Exercise 1.13. Prove that *Fib*(n) is the closest integer to $\phi^n/\sqrt{5}$, where $\phi = (1 + \sqrt{5})/2$. Hint: Let $\psi = (1 - \sqrt{5})/2$. Use induction and the definition of the Fibonacci numbers (see section 1.2.2) to prove that *Fib*(n) = $(\phi^n - \psi^n)/\sqrt{5}$.

$$Fib(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

$$Fib(n-1) = \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}}$$

$$\varphi^2 = \varphi + 1, \qquad \psi^2 = \psi + 1$$

$$\begin{aligned}
Fib(n+1) &= Fib(n) + Fib(n-1) \\
&= \frac{\varphi^n - \psi^n}{\sqrt{5}} + \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} \\
&= \frac{\varphi^n - \psi^n + \varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} \\
&= \frac{(\varphi^n + \varphi^{n-1}) - (\psi^n + \psi^{n-1})}{\sqrt{5}} \\
&= \frac{\varphi^{n-1}(\varphi + 1) - \psi^{n-1}(\psi + 1)}{\sqrt{5}} \\
&= \frac{\varphi^{n-1} \cdot \varphi^2 - \psi^{n-1} \cdot \psi^2}{\sqrt{5}} \\
&= \frac{\varphi^{n+1} - \psi^{n+1}}{\sqrt{5}}
\end{aligned}$$