Solving the Problem of the Vertical Movement of a String Using Separation of Variables

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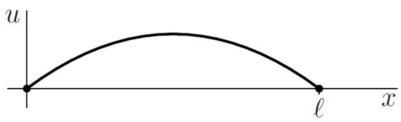
Using the method of separation of variables, we will solve the following problem:

$$\begin{aligned} u_{tt} &= c^2 u_{tt}, \\ u(0,t) &= 0 \\ u(\ell,t) &= 0 \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{aligned} \tag{1}$$

where $0 < x < \ell$, $t > 0$ and $\phi(0) = \phi(\ell) = \psi(0) = \psi(\ell) = 0$.

Preliminary Analysis

This problem describes the vertical movement of a string with length ℓ from the x axis at a position x and a time t. The ends of the string are held fix, where the left end of the string is $x = 0$ and the right end is $x = \ell$.



If the string undergoes small amplitude vibrations, then we can assume that $u(x,t)$ behaves under the wave equation:

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2}{\partial x^2}(x,t) \tag{2}$$

with $0 < x < \ell$ and $t > 0$.

The boundary conditions are given by

$$\begin{aligned} u(0,t) &= 0 \\ u(\ell,t) &= 0 \end{aligned} \tag{3}$$

meaning that the string is held at a height 0 in both its ends.

The initial conditions are given by

$$\begin{aligned} u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{aligned} \tag{4}$$

which refer to the position and the speed of the string at the time 0.

We want to find $u(t,x)$ for all x and t. To this end, we will use the method of separation of variables, following three steps:

1. Find all solutions of (2) that satisfy the form $u(x,t) = X(x)T(t)$, for some $X(x)$ that only depends on position and some $T(t)$ that only depends on time.
2. Evaluate the boundary conditions given in (3)
3. Evaluate the initial conditions given in (4)

Solving the equation

We will begin by rewriting the function $u(x,t) = X(x)T(t)$ as:

$$\begin{aligned} X(x)T''(t) &= c^2 X''(x)T(t) \\ \frac{X''(x)}{X(x)} &= \frac{1}{c^2}\frac{T''(t)}{T(t)} \end{aligned} \tag{5}$$

We notice that the right side of (5) is independent of X, whereas the left side is independent of t. Which means that both sides must be constant. We will call this constant ω:

$$\begin{aligned} \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} &= 0 \\ \Rightarrow X''(x) - \omega X(x) &= 0 \\ \Rightarrow T''(t) - c^2\omega T(t) &= 0 \end{aligned} \tag{6}$$

Now we have two ordinary differential equations, which we will proceed to solve:

1. $X''(x) - \omega X(x) = 0$

   Assuming $X(x) = e^{rx}$ for some unknown r

$$\begin{aligned} \frac{d^2}{dx^2}e^{rx} - \omega e^{rx} &= 0 \\ (r^2 - \omega)e^{rx} &= 0 \\ r^2 - \omega &= 0 \\ r &= \pm\sqrt{\omega} \end{aligned} \tag{7}$$

   Therefore, the general solution for X(x) is:

$$X(x) = p_1 e^{\sqrt{\omega}x} + p_2 e^{-\sqrt{\omega}x}$$

2. $T''(t) - c^2\omega T(t) = 0$

   Assuming $T(t) = e^{st}$ for some unknown s:

$$\begin{aligned} \frac{d^2}{dx^2}e^{st} - c^2\omega e^{st} &= 0 \\ (s^2 - c^2\omega)e^{st} &= 0 \\ s^2 - c^2\omega &= 0 \\ s &= \pm c\sqrt{\omega} \end{aligned} \tag{8}$$

   Therefore, the general solution for T(t) is:

$$T(t) = p_3 e^{c\sqrt{\omega}t} + p_4 e^{-c\sqrt{\omega}t}$$

In the previous solutions, p_1, p_2, p_3, p_4 are arbitrary constants and $\omega \neq 0$.

If $\omega = 0$, we get the trivial solutions where $X''(x) = 0$ and $T''(t) = 0$, plus the general solution:

$$\begin{aligned} X(x) &= p_1 + p_2 x \\ T(t) &= p_3 + p_4 t \end{aligned} \tag{9}$$

Putting both results back together, we get the set of solutions for equation (2):

$$\begin{aligned} u(x,t) &= (p_1 e^{\sqrt{\omega}x} + p_2 e^{-\sqrt{\omega}x})(p_3 e^{c\sqrt{\omega}t} + p_4 e^{-c\sqrt{\omega}t}),\ \omega \neq 0 \\ u(x,t) &= (p_1 + p_2 x)(p_3 + p_4 t) \end{aligned} \tag{10}$$

Evaluating the Boundary Conditions

Recall the boundary conditions defined in (3):

$$\begin{aligned} u(0,t) &= 0 \\ u(\ell,t) &= 0 \end{aligned}$$

Since $X_i(x)T_i(t)$, $i = 1, 2, 3, ..., n$ all solve the equation (2), it follows that $\sum_i a_i X_i(x) T_i(t)$ also will lead to solutions for any constant a. Therefore,

$$\sum_i a_i X_i(0) T_i(t) = 0 \qquad \text{for all } t > 0 \tag{11}$$

satisfies the boundary condition $u(0,t) = 0$.

Likewise,

$$\sum_i a_i X_i(\ell) T_i(t) = 0 \qquad \text{for all } t > 0 \tag{12}$$

satisfies the boundary condition $u(\ell,t) = 0$.

We are now interested in knowing which of the solutions that we found in (10) satisfy $X(0) = X(\ell) = 0$

Let's consider $\omega = 0$, so that $X(x) = p_1 + p_2 x$. The condition $X(0) = X(\ell) = 0$ is only satisfied if and only if $p_1 = p_2 = 0$. But this trivial solution does not interest us, so we are going to discard $\omega = 0$.

Now, consider $\omega \neq 0$, so that $p_1 e^{\sqrt{\omega}x} + p_2 e^{-\sqrt{\omega}x}$. Here we see that the condition $X(0) = 0$ can only be satisfied if $p_1 + p_2 = 0$, which means that $p_2 = -p_1$.

Similarly, the condition $X(\ell) = 0$ is satisfied if and only if

$$0 = p_1 e^{\sqrt{\omega}\ell} + p_2 e^{-\sqrt{\omega}\ell} = p_1 (e^{\sqrt{\omega}\ell} - e^{-\sqrt{\omega}\ell}) \tag{13}$$

Once again, we want to discard any trivial solutions: for example, when $p_1 = 0$. Therefore, we will only settle for values of ω that satisfy

$$\begin{aligned} e^{\sqrt{\omega}\ell} - e^{-\sqrt{\omega}\ell} = 0 &\iff e^{\sqrt{\omega}\ell} = e^{-\sqrt{\omega}\ell} \\ &\iff e^{2\sqrt{\omega}\ell} = 1 \end{aligned} \tag{14}$$

For this reason, we discard $\omega = 0$. So $e^{2\sqrt{\omega}\ell} = 1$ if there exists and integer k such that

$$\begin{aligned} 2\sqrt{\omega}l = 2p\pi i &\iff \sqrt{\omega} = k\frac{\pi}{\ell} i \\ &\iff \omega = -k^2 \frac{\pi^2}{\ell^2} \end{aligned} \tag{15}$$

Taking $\sqrt{\omega} = k\frac{\pi}{\ell}i$ and $p_2 = -p_1$, we get

$$\begin{aligned} X(x)T(t) &= p_1\left(e^{i\frac{k\pi}{\ell}x} - e^{-i\frac{k\pi}{\ell}x}\right)\left(p_3 e^{i\frac{ck\pi}{\ell}t} - p_4 e^{-i\frac{ck\pi}{\ell}t}\right) \\ &= 2ip_1 \sin\left(\frac{k\pi}{\ell}x\right)\left[(p_3+p_4)\cos\left(\frac{ck\pi}{\ell}t\right) + i(p_3-p_4)\sin\frac{ck\pi}{\ell}t\right] \\ &= \sin\left(\frac{k\pi}{\ell}x\right)\left[\alpha_k\cos\left(\frac{ck\pi}{\ell}t\right) + \beta_k\sin\left(\frac{ck\pi}{\ell}t\right)\right] \end{aligned}$$

where $\alpha_k = 2ip_1(p_3+p_4)$ and $\beta_k = -p_1(p_3-p_4)$, with $p_1, p_3, p_4, \alpha_k, \beta_k \in \mathbb{C}$

Imposing the Initial Conditions

Recall the initial conditions defined in (4):

$$\begin{aligned} u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{aligned}$$

We have seen that the wave equation (2) and the given boundary conditions (3) successfully adapt to

$$u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{\ell}x\right)\left[\alpha_k\cos\left(\frac{ck\pi}{\ell}t\right) + \beta_k\sin\left(\frac{ck\pi}{\ell}t\right)\right] \tag{16}$$

where α_k, β_k are arbitrary constants.

Now, we need to find values for α_k, β_k that satisfy the given initial conditions, such that

$$\phi(x) = u(x,0) = \sum_{k=1}^{\infty} \alpha_k \sin\left(\frac{k\pi}{\ell}x\right) \tag{17}$$

$$\psi(x) = u_t(x,0) = \sum_{k=1}^{\infty} \beta_k \frac{ck\pi}{\ell} \sin\left(\frac{k\pi}{\ell}x\right) \tag{18}$$

To achieve this, first note that any function defined on the interval $0 < x < \ell$ fits the following unique representation as a linear combination of $\sin\frac{k\pi x}{\ell}$:

$$\xi(x) = \sum_{k=1}^{\infty} b_k \sin\frac{k\pi x}{\ell} \tag{19}$$

Additionally, we know the formula for the coefficients

$$b_k = \frac{2}{\ell}\int_0^{\ell} \xi(x) \sin\frac{k\pi x}{\ell} dx$$

So, we can combine (17) and (19) by considering $\phi(x) = \xi(x)$ and $b_k = \alpha_k$:

$$\alpha_k = \frac{2}{\ell}\int_0^{\ell} \phi(x) \sin\frac{k\pi x}{\ell} dx \tag{20}$$

Similarly, we can make (18) and (19) match by considering $\psi(x) = \xi(x)$ and $b_k = \beta_k\frac{ck\pi}{\ell}$:

$$\beta_k = \frac{2}{ck\pi}\int_0^{\ell} \psi(x) \sin\frac{k\pi x}{\ell} dx \tag{21}$$

With these values of α_k (20) and β_k (21) associated to the sum in the equation (16)

$$u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{\ell}x\right)\left[\alpha_k \cos\left(\frac{ck\pi}{\ell}t\right) + \beta_k \sin\left(\frac{ck\pi}{\ell}t\right)\right]$$

we have reached a final solution.