0.1 General solution

We seek a general solution to Laplace's equation on a rectangle $0 \leq x \leq L$ and $0 \leq y \leq H$:

$$\nabla^2 V(x,y) = 0, \tag{1}$$

subject to

$$\begin{cases} V(x,0) = v_1(x) & \text{(I)} \\ V(x,H) = v_2(x) & \text{(II)} \\ V(0,y) = v_3(y) & \text{(III)} \\ V(L,y) = v_4(y) & \text{(IV)} \end{cases} \tag{2}$$

Because Laplace's equation is a linear equation, each boundary condition can be satisfied separately then the final solution can be superimposed.

First, consider the boundary conditions

$$\begin{cases} V(x,0) = v_1(x) & \text{(I)} \\ V(x,H) = 0 & \text{(II)} \\ V(0,y) = 0 & \text{(III)} \\ V(L,y) = 0 & \text{(IV)} \end{cases} \tag{3}$$

Beginning with the Laplace equation in 2D

$$\nabla^2 V = 0. \tag{4}$$

Assume a seperable solution $V(x,y) = X(x)Y(y)$. Then equation 1 becomes

$$0 = \nabla^2 X(x)Y(y) \tag{5}$$

$$= Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} \tag{6}$$

$$= \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} \tag{7}$$

Choosing a separation constant k^2 then the general solutions are

$$\frac{\partial^2 X}{\partial x^2} = -k^2 X \tag{8}$$

$$\frac{\partial^2 Y}{\partial y^2} = k^2 Y \tag{9}$$

Considering the x-part, the general solutions are of the form

$$X(x) = A\sin(kx) + B\cos(kx) \tag{10}$$

Imposing the boundary conditions (III) and (IV) gives the requirement

$$X(0) = X(L) = 0 \tag{11}$$

Which gives

$$X(x) = \sum_{n=0}^{\infty} A_n \sin\Big(\frac{n\pi}{L}x\Big) \tag{12}$$

Then the y-part has general solutions of the form

$$Y(y) = C\sinh\Big(\frac{n\pi}{L}(y-H)\Big) + D\cosh\Big(\frac{n\pi}{L}(y-H)\Big) \tag{13}$$

choosing the shift in y by H to satisfy boundary condition (II), $V(x,H)=0$, this implies that $D=0$ as well. This gives the solution

$$Y(y) = C\sinh\Big(\frac{n\pi}{L}(y-H)\Big) \tag{14}$$

And total solution

$$V(x,y) = \sum_{n=0}^{\infty} A_n \sinh\Big(\frac{n\pi}{L}(y-H)\Big)\sin\Big(\frac{n\pi}{L}x\Big) \tag{15}$$

To satisfy boundary condition (I), let $v_1(x,0)$ be decomposed into a fourier series

$$v_1(x,0) = \sum_{n=0}^{\infty} a_n \sin\Big(\frac{n\pi}{L}x\Big), \tag{16}$$

where

$$a_n = \frac{2}{L}\int_0^L \mathrm{d}x\, v_1(x)\sin\Big(\frac{n\pi}{L}x\Big). \tag{17}$$

Which gives

$$A_n = \frac{a_n}{\sinh\big(-\frac{n\pi H}{L}\big)}. \tag{18}$$

Thus for the first set of conditions

$$\boxed{V_1(x,y) = \sum_{n=0}^{\infty} A_n \sinh\Big(\frac{n\pi}{L}(y-H)\Big)\sin\Big(\frac{n\pi}{L}x\Big)}, \tag{19}$$

with

$$\boxed{A_n = \frac{2/L}{\sinh\big(-\frac{n\pi H}{L}\big)}\int_0^L \mathrm{d}x\, v_1(x)\sin\Big(\frac{n\pi}{L}x\Big)}. \tag{20}$$

Then for the second set of boundary conditions

$$\begin{cases} V(x,0)=0 & \text{(I)} \\ V(x,H)=v_2(x) & \text{(II)} \\ V(0,y)=0 & \text{(III)} \\ V(L,y)=0 & \text{(IV)} \end{cases} \tag{21}$$

$$\boxed{V_2(x,y) = \sum_{n=0}^{\infty} B_n \sinh\Big(\frac{n\pi}{L}y\Big)\sin\Big(\frac{n\pi}{L}x\Big)}, \tag{22}$$

with

$$\boxed{B_n = \frac{2/L}{\sinh\big(\frac{n\pi H}{L}\big)}\int_0^L \mathrm{d}x\, v_2(x)\sin\Big(\frac{n\pi}{L}x\Big)}. \tag{23}$$

By the symmetry of x and y upon interchange this gives for boundary condition set

$$\begin{cases} V(x,0)=0 & \text{(I)} \\ V(x,H)=0 & \text{(II)} \\ V(0,y)=v_3(y) & \text{(III)} \\ V(L,y)=0 & \text{(IV)} \end{cases} \tag{24}$$

$$\boxed{V_3(x,y) = \sum_{n=0}^{\infty} C_n \sinh\Big(\frac{n\pi}{H}(x-L)\Big)\sin\Big(\frac{n\pi}{H}y\Big)}, \tag{25}$$

with

$$\boxed{C_n = \frac{2/H}{\sinh\left(-\frac{n\pi L}{H}\right)} \int_0^H \mathrm{d}y\, v_3(y) \sin\left(\frac{n\pi}{H} y\right)}. \tag{26}$$

And finally

$$\begin{cases} V(x,0) = 0 & \text{(I)} \\ V(x,H) = 0 & \text{(II)} \\ V(0,y) = 0 & \text{(III)} \\ V(L,y) = v_4(y) & \text{(IV)} \end{cases} \tag{27}$$

$$\boxed{V_4(x,y) = \sum_{n=0}^{\infty} D_n \sinh\left(\frac{n\pi}{H} x\right) \sin\left(\frac{n\pi}{H} y\right)}, \tag{28}$$

with

$$\boxed{D_n = \frac{2/H}{\sinh\left(\frac{n\pi L}{H}\right)} \int_0^H \mathrm{d}y\, v_4(y) \sin\left(\frac{n\pi}{H} y\right)}. \tag{29}$$

0.2 Solution on a square

Now considering the solution to the problem on a square:

$$H \to H = L \tag{30}$$

and

$$\begin{cases} V(x,0) = v_1(x) & \text{(I)} \\ V(x,L) = v_2(x) & \text{(II)} \\ V(0,y) = v_3(y) & \text{(III)} \\ V(L,y) = v_4(y) & \text{(IV)} \end{cases} \tag{31}$$

Then

$$\boxed{\begin{aligned} V(x,y) = \sum_{n=0}^{\infty} \Bigg\{ & \left[A_n \sinh\left(\frac{n\pi}{L}(y-L)\right) + B_n \sinh\left(\frac{n\pi}{L} y\right)\right] \sin\left(\frac{n\pi}{L} x\right) \\ & + \left[C_n \sinh\left(\frac{n\pi}{L}(x-L)\right) + D_n \sinh\left(\frac{n\pi}{L} x\right)\right] \sin\left(\frac{n\pi}{L} y\right) \Bigg\} \end{aligned}} \tag{32}$$

where

$$A_n = \frac{2/L}{\sinh(-n\pi)} \int_0^L \mathrm{d}x\, v_1(x) \sin\left(\frac{n\pi}{L} x\right) \tag{33}$$

$$B_n = \frac{2/L}{\sinh(n\pi)} \int_0^L \mathrm{d}x\, v_2(x) \sin\left(\frac{n\pi}{L} x\right) \tag{34}$$

$$C_n = \frac{2/L}{\sinh(-n\pi)} \int_0^L \mathrm{d}y\, v_3(y) \sin\left(\frac{n\pi}{L} y\right) \tag{35}$$

$$D_n = \frac{2/L}{\sinh(n\pi)} \int_0^L \mathrm{d}y\, v_4(y) \sin\left(\frac{n\pi}{L} y\right). \tag{36}$$