summary of "stable fluids"

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1 abstract

This paper propose a method of stable fluid simulation.

2 Principle

2.1 Equation of motion

We will consider dynamics of incompressible fluids. Equation of motion is described by Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \tag{1}$$

and incompressible condition

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

where $\mathbf{u}$ is velocity field, p is pressure, ν is dynamic viscosity coefficient and $\mathbf{f}$ is external force. Each term of Navier-Stokes equation represent following physical meanings:

- 1st term of right hand side: advection
- 2nd term of right hand side: the effect of pressure
- 3rd term of right hand side: diffusion term
- 4th term of right hand side: external force

Navier-Stokes eq. (1) contains unknown variable p. So we will explain how to calculate pressure gradient term. Any vector field $\mathbf{w}$ can be decomposed as following

$$\mathbf{w} = \mathbf{u} + \nabla q \tag{3}$$

where q is scalar field and $\nabla \cdot \mathbf{u} = 0$. Let P be a map from vector field $\mathbf{w}$ to its divergence free term $\mathbf{u}$

$$P\mathbf{w} = \mathbf{u} \tag{4}$$

By appling P to Navier-Stokes eq. (1), we get

$$\frac{\partial \mathbf{u}}{\partial t} = P(-(\mathbf{u} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \mathbf{u} + \mathbf{f}) \tag{5}$$

where we used $P(\nabla p) = 0$ and $P\mathbf{u} = \mathbf{u}$ because $\mathbf{u}$ is divergence free. Now, calculation of pressure gradient term is replaced by calculation of P.

Fluids cast substances along with its dynamics. A typical example is fog. Equation of motion of such substances is following:

$$\frac{\partial a}{\partial t} = -\mathbf{u} \cdot \nabla a + \kappa_a \nabla^2 a - \alpha_a a + S_a \tag{6}$$

where a is density of substances, κ_a is diffusion constant, α_a is dissipation rate, S_a is source term.

2.2 discretization of Equation of motion

We will consider solve eq. (1) by discretizing in temporal and spatial way. Time evolution from t to $t + \Delta t$ is calculated by following 4 steps:

1. add force
2. advect
3. diffuse
4. project

$$w_0(\mathbf{x})(= u(\mathbf{x}, t)) \xrightarrow{addforce} w_1(\mathbf{x}) \xrightarrow{advect} w_2(\mathbf{x}) \xrightarrow{diffuse} w_3(\mathbf{x}) \xrightarrow{project} w_4(\mathbf{x})(= u(\mathbf{x}, t + \Delta t)) \tag{7}$$

and in each step, every physical quantities are defined in faces of grids. we will explain each steps. In the following, $a_{i,j,k}$ denotes the value of field a in the face of grid(i, j, k)

1. add force

In this step, we calculate external force term

$$w_1(\mathbf{x}) = w_0(\mathbf{x}) + \Delta t \mathbf{f}(\mathbf{x}, t). \tag{8}$$

In each grids, this equation is

$$w_{1,i,j,k} = w_{0,i,j,k} + \Delta t \mathbf{f}(t)_{i,j,k} \tag{9}$$

2. advect

In this step, we calculate advect term. we use method of characteristics for updating velocity field(see Appendix)

$$w_{2,x}(x, y, z) = w_{1,x}(x - u_x \Delta t, y - u_y \Delta t, z - u_z \Delta t) \tag{10}$$
$$w_{2,y}(x, y, z) = w_{1,y}(x - u_x \Delta t, y - u_y \Delta t, z - u_z \Delta t) \tag{11}$$
$$w_{2,z}(x, y, z) = w_{1,z}(x - u_x \Delta t, y - u_y \Delta t, z - u_z \Delta t) \tag{12}$$

where $w_{2,\alpha}(\alpha = x, y, z)$ is α component of $\mathbf{w}_2$. I each grids

$$w_{2,x,i,j,k}(x, y, z) = w_{1,x,i,j,k}(x - u_x \Delta t, y - u_y \Delta t, z - u_z \Delta t) \tag{13}$$
$$w_{2,y,i,j,k}(x, y, z) = w_{1,y,i,j,k}(x - u_x \Delta t, y - u_y \Delta t, z - u_z \Delta t) \tag{14}$$
$$w_{2,z,i,j,k}(x, y, z) = w_{1,z,i,j,k}(x - u_x \Delta t, y - u_y \Delta t, z - u_z \Delta t) \tag{15}$$

3. diffuse

In this step, we calculate diffuse term. We use implicit form to update velocity field.

$$\frac{\mathbf{w}_3(\mathbf{x}) - \mathbf{w}_2(\mathbf{x})}{\Delta t} = \nu \Delta^2 \mathbf{w}_3(\mathbf{x}) \tag{16}$$

This equation can be deformed to linear equation

$$(\mathbf{I} - \nu\Delta t\nabla^2)\mathbf{w}_3(\mathbf{x}) = \mathbf{w}_2(\mathbf{x}). \tag{17}$$

Laplacian is discretized as followings

$$(\nabla^2 a)_{i,j,k} = \frac{a_{i+1,j,k} + a_{i-1,j,k} + a_{i,j+1,k} + a_{i,j-1,k} + a_{i,j,k+1} + a_{i,j,k-1} - 6a_{i,j,k}}{h^2} \tag{18}$$

So, we finally get

$$\sum_{i',j',k'} (\delta_{(i,j,k),(i',j',k')} - \frac{\nu\Delta t}{h^2}(\delta_{(i+1,j,k),(i',j',k')} + \delta_{(i-1,j,k),(i',j',k')} + \delta_{(i,j+1,k),(i',j',k')} \tag{19}$$

$$+\delta_{(i,j-1,k),(i',j',k')} + \delta_{(i,j,k+1),(i',j',k')} + \delta_{(i,j,k-1),(i',j',k')} - 6\delta_{(i,j,k),(i',j',k')}))\mathbf{w}_{3,i',j',k'} = \mathbf{w}_{2,i,j,k} \tag{20}$$

and by solving this equation, we have $\mathbf{w}_3$.

4. project

In this step, we project $\mathbf{w}_3$ to divergence free space. $\mathbf{w}_4(= \mathbf{u}(\mathbf{x}, t + \Delta t))$ is defined as

$$\mathbf{w}_3 = \mathbf{w}_4 + \nabla q \tag{21}$$

where

$$\nabla \cdot \mathbf{w}_4 = 0. \tag{22}$$

So the task to get $\mathbf{w}_4$ comes down to evaluating q. We take divergence of both sides of eq. (21)

$$\nabla^2 q = \nabla \cdot \mathbf{w}_3. \tag{23}$$

This equation is discretized as

$$\sum_{i',j',k'} (\delta_{(i+1,j,k),(i',j',k')} + \delta_{(i-1,j,k),(i',j',k')} + \delta_{(i,j+1,k),(i',j',k')} \tag{24}$$

$$+\delta_{(i,j-1,k),(i',j',k')} + \delta_{(i,j,k+1),(i',j',k')} + \delta_{(i,j,k-1),(i',j',k')} - 6\delta_{(i,j,k),(i',j',k')})q_{i',j',k'} \tag{25}$$

$$= \frac{1}{2}\left(\frac{\mathbf{w}_{3,i+1,j,k} - \mathbf{w}_{3,i-1,j,k}}{h} + \frac{\mathbf{w}_{3,i,j+1,k} - \mathbf{w}_{3,i,j-1,k}}{h} + \frac{\mathbf{w}_{3,i,j,k+1} - \mathbf{w}_{3,i,j,k-1}}{h}\right) \tag{26}$$

and we can get q by solving this linear equation. We finally get

$$\mathbf{w}_{4,i,j,k} = \mathbf{w}_{3,i,j,k} - \begin{pmatrix} \frac{1}{2}\frac{q_{i+1,j,k} - q_{i-1,j,k}}{h} \\ \frac{1}{2}\frac{q_{i,j+1,k} - q_{i,j-1,k}}{h} \\ \frac{1}{2}\frac{q_{i,j,k+1} - q_{i,j,k-1}}{h} \end{pmatrix} \tag{27}$$

Calculation method of substances' advect term, diffuse term and external force term are same as fluids.

Dissipation term is discretized as

$$\frac{a(t + \Delta t) - a(t)}{\Delta t} = \alpha_a a(t + \Delta t) \tag{28}$$

where we used implicit form due to numerical stability. In each grid cell, a is updated as

$$a_{i,j,k}(t + \Delta t) = \frac{1}{(1 + \alpha_a\Delta t)} a_{i,j,k}(t) \tag{29}$$

As a result, the algorithm of one time step is summarize as followings.

procedure ONE TIME STEP
    $Updatefluidvelocityfield$
    $Movingsubstances$
end procedure

Update of fluid velocity field and Moving substances are

procedure UPDATE FLUID VELOCITY FIELD
    $addForce : eq.(9)$
    $advect : eq.(13)$
    $diffuse : eq.(19)$
    $project : eq.(27)$
end procedure

procedure MOVING SUBSTANCES
    $addForce : eq.(9)$
    $transport$
    $diffuse : eq.(19)$
    $dissipate : eq.(29)$
end procedure

付録 A　method of characteristics

method of characteristics is a technique for solving partial differential equation. We consider advect equation with initial condition

$$\frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, t) = -(\mathbf{u}(\mathbf{x}, t) \cdot \nabla)\mathbf{u}(\mathbf{x}, t) \tag{30}$$
$$\mathbf{u}(\mathbf{x}, 0) = \phi(\mathbf{x}) \tag{31}$$

We assume that $\mathbf{u}, t$ depend on a parameter s and we differentiate u with respect to s

$$\frac{du_x}{ds}(x(s), y(s), z(s), t(s)) = \frac{\partial u_x}{\partial x}\frac{dx}{ds} + \frac{\partial u_x}{\partial y}\frac{dy}{ds} + \frac{\partial u_x}{\partial z}\frac{dz}{ds} + \frac{\partial u_x}{\partial t}\frac{dt}{ds} \tag{32}$$
$$\frac{du_y}{ds}(x(s), y(s), z(s), t(s)) = \frac{\partial u_y}{\partial x}\frac{dx}{ds} + \frac{\partial u_y}{\partial y}\frac{dy}{ds} + \frac{\partial u_y}{\partial z}\frac{dz}{ds} + \frac{\partial u_y}{\partial t}\frac{dt}{ds} \tag{33}$$
$$\frac{du_z}{ds}(x(s), y(s), z(s), t(s)) = \frac{\partial u_z}{\partial x}\frac{dx}{ds} + \frac{\partial u_z}{\partial y}\frac{dy}{ds} + \frac{\partial u_z}{\partial z}\frac{dz}{ds} + \frac{\partial u_z}{\partial t}\frac{dt}{ds} \tag{34}$$
$$\tag{35}$$

We take s as

$$\frac{dx}{ds} = u_x \tag{36}$$
$$\frac{dy}{ds} = u_y \tag{37}$$
$$\frac{dz}{ds} = u_z \tag{38}$$
$$\frac{dt}{ds} = 1 \tag{39}$$

Then, eq. (32), (33), (34) becomes

$$\frac{du_x}{ds}(x(s),y,z,t(s)) = \frac{\partial u_x}{\partial x}u_x + \frac{\partial u_x}{\partial y}u_y + \frac{\partial u_x}{\partial z}u_z + \frac{\partial u_x}{\partial t}. \tag{40}$$

$$\frac{du_y}{ds}(x(s),y,z,t(s)) = \frac{\partial u_y}{\partial x}u_x + \frac{\partial u_y}{\partial y}u_y + \frac{\partial u_y}{\partial z}u_z + \frac{\partial u_y}{\partial t}. \tag{41}$$

$$\frac{du_z}{ds}(x(s),y,z,t(s)) = \frac{\partial u_z}{\partial x}u_x + \frac{\partial u_z}{\partial y}u_y + \frac{\partial u_z}{\partial z}u_z + \frac{\partial u_z}{\partial t}. \tag{42}$$

Because of right hand side of this equation equals original advection equation

$$\frac{du_x}{ds}(x(s),y(s),z(s),t(s)) = 0. \tag{43}$$

$$\frac{du_y}{ds}(x(s),y(s),z(s),t(s)) = 0. \tag{44}$$

$$\frac{du_z}{ds}(x(s),y(s),z(s),t(s)) = 0. \tag{45}$$

This means that u_x, u_y, u_z does not depend on s.

Now we can update velocity field from this equation. First, from eq. (39), $t = s + t_0$. We take initial time as $t_0 = 0$, so we have

$$t = s. \tag{46}$$

Second, from eq. (36), (37), (38), we have

$$x(s) = u_x s + x_0 \tag{47}$$

$$y(s) = u_y s + y_0 \tag{48}$$

$$z(s) = u_z s + z_0 \tag{49}$$

where we used the fact that u_x does not depend on s. Third, we integrate eq. (43) from 0 to t

$$\begin{aligned}
u_x(x(t),y(t),z(t),t(t)) &= u_x(x(0),y(0),z(0),t(0)) && (50)\\
&= u_x(x_0,y_0,z_0,0) && (51)\\
&= \phi_x(x-u_xt, y-u_yt, z-u_zt) && (52)\\
u_y(x(t),y(t),z(t),t(t)) &= \phi_y(x-u_xt, y-u_yt, z-u_zt) && (53)\\
u_z(x(t),y(t),z(t),t(t)) &= \phi_z(x-u_xt, y-u_yt, z-u_zt) && (54)\\
& && (55)
\end{aligned}$$

where we used eq. (46), eq. (47), eq. (48) and eq. (49). In fact, we can confirm that eq. (50) is solution of original advect equation

$$\begin{aligned}
&\frac{\partial\phi(x-u_xt, y-u_yt, z-u_zt)}{\partial t} + u_x\frac{\partial\phi(x-u_xt, y-u_yt, z-u_zt)}{\partial x} && (56)\\
&+u_y\frac{\partial\phi(x-u_xt, y-u_yt, z-u_zt)}{\partial y} + u_z\frac{\partial\phi(x-u_xt, y-u_yt, z-u_zt)}{\partial z} && (57)\\
&= -u_x\frac{\partial\phi}{\partial x} - u_y\frac{\partial\phi}{\partial y} - u_z\frac{\partial\phi}{\partial z} + u_x\frac{\partial\phi}{\partial x} + u_y\frac{\partial\phi}{\partial y} + u_z\frac{\partial\phi}{\partial z} = 0 && (58)
\end{aligned}$$

In the notation of Chapter 2, we have

$$w_{2,x}(x,y,z) = w_{1,x}(x-u_x\Delta t, y-u_y\Delta t, z-u_z\Delta t) \tag{59}$$

$$w_{2,y}(x,y,z) = w_{1,y}(x-u_x\Delta t, y-u_y\Delta t, z-u_z\Delta t) \tag{60}$$

$$w_{2,z}(x,y,z) = w_{1,z}(x-u_x\Delta t, y-u_y\Delta t, z-u_z\Delta t) \tag{61}$$