A Preliminary Introduction to Copula Theory

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November 10, 2022

Abstract

Copula[**?**]

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[*] https://github.com/nymath/notes4master

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FX

1.1 Theorem

$U \sim U(0,1)$UFX(), F

$$F^{-1} : [0,1] \mapsto \mathbb{R}$$

$F^{-1}(U)X$

Remark: XFcdf$F \circ X = F(X) Uniform(0,1)$

FF^{-1}

1.1U $(U_1, \cdots, U_p)$ Copula

Copulalink function alpha$Copula$

[redacted]

Example 1.3 *(Copula)*:

1. Copula: $C(u_1, \cdots, u_d) = \Pi_{k=1}^{p} u_k$
2. (Comonotonicity)Copula: $C(u_1, \cdots, u_d) = \min\{u_1, \cdots, u_d\}$
3. Copula()
4. Gaussian Copula:

[redacted]

1.5 Theorem

$(U_1, \cdots, U_p)C$

$$C(u_1, \cdots, u_p) = \Pr(U_1 \le u_1, \cdots, U_p \le u_p) = C(u_1, \cdots, u_p)$$

C

Remark: Copula

2 Copula

2.1 Copula

$(X_1, \cdots, X_p)$, $(U_1, \cdots, U_p)$

2.1 Theorem

X

$$= +$$

Copula$X$

$$= +\ \text{Copula}$$

2.2 Theorem ***Sklar's Theorem***

Copula()$(U_1, \cdots, U_p)F_i$

$$(X_1, \cdots, X_p) = (F_1^{-1}(U_1), \cdots, F_p^{-1}(U_p))$$

$(X_1, \cdots, X_p)F$

$$F(x_1, \cdots, x_p) = C(F_1(x_1), \cdots, F_p(x_p))$$

Sklar's TheoremCopula

[redacted]

2.4 Theorem ***Sklar's Theorem***

XcdfFF_iXCopula C which is define as follows:

$$C(u_1, \cdots, u_p) = F(F_1^{-1}(u_1), \cdots, F_p^{-1}(u_p))$$

2.5 Theorem ***Copula***

f, g(), (X, Y)copula C,

$$f(X), g(Y)\text{Copula C.}$$

Remark: X, Y

$$(U_1, U_2) = (F_X(X), F_Y(Y))$$

copula C.

$U X$

3 Dependence Measures

Measure

3.1

3.1.1 Pearson rho

[redacted]

Remark: Pearson rhoX, YX-E(X)Y-E(Y)XL^2XE(X)

[redacted]

Remark:

3.1.2 Spearman rho

[redacted]

Remark: Spearman rho(X,Y)Copulaf(X),g(Y)XYCopulaf(X), g(Y)spearman rhoSpearman rhocopulaCare

[redacted]

Remark: spearman rho$n$ $\frac{r_{i1}}{n}$$U_1$$U_1, U_2$Pearson rho

3.5 Theorem

$$\rho_s = 12 \int_{\mathbb{R}^2} C(u, v) - uv \;\mathrm{d}u\mathrm{d}v$$

3.1.3 Kendall tau

[redacted]

kendall tauCopulaKendall Kendall tauCopula

3.7 Theorem

$$\tau = 4 \int_{\mathbb{R}^2} C(u, v) \;\mathrm{d}C(u, v) - 1$$

Family	Kendall's τ	Range of τ
Gaussian	$\tau = \frac{2}{\pi}\arcsin(\rho)$	$[-1, 1]$
t	$\tau = \frac{2}{\pi}\arcsin(\rho)$	$[-1, 1]$
Gumbel	$\tau = 1 - \frac{1}{\delta}$	$[0, 1]$
Clayton	$\tau = \frac{\delta}{\delta+2}$	$[0, 1]$
Frank	$\tau = 1 - \frac{4}{\delta} + 4\frac{D_1(\delta)}{\delta}$ with	$[-1, 1]$
$D_1(\delta) = \int_0^\delta \frac{x/\delta}{e^x-1}dx$ (Debye function)		

Table 1:

3.2

3.8 Theorem

X, Y

$$\rho_p = 2\sin\left(\frac{\pi}{6}\rho_s\right) \text{ and } \tau = \frac{2}{\pi}\arcsin(\rho)$$

3.3

[redacted]

[redacted]

Remark: $[0, 1]$

3.4

In the case of d variables, we consider the dependence of any pair of variables. Additionally, we are interested in the dependence of two variables after the effect of the remaining variables

Family	Upper tail dependence	Lower tail dependence
Gaussian	–	–
t	$2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right)$	$2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right)$
Gumbel	$2-2^{1/\delta}$	–
Clayton	–	$2^{-1/\delta}$
Frank	–	–
Joe	$2-2^{1/\delta}$	–
BB1	$2-2^{1/\delta}$	$2^{-1/(\delta\theta)}$
BB7	$2-2^{1/\theta}$	$2^{-1/\delta}$
Galambos	$2^{-1/\delta}$	–
BB5	$2-\left(2-2^{-1/\delta}\right)^{1/\theta}$	–
Tawn	$(\psi_1+\psi_2)-\left(\psi_1^\theta+\psi_2^\theta\right)^{1/\theta}$	–
t-EV	$2\left[1-T_{\nu+1}\left(z_{1/2}\right)\right]$	–
Hsler-Reiss	$2\left[1-\Phi\left(\frac{1}{\lambda}\right)\right]$	–
Marshall-Olkin	$\min\{\alpha_1,\alpha_2\}$	–

Table 2:

are removed (partial correlations) or the dependence when we fix the values of the remaining variables (conditional correlations).

At first, we introduce some notations for convenience.

- $I^d := \{1, 2, \cdots, d\}$
- $I^d_{-i} := I^d \backslash \{i\}$.

4 Bivariate(Explicit) Copula Classes

copula

4.1 Gaussian Copula

CopulaCopula

[redacted]

Copula$\Sigma(Z_1, \cdots, Z_p)$

$$(U_1, \cdots, U_p) = (\varphi(Z_1), \cdots, \varphi(Z_p)).$$

ΣCopula 000651.SZ601318.SH Copula()CopulaGammaCopula+Gamma

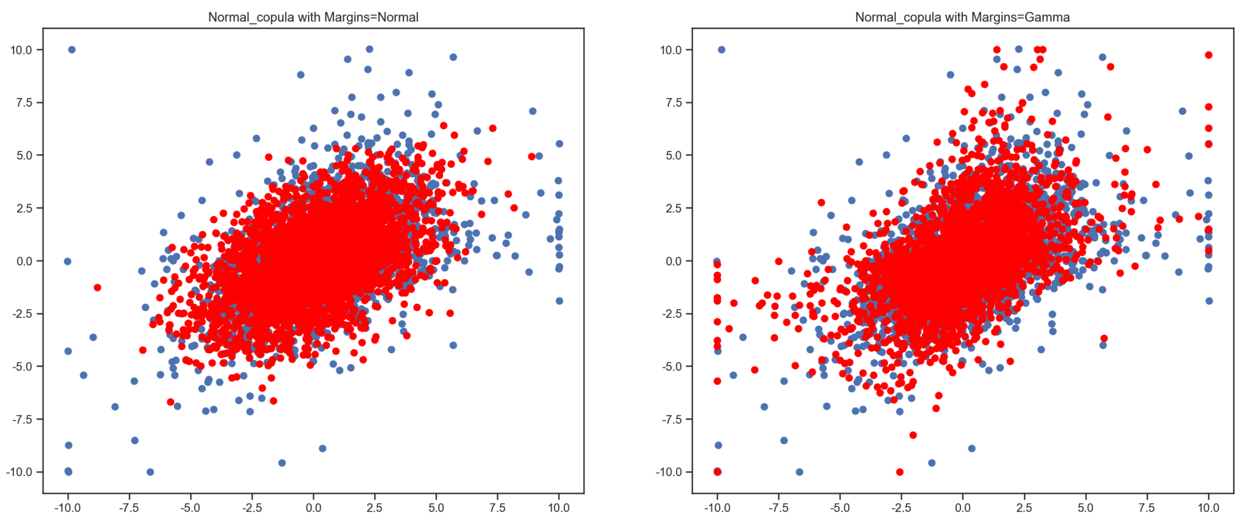

Figure 1: Copula

Algorithm 4.2 *(Gaussian Copula)*:

1. Sample i.i.d $(Z_1, \cdots, Z_p)$ from $N(0, 1)$.
2. Using Cholesky Decomposition to $\Sigma = AA'$ and obtain $[X_1, \cdots, X_p] = [Z_1, \cdots, Z_p]A'$
3. Return $(U_1, \cdots, U_p) = (\varphi(X_1), \cdots, \varphi(X_p))$

4.2 t Copula

[redacted]

[redacted]

Algorithm 4.4 *(Simulate a t Copula)*:

1. Generate $(Z_1, \cdots, Z_p)$ from $N_p(0, \Sigma)$.
2. Let $(X_1, \cdots, X_p) = (\frac{Z_1}{\sqrt{\frac{S}{\nu}}}, \cdots, \frac{Z_p}{\sqrt{\frac{S}{\nu}}})$ where $S \sim \chi_v$ independent of Z_i.
3. Return $(U_1, \cdots, U_p) = (t_\nu(X_1), \cdots, t_\nu(X_p))$.

5 Archimedean Copulas

In practice, Archimedean copulas are popular because they allow modeling dependence in arbitrarily high dimensions with only one parameter, governing the strength of dependence. In addition, there exists a close-form expresssion of the relationship between this parameter and kendall tau.

[redacted]

Name of copula	Bivariate copula $C_\theta(u,v)$	parameter θ	generator $\psi_\theta(t)$	generator inverse $\psi_\theta^{-1}(t)$
Ali-Mikhail-Haq	$\frac{uv}{1-\theta(1-u)(1-v)}$	$\theta \in [-1,1]$	$\log\left[\frac{1-\theta(1-t)}{t}\right]$	$\frac{1-\theta}{\exp(t)-\theta}$
Clayton	$\left[\max\left\{u^{-\theta}+v^{-\theta}-1;0\right\}\right]^{-1/\theta}$	$\theta \in [-1,\infty)\backslash\{0\}$	$\frac{1}{\theta}\left(t^{-\theta}-1\right)$	$(1+\theta t)^{-1/\theta}$
Frank	$-\frac{1}{\theta}\log\left[1+\frac{(\exp(-\theta u)-1)(\exp(-\theta v)-1)}{\exp(-\theta)-1}\right]$	$\theta \in \mathbb{R}\backslash\{0\}$	$-\log\left(\frac{\exp(-\theta t)-1}{\exp(-\theta)-1}\right)$	$-\frac{1}{\theta}\log(1+\exp(-t)(\exp(-\theta)-1))$
Gumbel	$\exp\left[-\left((-\log(u))^\theta+(-\log(v))^\theta\right)^{1/\theta}\right]$	$\theta \in [1,\infty)$	$(-\log(t))^\theta$	$\exp\left(-t^{1/\theta}\right)$
Independence	uv		$-\log(t)$	$\exp(-t)$
Joe	$1-\left[(1-u)^\theta+(1-v)^\theta-(1-u)^\theta(1-v)^\theta\right]^{1/\theta}$	$\theta \in [1,\infty)$	$-\log\left(1-(1-t)^\theta\right)$	$1-(1-\exp(-t))^{1/\theta}$

Table 3: Archimedean Generator

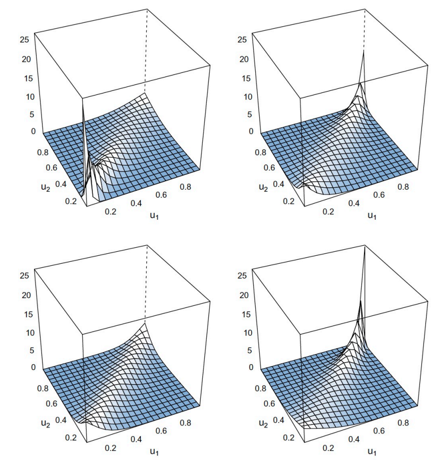

Figure 2: Top left: Clayton, top right: Gumbel, bottom left: Frank, bottom right: Joe.

5.1 Simple Archimedean Copula

5.2 Laplace-Lebesgue-Stieltjes Transform

[redacted]

5.3 Nested Archimedean Copula

5.4 Simulations

Algorithm 5.3 *(Marshall and Olkin)*:

1. Sample V from $F = LS^{-1}(\psi^{-1})$
2. Sample i.i.d $E_1, \cdots, E_p$ from $Exp(1)$, independent of V.
3. Return $U = (\psi^{-1}(\frac{E_1}{V}), \cdots, \psi^{-1}(\frac{E_p}{V}))$

6 test

7 Applications in Assurance

8 Applications in Derivatives