc$? Justify your answer.
3. a and b are two odd integers. Can we find an integer x such that $x^2 = a^2 + b^2$? Justify your answer.
4. Find all integers n, if exists, so that $5n + 3$ becomes a prime given $2n + 1$ and $3n + 1$ both are squares.
5. Find all prime numbers p such that $p^2 + 2$ and $p^3 + 2$ also become prime numbers.
6. Prove that $4^n + n^4$ is not a prime for $n > 1$.
7. Assuming that $\gcd(a, b) = 1$, prove the following:
   (a) $\gcd(a + b, a - b) = 1$ or 2.
   (b) $\gcd(a + b, a^2 + b^2) = 1$ or 2.
8. Which of the following Diophantine equations cannot be solved? Justify your answer.
   (a) $6x + 51y = 22$.
   (b) $33x + 14y = 115$.
   (c) $14x + 35y = 93$.
   (d) $33x + 35y = 71$.
   (e) $12x + 16y = 8$.
9. Determine all solutions in the integers of the following Diophantine equations:
   (a) $24x + 138y = 18$.
   (b) $221x + 35y = 11$.
10. Determine all solutions in the positive integers of the following Diophantine equations:
    (a) $54x + 21y = 906$.
    (b) $158x - 57y = 7$.
11. Prove the following theorem:

    The linear Diophantine equation $ax + by = c$ has a solution if and only if $d \mid c$, where $d = \gcd(a, b)$. If x_0, y_0 is any particular solution of this equation, then all solutions are given by
    $$x = x_0 + \left(\frac{b}{d}\right) t \quad y = y_0 - \left(\frac{a}{d}\right) t$$
    where t is any arbitrary integer.

12. There were 63 equal piles of plantain fruit put together and 7 single fruits. They were divided evenly among 23 travellers. What is the number of fruits in each pile?

13. Prove that $\sqrt{p}$ is an irrational number for any prime p.

14. Is it true that every odd integer can be written in the form $p + 2a^2$, where p is a prime or 1 and $a \geq 0$. Justify your answer.

15. Is it true that $p^3 + 4$ is also a prime number, given p and $p^2 + 8$ are also prime numbers.

16. Prove that $2^n \nmid n!$ for all $n \geq 1$.

17. Let p_n be the n^{th} prime number. Let $P_n = p_1 p_2 \cdots p_n + 1$. Prove that P_n cannot be a perfect square for any $n \geq 1$.

18. Assume that p_n is the n^{th} prime number. Prove that the sum

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n}$$

is never an integer for any $n \geq 1$.

19. Solve the following linear congruences:
    (a) $25x \equiv 15 \pmod{2}9$.
    (b) $5x \equiv 2 \pmod{2}6$.
    (c) $6x \equiv 15 \pmod{2}1$.

20. Solve the following system of linear congruences:
    (a) $x \equiv 1 \pmod{3}$, $x \equiv 2 \pmod{5}$, $x \equiv 3 \pmod{7}$.
    (b) $x \equiv 5 \pmod{1}1$, $x \equiv 14 \pmod{2}9$, $x \equiv 15 \pmod{3}1$.
    (c) $2x \equiv 1 \pmod{5}$, $3x \equiv 9 \pmod{6}$, $4x \equiv 1 \pmod{7}$, $5x \equiv 9 \pmod{1}1$.

21. Solve the linear congruence $17x \equiv 3 \pmod{2 \cdot 3 \cdot 5 \cdot 7}$ by solving the system

$$\begin{array}{ll} 17x \equiv 3 \pmod{2} & 17x \equiv 3 \pmod{3} \\ 17x \equiv 3 \pmod{5} & 17x \equiv 3 \pmod{7} \end{array}$$

22. (a) Obtain three consecutive integers, each having a square factor.
    (b) Obtain three consecutive integers, the first of which is divisible by a square, the second by a cube, and the third by a fourth power.

23. A band of 17 pirates stole a sack of gold coins. When they tried to divide the fortune into equal portions, 3 coins remained. In the ensuing brawl over who should get the extra coins, one pirate was killed. The wealth was redistributed, but this time an equal division left 10 coins. Again an argument developed in which another pirate was killed. But now the total fortune was evenly distributed among the survivors. What was the least number of coins that could have been stolen?

24. Find the last two digits of $3^{835845348763467638453451515153457}$.