A Miscellany of Observations regarding Cardinal Characteristics of the Continuum

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Definition

Given an ideal $\mathcal{I}$ on a set X,

$\mathrm{add}(\mathcal{I})$ is the minimal cardinality of a subfamily of $\mathcal{I}$ whose union is not in $\mathcal{I}$,

$\mathrm{cov}(\mathcal{I})$ is the minimal cardinality of a subfamily of $\mathcal{I}$ whose union is X,

$\mathrm{non}(\mathcal{I})$ is the minimal cardinality of a an element of $\mathcal{P}(X) \setminus \mathcal{I}$,

$\mathrm{cof}(\mathcal{I})$ is the minimal cardinality of a base of $\mathcal{I}$.

Notation

$\mathcal{M}$—the ideal of meagre sets of reals,

$\mathcal{N}$—the ideal of sets of reals having Lebesgue measure zero.

$[X]^k$ denotes the family of all subsets of X having exactly k elements. For functions $f, g \in {}^\omega\omega$ we write $f \leqslant^* g$ to say that $\{n < \omega | g(n) < f(n)\}$ is finite.

Definition (Rothberger (1939), Sierpiński (1939))

The *unbounding number* (sometimes called the *bounding number*) $\mathfrak{b}$ is the minimal cardinality of a family $\mathcal{F}$ of functions $f \in {}^\omega\omega$ such that there is no function $g \in {}^\omega\omega$ satisfying $f \leqslant^* g$ for all $f \in \mathcal{F}$.

Definition

A *tower* is a sequence $\langle X_\xi | \xi < \alpha \rangle$—$\alpha$ being an ordinal—of infinite sets of natural numbers such that $\gamma < \beta < \alpha$ implies that $X_\beta \setminus X_\gamma$ is finite. It is *extendable* if there is an infinite X with $X_\xi \setminus X$ finite for all $\xi < \alpha$, otherwise *unextendable*.

Definition (Rothberger (1939 & 1948))

The *tower number* is the minimal length of an unextendable tower.

We call a pair $\pi = (D, \langle \pi_n | n \in D \rangle)$ where D is an infinite set of natural numbers and $\pi_n : {}^n\omega \longrightarrow \omega$ for every $n \in D$, a *predictor*. We say that π *predicts* a sequence $s \in {}^\omega\omega$ if $\{n \in D | \pi_n(s \restriction n) \neq s(n)\}$ is finite, otherwise s *evades* π. A $\mathbb{Z}$-valued predictor is *linear* if all $\pi_n : \mathbb{Z} \to \mathbb{Q}$ are $\mathbb{Q}$-linear maps.

Definition (Blass(1994), Brendle & Shelah(1996))

The *evasion number* $\mathfrak{e}$ is the minimal cardinality of a family $\mathcal{E}$ of functions $f \in {}^\omega\omega$ such that every predictor is evaded by a member of $\mathcal{E}$. The *linear evasion number* $\mathfrak{e}_\ell$ is the minimal cardinality of a family $\mathcal{E}$ of functions $f \in {}^\omega\omega$ such that every $\mathbb{Z}$-valued linear predictor is evaded by a member of $\mathcal{E}$.

We also say that a set A of pairs of natural numbers is *unbounded* if $A \not\subset [\omega]^2 \setminus [\omega \setminus k]^2$ for every natural number k.

Definition

For infinite sets S and X of natural numbers and an unbounded $A \subset [\omega]^2$, we say

$$S \text{ } \textit{splits} \text{ } X \text{ if both } X \cap S \text{ and } X \setminus S \text{ are infinite,}$$

$$S \text{ } \textit{bisects} \text{ } X \text{ if } \lim_{n \nearrow \infty} \frac{|S \cap X \cap n|}{|X \cap n|} = \frac{1}{2},$$

$$S \text{ } \textit{pair-splits} \text{ } A \subset [\omega]^2 \text{ if } A \setminus ([S]^2 \cup [\omega \setminus S]^2) \text{ is infinite.}$$

Definition

A family $\mathcal{F}$ of infinite sets of natural numbers is called

splitting if for every infinite set X of natural numbers there is a member of $\mathcal{F}$ splitting X,

pair-splitting if for every unbounded $A \subset [\omega]^2$ there is a member of $\mathcal{F}$ pair-splitting it.

unbisected if there is no set S of natural numbers bisecting every member of $\mathcal{F}$.

Definition (Booth (1974))

The *splitting number* $\mathfrak{s}$ is the minimal cardinality of a splitting family.

Definition (Minami (2010))

The *pair-splitting number* $\mathfrak{s}_{pair}$ is the minimal cardinality of a pair-splitting family.

Definition (Brendle, Halbeisen, Klausner, Lischka & Shelah (2018))

The *unbisecting* (or *semirefining* (or *semireaping*)) number $\mathfrak{r}_{1/2}$ is the minimal cardinality of an unbisected family.

Definition (Brendle, Brian & Hamkins (2018))

The *subseries number* ß is the minimal cardinality of a family $\mathcal{I}$ of sets of natural numbers such that for every conditionally convergent series $\sum_{i\nearrow\infty} r_i$ there is an index set $I \in \mathcal{I}$ such that $\sum_{j\in I} r_j$ fails to converge to a real number.

Definition (Brendle, Brian & Hamkins (2018))

The *oscillating subseries number* ß_o is the minimal cardinality of a family $\mathcal{I}$ of sets of natural numbers such that for every conditionally convergent series $\sum_{i\nearrow\infty} r_i$ there is an index set $I \in \mathcal{I}$ such that $\sum_{j\in I} r_j$ neither converges to a real number nor tends to either $-\infty$ or $+\infty$.

Theorem (Truss(1977), Miller(1981))

$\mathrm{add}(\mathcal{M}) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M}))$.

Theorem (Blass (1994))

$\mathrm{add}(\mathcal{N}) \leqslant \mathfrak{e}$.

Theorem (Brendle & Shelah (1996))

$\mathfrak{e}_\ell = \min(\mathfrak{b}, \mathfrak{e})$.

Theorem (Kada (1998))

$\mathfrak{e} \leqslant \mathrm{cov}(\mathcal{M})$.

Theorem (Brendle, Brian, Hamkins (2018))

$\mathrm{cov}(\mathcal{N}) \leqslant \mathbf{ß} \leqslant \mathbf{ß}_o \leqslant \mathrm{non}(\mathcal{M})$.

Theorem (Brendle, Halbeisen, Klausner, Lischka, Shelah (2018))

$\mathrm{cov}(\mathcal{N}) \leqslant \mathfrak{r}_{1/2} \leqslant \mathrm{non}(\mathcal{M}), \mathfrak{r}$.

Theorem

$\mathfrak{e} \leqslant \mathfrak{s}_{pair}$.

The following proof uses an interval partition with intervals of sufficiently quickly growing lengths such that, regardless of the behaviour of a set S of integers on previous intervals, a predictor will still narrow down the possibilities of that set on the next interval sufficiently to point to two elements in the current interval which are either both in S or both outside of S.

Proof. Assume towards a contradiction that $\mathfrak{s}_{pair} < \mathfrak{e}$ and let $\mathcal{S}$ be a pair-splitting family of cardinality $\mathfrak{s}_{pair}$. We inductively define a function $f : \omega \longrightarrow \omega$ by $f(0) := 3$ and $f(i+1) := 2f(i)\left(1 + 2^{2^{f(i)}}\right)$ and use it to define an interval partition $\langle J_i | i < \omega\rangle$ by setting $J_i := f(i) \setminus \bigcup_{k<i} J_k$ for all natural numbers i. Note the all intervals in this partition have odd length. Let $\langle F_i | i < \omega\rangle$ be an enumeration of all finite sets of natural numbers. We define a family

$$\mathcal{E} := \{g_S | S \in \mathcal{S}\}\text{, where}$$
$$g_S := \langle m_k^S | k < \omega\rangle \text{ for } S \in \mathcal{S}\text{, and}$$
$$m_k^S \text{ is such that } F_{m_k^S} = S \cap J_k.$$

As $|\mathcal{E}| \leqslant |\mathcal{S}| = \mathfrak{s}_{pair} < \mathfrak{e}$, there is a predictor $\pi = (B, \langle \pi_i | i \in B\rangle)$ predicting every member of $\mathcal{E}$.

Claim

For every natural number k there is a pair $a_k = \{b_{2k}, b_{2k+1}\} \subset J_k$ such that for every $x \in \mathcal{E}$, we have $a_k \cap F_{\pi_k(x \restriction k)} \in \{0, a_k\}$.

Proof of Claim. For any natural number k there are $2^{f(k)}$ subsets of $f(k)$ and hence just as many initial segments $x \restriction k$ for $x \in \mathcal{E}$. Let $\langle t_i | i < 2^{f(k)} \rangle$ be an enumeration of these initial segments. Set $K_{2^{f(k)}} := J_k$ and for every $i \leqslant 2^{f(k)}$, let K_i be the bigger one of the two sets $K_{i+1} \cap F_{\pi_k(t_i)}$ and $K_{i+1} \setminus F_{\pi_k(t_i)}$. Inductively we have $|K_i| \geqslant 2^i + 1$ for every $i \leqslant 2^{f(k)}$. Now we let $b_{2k} := \min(K_0)$ and $b_{2k+1} := \max(K_0)$. Then $a_k := \{b_{2k}, b_{2k+1}\}$ provides what was demanded. ©

Now we let $A := \{a_k | k < \omega\}$. The set A is an unbounded set of pairs, therefore there is an $S \in \mathcal{S}$ pair-splitting it. Let ℓ be a natural number such that $\pi_i(g_S \restriction i) = g_S(i)$ for all $i \in B \setminus \ell$ and let $a \in A \setminus [g(\ell)]^2$ be split by S. Let n be such that $a \subset J_n$. Then

$$\begin{aligned} \{0, a\} \ni a \cap F_{\pi_n(g_S \restriction n)} &= a \cap F_{g_S(n)} = a \cap F_{m_n^S} = a \cap J_n \cap S \\ &= a \cap S \notin \{0, a\}, \text{ a contradiction.} \quad \square \end{aligned}$$

Theorem

$\mathfrak{s}_{pair} \leqslant \text{ß}$.

The following proof depends on the observation that for any unbounded set A of pairs of natural numbers one can define an alternating harmonic series padded with zeros(a technique employed in Blass et al. [2020] and Brendle et al. [a]) such that any index set rendering the respective subseries divergent will also pair-split A.

Proof. Assume towards a contradiction that $\text{ß} < \mathfrak{s}_{pair}$ and let $\mathcal{I}$ be an ß-sized family of infinite index sets of natural numbers such that for every conditionally convergent series $\sum_{i \nearrow \infty} r_i$ there is an index set $I \in \mathcal{I}$ such that $\sum_{i \in I} r_i$ is diverging. As $|\mathcal{I}| = \text{ß} < \mathfrak{s}_{pair}$, there is an unbounded collection A of pairs of natural numbers which is not pair-split by any member of the family $\mathcal{I}$. We may suppose without loss of generality that $a_1 < a_2$ or $a_3 < a_0$ for $\{\{a_0, a_1\}_<, \{a_2, a_3\}_<\} \in [A]^2$. Let $\langle b_i | i < \omega \rangle$ be an enumeration of $\bigcup A$. Clearly, $A = \{\{b_{2i}, b_{2i+1}\} | i < \omega\}$. Let $n(i) := \min(\{j < \omega | b_j \geqslant i\})$ for all natural numbers i.

We now consider the alternating harmonic series

$$\sum_{i<\nearrow\infty} r_i,$$

$$\text{where } r_i := \begin{cases} \frac{(-1)^{n(i)}}{n(i)+1} & \text{if } i \in \bigcup A, \\ 0 & \text{else.} \end{cases}$$

which is known to be conditionally convergent. Now let $I \in \mathcal{I}$ be such that $\sum_{i \in I} r_i$ diverges. Let k be a natural number such that $A \subset [k]^2 \cup [\omega \setminus k]^2$ and $a \subset I$ or $a \subset \omega \setminus I$ for all $a \in A \setminus [k]^2$. As $\sum_{i \in I} r_i$ diverges, so does $\sum_{i \in I \setminus k} r_i$. But $\sum_{i \in I \setminus k} r_i = \sum_{i \in I \cap \bigcup A \setminus k} r_i$ which, being divergent, by the Leibniz criterion cannot be an alternating series. Then $a \not\subset I$ or $a \not\subset \omega \setminus I$ for some $a \in A \setminus [k]^2$, a contradiction. □

Theorem

$\min(\mathfrak{b}, \mathfrak{r}_{1/2}) \leqslant \text{ß}$.

The idea of the following proof is to use a bound for the distances between elements of the index set of a subseries and observe that a bisecting sequence implies convergence of the respective subseries.

Proof. Assume towards a contradiction that $\mathfrak{ß} < \min(\mathfrak{b}, \mathfrak{r}_{1/2})$. Let $\mathcal{I}$ be a $\mathfrak{ß}$-sized family of infinite index sets of natural numbers such that for every conditionally convergent series $\sum_{i \to \infty} r_i$ there is an index set $I \in \mathcal{I}$ such that $\sum_{i \in I} r_i$ is diverging. As $|\mathcal{I}| = \mathfrak{ß} < \mathfrak{r}_{1/2}$, there is an $S \in [\omega]^\omega$ bisecting all members of $\mathcal{I}$. We define a family

$$\mathcal{B} := \{f_I | I \in \mathcal{I}\}, \text{ where}$$

$$f_I := \langle n_j^I | j < \omega \rangle, \text{ and}$$

$$n_j^I := \min\left(\left\{k < \omega \middle| \forall \ell \in \omega \setminus k : \frac{|S \cap I \cap \ell|}{|I \cap \ell|} \in \left]\frac{1}{2} - \frac{1}{2j}, \frac{1}{2} + \frac{1}{2j}\right[\right\}\right).$$

As $|\mathcal{B}| \leqslant |\mathcal{I}| = \mathfrak{ß} < \mathfrak{b}$, there is a $g \in {}^\omega\omega$ such that $f \leqslant^* g$ for all $f \in \mathcal{B}$. We may suppose without loss of generality that $g(i)2(i+1) \leqslant g(i+1)$ for all natural numbers i.

We define a conditionally convergent series

$$\sum_{k\to\infty} t_k$$

$$\text{with } t_k = \begin{cases} \frac{(-1)^i}{g(i)(i+1)(2i+1)} & \text{for } k \in g(i)2(i+1) \setminus g(i), \\ 0 \text{ else.} \end{cases}$$

By assumption, there is an index set $I \in \mathcal{I}$ such that $\sum_{j\in I} t_j$ is diverging.

Let $k \in \omega \setminus 30$ be such that $f_I(\ell) \leqslant g(\ell)$ for all $\ell \in \omega \setminus k$. Since

$$\sum_{j\in I} t_j = \sum_{j\nearrow\infty} u_j \text{ with}$$

$$u_j := \sum_{\ell\in I\cap g(j)2(j+1)\setminus g(j)} t_\ell,$$

and $u_j < 0$ if and only if j is odd, the Leibniz criterion implies that there has to be a $j \in \omega \setminus k$ such that $|u_j| \leqslant |u_{j+1}|$.

But then

$$\sum_{\ell \in I \cap g(j)2(j+1) \setminus g(j)} |t_\ell| = |u_j| \leqslant |u_{j+1}| = \sum_{\ell \in I \cap g(j+1)2(j+2) \setminus g(j+1)} |t_\ell|,$$

$$\iff \sum_{\ell \in I \cap g(j)2(j+1) \setminus g(j)} \frac{1}{g(j)(j+1)(2j+1)}$$

$$\leqslant \sum_{\ell \in I \cap g(j+1)2(j+2) \setminus g(j+1)} \frac{1}{g(j+1)(j+2)(2j+3)}$$

$$\iff A := \frac{|I \cap g(j)2(j+1) \setminus g(j)|}{g(j)(j+1)(2j+1)} \leqslant \frac{|I \cap g(j+1)2(j+2) \setminus g(j+1)|}{g(j+1)(j+2)(2j+3)} =: B$$

Now the values of A and B can be estimated as follows

$$A \geqslant \frac{\left(1/2 - 1/2j\right) g(j)2(j+1) - \left(1/2 + 1/2j\right) g(j)}{g(j)(j+1)(2j+1)}$$

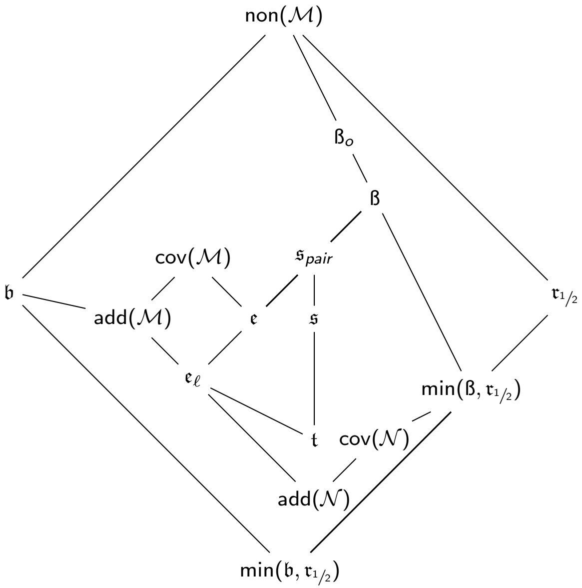
$$= \frac{(j-1)2(j+1) - (j+1)}{2j(j+1)(2j+1)} = \frac{2j^2 - j - 3}{j(4j^2 + 6j + 2)},$$

$$B \leqslant \frac{\left(1/2 + 1/2j\right) g(j+1)2(j+2) - \left(1/2 - 1/2j\right) g(j+1)}{g(j+1)(j+2)(2j+3)}$$

$$= \frac{2(j+1)(j+2) - (j-1)}{2j(j+2)(2j+3)} = \frac{2j^2+5j+5}{j(4j^2+14j+12)}.$$

Therefore $(2j^2 - j - 3)(4j^2 + 14j + 12) \leqslant (2j^2 + 5j + 5)(4j^2 + 6j + 2)$

$$\begin{aligned}
&\iff 8j^4 + 24j^3 - 2j^2 - 54j - 36 \leqslant 8j^4 + 22j^3 + 54j^2 + 40j + 10\\
&\iff 2j^3 - 56j^2 - 94j - 46 \leqslant 0\\
&\iff 27j^3 - 756j^2 - 1269j - 621 \leqslant 0\\
&\implies 27j^3 - 756j^2 - 1269j - 1958 \leqslant 0\\
&\iff (3j - 89)(9j^2 + 15j + 22) \leqslant 0\\
&\iff 9j^2 + 15j + 22 \leqslant 0\\
&\iff 36j^2 + 60j + 88 \leqslant 0\\
&\iff (6j + 5)^2 \leqslant -63, \text{ a contradiction.}
\end{aligned}$$


$\mathrm{non}(\mathcal{M})$
$ß_o$
$ß$
$\mathfrak{s}_{pair}$
$\mathrm{cov}(\mathcal{M})$
$\mathfrak{b}$
$\mathrm{add}(\mathcal{M})$
$\mathfrak{e}$
$\mathfrak{s}$
$\mathfrak{r}_{1/2}$
$\mathfrak{e}_\ell$
$\min(ß, \mathfrak{r}_{1/2})$
$\mathfrak{t}$
$\mathrm{cov}(\mathcal{N})$
$\mathrm{add}(\mathcal{N})$
$\min(\mathfrak{b}, \mathfrak{r}_{1/2})$

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