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Introduction to Differential Equations

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Contents

1 First Order Differential Equations (FODEs)

FODEs take the form $y' + p(t)y = 0$ where y is ...

1.1 Summary

Solved by integrating factors. $\mu(t) = \exp\left(\int p(t)dt\right)$

1.2 Questions

TBD

2 Existence & Uniqueness

Suppose that $f(t, y)$ is a cont. function defined on an open domain $D \subseteq \mathbb{R}$ with $(t_0, \alpha_0) \in D$. Further suppose that f is Lipschitz w.r.t y. Then there is an open interval $I \subset \mathbb{R}$ with $t_0 \in I$ and a continuously differentiable function $\phi : I \to \mathbb{R}$ that solves $y' = f(t, y), y(t_0) = \alpha_0$. (Existence)

Furthermore, if $\psi : I \to \mathbb{R}$ is any other continuously differentiable function that solves the above, then $\psi = \psi$. (Uniqueness)

Lipschitz: $|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$.

Lipschitz is a condition stronger than continuous but weaker than differentiable. If $\frac{\partial f}{\partial y}$ exists and is continuous, then f is Lipschitz w.r.t. y.

Lipschitz is a condition for ensuring "niceness" of a function.

Example:

$$f(y) = \frac{3}{2} y^{\frac{1}{3}} \implies \frac{\partial f}{\partial y} = \frac{1}{2} y^{\frac{-2}{3}} \tag{1}$$

(2) blows up at 0: vertical tangent near 0, cannot be contained within Lipschitz bounds. Thus, f does not satisfy niceness around 0.

$$\tag{2}$$

Above, f is separable.

Generally, existence-uniqueness fails when no solutions exist, or they exist but are not unique.

In this example, $y(0) = 0$ makes the equation fail uniqueness. Multiple solutions exist for $y(0) = 0$. (?)

Example 2:

$$y' = f(t, y) = 1 + y^2, y(0) = 0 \tag{3}$$

Note: Separable ODE and can be solved as such. Note that $\frac{\partial f}{\partial y} = 2y$, which is continuous, so we can expect to solve this ODE uniquely. It has a solution $y(t) = \tan(t)$ which is defined on the interval $(-\pi/2, \pi/2)$

3 SODEs

$y'' = f(t, y, y')$. Need to specify $y(t_0) = \alpha_0, y'(t_0) = \alpha_1$ an initial value problem.

Theorem: E&U for SODEs $y'' + p_1(t)y' + p_2(t)y = g(t)$

Suppose that p_1, p_2, g are continuous on an open interval I, $t_0 \in I$, and $\alpha_0, \alpha_1 \in \mathbb{R}$, then there is a unique solution to the above equation with $y(t_0) = \alpha_0, y(t_1) = \alpha_1$.

For this course, we will pay special attention to constant coefficient ODEs, i.e. $ay'' + by' = g(t), a, b, c \in \mathbb{R}, a \neq 0$. Through variation of parameters, we can deal with the homegeneous equation $ay'' + by' + cy = 0$.

Example: $y'' + 3y' - 4y = 0$

We will try to solve this with an educated guess: $y = e^{rt}$ for some unknown constant. Plugging into the equation, we get that $r^2 + 3r - 4 = 0$ for some r. This is a quadratic equation, so we can solve for r using the quadratic formula. We get that $r = -4, r = 1$. Thus, $y_1 = e^{-4t}, y_2 = e^t$ are solutions to the ODE.

More generally, guess $y = e^{rt}$ for the SODE $ay'' + by' + cy = 0$ which will be a solution iff $ar^2 + br + c = 0$. This is called the characteristic equation of the ODE.

Theorem: suppose $p_1(t), p_2(t)$ are continuous on an open interval I, then the set of solutions to the ODE $y'' + p_1(t)y' + p_2(t)y = 0$ is a 2-dimensional subspace of $C^2(I)$. $C^k(I)$ is the set of k-times continuously differentiable functions on I. Being a linear ODE gives the solutions a linear structure.

A fundamental set of solutions to the ODE is a basis for the set of solutions to the ODE, i.e. a set of solutions $\{y_1, y_2\}$ such that any other solution y can be written as a linear combination of y_1, y_2.

Essentially, the given equation is an operation on y that produces some function y. (?)

Differential operation: $L[y] = y'' + p_1(t)y' + p_2(t)y$

and the given equation is $L[y] = 0$.

Note that L is linear, i.e. $L[\lambda_1 y + \lambda_2 y] = \lambda_1 L[y] + \lambda_2 L[y]$ which implies its solutions are a vector space. E&U theorem specifies an isomorphism between the set of solutions and $\mathbb{R}^2$, whatever that means.

Example: $y'' + 3y' - 4y = 0$ has solutions e^{-4t}, e^t. More solutions would be $c_1 e^{-4t}, c_2 e^t$ from the linear structure of the solutions.

Is $\{e^{-4t}, e^t\}$ the fundamental set of solutions?

Suppose $\phi(t)$ is the solution with $y(0) = \alpha_0, y'(0) = \alpha_1$. Say $\psi(t) = c_1 e^{-4t} + c_2 e^t$. Then $\psi(0) = c_1 + c_2, \psi'(0) = -4c_1 + c_2$. We will solve $c_1 + c_2 = \alpha_0, -4c_1 + c_2 = \alpha_1$.

...

So, $\psi(t) = \frac{1}{5}(\alpha_0 - \alpha_1)e^{-4t} + \frac{1}{5}(4\alpha_0 - \alpha_1)e^t$.

Solve $y(0) = \alpha_0, y'(0) = \alpha_1$, which implies $\phi(t) = \psi(t)$. More generally, the fundamental set of solutions for a constant coefficient SODE is its characteristic polynomial.

How to check if a set of solutions is the fundamental set?

Use Wronskian operations:

W: function x function $\rightarrow$ function

$W[y_1, y_2](t) = \det(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix})$

If $W[y_1, y_2] \neq 0$, then $\{y_1, y_2\}$ is lienarly independent. The converse is not true.

Theorem: suppose $p_1(t), p_2(t)$ are continuous on open interval I, and ϕ_1, ϕ_2 solve $y'' + p_1(t)y' + p_2(t)y = 0$. Then, the following are equivalent:

1. $\{\phi_1, \phi_2\}$ are a fundamental set of solutions.

2. $\begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix}$ is an invertible matrix for some $t_0 \in I$. Justification? Consider characteristic polynomial example, essentially multiplying by this matrix, and solve for arbitrary solutions as combinations of the fundamental set.

3. $W[\phi_1, \phi_2](t_0) \neq 0$ for some $t_0 \in I$ (follows from above equivalence).

4. $W[\phi_1, \phi_2](t) \neq 0$ for all $t \in I$.

5. if ψ_1, ψ_2 solve as well, then $W[\psi_1, \psi_2] = kW[\phi_1, \phi_2]$ for some $k \in \mathbb{R}$.

Essentially, Wronskian is a tool that tells us whether we have the fundamental set of solutions. (2) and (3) are most useful for this purpose.

Motivation: it's very difficult to solve arbitrary 2nd diff. eqns., but we can calculate their Wronskian fairly simply.

Abel's formula:

If ϕ_1, ϕ_2 solve $y'' + p_1(t)y' + p_2(t)y = 0$, then $W[\phi_1, \phi_2] = C\exp(-\int p_1(t)dt)$ where C is the determinant mess.