Lecture 3: Stochastic Differential Equations

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Strong solutions

- Let $B = \{B_t^j, t \geq 0, j = 1, \ldots, d\}$ be a d-dimensional Brownian motion and ξ an m-dimensional random vector independent of B.
- Let $\mathcal{F}_t$ be the σ-field generated by $\{B_s, 0 \leq s \leq t\}$, ξ and the null sets.
- Consider measurable coefficients $b_i(t,x)$ and $\sigma_{ij}(t,x)$, $1 \leq i \leq m$, $1 \leq j \leq d$ from $[0,\infty) \times \mathbb{R}^m$ to $\mathbb{R}$.
- Our aim to give a meaning to the *stochastic differential equation* on $\mathbb{R}^m$:

$$\boxed{dX_t^i = b_i(t, X_t)dt + \sum_{j=1}^{d} \sigma_{ij}(t, X_t)dB_t^j, \quad 1 \leq i \leq m} \tag{1}$$

with initial condition $X_0 = \xi$.

Definition

We say that an adapted and continuous process $X = \{X_t, t \geq 0\}$ is a solution to equation (1) if for all $t \geq 0$,

$$X_t = \xi + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \quad \text{a.s.}$$

or

$$X_t^i = \xi_i + \int_0^t b_i(s, X_s)ds + \sum_{j=1}^d \int_0^t \sigma_{ij}(s, X_s)dB_s^j, \quad 1 \leq i \leq m, \text{ a.s.}$$

- b is called the drift and σ is called the diffusion coefficient.

Theorem

Suppose that the coefficients are locally Lipschitz in the space variable, that is, for each $N \geq 1$, there exists $K_N > 0$ such that for each $\|x\|, \|y\| \leq N$ and $t \geq 0$

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq K_N \|x - y\|.$$

Then, two solutions with the same initial condition coincide almost surely (strong uniqueness holds).

- In the absence of the locally Lipschitz condition equation (1) might fail to be solvable or have multiple solutions.

  Example : $X_t = \int_0^t |X_s|^\alpha ds$, where $\alpha \in (0,1)$. Then $X_t = 0$ is a solution and also, for any $s \geq 0$,

  $$X_t = \left(\frac{t-s}{\beta}\right)^\beta \mathbf{1}_{[s,\infty)}(t), \quad \beta = 1/(1-\alpha),$$

  is also a solution.

Lemma (Gronwall lemma)

Let u be a nonnegative continuous function on $[0, \infty)$ *such that*

$$u(t) \leq \alpha(t) + \int_0^t \beta(s)u(s)ds, \quad t \geq 0$$

with $\beta \geq 0$ *and* α *non-decreasing. Then,*

$$u(t) \leq \alpha(t) \exp\left(\int_0^t \beta(s)ds\right), \quad t \geq 0.$$

- In particular, if α and β are constant, we get

$$\boxed{u(t) \leq \alpha e^{\beta t}.}$$

Proof :

- Let X and $\tilde{X}$ two solutions. Define

$$S_n = \inf\{t \geq 0 : \|X_t\| \geq n \quad \text{or} \quad \|\tilde{X}_t\| \geq n\}.$$

  Clearly S_n are stopping times such that $S_n \uparrow \infty$.

- Then,

$$\begin{aligned}
E\|X_{t\wedge S_n} - \tilde{X}_{t\wedge S_n}\|^2 &\leq 2E\left[\int_0^{t\wedge S_n} \|b(u, X_u) - b(u, \tilde{X}_u\| du\right]^2 \\
&\quad +2E\sum_{i=1}^{m}\left|\sum_{j=1}^{d}\int_0^{t\wedge S_n}(\sigma_{ij}(u, X_u) - \sigma_{ij}(u, \tilde{X}_u))dB_u^j\right|^2 \\
&\leq 2tE\int_0^{t\wedge S_n} \|b(u, X_u) - b(u, \tilde{X}_u\|^2 du \\
&\quad +2E\int_0^{t\wedge S_n} \|\sigma(u, X_u) - \sigma(u, \tilde{X}_u\|^2 du.
\end{aligned}$$

- Using the local Lipschitz property, we obtain for any $t \geq 0$,

$$E\|X_{t\wedge S_n} - \tilde{X}_{t\wedge S_n}\|^2 \leq 2(t+1)K_n^2 \int_0^t E\|X_{u\wedge S_n} - \tilde{X}_{u\wedge S_n}\|^2 du.$$

- Then $g(t) = E\|X_{t\wedge S_n} - \tilde{X}_{t\wedge S_n}\|^2$ satisfies

$$g(t) \leq 2(t+1)K_n^2 \int_0^t g(u)du,$$

which, by Gronwall's lemma, implies that $g = 0$.

- Letting $n \to \infty$ we conclude that $X_t = \tilde{X}_t$.

- A local Lipschitz condition is not sufficient to guarantee global existence of a solution.

  Example :

  $$X_t = 1 + \int_0^t X_s^2 ds.$$

  the solution is $X_t = \frac{1}{1-t}$, which explodes as $t \uparrow 1$.

- *Exercise :* Given $x \in \mathbb{R}^m$, we can find a strictly positive stopping time τ and a stochastic process $\{X_t, t < \tau\}$ such that

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \quad t < \tau.$$

The process $\{X_t, t < \tau\}$ is unique in the sense that if ρ is another strictly positive stopping time and $\{Y_t, t < \rho\}$ satisfies

$$Y_t = x + \int_0^t b(s, Y_s)ds + \int_0^t \sigma(s, Y_s)dB_s, \quad t < \rho,$$

then $\rho \leq \tau$ and for every $t \geq 0$, $Y_t \mathbf{1}_{\{t<\rho\}} = X_t \mathbf{1}_{\{t<\rho\}}$.

Theorem

Suppose that the coefficients b and σ satisfy the global Lipschitz and linear growth conditions :

$$\begin{aligned} \|b(t,x)-b(t,y)\| + \|\sigma(t,x)-\sigma(t,y)\| &\leq K(\|x-y\|, \\ \|b(t,x)\|^2 + \|\sigma(t,x)\|^2 &\leq K^2(1+\|x\|^2), \end{aligned}$$

for every $x, y \in \mathbb{R}^m$, $t \geq 0$. Suppose also that

$$E\|\xi\|^2 < \infty.$$

Then, there exist a unique solution such that for any $T > 0$

$$E\left(\sup_{0\leq t\leq T} \|X_t\|^2\right) \leq C_{T,K}(1+E\|\xi\|^2),$$

where $C_{T,K}$ depends on T and K.

Proof :

(i) Define the Picard iterations by putting $X_t^{(0)} = \xi$ and for $k \geq 0$,

$$X_t^{(k+1)} = \xi + \int_0^t b(s, X_s^{(k)})ds + \int_0^t \sigma(s, X_s^{(k)})dB_s.$$

It is easy to check that

$$E\left(\sup_{0\leq t\leq T} \|X_t^{(1)}\|^2\right) \leq C_{T,K}(1 + E\|\xi\|^2).$$

Then $X_t^{(k+1)} - X_t^{(k)} = V_t + M_t$, where

$$V_t = \int_0^t [b(s, X_s^{(k)}) - b(s, X_s^{(k-1)})]ds$$

and

$$M_t = \int_0^t [\sigma(s, X_s^{(k)}) - \sigma(s, X_s^{(k-1)})]dB_s.$$

(ii) By the maximal inequality for square integrable martingales,

$$
\begin{aligned}
E\left[\sup_{0\le t\le T}\|M_t\|^2\right] &\le 4E\int_0^T \|\sigma(s,X_s^{(k)})-\sigma(s,X_s^{(k-1)})\|^2 ds \\
&\le 4K^2\int_0^T E\|X_s^{(k)}-X_s^{(k-1)}\|^2 ds.
\end{aligned}
$$

On the other hand,

$$
E\left[\sup_{0\le t\le T}\|V_t\|^2\right]\le K^2T\int_0^T E\|X_s^{(k)}-X_s^{(k-1)}\|^2 ds,
$$

which leads to

$$
E\left[\sup_{0\le t\le T}\|X_t^{(k+1)}-X_t^{(k)}\|^2\right]\le L\int_0^T E\|X_s^{(k)}-X_s^{(k-1)}\|^2 ds,
$$

where $L = 2K^2(4+T)$.

(iii) By iteration,

$$E\left[\sup_{0\le t\le T}\|X_t^{(k+1)}-X_t^{(k)}\|^2\right]\le C^*\frac{(LT)^k}{k!},$$

where $C^*=E\left[\max_{0\le t\le T}\|X^{(1)}-\xi\|^2\right]<\infty$. Consider the Banach space $\mathcal{E}_T$ of continuous adapted processes $X=\{X_t,t\in[0,T]\}$ such that

$$\|X\|_{\mathcal{E}_T}:=\left(E\left(\sup_{0\le t\le T}\|X_t\|^2\right)\right)^{\frac{1}{2}}<\infty.$$

Then the sequence $X^{(k)}$ converges in $\mathcal{E}_T$ to a limit X which satisfies the equation. $\square$

- Under the assumptions of the theorem, if $E\|\xi\|^p < \infty$ for some $p \geq 2$, then the solution satisfies the following moments estimate,

$$E\left[\sup_{t\in[0,T]} \|X_t\|^p\right] \leq C_{T,K,p}(1 + E[\|\xi\|^p]),$$

where C depends on K, T and p.

The proof uses Burkholder-David-Gundy inequality.

Linear stochastic differential equations

- The geometric Brownian motion

$$X_t = \xi e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t}$$

solves the linear SDE

$$dX_t = \mu X_t dt + \sigma X_t dB_t.$$

- More generally, the solution of the homogeneous linear SDE

$$dX_t = \ b(t) X_t dt + \sigma(t) X_t dB_t,$$

where $b(t)$ and $\sigma(t)$ are continuous functions, is

$$\boxed{X_t = \xi \exp\left[\int_0^t \left(b(s) - \frac{1}{2}\sigma^2(s)\right) ds + \int_0^t \sigma(s) dB_s\right].}$$

Ornstein-Uhlenbeck process

- Consider the SDE (*Langevin equation*)

$$dX_t = a(\mu - X_t)\,dt + \sigma dB_t$$

with initial condition $X_0 = x$, where $a, \sigma > 0$ and μ is a real number.

- The process $Y_t = X_t e^{at}$, satisfies

$$dY_t = aX_t e^{at} dt + e^{at} dX_t = a\mu e^{at} dt + \sigma e^{at} dB_t.$$

Thus,

$$Y_t = x + \mu(e^{at} - 1) + \sigma \int_0^t e^{as} dB_s,$$

which implies

$$\boxed{X_t = \mu + (x - \mu)e^{-at} + \sigma e^{-at} \int_0^t e^{as} dB_s.}$$

- The process X_t is Gaussian with mean and covariance given by :

$$\begin{aligned}
E(X_t) &= \mu + (x-\mu)e^{-at}, \\
\mathrm{Cov}\,(X_t, X_s) &= \sigma^2 e^{-a(t+s)} E\left[\left(\int_0^t e^{ar} dB_r\right)\left(\int_0^s e^{ar} dB_r\right)\right] \\
&= \sigma^2 e^{-a(t+s)} \int_0^{t\wedge s} e^{2ar} dr \\
&= \frac{\sigma^2}{2a}\left(e^{-a|t-s|} - e^{-a(t+s)}\right).
\end{aligned}$$

- The law of X_t is the normal distribution

$$N\left(\mu + (x-\mu)e^{-at}, \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)\right)$$

and it converges, as t tends to infinity to the normal law

$$\nu = N(\mu, \frac{\sigma^2}{2a}).$$

This distribution is called invariant or *stationary.*

Exercise : Show that if $\mathcal{L}(\xi) = \nu$, then X_t has law ν, $\forall t \geq 0$.

General linear SDEs

- Consider the equation

$$dX_t = (a(t) + b(t)X_t)\,dt + (c(t) + d(t)X_t)\,dB_t,$$

with initial condition $\xi = x$, where a, b, c and d are continuous functions. The solution to this equation is given by

$$\boxed{X_t = U_t\left(x + \int_0^t [a(s) - c(s)d(s)]\,U_s^{-1}ds + \int_0^t c(s)\;U_s^{-1}dB_s\right),}$$

where

$$U_t = \exp\left(\int_0^t b(s)ds + \int_0^t d(s)dB_s - \frac{1}{2}\int_0^t d^2(s)ds\right).$$

Proof : Write $X_t = U_tV_t$, where $dV_t = \alpha(t)dt + \beta(t)dB_t$, and find α and β. $\square$

Stochastic flows

- Suppose that the coefficients are globally Lipschitz with linear growth.
- Denote by X_t^x the solution with initial condition $x \in \mathbb{R}^m$.

Proposition

Let $T > 0$. For every $p \geq 2$, there exists a constant $C_{p,T}$ such that for each $0 \leq s \leq t \leq T$ and $x, y \in \mathbb{R}^m$,

$$E\left(\|X_t^x - X_s^y\|^p\right) \leq C_{p,T}(\|x - y\|^p + |t - s|^{p/2}).$$

As a consequence, there exists a version $\{\widetilde{X}_t^x, t \geq 0, x \in \mathbb{R}^m\}$ of the process $\{X_t^x, t \geq 0, x \in \mathbb{R}^m\}$ which is continuous in $(t, x) \in [0, \infty) \times \mathbb{R}^m$.

- The continuous process of continuous maps $\Psi_t : x \to X_t^x$ is called the *stochastic flow* associated to equation (1).

- Denote by $\{X_s^{t,x}, s \geq t\}$ the solution to equation (1) starting at time t with initial condition x :

$$X_s^{t,x} = x + \int_t^s b(\theta, X_\theta^{t,x})d\theta + \int_t^s \sigma(\theta, X_\theta^{t,x})dB_\theta, \quad s \geq t.$$

- One can also snow that there is a version of the process $\{X_s^{t,x}, s \geq t \geq 0, x \in \mathbb{R}^m\}$ which is continuous in all its variables.

Proposition (Flow property)

If $s \geq t$,

$$X_s^{0,x} = X_s^{t, X_t^{0,x}}, \quad \text{a.s.}$$

Proof :

- Almost surely, for any $y \in \mathbb{R}^m$,

$$X_s^{t,y} = y + \int_t^s b(\theta, X_\theta^{t,y})d\theta + \int_t^s \sigma(\theta, X_\theta^{t,y})dB_\theta.$$

  Substituting y by $X_t^{0,x}$ yields

$$X_s^{t,X_t^{0,x}} = X_t^{0,x} + \int_t^s b(\theta, X_\theta^{t,X_t^{0,x}})d\theta + \int_t^s \sigma(\theta, X_\theta^{t,X_t^{0,x}})dB_\theta.$$

- On the other hand, $X_s^{0,x}$ is also a solution to this equation for $s \geq t$ because

$$X_s^{0,x} = X_t^{0,x} + \int_t^s b(\theta, X_\theta^{0,x})d\theta + \int_t^s \sigma(\theta, X_\theta^{0,x})dB_\theta.$$

  Then, the uniqueness of the solution implies the result. □

Markov property

Theorem

The solution X_t is a Markov process with respect to the Brownian filtration $\mathcal{F}_t$. Furthermore, for any $f \in C_b(\mathbb{R}^m)$ and $t \geq s$, we have

$$E[f(X_t)|\mathcal{F}_s] = (P_{s,t}f)(X_s),$$

where $P_{s,t}f(x) = E[f(X_t^{s,x})]$.

- If the coefficients are time independent, $P_{s,t}$ can be written as P_{t-s}, where $\{P_t, t \geq 0\}$ is the semigroup of operators with infinitesimal generator

$$L = \frac{1}{2} \sum_{i,k=1}^{m} a_{ik} \frac{\partial^2 f}{\partial x_i \partial x_k} + \sum_{i=1}^{m} b_i \frac{\partial f}{\partial x_i},$$

where $a_{ik} = \sum_{j=1}^{d} \sigma_{ij} \sigma_{kj}$.

Sketch of the proof :

(i) $X_t^{s,x}$ is a measurable function of x and the Brownian increments $\{B_{s+u} - B_s, u \geq 0\}$, that is

$$X_t^{s,x} = \Phi(x, B_{s+u} - B_s, u \geq 0).$$

(ii) This implies, by the flow property, that

$$X_t^{0,x} = \Phi(X_s^{0,x}, B_{s+u} - B_s, u \geq 0),$$

where $X_s^{0,x}$ is $\mathcal{F}_s$-measurable and $\{B_{s+u} - B_s, u \geq 0\}$ is independent of $\mathcal{F}_s$.

(iii) Therefore,

$$E[f(\Phi(X_s^{0,x}, B_{s+u} - B_s, u \geq 0))|\mathcal{F}_s] = E[f(\Phi(y, B_{s+u} - B_s, u \geq 0))]|_{y=X_s^{0,x}},$$

which yields the result. □

Numerical approximations

Euler's scheme :

- Fix $T > 0$ and set $t_i = \frac{iT}{n}, = 0, 1, \ldots, n$. The *Euler's method* consists in the recursive scheme :

$$X^{(n)}(t_i) = X^{(n)}(t_{i-1}) + b(t_{i-1}, X^{(n)}(t_{i-1}))\frac{T}{n} + \sigma(t_{i-1}, X^{(n)}(t_{i-1}))\Delta B_i,$$

  $i = 1, \ldots, n$, where $\Delta B_i = B_{t_i} - B_{t_{i-1}}$.

  The initial value is $X_0^{(n)} = x_0$.

- Inside the interval (t_{i-1}, t_i) the value of the process $X_t^{(n)}$ is given by linear interpolation, or by the equation

$$X_t^{(n)} = x_0 + \int_0^t b(\kappa_n(s), X_{\kappa_n(s)}) + \int_0^t \sigma(\kappa_n(s), X_{\kappa_n(s)})dB_s,$$

  where $\kappa_n(s) = t_{i-1}$ if $s \in [t_{i-1}, t_i)$.

Proposition

The error of the Euler's method is of order $n^{-\frac{1}{2}}$:

$$\sqrt{E\left[\left(X_T - X_T^{(n)}\right)^2\right]} \le C\sqrt{\frac{T}{n}}.$$

- In order to simulate a trajectory of the solution using Euler's method, it suffices to simulate the values of n independent random variables $\xi_1, \ldots, \xi_n$ with distribution $N(0,1)$, and replace ΔB_i by $\sqrt{\frac{T}{n}}\xi_i$.

Milstein scheme :

- Euler's method can be improved by adding a correction term. To simplify we assume $m = d = 1$ and that the coefficients are time independent. We can write

$$X(t_i) = X(t_{i-1}) + \int_{t_{i-1}}^{t_i} b(X_s)ds + \int_{t_{i-1}}^{t_i} \sigma(X_s)dB_s. \tag{2}$$

Euler's method is based on the approximations

$$\begin{aligned} \int_{t_{i-1}}^{t_i} b(X_s)ds &\approx b(X(t_{i-1}))\frac{T}{n}, \\ \int_{t_{i-1}}^{t_i} \sigma(X_s)dB_s &\approx \sigma(X(t_{i-1}))\Delta B_i. \end{aligned}$$

- Applying Itô's formula to the processes $b(X_s)$ and $\sigma(X_s)$, we obtain

$$\begin{aligned}
& X(t_i) - X(t_{i-1}) \\
= \; & \int_{t_{i-1}}^{t_i} \Bigg[b(X(t_{i-1})) + \int_{t_{i-1}}^{s} \left(bb' + \frac{1}{2} b'' \sigma^2 \right) (X_r) dr \\
& + \int_{t_{i-1}}^{s} (\sigma b') (X_r) dB_r \Bigg] ds \\
& + \int_{t_{i-1}}^{t_i} \Bigg[\sigma(X(t_{i-1})) + \int_{t_{i-1}}^{s} \left(b\sigma' + \frac{1}{2} \sigma'' \sigma^2 \right) (X_r) dr \\
& + \int_{t_{i-1}}^{s} (\sigma \sigma') (X_r) dB_r \Bigg] dB_s \\
= \; & b(X(t_{i-1})) \frac{T}{n} + \sigma(X(t_{i-1})) \Delta B_i + \int_{t_{i-1}}^{t_i} \left(\int_{t_{i-1}}^{s} (\sigma \sigma') (X_r) dB_r \right) dB_s + R_{i,n},
\end{aligned}$$

where the term $R_{i,n}$ is of lower order.

- This double stochastic integral can also be approximated by

$$(\sigma\sigma')\,(X(t_{i-1}))\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s} dB_r\right) dB_s.$$

The rules of Itô stochastic calculus lead to

$$\begin{aligned}
\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s} dB_r\right) dB_s &= \int_{t_{i-1}}^{t_i}\left(B_s - B_{t_{i-1}}\right) dB_s \\
&= \frac{1}{2}\left(B_{t_i}^2 - B_{t_{i-1}}^2\right) - B_{t_{i-1}}\left(B_{t_i} - B_{t_{i-1}}\right) - \frac{T}{2n} \\
&= \frac{1}{2}\left[(\Delta B_i)^2 - \frac{T}{n}\right].
\end{aligned}$$

- The Milstein's method consists in the recursive scheme :

$$
\begin{aligned}
X^{(n)}(t_i) &= X^{(n)}(t_{i-1}) + b(X^{(n)}(t_{i-1}))\frac{T}{n} + \sigma(X^{(n)}(t_{i-1}))\,\Delta B_i \\
&\quad + \frac{1}{2}(\sigma\sigma')(X^{(n)}(t_{i-1}))\left[(\Delta B_i)^2 - \frac{T}{n}\right].
\end{aligned}
$$

- One can show that the error is of order $\frac{T}{n}$, that is,

$$
\sqrt{E\left[\left(X_T - X_T^{(n)}\right)^2\right]} \le C\frac{T}{n}.
$$

Proposition (Yamada-Watanabe '71)

Consider the 1-dimensional SDE

$$dX_t = b(t, X_t) + \sigma(t, X_t) dB_t,$$

where the coefficients have linear growth and satisfy

$$\begin{aligned} |b(t,x) - b(t,y)| &\leq K|x-y| \\ |\sigma(t,x) - \sigma(t,y)| &\leq h(|x-y|), \end{aligned}$$

with $h : [0, \infty) \to [0, \infty)$ *is strictly increasing,* $h(0) = 0$ *and*

$$\int_0^\epsilon h^{-2}(x) dx = \infty, \quad \forall \epsilon > 0.$$

Then strong uniqueness holds.

- *Example :* $\sigma(x) = |x|^\alpha$ with $\alpha \geq \frac{1}{2}$ (Girsanov '62).

Proof in the case $b = 0$, $\sigma(x) = |x|^\alpha$, $\alpha \in (\frac{1}{2}, 1]$:

- Let X and $\tilde{X}$ be two solutions with the same initial condition. Then $Y = X - \tilde{X}$ satisfies

$$Y_t = \int_0^t \left[|X_s|^\alpha - |\tilde{X}_s|^\alpha \right] dB_s.$$

- Applying Ito's formula to $\psi_n(x)$ such that $\psi''(x) = n\mathbf{1}_{[-\frac{1}{n}, \frac{1}{n}]}(x)$, yields

$$E[\psi_n(Y_t)] = \frac{n}{2} E\left[\int_0^t \mathbf{1}_{[-\frac{1}{n}, \frac{1}{n}]}(Y_s) \left[|X_s|^\alpha - |\tilde{X}_s|^\alpha \right]^2 ds \right].$$

which implies

$$E[\psi_n(Y_t)] \le \frac{n}{2} E\left[\int_0^t \mathbf{1}_{[-\frac{1}{n}, \frac{1}{n}]}(Y_s) |Y_s|^{2\alpha} ds \right] \le \frac{t}{2} n^{1-2\alpha} \to 0.$$

Therefore, $E[|Y_t|] = 0$.

Weak solutions

Definition

A *weak solution* is a triple (X, B), $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_t$, such that :

(i) $\mathcal{F}_t$ is a filtration in a probability space $(\Omega, \mathcal{F}, P)$, right-continuous and containing all P-null sets.

(ii) X_t is a continuous m-dimensional adapted process and B_t is an $\mathcal{F}_t$-Brownian motion on $\mathbb{R}^d$.

(iii) Equation (1) is satisfied.

- The filtration $\mathcal{F}_t$ may not be the augmentation of the filtration generated by B and the initial condition.

Example :

- Consider the SDE
$$X_t = \int_0^t \mathrm{sgn}(X_s) dB_s$$
where $\mathrm{sgn}(x) = \mathbf{1}_{(0,\infty)}(x) - \mathbf{1}_{(-\infty,0]}(t)$.
- One can construct a weak solution by choosing a Brownian motion X_t and
$$B_s = \int_0^t \mathrm{sgn}(X_s) dX_s.$$
- In this case, strong uniqueness does not hold, but there is uniqueness in law of all weak solutions.
- The filtration generated by X_t is strictly larger than the filtration generated by B_t (which is the filtration generated by $|X_t|$).

Proposition

Consider the SDE

$$dX_t = b(t, X_t)dt + dB_t, \quad t \in [0, T]$$

where B_t *is a* d*-dimensional Brownian motion and* $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ *satisfies*

$$\|b(t, x)\| \leq K(1 + \|x\|).$$

Then there is a weak solution for any initial distribution μ*.*

Proof :

- To simplify we assume that the initial condition is constant $\xi = x$.
- By Girsanov theorem, if X_t is a Brownian motion starting from x, the process

$$B_t = X_t - x - \int_0^t b(s, X_s) ds$$

is a Brownian motion starting from zero under the probability Q such that

$$Z_T = \frac{dQ}{dP} = \exp\left\{ \sum_{j=1}^{d} \int_0^T b_j(s, X_s) dX_s^j - \frac{1}{2} \int_0^T \|b(s, X_s)\|^2 ds \right\}.$$

- For each $t \geq 0$, consider the second-order differential operator

$$L_t f = \frac{1}{2} \sum_{i,k=1}^{m} a_{ik} \frac{\partial^2 f}{\partial x_i \partial x_k} + \sum_{i=1}^{m} b_i \frac{\partial f}{\partial x_i},$$

where $a_{ik} = \sum_{j=1}^{d} \sigma_{ij} \sigma_{kj}$.

Proposition

Let (X, B), $(\Omega, \mathcal{F}, P)$, $\mathcal{F}_t$, be a weak solution to equation (1). Then, for any $f \in C^{1,2}([0, \infty) \times \mathbb{R}^m)$, the process

$$M_t^f = f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial f}{\partial s} + L_s f \right) (s, X_s) ds$$

is a continuous local martingale, such that

$$\langle M^f, M^g \rangle_t = \sum_{i,k=1}^{m} \int_0^t a_{ik}(s, X_s) \frac{\partial f}{\partial x_i}(s, X_s) \frac{\partial g}{\partial x_i}(s, X_s) ds.$$

Proof : Use Ito formula and the stopping times

$$S_n = \inf \left\{ t \geq 0, \|X_t\| \geq n \text{ or } \int_0^t \sigma_{ij}^2(s, X_s) ds \geq n \text{ for some } (i,j) \right\}.$$

□

- If f has compact support and the coefficients σ_{ij} are bounded in the support of f, then M_t^f is a square integrable martingale.

Martingale problem

Definition

A probability P on $C([0,\infty);\mathbb{R}^m)$ under which

$$M_t^f = f(y(t)) - f(y(0)) - \int_0^t (L_s f)(y(s))ds$$

is a continuous local martingale for every $f \in C^2(\mathbb{R}^m)$ is called a solution to the *martingale problem* associated with L_t.

- The existence of solution to the martingale problem is equivalent to the existence of a weak solution.
- If the coefficients b and σ are bounded and continuous, then there exist a solution to the martingale problem for any initial distribution μ such that $\int_{\mathbb{R}^m} \|x\|^{2m}\mu(dx) < \infty$ for some $m > 1$.

Feynman -Kac formula

- Fix $T > 0$. Consider functions $f : \mathbb{R}^m \to \mathbb{R}$, $k : [0, T] \times \mathbb{R}^m \to [0, \infty)$ such that $|f(x)| \leq L(1 + \|x\|^{2\lambda})$ for some $\lambda \geq 1$.

Theorem

Let $v : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ of class $C^{1,2}$, bounded by $M(1 + \|x\|^{2\mu})$, where $\mu \geq 1$, that satisfies the Cauchy problem

$$\boxed{\frac{\partial v}{\partial t} + L_t v = kv}, \quad (t, x) \in [0, \infty) \times \mathbb{R}^m$$

with terminal condition $v(T, x) = f(x)$, $x \in \mathbb{R}^m$. Then $v(t, x)$ admits the stochastic representation

$$v(t, x) = E^{t,x}\left[f(X_T)\exp\left\{-\int_t^T k(\theta, X_\theta)d\theta\right\}\right],$$

where we denote by $E^{t,x}$ the expectation of X_s starting at time t at the point x.

Proof :

- Applying Itô's formula we obtain that the process

$$Y_s = v(s, X_s) \exp\left\{-\int_t^s k(\theta, X_\theta) d\theta\right\}$$

is a continuous local martingale localized by the sequence of stopping times $S_n = \inf\{s \geq t : \|X_s\| \geq n\}$.

- Therefore, $v(t,x) = E[Y_{T \wedge S_n}]$ and we obtain

$$\begin{aligned} v(t,x) &= E^{t,x}\left[v(S_n, X_{S_n}) \exp\left\{-\int_t^{S_n} k(\theta, X_\theta) d\theta\right\} \mathbf{1}_{\{S_n \leq T\}}\right] \\ &\quad + E^{t,x}\left[f(X_T) \exp\left\{-\int_t^T k(\theta, X_\theta) d\theta\right\} \mathbf{1}_{\{S_n > T\}}\right]. \end{aligned}$$

- We know that

$$E^{t,x}\left[\sup_{t \leq s \leq T} \|X_s\|^{2n}\right] \leq C(1 + \|x\|^{2n}).$$

- By dominated convergence, the second term converges to

$$E^{t,x}\left[f(X_T)\exp\left\{-\int_t^T k(\theta, X_\theta)d\theta\right\}\right].$$

- The first term can be estimated by

$$E^{t,x}[|v(S_n, X_{S_n})|\mathbf{1}_{\{S_n \le T\}}] \le M(1+n^{2\mu})P^{t,x}(S_n \le T).$$

and

$$\begin{aligned} P^{t,x}(S_n \le T) &= P^{t,x}\left(\sup_{t\le s\le T}\|X_s\| \ge n\right) \le n^{-2N}E^{t,x}\left[\sup_{t\le s\le T}\|X_s\|^{2N}\right] \\ &\le Cn^{-2N}(1+\|x\|^{2N}), \end{aligned}$$

and it suffices to choose $N > \mu$ to show that the second term tends to zero. $\square$

The Malliavin calculus

- Consider a d-dimensional Brownian motion $B = \{B_t, 0 \le t \le T\}$ and let $\mathcal{F}_t$ be its filtration augmented with the null sets.
- An $\mathcal{F}_T$-measurable random variable F is said to be *cylindrical* if it can be written as
$$F = f\left(\int_0^T h_s^1 dB_s, \dots, \int_0^T h_s^n dB_s\right),$$
where $h^i \in L^2([0,T]; \mathbb{R}^d)$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a C^∞ function such that f all its partial derivatives have polynomial growth.
- The space $\mathcal{S}$ of cylindrical random variables is dense in $L^p(\Omega, \mathcal{F}_T, P)$ for any $p \ge 1$.

Definition

The Malliavin derivative of $F \in \mathcal{S}$ is the $\mathbb{R}^d$-valued process given by
$$D_t F = \sum_{i=1}^n h_t^i \frac{\partial f}{\partial x_i}\left(\int_0^T h_s^1 dB_s, \dots, \int_0^T h_s^n dB_s\right).$$

Proposition (Integration by parts formula)

Let $F \in \mathcal{S}$ and let $\{u_t, t \in [0,T]\}$ be an m-dimensional progressively measurable process that satisfies Novikov condition. Then

$$E\left(\int_0^T \langle D_s F, u_s \rangle ds\right) = E\left(F \int_0^T u_s dB_s\right).$$

Proof :

(i) Let $F = f(\int_0^T h_s^1 dB_s, \dots, \int_0^T h_s^n dB_s)$. Fix $\epsilon > 0$ and write

$$F_\epsilon = f\left(\int_0^T h_s^1 d\left(B_s + \epsilon \int_0^s u_r dr\right), \dots, \int_0^T h_s^n d\left(B_s + \epsilon \int_0^s u_r dr\right)\right).$$

(ii) From Girsanov's theorem, we have

$$E[F_\epsilon] = E\left(\exp\left(\epsilon \int_0^T u_r dB_s - \frac{\epsilon^2}{2} \int_0^T u_r^2 dr\right) F\right),$$

which implies

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}(E[F_\epsilon] - E[F]) = E\left(F \int_0^T u_s dB_s\right).$$

(iii) On the other hand,

$$
\begin{aligned}
& \lim_{\epsilon \to 0} \frac{1}{\epsilon}(E[F_\epsilon] - E[F]) \\
= \quad & E\left(\int_0^T \sum_{i=1}^n \frac{\partial f}{\partial x_i} \left(\int_0^T h_s^1 dB_s, \ldots, \int_0^T h_s^n dB_s \right) \langle h_s^i, u_s \rangle ds \right) \\
= \quad & E\left(\int_0^T \langle D_s F, u_s \rangle ds \right). \quad \square
\end{aligned}
$$

- For any $p \geq 1$ we denote by $\mathcal{L}_T^p$ the space of d-dimensional measurable processes $\{X_t, t \in [0, T]\}$ such that

$$E\left(\left(\int_0^T \|X_t\|^2 dt\right)^{\frac{p}{2}}\right) < \infty.$$

Proposition

The operator D is closable from $L^p(\Omega, \mathcal{F}_T, P)$ into $\mathcal{L}_T^p$, for any $p \geq 1$.

Proof in the case $p > 1$:

(i) Let $F_n \in \mathcal{S}$, $F_n \xrightarrow{L^p} 0$ and such that $DF_n \xrightarrow{\mathcal{L}^p_T} X$. We claim that $X = 0$.

(ii) For any $h \in L^2([0,T];\mathbb{R}^d)$ and $G \in \mathcal{S}$, we have

$$\lim_{n\to\infty} E\left(\int_0^T G\langle D_sF_n, h_s\rangle ds\right) = E\left(G\int_0^T \langle X_s, h_s\rangle ds\right)$$

and

$$\begin{aligned} E\left(\int_0^T G\langle D_sF_n, h_s\rangle ds\right) &= E\left(\int_0^T \langle D_s(GF_n), h_s\rangle ds\right) \\ &\quad - E\left(\int_0^T F_n\langle D_sG, h_s\rangle ds\right) \\ &= E\left(F_n\left[G\int_0^T h_s dB_s - \int_0^T \langle D_sG, h_s\rangle ds\right]\right) \to 0. \end{aligned}$$

As a consequence, we obtain $E\left(G\int_0^T \langle X_s, h_s\rangle ds\right) = 0$, which implies $X = 0$. $\square$

- The domain of D, denoted by $\mathbb{D}^{1,p}$ is the closure of $\mathcal{S}$ under the norm

$$\|F\|_{1,p} = \left(E(|F|^p) + E(\|DF\|^p_{L^2([0,T];\mathbb{R}^d)} \right)^{\frac{1}{p}}.$$

- For $p > 1$ we can consider the adjoint operator δ of D. It is a densely defined operator from $\mathcal{L}^p_T$ into $L^p(\Omega, \mathcal{F}_T, P)$, characterized by the duality relation

$$E(F\delta(u)) = E\left(F \int_0^T u_s dB_s \right), \quad F \in \mathbb{D}^{1,p}.$$

- The domain of δ in $\mathcal{L}^p_T$ contains the space of d-dimensional progressively measurable processes u in $\mathcal{L}^p_T$ and

$$\boxed{\delta(u) = \int_0^T u_s dB_s.}$$

Clark-Ocone formiula

Proposition

Let $F \in \mathbb{D}^{1,2}$. Then,

$$F = E(F) + \int_0^T E(D_t F | \mathcal{F}_t) dB_t.$$

Proof :

- Assume $d = 1$. For any $v \in L^2(\mathcal{P})$ we can write, using the duality relationship

$$E\left(F\int_0^T v_t dB_t\right) = E(F\delta(v)) = E\left(\int_0^T D_t F v_t dt\right) = \int_0^T E[E(D_t F|\mathcal{F}_t)v_t]dt.$$

- If we assume that $F = E(F) + \int_0^T u_t dB_t$, then by the Itô isometry

$$E\left(F\int_0^T v_t dB_t\right) = \int_0^T E(u_t v_t)dt.$$

Comparing these two expressions we deduce that

$$u_t = E(D_t F|\mathcal{F}_t)$$

almost everywhere in $\Omega \times [0, T]$.

- If $F \in \mathcal{S}$, the kth derivative of F is the k-parameter process with values in $\mathbb{R}^{d\times k}$ given by

$$D^k_{t_1,\dots,t_k}F = D_{t_1}\cdots D_{t_k}F.$$

- For any $p \geq 1$ the operator D^k is closable on $\mathcal{S}$. We denote by $\mathbb{D}^{k,p}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p} = \left(E[|F|^p] + \sum_{j=1}^{k} E(\|DF\|^p_{L^2([0,T]^j;\mathbb{R}^d)}) \right)^{\frac{1}{p}}.$$

- Set

$$\mathbb{D}^\infty = \cap_{p\geq 1} \cap_{k\geq 1} \mathbb{D}^{k,p}.$$

Existence and regularity of densities

- Let $F = (F^1, \ldots, F^m)$ be such that $F^i \in \mathbb{D}^{1,2}$ for $i = 1, \ldots, m$.
- The *Malliavin matrix* of F is

$$\gamma_F = (\langle DF^i, DF^j \rangle_{L^2([0,T];\mathbb{R}^d)})_{1 \le i,j \le m}.$$

Theorem (Criterion for absolute continuity)

If $\det \gamma_F > 0$ *a.s., then the law of* F *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^m$*.*

Theorem (Criterion for smoothness of the density)

If $F_i \in \mathbb{D}^\infty$ *and* $E[(\det \gamma_F)^{-p}] < \infty$ *for all* $p \ge 1$*, then the law of* F *possesses and infinitely differentiability density.*

Let $F = X_t$, where $\{X_t, t \geq 0\}$ is the diffusion process on $\mathbb{R}^m$

$$dX_t = b(X_t)dt + \sum_{k=1}^{d} \sigma_k(X_t) dB_t^k, \qquad X_0 = x_0.$$

Theorem

If the Lie algebra spanned by $\{\sigma_1, \ldots, \sigma_d\}$ *at* $x = x_0$ *is* $\mathbb{R}^m$*, where* $\sigma_k = \sum_{i=1}^m \sigma_{ik} \frac{\partial}{\partial x_i}$*, then for any* $t > 0$ $(\det \gamma_{X_t})^{-1} \in \cap_{p \geq 2} L^p(\Omega)$ *and the density* $p_t(x)$ *of* X_t *is* $\mathcal{C}^\infty$*.*

- $p_t(x)$ satisfies the Fokker-Planck equation

$$\left(-\frac{\partial}{\partial t} + L^* \right) p_t = 0,$$

where

$$L = \frac{1}{2} \sum_{i,j=1}^{m} (\sigma\sigma^T)_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i \frac{\partial}{\partial x_i}.$$

Then, $p_t \in C^\infty$ means that $\frac{\partial}{\partial t} - L^*$ is hypoelliptic (Hörmander's theorem).

References :

1. F. Baudoin : *Diffusion Processes and Stochastic Calculus*. EMS, 2010.
2. I. Karatzas and S. E. Shreve : *Brownian Motion and Stochastic Calculus*. Springer, 1998.
3. D. Nualart : *Lecture Notes on Stochastic Processes*.
4. D. Nualart : *The Malliavin calculus and related topics*. Springer, 2005.