UNIQUE EQUILIBRIUM STATES FOR GEODESIC FLOWS ON SURFACES WITHOUT FOCAL POINTS

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ABSTRACT. In this paper we prove the uniqueness of the equilibrium states for a class of nonzero potentials over geodesics flows on compact rank 1 surface without focal points. This class includes scalar multiples of the geometric potential and Hölder potentials have unique equilibrium states provided they do not carry full pressure on the singular set. This result partially extends, and is inspired by, Burns-Climenhaga-Fisher-Thompson's work [BCFT17] on the uniqueness of the equilibrium states for the same potentials over geodesic flows on compact rank 1 nonpositively curved manifolds.

CONTENTS

1. INTRODUCTION

This paper is devoted to

The outline of this paper is as follows. In Section

Need To Define

- M is a closed rank 1 surface without focal points.
- Rank 1 = genus ≥ 2

Date: June 4, 2018.

This material is based upon work supported by the National Science Foundation (NSF) under Grant Number DMS 1641020. The second author gratefully acknowledges support from the NSF Postdoctoral Research Fellowship under Grant Number DMS 1703554.

- $f_t : T^1M \to T^1M$ is the geodesic flow.
- *Singular set* $\mathrm{Sing} := \{v \in T^1M : \ \mathrm{rank}(v) > 1\} = \{v \in T^1M : \ E^u(v) \cap E^u(v) \neq \emptyset\}$.
- *Regular set* $\mathrm{Reg} := T^1M \backslash \mathrm{Sing}$.
- X a compact metric space.
- geometric potential $\varphi^u(v) := -\lim_{t\to\infty} \frac{1}{t} \log\det(df_t|_{E^u(v)})$

Theorem A. Let M be a rank 1 surface without focal points and $\mathcal{F}$ be the geodesic flow over M. Let $\varphi : T^1M \to \mathbb{R}$ be a Hölder continuous potential or a scalar multiple of the geometrical potential. Suppose φ verifies the pressure gap property $P(\mathrm{Sing}, \varphi) < P(\varphi)$, then φ has a unique equilibrium state μ_φ.

Theorem B. Suppose φ satisfies the same assumptions in Theorem A. Then the unique equilibrium state μ_φ is fully supported, $\mu_\varphi(\mathrm{Reg}) = 1$, Bernoulli, and is the weak* limit of the weighted regular periodic orbits.

Theorem C. With the same $\mathcal{F}$ and M as above. Suppose $\varphi = q\varphi^u$ is a scalar multiple of the geometric potential. Then, for $q \in (-\infty, 1)$, φ has a unique equilibrium state μ_q which is, fully supported, $\mu_\varphi(\mathrm{Reg}) = 1$, Bernoulli, and is the weak* limit of the weighted regular periodic orbits. Furthermore, the map $q \mapsto P(q\varphi^u)$ is C^1 for $q \in (-\infty, 1)$; and $P(q\varphi^u) = 0$ for $q \in [0, 1)$ when $\mathrm{Sing} \neq \emptyset$.

Emphasis where we use NO FOCAL POINTS in our argument. Why our proof can apply to NO CONJUGATE POINTS. WHAT HAVE WE EXTENDED FROM NON-POSITIVELY CURVED.

Acknowledgement. The author

2. Preliminaries of non-uniform hyperbolic dynamics

Need To Define

- topological pressure for orbit segments and for sets
- finite orbit segments $\mathcal{C} \subset X \times [0, \infty)$ and $[\mathcal{C}]$
- specification
- Bowen property
- C-T decomposition terms
- obstructions of expansivity

2.1. Climenhaga-Thompson's criteria for uniqueness of equilibrium states.

Theorem 2.1 ([CT16], Theorem A)**.** *Let $(X, \mathcal{F})$ be a flow on a compact metric space, and $\varphi : X \to \mathbb{R}$ be a continuous potential. Suppose that $P^\perp_{\exp}(\varphi) < P(\varphi)$ and $X \times [0, \infty)$ admits a decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ with the following properties:*

(I) *$\mathcal{G}$ has specification;*
(II) *φ has Bowen property on $\mathcal{G}$;*
(III) *$P([\mathcal{P}] \cup [\mathcal{S}], \varphi) < P(\varphi)$.*

Then $(X, \mathcal{F}, \varphi)$ has a unique equilibrium state μ_φ.

3. Preliminaries of surfaces without focal points

Need To Define

- Jacobi fields and foliations $\mathcal{J}^s, \mathcal{J}^u, E^u, E^s, E^{cu}, E^{cs}, W^u, W^s, W^{cu}, W^{cs}$
- Horoshperes $H^s(v) := \pi W^s(v)$ and $H^u(v) := \pi W^u(v)$
- $\mathcal{U}^s, \mathcal{U}^u$
- λ

- $W^{cs}(v) := \bigcup_{t\in\mathbb{R}} W^s(f_t(v))$; $W^{cu}(v) := \bigcup_{t\in\mathbb{R}} W^u(f_t(v))$.
- Busemann function $b_v(x) := \lim_{t\to\infty} d(x, \gamma_v(t)) - t$
- d_S Sasaki metric on TM
- $d_K(u, v) := \max\{d(\gamma_u(t), \gamma_v(t)) : t \in [0, 1]\}$ Knieper metric on T^1M
- Bowen ball w.r.t. d_K: $B_T(v, \varepsilon) := \{w \in T^1M : d_K(f_t w, f_t v) < \varepsilon \textit{ for all } 0 \le t \le T\}$
- Intrinsic metric on $W^s(v)$: $d^s(u, w) := \inf\{l(\pi\gamma) : \gamma : [0, 1] \to W^s(v), \ \gamma(0) = u, \ \gamma(1) = w\}$

- Intrinsic metric on $W^{cs}(v)$: locally, $d^s(u, w) := |t| + d^s(f_t u, w)$ where t is the unique value so that $f_t u \in W^s(w)$, which extends to whole $W^{cs}(v)$
- *local stable leaf* through v of size ρ: $W^s_\rho(v) := \{w \in W^s(s) : d^s(u, v) \le \rho\}$
- d^u, d^{cu}, $W^{cs}_\rho(v), W^{cu}_\rho(v)$ are defined in the same manner
- $v \in \mathrm{Sing} \iff K(\pi f_t v)$ <0 for some $t \in \mathbb{R}$ (by [Ebe73, Corollary 3.3, 3.6])
- Rank 1 $\iff g \ge 2$ for surface by D. Burago (closed flat g=1 =¿ conjugate point)

Remark 3.1. When ρ is small enough, the intrinsic metrics are uniformly equivalent to d_S and d_K.

3.1. **Surfaces without focal points.**

Fact 3.2 ([Esc77, Pes77])**.**

(1) ([Pes77, Theorem 4.7, 6.7]) $T_v(T^1M) = E^s(v) \oplus E^u(v) \oplus E^c(v)$, $\dim E^u(v) = E^s(v) = n-1$, *and* $\dim E^c(v) = 1$.
(2) ([Pes77, Theorem 4.11]) *The subbundles $E^u(v)$, $E^s(v)$, $E^{cu}(v)$ and $E^{cs}(v)$ are flow invariant.*
(3) ([Pes77, Theorem 6.1, 6.4]) *The subbundles $E^u(v)$, $E^s(v)$, $E^{cu}(v)$ and $E^{cs}(v)$ are integrable to flow invariant foliations $W^u(v)$, $W^s(v)$, $W^{cu}(v)$ and $W^{cs}(v)$, respectively.*
(4) ([Pes77, Theorem 6.4])
$$W^{cs}(v) = \{w \in T^1M : \exists C > 0 \text{ such that } d(\gamma_v(t), \gamma_w(t)) \le C \text{ for } t > 0\},$$
$$W^{cu}(v) = \{w \in T^1M : \exists C > 0 \text{ such that } d(\gamma_v(t), \gamma_w(t)) \le C \text{ for } t < 0\}.$$
(5) ([Esc77, Lemma, p. 246]) $E^u(v) \cap E^s(v) \ne \emptyset$ *if and only if* $v \in \mathrm{Sing}$.

Remark 3.3. Both Eschenburg [Esc77] and Pesin [Pes77] proved that the stable foliation E^s and unstable foliation E^u are given by the stable and unstable Jacobi fields (as in Eberlein [Ebe73]). In both work recalled above, they also showed when M has no focal points then $W^s(v)$ (resp. $W^u(v)$) is indeed the stable (resp. unstable) horoshpere on T^1M. More precisely,
$$\widetilde{W^s}(v) = \{(x, -\nabla_x b_v^+) : x \in H^s(v)\} \text{ and } \widetilde{W^u}(v) = \{(x, -\nabla_x b_{-v}) : x \in H^u(v)\}.$$

Fact 3.4.

(1) ([Hur86]) *The geodesic flow $\mathcal{F}$ is topologically transitive.*
(2) *(copy from [GR17]) The geodesic flow $\mathcal{F}$ is topologically mixing.*
(3) ([GR17, Lemma 6.8]) *For surface cases,* $h_{top}(\mathrm{Sing}) = 0$.
(4) ([GR17, Corollay 6.12]) *Let M be a closed surface without focal points and $\mathcal{F} = (f_t)_{t\in\mathbb{R}}$ be the geodesic flow on T^1M. Then* $h_{top}(\mathcal{F}) > 0 = h_{top}(\mathrm{Sing})$.
(5) $v \in \mathrm{Sing} \iff K(\pi f_t v) < 0$ *for some* $t \in \mathbb{R}$ *(by* [Ebe73, Corollary 3.3, 3.6]*)*
(6) *Rank 1 $\iff g \ge 2$ for surface by D. Burago (closed flat g=1 =¿ conjugate point)*

3.2. **A hyperbolicity index λ.**

Fact 3.5 ([Esc77])**.**

(1) ([Esc77, Theorem 1 (i)]) *The symmetric linear operator of* $\mathcal{U}^s(v) : T_{\pi v} H^s(v) \to T_{\pi v} H^s(v)$ *given by* $v \mapsto \nabla_v N$, *i.e., the shape operator on* $H^s(v)$, *is well-defined, where* N *is the unit normal vector field of* $H^s(v)$ *toward the same side as* v. *Moreover, we have* $\mathcal{U}^s(v) = -\nabla_v \nabla b_v$.

(2) ([Esc77, Theorem 2 (i)] $H^u(v)$, $H^s(v)$ *are* C^2*-embedded hypersurface in* $\widetilde{M}$. $\mathcal{U}^u$ *is positively semidefinite and and* $\mathcal{U}^s$ *is negatively semidefinite.*

Definition 3.6. For $v \in T^1M$, let $\lambda^u(v)$ be minimum eigenvalue of $\mathcal{U}^u(v)$ and $\lambda^s(v) = \lambda^u(-v)$. We define $\lambda(v) = \min(\lambda^u(v), \lambda^s(v))$.

Let Λ be the maximum eigenvalue of $\mathcal{U}^u(v)$ for $v \in T^1M$.

Lemma 3.7 ([BCFT17], Lemma 2.9)**.** *Let* $v \in T^1M$ *and* J^u *(resp.* J^s*) be a unstable (resp. stable) Jacobi field along* γ_v. *Then*

$$||J^u(t)|| \geq e^{\int_0^t \lambda^u(f_s v)ds} ||J^u(0)|| \text{ and } ||J^s(t)|| \leq e^{-\int_0^t \lambda^s(f_s v)ds} ||J^s(0)||.$$

A handy lemma for computation:

Lemma 3.8. *Let* $\psi : \mathbb{R} \to \mathbb{R}$ *be a non-negative integrable function and* $\psi_T(t) := \int_{-T}^T \psi(t+\tau)d\tau$. *For every* $a \leq b$,

$$\int_a^b \psi_T(t)dt \leq 2T \int_{a-T}^{b+T} \psi(t)dt.$$

In particular, $0 \leq \int_0^t \lambda_T^\sigma(f_s v)ds \leq 2T \int_0^t \lambda^\sigma(f_s v)ds$ *for all* $t, T > 0$ *and* $\sigma \in \{s, u\}$.

Proof. For $b - a \leq 2T$,

$$\begin{aligned}
\int_a^b \psi_T(t)dt &= \int_a^b \int_{-T}^T \psi(t+\tau)d\tau dt \\
&= \int_{a-T}^{b-T} (\tau + T - a)\psi(\tau)d\tau + \int_{b-T}^{a+T} (b-a)\psi(\tau)d\tau + \int_{a+T}^{b+T} (b+T-\tau)\psi(\tau)d\tau \\
&\leq (b-a)\int_{a-T}^{b-T} \psi(\tau)d\tau + (b-a)\int_{b-T}^{a+T} \psi(\tau)d\tau + (b-a)\int_{a+T}^{b+T} \psi(\tau)d\tau \\
&= (b-a)\int_{a-T}^{b+T} \psi(\tau)d\tau \leq 2T \int_{a-T}^{b+T} \psi(\tau)d\tau.
\end{aligned}$$

For $b - a \geq 2T$,

$$\begin{aligned}
\int_a^b \psi_T(t)dt &= \int_a^b \int_{-T}^T \psi(t+\tau)d\tau dt \\
&= \int_{a-T}^{a+T} (\tau + T - a)\psi(\tau)d\tau + \int_{a+T}^{b-T} 2T\psi(\tau)d\tau + \int_{b-T}^{b+T} (s+T-\tau)\psi(\tau)d\tau \\
&\leq 2T\int_{a-T}^{a+T} \psi(\tau)d\tau + 2T\int_{a+T}^{b-T} \psi(\tau)d\tau + 2T\int_{b-T}^{b+T} \psi(\tau)d\tau = 2T\int_{a-T}^{b+T} \psi(\tau)d\tau.
\end{aligned}$$

□

As in the paper, let δ be small enough so that if $d_K(v,w) < \delta e^\Lambda$, then $|\lambda_T(v) - \lambda_T(w)| < \frac{\eta}{2}$, and define $\tilde{\lambda} = \max\{0, \lambda(v) - \frac{\eta}{2}\}$ and $\tilde{\lambda}_T = \max\{0, \lambda_T(v) - \frac{\eta}{2}\}$. If $w \in B_t(v,\delta)$ then $d_K(f_s v, f_s w) \leq \delta e^\Lambda$ and

$$\int_0^t \lambda(f_\tau w)d\tau \geq \int_0^t \widetilde{\lambda}(f_\tau v)d\tau \geq \int_0^t \lambda(f_\tau v)d\tau - \frac{\eta}{2}t.$$

$$\int_0^t \lambda_T^u(f_\tau w)d\tau \geq \int_0^t \widetilde{\lambda_T^u}(f_\tau v)d\tau \geq \int_0^t \lambda_T^u(f_\tau v)d\tau - \frac{\eta}{2}t.$$
$$\int_0^t \lambda_T^s(f_\tau w)d\tau \geq \int_0^t \widetilde{\lambda_T^s}(f_\tau v)d\tau \geq \int_0^t \lambda_T^s(f_\tau v)d\tau - \frac{\eta}{2}t.$$
$$\int_0^t \lambda_T(f_\tau w)d\tau \geq \int_0^t \tilde{\lambda}_T(f_\tau v)d\tau \geq \int_0^t \lambda_T(f_\tau v)d\tau - \frac{\eta}{2}t.$$

Need To Define

- λ_T^u be the minimal eigenvalue (in surface case, it is "the" eigenvalue) of the operator $\mathcal{U}_T^u(v)$: $T_{\pi v}H^u(v) \to T_{\pi v}H^u(v)$ by $T_{\pi v}H^u(v) \ni w \mapsto \int_{-T}^T P_{-\tau}\mathcal{U}^u(f_\tau v)(P_\tau v) \in T_{\pi v}H^u(v)$ where P_t is the parallel transportation along the geodesic $\gamma_v(t)$.
- $\lambda_T^s(v) := \lambda_T^u(-v)$
- $\lambda_T(v) := \min\{\lambda_T^s(v), \lambda_T^u(v)\}$

Lemma 3.9 ([BCFT17], Lemma 3.8)**.** *Given η, δ as above, $v \in T^1M$, and $w, w' \in W_\delta^s(v)$, we have the following for every $t \geq 0$:*

$$d^s(f_t w, f_t w') \leq d^s(w, w')e^{-\int_0^t \widetilde{\lambda^s}(f_\tau v)d\tau}.$$

Similarly, if $w, w' \in W_\delta^u(v)$, then for any $t \geq 0$,

$$d^u(f_{-t} w, f_{-t} w') \leq d^u(w, w')e^{-\int_0^t \widetilde{\lambda^u}(f_{-\tau} v)d\tau}.$$

4. Decompositions for geodesic flow

4.1. Sing, λ, λ_T, and decompositions. Need To Define

- $\mathcal{P}_T(\eta) := \{(v,t)\colon\ \int_0^\tau \lambda_T(f_s v)ds < \tau\eta$ for some $\tau \in [0,\tau]\}$.
- $\mathcal{S}_T(\eta) := \{(v,t)\colon\ \int_0^\tau \lambda_T(f_{-s} f_t v)ds \geq \tau\eta$ for some $\tau \in [0,\tau]\}$.
- $\mathcal{B}_T(\eta) = \mathcal{P}_T(\eta) \cup \mathcal{S}_T(\eta)$
- $\mathcal{G}_T(\eta) := \{(v,t)\colon\ \int_0^\tau \lambda_T(f_s v)ds \geq \tau\eta$ and $\int_0^\tau \lambda_T(f_{-s} f_t v)ds \geq \tau\eta\ \forall \tau \in [0,t]\}$.
- We have the following decomposition:

$$\mathcal{G} := \mathcal{G}_T(\eta) \quad \text{and} \quad \mathcal{P} \cup \mathcal{S} := \mathcal{B}_T(\eta). \tag{4.1}$$

  Given an orbit segment $(x,t) \in T^1S \times \mathbb{R}^+$, there exists a unique $p, g, s \geq 0$ such that $t = p + g + s$, and

$$(x,p) \in \mathcal{P}_T(\eta), \quad (f^p x, g) \in \mathcal{G}_T(\eta), \text{ and } (f^{p+g}x, s) \in \mathcal{S}_T(\eta).$$

- $\mathrm{Reg}_T(\eta) := \{v \in T^1M :\ \lambda_T(v) \geq \eta\}$.
- stable and unstable Jacobi fields estimates along $\mathcal{G}_T(\eta)$, $\mathcal{C}_T(\eta)$
- expending and contracting estimates along $\mathcal{G}_T(\eta)$
- angle between $E^u(v)$ and $E^s(v)$ for $v \in \mathcal{G}_T(\eta)$

Fact 4.1.

(1) Sing *is closed and flow invariant.*
(2) $\mathrm{Reg}_T(\eta) \subset T^1M$ *is compact.*
(3) $\mathcal{G}_T(\eta) \subset T^1M \times \mathbb{R}$ *is closed.*
(4) Reg *is dense in T^1M. (Need a lemma to prove it.)*

Theorem 4.2. *Let $\varphi : T^1M \to \mathbb{R}$ be continuous, If $P(\mathrm{Sing}) < P(\varphi)$, and for all $\eta > 0$ the potential φ has the Bowen property on*

$$G_T(\eta) = \{(v,t)\colon\ \int_0^\tau \lambda_T(f_s v)ds \geq \tau\eta \text{ and } \int_0^\tau \lambda_T(f_{-s} f_t v)ds \geq \tau\eta\ \forall \tau \in [0,t]\},$$

then the geodesic flow has a unique equilibrium stat for φ.

4.2. $\lambda, \lambda^s, \lambda^u, \lambda_T$, and the singular set Sing.

Lemma 4.3. *For any $T > 0$ and $v \in T^1M$, we have $\lambda_T^\sigma(v) = \int_{-T}^{T} \lambda^\sigma(f_\tau v) d\tau$ for $\sigma \in \{s, u\}$. (Can be simplified!!!)*

Proof. Let us only check for λ_T^u, and for λ_T^s follows similarly. It is enough to show $\int_{-T}^{T} \lambda^u(f_\tau v) d\tau$ that is the eigenvalue of $\mathcal{U}_T^u(v)$.

To see this, we first notice that since $\dim M = 2$ we have, for all $\tau > 0$ and $w \in T_{\pi v} H^u(v)$ with $||w|| = 1$, $P_\tau(w) \in T_{\pi f_\tau v} H^u(f_\tau v) = \dot{\gamma}_v^\perp(\tau)$ and $P_\tau(w)$ is the eigenvector of $\mathcal{U}^u(f_\tau v)$.

Thus
$$P_{-\tau} \mathcal{U}^u(f_\tau v) P_\tau(w) = \lambda^u(f_\tau v) w.$$
Hence,
$$\begin{aligned} \mathcal{U}_T^u(v)(w) &= \int_{-T}^{T} P_{-\tau} \mathcal{U}^u(f_\tau v) P_{-\tau}(w) d\tau \\ &= \left(\int_{-T}^{T} \lambda^u(f_\tau v) d\tau \right) \cdot w. \end{aligned}$$
□

Recall: The following lemma works on no focal points setting.

Lemma 4.4. *The following are equivalent for $v \in T^1M$. (Can be simplified!!!)*

(1) *$v \in$ Sing.*
(2) *$\lambda^u(f_t v) = 0$ for all $t \in \mathbb{R}$.*
(3) *$\lambda^v(f_t v) = 0$ for all $t \in \mathbb{R}$*

Proof. It is clear that (1) $\implies$ (2) and (1) $\implies$ (3). We will prove (2) $\implies$ (1), and (3) $\implies$ (2) follows similarly.

To see (2) $\implies$ (1), it is enough to show J^u is a parallel Jacobi field where $J^u(t)$ is a unstable Jacobi field along $\gamma_v(t)$. Notice that $J^u(t)$ satisfies $J^u(0) = w \in T_{\pi v} H^u(v)$ and $(J^u)'(0) = 0$ (because $\lambda^u(v) = 0$). Similarly, for any $\tau \in \mathbb{R}$ we have a unstable Jacobi field $J_\tau^u(t)$ along $\gamma_{f_\tau v}$ such that $J_\tau^u(0) = w_\tau = P_\tau w \in T_{\pi f_\tau v} H^u(f_\tau v)$ and $(J_\tau^u)'(0) = 0$.

Claim: $J_\tau^u(t) = J^u(\tau + t)$ for any $\tau, t \in \mathbb{R}$.

We first recall that J^u satisfies the equation $(J^u)'(t) = \mathcal{U}^u(t) J^u(t)$ where $\mathcal{U}^u(t)$ is the second fundamental form of $H^u(v)$, and similarly we have $(J_\tau^u)'(t) = \mathcal{U}^u(\tau + t) J_\tau^u(t)$. This shows that $J^u(t + \tau)$ and $J_\tau^u(t)$ satisfies the same equation $\frac{J'(x)}{J(x)} = \mathcal{U}^u(x)$; moreover, since $\dim M = 2$ we also know $J^u(\tau) = P_\tau w = J_\tau^u(0)$. Thus, we have $J_\tau^u(t) = J^u(\tau + t)$ for any $\tau, t \in \mathbb{R}$.

By the claim, we have $(J^u)''(t) = (J_t^u)''(0) = 0$ for all $t \in \mathbb{R}$. □

Remark 4.5. It is also not hard to see that $v \in$ Sing $\implies K(\pi v) = 0$ where K is the Gaussian curvature.

Lemma 4.6. *$\lambda_T(v) = 0$ for all T if and only if $v \in$ Sing. (Can be simplified!!!)*

Proof. The if direction is clear. In the following we prove the only if direction.

First we notice that since λ is nonnegative, continuous, and $\lambda_T(v) \geq \int_{-T}^{T} \lambda(f_t v) dt \geq 0$, we have that $\lambda_T(v) = 0$ for all $T \in \mathbb{R}$ implies $\lambda(f_t v) = 0$ for all $t \in \mathbb{R}$.

Claim: There are only three possible cases such that $\lambda(f_t v) = 0$ for all $t \in \mathbb{R}$:

(i) $\lambda^s(f_t v) = 0$ for all $t \in \mathbb{R}$.
(ii) $\lambda^u(f_t v) = 0$ for all $t \in \mathbb{R}$.
(iii) There exists $t_0 \in \mathbb{R}$ such that $\lambda^s(f_{t_0} v) = \lambda^u(f_{t_0} v) = 0$.

It is clear from Lemma 4.4 that both (i) and (ii) give $v \in \mathrm{Sing}$. To see (iii) also implies $v \in \mathrm{Sing}$, we recall that, for $\sigma \in \{s,u\}$, $\lambda^\sigma(f_{t_0}v) = 0$ implies that there exists $0 \neq w^\sigma \in T_{\pi(f_{t_0}v)}H^\sigma(f_{t_0}v)$ such that $\mathcal{U}^\sigma(w^\sigma) = 0$. Since both w^u, w^s are tangent to $f_{t_0}v$ and $\dim M = 2$, we know $w^u = w^s$ (by taking same length). It is not hard to see that the $H^u(f_{t_0}v)-$ Jacobi field J^u matches the $H^s(f_{t_0}v)-$ Jacobi field J^s, that implies, $E^u(f_{t_0}v) \cap E^v(f_{t_0}v) \neq 0$. Thus we have $f_{t_0}v \in \mathrm{Sing}$, and because Sing is flow invariant we have $v \in \mathrm{Sing}$.

To see the claim, let $U := \{t \in \mathbb{R} : \ \lambda^u(f_tv) = 0\}$ and $S := \{t \in \mathbb{R} : \ \lambda^s(f_tv) = 0\}$. Since both λ^u, λ^s are continuous, U and S are closed sets in $\mathbb{R}$. Notice that if $U \cap S = \emptyset$ then $U = \mathbb{R}\backslash S$; thus U, S are clopen sets. Since $\mathbb{R}$ is connected, we have when $U \cap S = \emptyset$ then $U = \mathbb{R}$ or $S = \mathbb{R}$. □

Lemma 4.7. *Let μ be a $\mathcal{F}$-invariant probability measure on T^1M. Suppose $\lambda(v) = 0$ for μ almost every $v \in T^1M$, then* $\mathrm{supp}(\mu) \subset \mathrm{Sing}$.

Proof. Suppose not, i.e., $\mathrm{supp}(\mu) \not\subseteq \mathrm{Sing}$. Since μ is Borel, there exists $v \in \mathrm{Reg}$ such that for all $r > 0$ $\mu(B(v,r)) > 0$ and $\lambda(w) = 0$ for $\mu - a.e.$ $w \in B(v,r)$. We notice that since $v \in \mathrm{Reg}$ there exists t_0 such that $\lambda(f_{t_0}v) > 0$ (otherwise $v \in \mathrm{Sing}$ by Lemma 4.6). By the continuity of λ, there exists a neighborhood $B(f_{t_0}v, r_0)$ of $f_{t_0}v$ such that $\lambda|_{B(f_{t_0}v,r_0)} > 0$.

Moreover, there exists $r > 0$ such $B(v,r) \supset f_{-t_0}(B(f_{t_0}v, r_0))$; hence

$$0 = \mu(B(v,r)) = \mu(f_{-t_0}(B(f_{t_0}v, r_0))) = \mu(B(f_{t_0}v, r_0)),$$

which leads a contradiction because $\lambda|_{B(f_{t_0}v,r_0)} > 0$. □

4.3. Uniform distance estimates on $\mathcal{G}_T(\eta)$. Need To Define

Distance on W^u and W^s for $v \in \mathcal{G}_T(\eta)$

- $\mathrm{Reg}_T(\eta)$

Lemma 4.8 (Lemma 3.9)**.** *Given η, T, δ as above, and $(v,t) \in \mathcal{G}_T(\eta)$, then every $v' \in B_t(v,\delta)$ satisfy $(v',t) \in \mathcal{G}_T(\frac{\eta}{2})$. Moreover, for any $(v,t) \in \mathcal{G}_T(\eta)$, any $w, w' \in W^s_\delta(v)$, and any $0 \leq \tau \leq t$,*

$$d^s(f_\tau w, f_\tau w') \leq d^s(w,w')e^{-\frac{\eta}{4T}\tau}.$$

Similarly, for $w, w' \in f_{-\tau}W^u(f_\tau v)$ and $0 \leq \tau \leq t$, we have

$$d^u(f_\tau w, f_\tau w') \leq d^u(f_t w, f_t w')e^{-\frac{\eta}{4T}(t-\tau)}.$$

Proof. By Lemma 3.8 and 3.9, we have

$$\begin{aligned} d^s(f_\tau w, f_\tau w') &\leq d^s(w,w')e^{-\int_0^\tau \widetilde{\lambda^s}(f_s v)ds} \leq d^s(w,w')e^{\frac{-1}{2T}\int_0^\tau \widetilde{\lambda^s_T}(f_s v)ds} \\ &\leq d^s(w,w')e^{\frac{-1}{2T}(\int_0^\tau \lambda^s_T(f_s v)ds - \frac{\eta\tau}{2})} && \because v \in \mathcal{G}_T(\eta) \\ &\leq d^s(w,w')e^{\frac{-\eta\tau}{2T} + \frac{\eta\tau}{4T}} = d^s(w,w')e^{\frac{-\eta\tau}{4T}} \end{aligned}$$

Similarly, we have the other inequality. □

Lemma 4.9 (Lemma 3.10)**.** *Given $\eta, T > 0$, there exists $\theta > 0$ so that for any $v \in \mathrm{Reg}_T(\eta)$, we have $\measuredangle(E^u(f_tv), E^s(f_tv)) \geq \theta$ for any $-T \leq t \leq T$.*

Proof. Assume the contrary. Then there exists $\{(v_i,t_i)\}_{i\in\mathbb{N}} \subset \mathrm{Reg}_T(\eta) \times [-T,T]$ such that

$$\measuredangle(E^s(f_{t_i}v_i), E^u(f_{t_i}v_i)) \to 0.$$

Since $\mathrm{Reg}_T(\eta) \times [-T,T]$ is compact, there exist subsequences $t_{i_j} \to t_0$, and $v_{i_j} \to v_0$ such that $\measuredangle(E^s(f_{t_0}v_0), E^u(f_{t_0}v_0)) = 0$. Then, $f_{t_0}v_0 \in \mathrm{Sing}$. On the other hand, $\mathrm{Reg}_T(\eta)$ is closed so $v_0 \in \mathrm{Reg}_T(\eta)$. However, this is a contradiction because Sing is f_t-invariant. □

5. The specification property

Need To Define

- Closing lemma

Definition 5.1. A collection $\mathcal{C} \subset X \times [0, \infty)$ of orbit segments has *specification* at scale $\rho > 0$ if there is $\tau = \tau(\rho)$ such that for every $(x_1, t_1), \ldots, (x_n, t_n) \in \mathcal{C}$ there exists $y \in X$ and a sequence of jumping times $\tau_1, \ldots, t_{n-1} \in [0, \tau]$ such that for $s_0 = \tau_0 = 0$ and $s_j = \sum_{i=1}^{j} t_i + \sum_{i=1}^{j-1} \tau_i$, we have

$$f_{s_{j-1}+\tau_{j-1}}(y) \in B_{(t_j)}(x_j, \rho)$$

for every $j \in \{1, \ldots, n\}$. If C has specification at all scale, then we say $\mathcal{C}$ has specification.

For any $T, \eta > 0$, we define $\mathcal{C}_T(\eta) := \{(v, t) \colon v, f_t v \in \mathrm{Reg}_T(\eta)\}$. We prove that $\mathcal{C}_T(\eta)$ has specification for all $\eta, T > 0$.

Proposition 5.2. *For any $\eta, T > 0$, $\mathcal{C}_T(\eta)$ has specification. Hence, so does $\mathcal{G}_T(\eta)$.*

We collect relevant lemmas to prove the specification.

Definition 5.3. The foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^u$ have *local product structure* at scale $\delta > 0$ with constant $\kappa \geq 1$ at v if for any $w_1, w_2 \in B(v, \delta)$, the intersection $[w_1, w_2] := \mathcal{W}^u_{\kappa\delta}(w_1) \cap \mathcal{W}^{cs}_{\kappa\delta}(w_2)$ is a unique point and satisfies

$$d^u(w_1, [w_1, w_2]) \leq \kappa d_K(w_1, w_2),$$
$$d^{cs}(w_2, [w_1, w_2]) \leq \kappa d_K(w_1, w_2).$$

Corollary 5.4. *There exists $\delta_0 > 0$ and $\kappa \geq 1$ such that the foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^u$ have LPS with scale $\delta \in (0, \delta_0)$ with constant κ.*

Proof. It is a direct consequence of Lemma 4.9. □

Lemma 5.5. *Let $\eta, T > 0$ be given. Then, there exists $\delta > 0$ such that for any $\rho \in (0, \delta)$, there exists $a = a(\delta, \rho) > 0$ such that the following holds: for any $u, v \in N^K_\delta(Reg_T(\eta))$, there exists $\tau \in (0, a)$ and $[v, w]_\tau \in T^1M$ such that $f_\tau([v, w]_\tau) \in \mathcal{W}^{cs}_\rho(w)$ and $[v, w]_\tau \in \mathcal{W}^u_\rho(v)$.*

Proof. Need a proof? □

We now sketch the proof of Proposition 5.2. The method is quite similar to the BCFT; the only difference is that we choose the reference orbit (v_0, t_0) sufficiently long with t_0 depending on the prescribed T.

Define the distance d_t.

Proof sketch of Proposition 5.2. We begin by fixing the reference orbit $(v_0, t_0) \in \mathcal{C}_T(\eta)$, $\alpha > 1$, and $\epsilon > 0$ such that $\mathcal{W}^s_\epsilon(v_0)$ and $\mathcal{W}^u_\epsilon(f_{t_0}v_0)$ belong to $\mathrm{Reg}_T(\eta/2)$ and that for any $w, w' \in T^1S$ with $d_{t_0}(w, v_0) < \epsilon$ and $f_{t_0}w' \in \mathcal{W}^u_\epsilon(f_{t_0}w)$, we have

$$d^u(f_{t_0}w, f_{t_0}w') \geq \alpha d^u(w, w').$$

The existence of such orbit segment is guaranteed because we can simply choose $(v_0, t_0) \in G_T(\eta)$ with t_0 sufficiently long. To see existence of such a orbit segment, we notice that for v such that γ_v is a rank 1 geodesic, we know $\lambda(v) \geq \eta > 0$ for some η. Moreover, when we pick t_0 small enough we have $\lambda^u(v') > \frac{\eta}{2}$ and $\lambda^s(v') > \frac{\eta}{2}$ for $v' \in B_{t_o}(v)$, then by Lemma 4.8 there exists such $\alpha > 1$.

Comment of why we need v_0

Given any scale $\rho > 0$, we claim that we can choose $0 < \rho' \ll \rho$ such that $\mathcal{C}_T(\eta)$ has specification with scale ρ with the jumping time at most $t_0 + 2a$ with $a = a(\delta, \rho')$ from Lemma 5.5.

Let $(v_1, t_1), \ldots, (v_n, t_n) \in \mathcal{C}_T(\eta)$ be given. We set $(w_1, s_1) = (v_1, t_1)$, and we will inductively define orbit segments (w_j, s_j) for $j \geq 2$ with $w_j \in \mathcal{W}^u_\rho(v_1)$ and $f_{s_j}w_j \in \mathcal{W}^{cs}_{\rho'}(f_{t_j}v_j)$ such that

(w_j, s_j) shadows the orbit segments $(v_1, t_1) \to (v_0, t_0) \to (v_2, t_2) \to (v_0, t_0) \to (v_3, t_3) \to (v_0, t_0) \to \dots, (v_j, t_j)$ with scale ρ.

Suppose (w_j, s_j) with the properties listed above is given. We want to define (w_{j+1}, s_{j+1}) such that it w_{j+1} closely follows the orbit of w_j for time s_j, then (1) jumps to (with transition time $\leq a$) and shadows v_0 for time t_0, and then (2) jumps to (again with transition time $\leq a$) to and shadows v_{j+1} for time t_{j+1}. Since Lemma 5.5 allows only one jump at a time, we define auxiliary orbit segments (u_j, l_j) by applying Lemma 5.5 to $f_{s_j} w_j$ and v_0 which then satisfies (1). We then obtain (w_{j+1}, s_{j+1}) satisfying (2) again from Lemma 5.5 applied to $f_{l_j} u_j$ and v_{j+1}. In all of these joining orbits, we take the scale to be ρ'. If ρ' is initially chosen sufficiently small, then $f_{s_j} w_j$ and $f_{l_j} u_j$ are respectively close enough to $f_{t_j} v_j, f_{t_0} v_0 \in \mathrm{Reg_T}(\eta)$, and hence the existence of u_j and w_{j+1} via Lemma 5.5 is guaranteed. Clearly, (w_{j+1}, s_{j+1}) defined inductively as such has transition time between consecutive v_i's bounded above by $t_0 + 2a$.

We would be done if each (w_j, s_j) ρ-shadows every (v_i, t_i) up to $i \leq j$. It is sufficient to show that for each $i \leq j$, the d^u distance between $f_{s_i} w_j$ and $f_{s_i} w_i$ (note both w_i and w_j lie in $\mathcal{W}^u(v_1)$, so the d^u makes sense) and that d^{cs} distance between $f_{s_i - t_i} w_i$ and v_i are uniformly bounded depending only on ρ'. The latter is immediate because $f_{s_i - t_i} w_i \in \mathcal{W}^{cs}_{\rho'}(v_i)$ by construction and d^{cs} doesn't increase in forward time. For the former, notice that each $i \leq m \leq j$, $d^u(f_{s_i} w_m, f_{s_i} u_m)$ is at most $\rho' \alpha^{-(m-i)}$ because $d^u(f_{s_m} w_m, f_{s_m} u_m) \leq \rho'$ from the construction of u_m and each time $f_{s_m} u_m$ and $f_{s_m} w_m$ passes through the reference orbit (v_0, t_0) in backward time, their d^u distance decrease by factor of at least α. The same argument applied to $f_{s_i} u_m$ and $f_{s_i} w_{m+1}$ gives $d^u(f_{s_i} u_m, f_{s_i} w_{m+1}) \leq \rho' \alpha^{1+m-i}$. It then follows that the sum appearing in the right hand side of the inequality

$$d^u(f_{s_i} w_j, f_{s_i} w_i) \leq \sum_{m=i}^{j-1} d^u(f_{s_i} w_m, f_{s_i} u_m) + d^u(f_{s_i} u_m, f_{s_i} w_{m+1})$$

is uniformly bounded depending only on ρ'. Hence, with a sufficiently small initial choice of ρ', the orbit segment (w_n, s_n) ρ-shadows (with transition time $t_0 + 2a$) all (v_i, t_i) with $i \leq n$. □

!!!CLOSING LEMMA!!!

Lemma 5.6 (Closing Lemma; [BCFT17], Lemma 4.7)**.** *For $\varepsilon, \eta > 0$, there exists $s = s(\varepsilon) > 0$ such that for $(v, t) \in \mathcal{G}_T(\eta)$ there are $w \in B_t(v, \varepsilon)$ and $\tau \in [0, s(\varepsilon)]$ satisfying $f_{t+\tau} w = w$.*

6. The Bowen property

Need To Define

- Hölder potentials
- geometric potentials

6.1. **Bowen property for Hölder potentials.** Lemmas in this section are exactly the same as their corresponding in [BCFT17]. The proofs follow verbatim. However, for the completeness we still give proofs there.

Lemma 6.1. *If φ is Hölder along stable leaves (resp. unstable leaves), then φ has the Bowen property along stable leaves (resp. unstable leaves) with respect to $\mathcal{G}_T(\eta)$.*

Proof. It is a direct consequence of Lemma 4.8. We prove the stable leaves case, and for unstable leaves one uses the same argument.

Let $(v, t) \in \mathcal{G}_T(\eta)$, $\delta_1 > 0$ be as in Lemma 4.8 and $\delta_2 > 0$ be given by the Hölder continuity along stable leaves. Then for $\delta = \min\{\delta_1, \delta_2\}$ and $w \in W^s_\delta(v)$, we have

$$
\begin{aligned}
|\Phi(v,t)-\Phi(w,t)| &\le \int_0^t |\varphi(f_\tau v)-\varphi(f_\tau w)|\,\mathrm{d}\tau \le \int_0^t C_1\cdot d^s(f_\tau v, f_\tau w)^\theta \mathrm{d}\tau \\
&\le \int_0^t C_1\cdot \left(d^s(v,w)\cdot e^{-\frac{\eta}{4T}\tau}\right)^\theta \mathrm{d}\tau \le C_1\cdot d^s(v,w)^\theta \int_0^t e^{\frac{-\eta\theta}{4T}\tau}\mathrm{d}\tau \\
&\le C_1\delta^\theta \frac{4T}{\eta\theta}.
\end{aligned}
$$

□

Lemma 6.2. *Given $\eta > 0$, suppose φ has the Bowen property along stable leaves and unstable leaves with respect to $\mathcal{G}_T(\eta)$. Then $\varphi : T^1M \to \mathbb{R}$ has the Bowen property on $\mathcal{G}_T(\eta)$.*

Proof. We first notice that since the curvature of horoshperes is uniformly bounded, d_K and d^u are equivalent on W^u_δ when δ small enough. Hence, there exist $\delta_0, C > 0$ such that $d^u(u,v) \le C d_K(u,v)$ for $v \in T^1M$ and $u \in W^u_{\delta_0}(v)$. Let $\delta_1 > 0$ be the radius that guarantees any $(v,t) \in G_T(\eta)$ the foliations W^u and W^{cs} have local product structure at scale δ_1 with constant κ. Let $\delta_2 > 0$ be the radius given in Lemma 4.8 that if $(v,t) \in \mathcal{G}_T(\eta)$, then for $w \in W^u_{\delta_2}(v)$ we have $(w,t) \in \mathcal{G}_T(\frac{\eta}{2})$. Let $\delta_3, K > 0$ be the constants from the Bowen property for φ along stable and unstable leaves with respect to $\mathcal{G}_T(\frac{\eta}{2})$. Without loss generality we may assume $\delta_3 < \delta_0$.

Let $\delta = \min\{\delta_0, \delta_1, \delta_2, \frac{\delta_3}{2\kappa C}, \frac{\delta_3}{\kappa}\}$, $(v,t) \in \mathcal{G}_T(\eta)$, and $w \in B_t(v,\delta)$.

By the LPS there exists $v' \in W^u_{\kappa\delta} \cap W^{cs}_{\kappa\delta}(v)$. We claim: $f_t(v') \in W^u_{\delta_3}(f_t w)$. However, we will prove this claim in the end of this proof. Moreover, there exists $\rho \in [-\kappa\delta, \kappa\delta]$ such that $f_\rho(v') \in W^s_{\kappa\delta}(v) \subset W^s_{\delta_3}(v)$. Thus assuming the claim and by the Bowen property we have

$$\left|\Phi(v,t) - \Phi(f_\rho v', t)\right| \le K \text{ and } \left|\Phi(v',t)-\Phi(w,t)\right| \le K.$$

Hence, assuming the claim we have

$$
\begin{aligned}
|\Phi(v,t)-\Phi(w,t)| &\le \left|\Phi(v,t)-\Phi(f_\rho v',t)\right| + \left|\Phi(f_\rho v',t)-\Phi(v',t)\right| + \left|\Phi(v',t)-\Phi(w,t)\right| \\
&\le 2K + 2||\varphi||\cdot|\rho|.
\end{aligned}
$$

To see the claim, i.e., $f_t(v') \in W^u_{\delta_3}(f_t w)$: suppose it is false, then there exists $\sigma \in [0,t]$ such that

$$\delta_0 < d^u(f_\sigma v', f_\sigma w) \le \delta_3. \tag{6.1}$$

Notice that $v' \in W^{cs}_{\kappa\delta}(v) \subset B_t(v,\kappa\delta)$, so

$$d_K(f_\sigma v', f_\sigma w) \le d_K(f_\sigma v', f_\sigma v) + d_K(f_\sigma v, f_\sigma w) \le 2\kappa\delta.$$

Thus, $d^u(f_\sigma v', f_\sigma w) \le 2C\kappa\delta < \delta_3$ and we have derived a contradiction (to (6.1)). □

Theorem 6.3. *If φ is Hölder continuous, then it has the Bowen property with respect to $\mathcal{G}_T(\eta)$ for any $\eta > 0$.*

Proof. It follows Lemma 6.1, 6.2. □

6.2. **Bowen property for the geometric potential.** ???Change Notation???

We denote by J^u_v the unstable Jacobi field along γ_v with $J^u_v(0) = 1$. Let $U^u_v := (J^u_v)'/J^u_v$, then U^u_v is a solution to the Riccati equation

$$U' + U^2 + K(f_t v) = 0.$$

Notice that we also have $U^u_v(t) = \lambda^u(f_t v)$. From Lemma 7.6 in BCFT, in order to prove φ^u has Bowen property on $\mathcal{G}_T(\eta)$, we have only to prove the following proposition:

Lemma 6.4 (Prop 7.7)**.** *For every $\eta > 0$, there are $\delta, Q, \xi > 0$ such that given any $(v, T_0) \in \mathcal{G}_T(\eta), w_1 \in W^s_\delta(v)$ and $w_2 \in f_{-T_0}W^u_\delta(f_{T_0}v)$, for every $0 \le t \le T_0$ we have*

$$|U^u_v(t) - U^u_{w_1}(t)| \le Qe^{-\xi t},$$

$$|U^u_v(t) - U^u_{w_2}(t)| \le Q(e^{-\xi t} + e^{-\xi(T_0-t)}).$$

Lemma 6.5. *For every $\eta > 0$, there are δ, Q such that given any $(v, T_0) \in \mathcal{G}_T(\eta), w \in B_{T_0}(v, \delta)$, for every $0 \le t \le T_0$ we have*

$$|U^u_v(t) - U^u_w(t)| \le Q \exp\left(-\frac{\eta t}{T}\right) + \int_0^t \exp\left(-\frac{\eta(t-\tau)}{4T}\right)|K(f_\tau v) - K(f_\tau w)|d\tau.$$

Proof of Lemma 6.4. We may choose small δ so that $w_1, w_2 \in \mathcal{G}_T(\eta/2)$.

Since $w_1 \in W^s_\delta(v)$, the smoothness of K together with Lemma 4.8 implies

$$|K(f_\tau v) - K(f_\tau w_1)| \le Q' d_K(f_\tau v, f_\tau w_1) \le Q' d^s(f_\tau v, f_\tau w_1) \le Q \exp\left(-\frac{\eta\tau}{4T}\right),$$

for any $\tau \in [0, T_0]$. Thus by Lemma 6.5:

$$|U^u_v(t) - U^u_{w_1}(t)| \le Q \exp\left(-\frac{\eta t}{T}\right) + Qt \exp\left(-\frac{\eta t}{4T}\right) \le Qe^{-\xi t}.$$

once we fix $\xi < \eta/2T$. Hence $|U^u_v(t) - U^u_{w_1}(t)| \le Qe^{-\xi t}$.

For $w_2 \in f_{-T_0}W^u_\delta(f_{T_0}v)$, we have the following estimation for K:

$$|K(f_\tau v) - K(f_\tau w_2)| \le Qd_K(f_\tau v, f_\tau w_2) \le Qd^u(f_{\tau-T_0}f_{T_0}v, f_{\tau-T_0}f_{T_0}w_2) \le Q \exp\left(-\frac{\eta(T_0-\tau)}{4T}\right)$$

for any $\tau \in [0, T_0]$. We use Lemma 6.5 again and get:

$$\begin{aligned}
|U^u_v(t) - U^u_{w_2}(t)| &\le Q \exp\left(-\frac{\eta t}{2T}\right) + Q\int_0^t \exp\left(-\frac{\eta(T_0-\tau)}{4T}\right)d\tau \\
&\le Q \exp\left(-\frac{\eta t}{2T}\right) + Q \exp\left(-\frac{\eta(T_0-t)}{4T}\right)
\end{aligned}$$

□

Proof of Lemma 6.5. Without loss of generality, we may assume $U^u_w(0) \ge U^u_v(0)$ and let U_1 be the solution of the Riccati equation along γ_v with $U_1(0) = U^u_w(0)$. We have

$$|U^u_v(t) - U^u_w(t)| \le |U^u_v(t) - U_1(t)| + |U_1(t) - U^u_w(t)|.$$

Since $U^u_w(0) \ge U^u_v(0)$ and both U_1 and U^u_v are Riccati solutions along γ_v, we have $U_1(t) \ge U^u_v(t) = \lambda^u(f_t v)$ for all t. Hence

$$(U_1 - U^u_v)' = -(U_1 - U^u_v)(U_1 + U^u_v) \le -2\lambda^u(f_t v)(U_1 - U^u_v).$$

Thus $(U_1(t) - U^u_v(t))\exp(\int_0^t 2\lambda(f_s v)ds)$ is not increasing. By Lemma 4.8 we have

$$\begin{aligned}
0 &\le U_1(t) - U^u_v(t) \le (U^u_w(0) - U^u_v(0))\exp\left(-\int_0^t 2\lambda^u(f_s v)ds\right) \\
&\le Q \exp\left(-\frac{1}{T}\int_0^t \lambda^u_T(f_s v)ds\right) \le Q \exp\left(-\frac{\eta t}{T}\right).
\end{aligned}$$

Now we estimate $|U_1(t) - U^u_w(t)|$. We may assume $U_1(t) > U^u_w(t)$ (the other case is similar). Suppose $U_1(s_0) = U^u_w(s_0)$ at $s_0 < t$ and $U_1(\tau) > U^u_w(\tau)$ for any $\tau \in (s_0, t)$. By taking difference of the corresponding Riccati equations, for any $\tau \in (s_0, t)$, we have:

$$\begin{aligned}
(U_1 - U^u_w)'(\tau) &= -(U_1(\tau) - U^u_w(\tau))(U_1(\tau) + U^u_w(\tau)) + K(f_\tau v) - K(f_\tau w) \\
&\le -2\lambda^u(f_\tau w)(U_1 - U^u_w)(\tau) + |K(f_\tau v) - K(f_\tau w)|.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{d}{d\tau}\left((U_1(\tau)-U_v^u(\tau))\exp\left(\int_{s_0}^{\tau}2\lambda^u(f_s w)ds\right)\right)\\
=&\ \exp\left(\int_{s_0}^{\tau}2\lambda^u(f_s w)ds\right)((U_1-U_w^u)'(\tau)+2\lambda^u(f_\tau w)(U_1-U_w^u)(\tau))\\
\leq&\ \exp\left(\int_{s_0}^{\tau}2\lambda^u(f_s w)ds\right)|K(f_\tau v)-K(f_\tau w)|.
\end{aligned}$$

Together with Lemma 4.8, we have

$$\begin{aligned}
U_1(t)-U_v^u(t) &\leq \exp\left(-\int_{s_0}^{t}2\lambda^u(f_s w)ds\right)\int_{s_0}^{t}\exp\left(\int_{s_0}^{\tau}2\lambda^u(f_s w)ds\right)|K(f_\tau v)-K(f_\tau w)|d\tau\\
&= \int_{s_0}^{t}\exp\left(-\int_{\tau}^{t}2\lambda^u(f_s w)ds\right)|K(f_\tau v)-K(f_\tau w)|d\tau\\
&\leq \int_{s_0}^{t}\exp\left(-\frac{1}{T}\int_{\tau}^{t}\lambda_T^u(f_s w)ds\right)|K(f_\tau v)-K(f_\tau w)|d\tau\\
&\leq \int_{0}^{t}\exp\left(-\frac{\eta(t-\tau)}{4T}\right)|K(f_\tau v)-K(f_\tau w)|d\tau.
\end{aligned}$$

□

Theorem 6.6. *The geometric potential φ^u has the Bowen property with respect to $\mathcal{G}_T(\eta)$ for any $\eta > 0$.*

Proof. It follows from Lemma 6.4, 6.5. □

7. Pressure gap and the proof of Theorem A

Need to define

- $\mathcal{M}(\mathcal{C})$

Proposition 7.1. [BCFT17, Proposition 5.1] *Suppose φ is a continuous function, then*

$$P(\mathcal{C},\varphi) \leq \sup_{\mu\in\mathcal{M}(\mathcal{C})} P_\mu(\varphi)$$

where $P_\mu(\varphi) = h_\mu + \int \varphi d\mu$.

The following lemma establishes that the pressure of the obstruction to expansivity is strictly less than the entire pressure. It is a direct consequence of the flat strip theorem. Since the flat strip theorem is available in the no focal points setting (see ???), the proof in [BCFT17] holds here.

Proposition 7.2. [BCFT17, Proposition 5.4] *For a continuous potential φ, $P^{\perp}_{\exp}(\varphi) \leq P(\mathrm{Sing},\varphi)$.*

Proposition 7.3. *There exists T_0, $\eta_0 > 0$ such that*

$$P([B_{T_0}(\eta_0)]) < P(\varphi).$$

Proof. Let D be the metric compatible with the weak-$*$ topology on the space of $\mathcal{F}$-invariant probability measures $\mathcal{M}(T^1S)$. Fix $\delta < P(\varphi) - P(\mathrm{Sing},\varphi)$ and choose $\varepsilon > 0$ such that

$$\mu \in \mathcal{M}(T^1S) \text{ with } D(\mu,\mathcal{M}(\mathrm{Sing})) < \varepsilon \implies P_\mu(\varphi) - P(\mathrm{Sing}) < \delta.$$

The existence of such ε is guaranteed because the entropy map $\mathcal{M}(T^1S) \ni \mu \mapsto h_\mu(f)$ is upper semi-continuous $\mathcal{F} : T^1M \to T^1M$ is h-expansive (see Liu-Wang [LW16]). From Lemma 4.7, we have

$$\mathcal{M}(\mathrm{Sing}) = \bigcap_{\eta>0,T>0} \mathcal{M}_{\lambda_T}(\eta),$$

where $\mathcal{M}_{\lambda_T}(\eta) = \{\mu \in \mathcal{M}(T^1S)\colon \int \lambda_T d\mu \le \eta\}$. Hence, we can find $T_0, \eta_0 > 0$ such that

$$D(\mathcal{M}(\mathrm{Sing}), \mathcal{M}_{\lambda_{T_0}}(\eta_0)) < \varepsilon.$$

In particular, for any $\mu \in \mathcal{M}_{\lambda_T}(\eta_0)$, we have

$$P_\mu(\varphi) < P(\mathrm{Sing}, \varphi) + \delta.$$

We can verify that for such choice of η_0 and T_0, the pressure gap $P([\mathcal{B}_{T_0}(\eta_0)], \varphi) < P(\varphi)$ holds:

$$P([\mathcal{B}_{T_0}(\eta_0)], \varphi) \le \sup_{\mu \in \mathcal{M}([\mathcal{B}_{T_0}(\eta_0)])} P_\mu(\eta_0) \le \sup_{\mu \in \mathcal{M}_{\lambda_{T_0}}(\eta_0)} P_\mu(\varphi) \le \delta + P(\mathrm{Sing}, \varphi) < P(\varphi).$$

□

Theorem (Theorem A). *Let M be a rank 1 surface without focal points and $\mathcal{F}$ be the geodesic flow over M. Let $\varphi : T^1M \to \mathbb{R}$ be a Hölder continuous potential or $\varphi = q \cdot \varphi^u$ for some $q \in \mathbb{R}$. Suppose φ verifies the pressure gap property $P(\mathrm{Sing}, \varphi) < P(\varphi)$, then φ has a unique equilibrium state μ_φ.*

Proof. This follows from Theorem 2.1 (Climenhaga-Thompson's criteria for uniqueness of equilibrium states).

We first notice that by Proposition 7.2, φ satisfies the first assumption in 2.1. We take the decomposition $(\mathcal{P}_T(\eta), \mathcal{G}_T(\eta), \mathcal{S}_T(\eta))$, then by Proposition 5.2, Theorem 6.3, and Theorem 6.6, conditions (I) and (II) of Theorem 2.1 are verified.

Lastly, by Proposition 7.3, we know there exists $(T, \eta) = (T_0, \eta_0)$ such that bad orbits has strictly less pressure than φ, that is, $P([\mathcal{B}_{T_0}(\eta_0)]) < P(\varphi)$, which verifies the condition (III) of Theorem 2.1. □

8. Properties of the equilibrium states and the proof of Theorem B

Theorem (Theorem B). *Let $\varphi : T^1M \to \mathbb{R}$ be a Hölder continuous function or $\varphi = q\cdot\varphi^u$ satisfying $P(\mathrm{Sing}, \varphi) < P(\varphi)$. Then the equilibrium state μ_φ is fully supported, $\mu_\varphi(\mathrm{Reg}) = 1$, Bernoulli, and is the weak* limit of the weighted regular periodic orbits.*

Proof. The proof is separated into following propositions, namely, Proposition 8.1, 8.10, 8.4 and 8.13. □

Proposition 8.1. $\mu_\varphi(\mathrm{Reg}) = 1$.

Proof. Since μ_φ is the unique equilibrium state for φ, we have μ_φ is ergodic (cf. [CT16] Proposition 4.19). Because Sing is $\mathcal{F}-$invariant we have either $\mu_\varphi(\mathrm{Sing}) = 1$ or $\mu_\varphi(\mathrm{Sing}) = 0$. Suppose $\mu_\varphi(\mathrm{Sing}) = 1$, then

$$P(\mathrm{Sing}, \varphi) \ge h_{\mu_\varphi}(\mathcal{F}) + \int \varphi|_{\mathrm{Sing}}\, \mathrm{d}\mu_\varphi = P(\varphi),$$

which contradicts to the pressure gap condition. Thus $\mu_\varphi(\mathrm{Reg}) = 1$. □

Recall that $\mathcal{G}^M$ is the set of orbit segments whose bad parts are shorter than M.

Lemma 8.2 ([BCFT17], Lemma 6.1). *For M is large enough, $P(\mathcal{G}^M, \varphi) = P(\varphi)$. Moreover, the measure μ_ϕ has the lower Gibbs property on $\mathcal{G}^M$. More precisely, for any $\rho > 0$, there exists $Q, T, M > 0$ such that for every $(v, t) \in \mathcal{G}^M$ with $t \ge T$,*

$$\mu_\varphi(B_t(v, \rho)) \ge Qe^{-tP(\varphi) + \int_0^T \varphi(f_s v)\mathrm{d}s}.$$

Therefore, for $(v, t) \in \mathcal{G}_T$ and t is large we have $\mu_\varphi(B(v, \varphi)) > 0$.

Proof. The proof follows verbatim the proof of [BCFT17] Lemma 6.1. □

Lemma 8.3 ([BCFT17], Lemma 6.2). *Given $\rho, \eta > 0$, there exists $\eta_0 > 0$ so that for any $v \in \mathrm{Reg}_T(\eta)$ and all $t > 0$, there are $s \ge t$ and $w \in B(v, \rho)$ such that $(w, s) \in \mathcal{G}_T(\eta_0)$*

Proof. The proof almost follows verbatim the proof of [BCFT17] Lemma 6.2. One only needs to replace the [BCFT17] Lemma 3.9 in their proof by Lemma 4.8. ☐

Proposition 8.4. *The unique equilibrium state μ_φ is fully supported.*

Proof. We follow the same idea as in [BCFT17, Proposition 6.3]. We prove that for Reg, a dense set in T^1M, $\forall v \in \mathrm{Reg}$ and $\forall r > 0$ we have $\mu_\varphi(B(v, r)) > 0$.

Since $v \in \mathrm{Reg}$, there exists $t_0 \in \mathbb{R}$ such that $\lambda(f_{t_0}v) > 0$. For convenience, let's denote $v' = f_{t_0}v$. By the continuous of λ, there exists a neighborhood $v' \in B(v', 2\rho)$ such that $\lambda|_{B(v',2\rho)} > \eta$ for some $\eta > 0$, and we have $v' \in \mathrm{Reg}_T(\eta)$. By Lemma 8.3, there exists $\eta_0 > 0$ such that there is $w \in B(v', \rho)$ satisfying $(w, t) \in \mathcal{G}_T(\eta_0)$ for t big (depending on ρ, η).

Furthermore, the decomposition $(\mathcal{B}_T(\eta_0), \mathcal{G}_T(\eta_0), S_T(\eta_0))$ verifies Theorem 2.1, thus by Lemma 8.2 we know μ_φ satisfying the lower Gibbs property, i.e.,

$$\mu_\varphi(B(w', \rho)) > 0.$$

Last, by picking ρ small enough so that $f_{-t_0}B(v'\rho) \subset B(v, r)$, and because μ_φ is flow invariant, we have

$$\mu_\varphi(B(v, r)) \geq \mu_\varphi(B(v', 2\rho)) \geq \mu_\varphi(B(w, \rho)) > 0.$$

☐

Recall that

- Per the set of primitive periodic geometrically distinct geodesics
- $\mathrm{Per}(T)$ the set of primitive periodic geometrically distinct geodesics of length smaller than T
- $\mathrm{Per}(T-1, T)$ the set of primitive periodic geometrically distinct geodesics of length in $[T-1, T)$
- $\mathrm{Per}(T, \mathrm{Reg})$ is the subset of Per(T) containing all regular geodesics of $\mathrm{Per}(T)$
- $\mathrm{Per}(T-1, T, \mathrm{Reg})$ is the subset of $\mathrm{Per}(T-1, T)$ containing all regular geodesics of $\mathrm{Per}(T-1, T)$
- $P_{Gur}(\varphi) := \limsup_{T\to\infty} \frac{1}{T} \log \sum_{\tau \in \mathrm{Per}(T-1,T)} e^{\int_\tau \varphi}$
- $P_{\mathrm{Reg}}(\varphi) := \limsup_{T\to\infty} \frac{1}{T} \log \sum_{\tau \in \mathrm{Per}(T-1,T,\mathrm{Reg})} e^{\int_\tau \varphi}$

Lemma 8.5. *Notice that when $P_{Gur}(\varphi), P_{\mathrm{Reg}}(\varphi) \geq 0$ we have*

$$P^*_{Gur}(\varphi) := \limsup_{T\to\infty} \frac{1}{T} \log \sum_{\tau \in \mathrm{Per}(T)} e^{\int_\tau \varphi} = P_{Gur}(\varphi),$$

$$P^*_{\mathrm{Reg}}(\varphi) := \limsup_{T\to\infty} \frac{1}{T} \log \sum_{\tau \in \mathrm{Per}(T,\mathrm{Reg})} e^{\int_\tau \varphi} = P_{\mathrm{Reg}}(\varphi).$$

Remark 8.6. For any constant c, we have for $\mathcal{P}(\cdot) = P_{Gur}(\cdot), P_{\mathrm{Reg}}(\cdot), P(\cdot), P(\mathcal{G}^M, \cdot)$ we have $\mathcal{P}(\varphi + c) = \mathcal{P}(\varphi) + c$. However, for P^*_{Reg} and P^*_{Gur} we don't necessary have this equality.

In the case of nonpositive curved rank one Riemannian manifolds, Pollicott [Pol96, Lemma 4, Proposition 2] (also Gelfert-Schapira [GS14, Theorem 1.1]) proved that

$$P_{\mathrm{Reg}}(\varphi) \leq P_{Gur}(\varphi) \leq P(\varphi).$$

In Pollicott's proof, the nonpositive curved condition only used to guarantee the following two geometric properties: Let M be a Riemannian manifold

(1) For any two (unit speed) geodesics $\gamma_1,\gamma_2 : [0, L] \to M$ satisfy $d(\gamma_1(t), \gamma_2(t)) \leq d(\gamma_1(0), \gamma_2(0))+ d(\gamma_1(L), \gamma_2(L))$.
(2) For all $\eta > 0$ there exist $\rho > 0$ and $L_0 > 0$ such that given two geodesics $\gamma_1,\gamma_2 : [0, L] \to M$ with $L \geq L_0$ and $d_K(\dot\gamma_1(0), \dot\gamma_2(0))$, $d_K(\dot\gamma_1(0), \dot\gamma_2(0)) \leq \rho$ we have that $d_K(\dot\gamma_1(t), \dot\gamma_2(t)) \leq \eta$ for all $0 \leq t \leq L$.

Lemma 8.7. *Rank one Riemannian manifolds without focal points also satisfy the above two geometric conditions.*

Proof. The proof follows the same idea as [Pol96]. Let M be a Riemannian manifolds without focal points and $\gamma_1, \gamma_2 : [0, L] \to M$ be two geodesics. Since M has no focal points, there exists a unique geodesic $\gamma : [0, L] \to M$ such that $\gamma(0) = \gamma_1(0)$ and $\gamma(L) = \gamma_2(L)$ (by parametrizing it with an appropriate constant multiple of its arc length). Then the no focal point condition (consider the Jacobi field J along γ_1 given by the variation from γ_1 to γ) implies $d(\gamma(t), \gamma_1(t)) \leq d(\gamma_2(L), \gamma_1(L))$, and similarly, $d(g(t), g_2(t)) \leq d(\gamma_1(0), \gamma_2(0))$. Thus we have

$$d(\gamma_1(t), \gamma_2(t)) \leq d(\gamma(t), \gamma_1(t)) + d(g(t), g_1(t)) \leq d(\gamma_1(0), \gamma_2(0)) + d(\gamma_1(L), \gamma_2(L)).$$

One can prove the second condition using the same argument. □

Corollary 8.8. *For a compact rank one Riemannian manifold M without focal points, we have*

$$P_{\mathrm{Reg}}(\varphi) \leq P_{Gur}(\varphi) \leq P(\varphi)$$

where $\varphi : M \to \mathbb{R}$ is a continuous function.

Proof. By Lemma 8.7, we can follow the proof of Lemma 4 and Proposition 2 in [Pol96] without modification. □

Proposition 8.9. *We have*

$$P_{\mathrm{Reg}}(\varphi) = P(\varphi).$$

Proof. Since $P_{\mathrm{Reg}}(\varphi + c) = P_{\mathrm{Reg}}(\varphi) + c$ and $P(\varphi + c) = P(\varphi) + c$, w.o.l.g. we may assume $P_{\mathrm{Reg}}(\varphi), P(\varphi) > 0$.

With the assumption $P_{\mathrm{Reg}}(\varphi) > 0$ we have $P_{\mathrm{Reg}}(\varphi) = P^*_{\mathrm{Reg}}(\varphi)$. So, by Lemma 8.2 and Corollary 8.8, it is enough to show

$$P(G^M, \varphi) \leq P^*_{\mathrm{Reg}}(\varphi).$$

The see this, notice that for $(v, t) \in \mathcal{G}^M$, there exists $p, s > 0$ such that $(f_p v, t - s - p) := (v', t')$ is a good orbit.

By the uniform continuity of the flow f_t, we know that for $\varepsilon > 0$ (choosing the same ε as in the Bowen property) there exists $\varepsilon' > 0$, w.l.o.g. we may assume $\varepsilon > \varepsilon'$, such that for $v, w \in T^1M$ and $d_K(v, w) \leq \varepsilon'$ we have $d_K(f_s v, f_s w) \leq \varepsilon$ for all $s \in [-M, M]$.

By Lemma 5.6 (i.e., the closing lemma), for $\varepsilon' > 0$ there exists a regular periodic vector $w \in B_{t'}(v', \varepsilon')$ of period $t' + \tau$ for some $\tau \in [0, s(\varepsilon')]$.

Thus there exists $t_0 \in [-M, M]$ such that $d_K(f_{t_0} w, v) \leq \varepsilon$, and moreover, $f_{t_0} w \in B_t(v, \varepsilon)$.

Hence, $\left|\int_0^t \varphi(f_s v)\mathrm{d}s - \int_w \varphi\right| \leq 2M \cdot ||\varphi||_\infty + K$ where K is the constant given by the Bowen property.

Notice that if $d_K(v_1, v_2) \leq \varepsilon$ and $(v_1, t_1), (v_2, t_2)$ in a $(s, \delta)-$separated set where $\delta > 3\varepsilon$, then the corresponding regular periodic vectors w_1 and w_2, respectively, are distinct (because $\exists t_1 \in [0, s]$ such that $d_K(f_{t_1} w_1, f_{t_1} w_2) > \varepsilon$).

In other words, for any $(t,\delta)-$separated set $E \subset \mathcal{G}^M_t := \{(v,t) \in \mathcal{G}^M\}$, we have $E \subset \mathrm{Per}(s + s(\varepsilon'), \mathrm{Reg})$. Thus

$$\sum_{(v,t)\in E} e^{\int_0^t \varphi(f_s v)\mathrm{d}s} \leq \sum_{w\in \mathrm{Per}(t+s(\varepsilon'),\mathrm{Reg})} e^{2M\cdot ||\varphi||_\infty + K + \int_w \varphi}$$

Thus we have

$$\limsup_{t\to\infty} \frac{1}{t} \log \Lambda(\mathcal{G}^M, \varphi, \delta, t) \leq P^*_{\mathrm{Reg}}(\varphi),$$

and, hence,

$$P(\mathcal{G}^M, \varphi) = \lim_{\delta\to 0} \limsup_{t\to\infty} \frac{1}{t} \log \Lambda(\mathcal{G}^M, \varphi, \delta, t) \leq P^*_{\mathrm{Reg}}(\varphi).$$

□

Proposition 8.10. *The unique equilibrium state μ_φ is the weak* limit of the weighted regular periodic orbits. More precisely,*

$$\mu_\varphi = \lim_{T\to\infty} \frac{\sum_{\gamma\in \mathrm{Per}(T-1,T,\ \mathrm{Reg})} e^{\int_\gamma \varphi} \delta_\gamma}{\sum_{\gamma\in \mathrm{Per}(T-1,T,\ \mathrm{Reg})} e^{\int_\gamma \varphi}}$$

Proof. As pointed out in [GS14, Remark 3], in [Wal82, Theorem 9.10] Walters proved that the weak* limit of the weighted regular periodic orbits on (t,δ)-separated set is an equilibrium state for φ with respect to $P(\varphi, \delta, t)$.

Notice that in [Kni98, p. 310] Knieper proved that $\mathrm{Per}(\mathrm{Reg}, t)$ is a $(t,\delta)-$separated set for δ is small enough; hence, so is $\mathrm{Per}(T-1, T,\ \mathrm{Reg})$. Although his setting was nonpositively curved manifolds, his proof works for the no focal points setting as well (because Lemma 8.7 implies each homotopy class only contains one geometrically distinct closed geodesic).

Moreover, by Proposition 8.9 we have $P_{\mathrm{Reg}}(\varphi) = P(\varphi)$, thus the weak* limit of the weighted regular periodic orbits is an equilibrium state for φ, which is μ_φ by uniqueness. □

Remark 8.11. One can replace $\mathrm{Per}(T-1, T, \mathrm{Reg})$ by $\mathrm{Per}(T-\delta, T, \mathrm{Reg})$ for any $\delta > 0$.

Recall

- Lyapunov exponent $\chi(v)$
- positive Lyapunov exponent $\chi^+(v)$

Lemma 8.12. *Let $\mu \in \mathcal{M}(\mathcal{F})$. If for $\mu-$a.e. $v \in T^1S$ such that $\chi^+(v) = 0$ then* $\mathrm{supp}(\mu) \subset \mathrm{Sing}$.

Proof. We first recall that for $\xi \in T_vT^1M$ we have $||J_\xi(t)||^2 \leq ||df_t\xi||^2 \leq (1+\Lambda^2)||J_\xi(t)||^2$. Let $\mu \in \mathcal{M}(\mathcal{F})$ and w.l.o.g. we may assume v is a Lyapunov regular vector, then for $\xi \in E^u_v$ and suppose $\chi^+(v) = 0$, then by Lemma 3.7 and the Birkhoff ergodic theorem

$$\begin{aligned} 0 = \chi^+(v) &= \lim_{t\to\infty} \frac{1}{t} \log ||df_t|_{E^u_v}|| \\ &\geq \lim_{t\to\infty} \frac{1}{t} \log ||J^u_\xi(t)|| \\ &\geq \lim_{t\to\infty} \frac{1}{t} \log \left(e^{\int_0^t \lambda^u(f_\tau v)d\tau} ||J^u_\xi(0)|| \right) \\ &= \lim_{t\to\infty} \frac{1}{t} \int_0^t \lambda^u(f_\tau v)d\tau = \int \lambda^u(v)d\mu \end{aligned}$$

Therefore, $\lambda^u(v) = 0$ for $\mu - a.e.\ v \in T^1S$; hence, $\lambda(v) = 0$ for $\mu - a.e.\ v \in T^1S$. By Lemma 4.7, we are done.

□

Proposition 8.13. *The unique equilibrium state μ_φ is Bernoulli.*

Proof. By Lemma 8.12, we know if $\mu(\text{Reg}) = 1$ then $\chi^+(\mu) := \int \chi^+(v)d\mu > 0$, i.e., μ is hyperbolic. Therefore, by Proposition 8.1, we know that μ_φ is a hyperbolic measure, and by results in [LLS16], we have μ_φ is Bernoulli. One remark that in Ledrappier-Lima-Sarig [LLS16] the theorem requires $h_\mu(\mathcal{F}) > 0$; nevertheless, it has been improved in Lima-Sarig [LS17, Theorem 1.3] that one only needs to check $\chi^+(\mu) > 0$. □

9. The proof of Theorem C and examples

Lemma 9.1. *If M is a closed rank 1 surface without focal points, Then $P(q\varphi^u) > 0 = P(\text{Sing}, \varphi^u)$ for each $q \in (-\infty, 1)$; in particular $h_{\text{top}}(\text{Sing}) = 0$.*

Proof. It is a classical result proved in [Bur83, Theorem, p.6] that $\mu_L(\text{Reg}) > 0$. Thus by Lemma 8.12 we get

$$0 < \chi^+(\mu_L) = \int_{T^1M} \chi^+(v)d\mu_L = -\int_{T^1M} \varphi^u d\mu_L.$$

Moreover, by Pesin's entropy formula, we have

$$h_{\mu_L}(\mathcal{F}) = \int_{T^1M} \chi^+(v)d\mu_L.$$

Thus for $q \in (-\infty, 1)$

$$P(q\varphi^u) \geq h_{\mu_L}(\mathcal{F}) + \int q\varphi^u d\mu_L = (q-1)\int \varphi^u d\mu_L > 0.$$

Claim: $P(\text{Sing}, \varphi^u) = 0$
pf. for any $\mu \in \mathcal{M}(\text{Sing})$, $P_\mu(\varphi^u) := h_\mu(\mathcal{F}) + \int_{T^1M} \varphi^u d\mu = h_\mu(\mathcal{F}) + \int_{\text{Sing}} \varphi^u d\mu = h_\mu(\mathcal{F})$.
By Ruelle's inequality we have $h_\mu(\mathcal{F}) \leq \int \chi^+(v)d\mu = 0$ (because $\chi^+|_{\text{Sing}} = 0$). Therefore, $P(\text{Sing}, \varphi^u) = \sup_{\mu \in \mathcal{M}(\text{Sing})} P_\mu(\varphi^u) = 0$. □

Theorem 9.2 (Theorem C)**.** *If M is a closed rank 1 surface without focal points, then the geodesic flow has a unique equilibrium state μ_q for the potential $q\varphi^u$ fro each $q \in (-\infty, 1)$. This equilibrium state satisfies μ_q(Reg) $= 1$, is fully supported, Bernoulli, and is the weak $*$-limit of weighted regular periodic orbits. Moreover, the function $q \mapsto P(q\varphi^u)$ is C^1 for $q \in (-\infty, 1)$; and $P(q\varphi^u) = 0$ for $q \in [1, \infty)$ when* $\text{Sing} \neq \emptyset$.

Proof. By the above lemma, Theorem A, and Theorem B, it remains to show $q \mapsto P(q\varphi^u)$ is C^1 for $q \in (-\infty, 1)$ and $P(q\varphi^u) = 0$ for $q \geq 1$ when $\text{Sing} \neq \emptyset$. We first notice that when $\text{Sing} \neq \emptyset$, we have $P(q\varphi^u) \geq 0$ because pick any invariant measure μ such that $\text{supp}(\mu) \subset \text{Sing}$ we have

$$h_\mu(\mathcal{F}) + \int_{T^1M} \varphi^u d\mu = h_\mu(\mathcal{F}) + \int_{\text{Sing}} \varphi^u d\mu \geq 0.$$

Moreover, we know the positive Lyapunov exponent χ^+ is the Birkhoff average of $-\varphi^u$; thus together with Ruelle's inequality we have for any invariant measure $\nu \in \mathcal{M}(\mathcal{F})$:

$$h_\nu(\mathcal{F}) \leq \int_{T^1M} \chi^+(v)d\nu$$

and for $q \geq 1$

$$\begin{aligned} h_\nu(\mathcal{F}) + \int \varphi^u d\nu &= \underbrace{h_\nu(\mathcal{F}) - \int_{T^1M} \chi^+(v) d\nu}_{\leq 0} \\ &\geq h_\nu(\mathcal{F}) - q \int_{T^1M} \chi^+(v) d\nu \\ &= h_\nu(\mathcal{F}) + q \int_{T^1M} \varphi^u d\nu \end{aligned}$$

Therefore, we have for $q \geq 1$

$$P(q\varphi^u) = \sup\{h_\nu(\mathcal{F}) + q \int_{T^1M} \varphi^u d\nu : \ \nu \in \mathcal{M}(\mathcal{F})\} \leq 0;$$

hence we have $P(q\varphi^u) = 0$ for $q \geq 1$.

Lastly, Liu-Wang [LW16] proved that the geodesic flow is entropy expansive for manifolds without conjugates points. So by Walters [Wal92], we know that $q \mapsto P(q\varphi^u)$ is C^1 at where $q\varphi^u$ has a unique equilibrium state. In particular, we know $q \mapsto P(q\varphi^u)$ for $q < 1$. □

Corollary 9.3. [BCFT17, Lemma 9.1] *Let M be a closed rank 1 surface without focal points and $\varphi : T^1M \to \mathbb{R}$ continuous. If*

$$\sup_{v \in \mathrm{Sing}} \varphi(v) - \inf_{v \in T^1M} \varphi(v) < h_{\mathrm{top}}(\mathcal{F})$$

then $P(\mathrm{Sing}) < P(\varphi)$. In particular, constant functions have the pressure gap property.

Proof. It follows from the variational principle. More precisely,

$$\begin{aligned} &\sup_{v \in \mathrm{Sing}} \varphi(v) - \inf_{v \in T^1M} \varphi(v) < h_{\mathrm{top}}(\mathcal{F}) - \underbrace{h_{\mathrm{top}}(\mathrm{Sing})}_{=0} \\ \iff &\sup_{v \in \mathrm{Sing}} \varphi(v) + h_{\mathrm{top}}(\mathrm{Sing}) < h_{\mathrm{top}}(\mathcal{F}) + \inf_{v \in T^1M} \varphi(v) \end{aligned}$$

and

$$P(\mathrm{Sing}, \varphi) \leq h_{\mathrm{top}}(\mathrm{Sing}) + \sup_{v \in \mathrm{Sing}} \varphi(v) < h_{\mathrm{top}}(\mathcal{F}) + \inf_{v \in T^1M} \varphi(v) \leq P(\varphi).$$

□

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