

A Preliminary Introduction to Copula Theory

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Abstract

Copula[?]

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* <https://github.com/nymath/notes4master>

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1

FX

1.1 Theorem

$U \sim U(0, 1) \text{ UFX}(), F$

$$F^{-1} : [0, 1] \mapsto \mathbb{R}$$

$$F^{-1}(U)X$$

Remark: $XF \text{cdf} F \circ X = F(X) \text{Uniform}(0, 1)$

$$FF^{-1}$$

$\square 1.1U \square (U_1, \dots, U_p) \square \text{Copula}$

Copula link function αCopula

Example 1.3 (*Copula*):

1. Copula: $C(u_1, \dots, u_d) = \prod_{k=1}^p u_k$
2. (Comonotonicity) Copula: $C(u_1, \dots, u_d) = \min\{u_1, \dots, u_d\}$
3. Copula()
4. Gaussian Copula:

1.5 Theorem

$(U_1, \dots, U_p)C$

$$C(u_1, \dots, u_p) = \Pr(U_1 \leq u_1, \dots, U_p \leq u_p) = C(u_1, \dots, u_p)$$

C

Remark: Copula

2 Copula

2.1 Copula

$(X_1, \dots, X_p), (U_1, \dots, U_p)$

2.1 Theorem

\square^X

$= +$

Copula X

$= + \text{Copula}$

2.2 Theorem *Sklar's Theorem*

Copula $(U_1, \dots, U_p)F_i$

$$(X_1, \dots, X_p) = (F_1^{-1}(U_1), \dots, F_p^{-1}(U_p))$$

$(X_1, \dots, X_p)F$

$$F(x_1, \dots, x_p) = C(F_1(x_1), \dots, F_p(x_p))$$

Sklar's Theorem Copula

2.4 Theorem *Sklar's Theorem*

$X_{cdf} F F_i X$ Copula C which is define as follows:

$$C(u_1, \dots, u_p) = F(F_1^{-1}(u_1), \dots, F_p^{-1}(u_p))$$

2.5 Theorem *Copula*

$f, g(), (X, Y)$ copula C ,

$f(X), g(Y)$ Copula C .

Remark: X, Y

$$(U_1, U_2) = (F_X(X), F_Y(Y))$$

copula C .

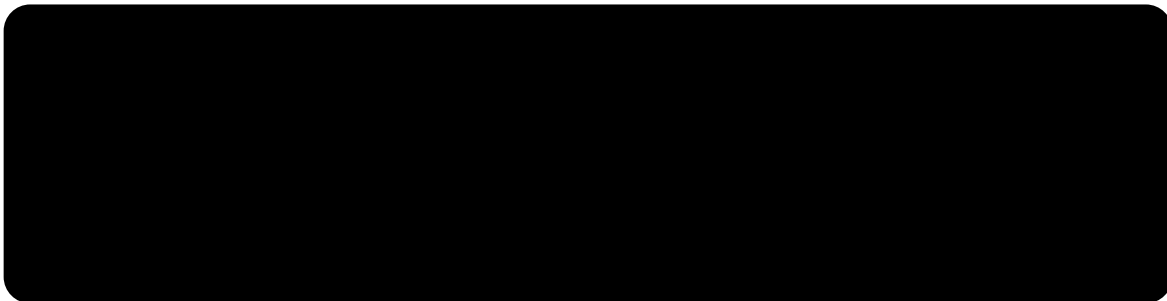
UX

3 Dependence Measures

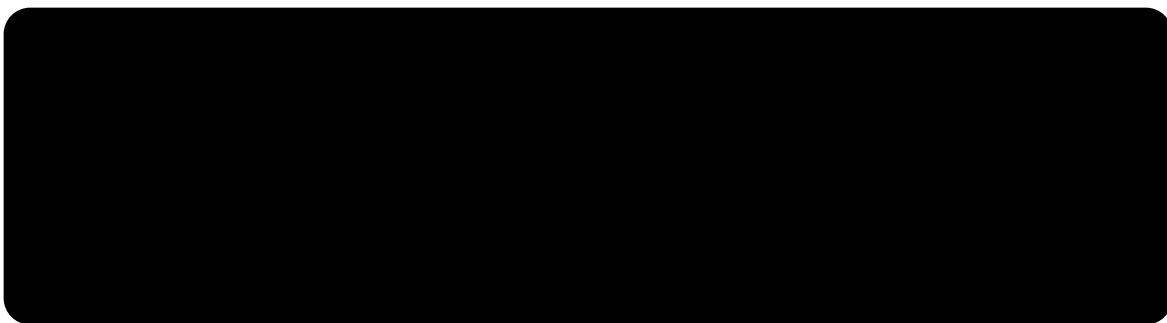
Measure

3.1

3.1.1 Pearson rho



Remark: Pearson rho $\frac{E[(X - E(X))(Y - E(Y))]}{\sqrt{E[(X - E(X))^2]E[(Y - E(Y))^2]}}$



Remark:

3.1.2 Spearman rho

Remark: Spearman rho(X,Y) = $\text{Corr}(F_X(X), F_Y(Y))$ = $\text{Corr}(F_X(X), F_Y(Y))$ = $\text{Corr}(F_X(X), F_Y(Y))$

Remark: spearman rho = $\frac{r_{s1}}{n} U_1 U_2$ Pearson rho

3.5 Theorem

$$\rho_s = 12 \int_{\mathbb{R}^2} C(u, v) - uv \, dC(u, v)$$

3.1.3 Kendall tau

kendall tau = $\text{Corr}(F_X(X), F_Y(Y))$ = $\text{Corr}(F_X(X), F_Y(Y))$ = $\text{Corr}(F_X(X), F_Y(Y))$

3.7 Theorem

$$\tau = 4 \int_{\mathbb{R}^2} C(u, v) \, dC(u, v) - 1$$

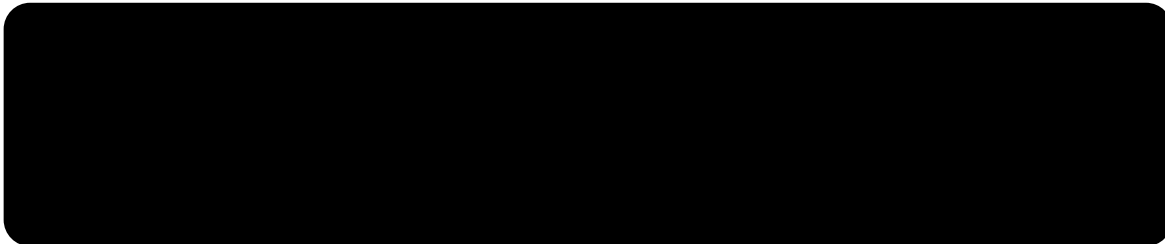
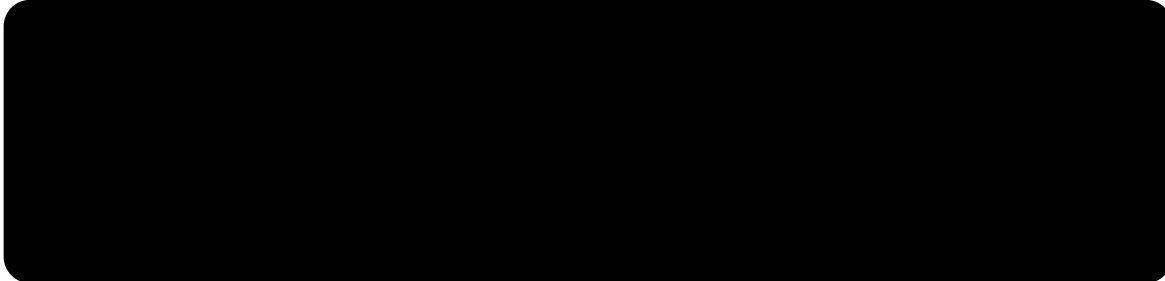
Family	Kendall's τ	Range of τ
Gaussian	$\tau = \frac{2}{\pi} \arcsin(\rho)$	$[-1, 1]$
t	$\tau = \frac{2}{\pi} \arcsin(\rho)$	$[-1, 1]$
Gumbel	$\tau = 1 - \frac{1}{\delta}$	$[0, 1]$
Clayton	$\tau = \frac{\delta}{\delta+2}$	$[0, 1]$
Frank	$\tau = 1 - \frac{4}{\delta} + 4 \frac{D_1(\delta)}{\delta}$ with	$[-1, 1]$
$D_1(\delta) = \int_0^\delta \frac{x/\delta}{e^x - 1} dx$ (Debye function)		

Table 1:

3.2

3.8 Theorem	
X, Y	$\rho_p = 2 \sin\left(\frac{\pi}{6} \rho_s\right)$ and $\tau = \frac{2}{\pi} \arcsin(\rho)$

3.3



Remark: $[0, 1]$

3.4

In the case of d variables, we consider the dependence of any pair of variables. Additionally, we are interested in the dependence of two variables after the effect of the remaining variables

Family	Upper tail dependence	Lower tail dependence
Gaussian	—	—
t	$2t_{\nu+1} \left(-\sqrt{\nu+1} \sqrt{\frac{1-\rho}{1+\rho}} \right)$	$2t_{\nu+1} \left(-\sqrt{\nu+1} \sqrt{\frac{1-\rho}{1+\rho}} \right)$
Gumbel	$2 - 2^{1/\delta}$	—
Clayton	—	$2^{-1/\delta}$
Frank	—	—
Joe	$2 - 2^{1/\delta}$	—
BB1	$2 - 2^{1/\delta}$	$2^{-1/(\delta\theta)}$
BB7	$2 - 2^{1/\theta}$	$2^{-1/\delta}$
Galambos	$2^{-1/\delta}$	—
BB5	$2 - (2 - 2^{-1/\delta})^{1/\theta}$	—
Tawn	$(\psi_1 + \psi_2) - (\psi_1^\theta + \psi_2^\theta)^{1/\theta}$	—
t-EV	$2 [1 - T_{\nu+1} (z_{1/2})]$	—
Hsler-Reiss	$2 [1 - \Phi (\frac{1}{\lambda})]$	—
Marshall-Olkin	$\min \{\alpha_1, \alpha_2\}$	—

Table 2:

are removed (partial correlations) or the dependence when we fix the values of the remaining variables (conditional correlations).

At first, we introduce some notations for convenience.

- $I^d := \{1, 2, \dots, d\}$
- $I_{-i}^d := I^d \setminus \{i\}$.

4 Bivariate(Explicit) Copula Classes

copula

4.1 Gaussian Copula

CopulaCopula

$$\text{Copula}\Sigma(Z_1, \dots, Z_p)$$

$$(U_1, \dots, U_p) = (\varphi(Z_1), \dots, \varphi(Z_p)).$$

Σ Copula 000651.SZ601318.SH Copula()CopulaGammaCopula+Gamma

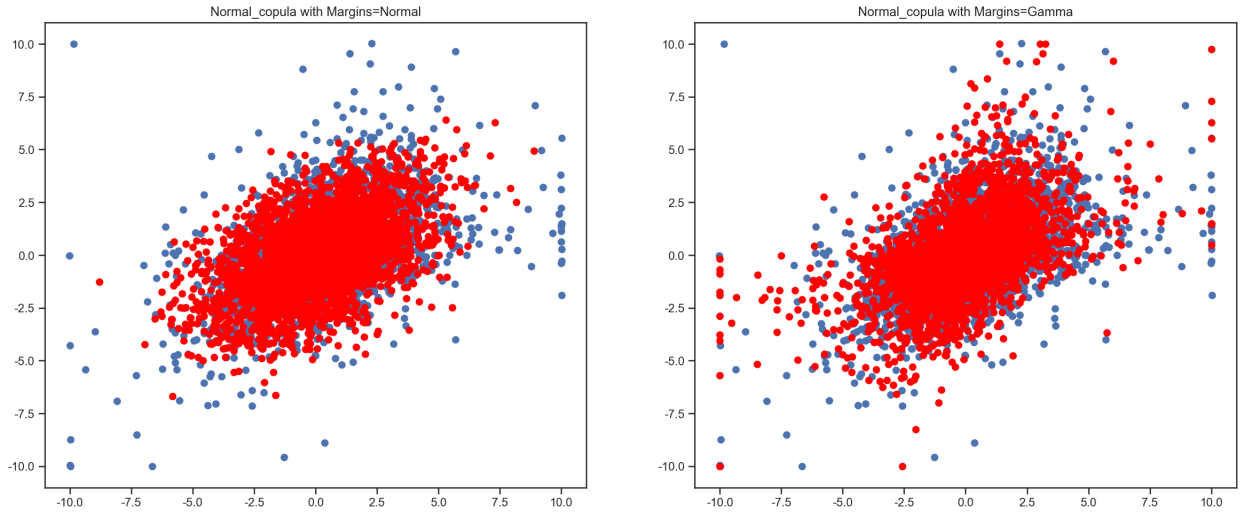


Figure 1: Copula

Algorithm 4.2 (*Gaussian Copula*):

1. Sample i.i.d (Z_1, \dots, Z_p) from $N(0, 1)$.
2. Using Cholesky Decomposition to $\Sigma = AA'$ and obtain $[X_1, \dots, X_p] = [Z_1, \dots, Z_p]A'$
3. Return $(U_1, \dots, U_p) = (\varphi(X_1), \dots, \varphi(X_p))$

4.2 t Copula

Algorithm 4.4 (*Simulate a t Copula*):

1. Generate (Z_1, \dots, Z_p) from $N_p(0, \Sigma)$.
2. Let $(X_1, \dots, X_p) = (\frac{Z_1}{\sqrt{\frac{S}{\nu}}}, \dots, \frac{Z_p}{\sqrt{\frac{S}{\nu}}})$ where $S \sim \chi_v$ independent of Z_i .
3. Return $(U_1, \dots, U_p) = (t_\nu(X_1), \dots, t_\nu(X_p))$.

5 Archimedean Copulas

In practice, Archimedean copulas are popular because they allow modeling dependence in arbitrarily high dimensions with only one parameter, governing the strength of dependence. In addition, there exists a close-form expression of the relationship between this parameter and kendall tau.

Name of copula	Bivariate copula $C_\theta(u, v)$	parameter θ	generator $\psi_\theta(t)$	generator inverse $\psi_\theta^{-1}(t)$
Ali-Mikhail-Haq	$\frac{uv}{1-\theta(1-u)(1-v)}$	$\theta \in [-1, 1]$	$\log \left[\frac{1-\theta(1-t)}{t} \right]$	$\frac{1-\theta}{\exp(t)-\theta}$
Clayton	$[\max \{u^{-\theta} + v^{-\theta} - 1; 0\}]^{-1/\theta}$	$\theta \in [-1, \infty) \setminus \{0\}$	$\frac{1}{\theta} (t^{-\theta} - 1)$	$(1 + \theta t)^{-1/\theta}$
Frank	$-\frac{1}{\theta} \log \left[1 + \frac{(\exp(-\theta u)-1)(\exp(-\theta v)-1)}{\exp(-\theta)-1} \right]$	$\theta \in \mathbb{R} \setminus \{0\}$	$-\log \left(\frac{\exp(-\theta t)-1}{\exp(-\theta)-1} \right)$	$-\frac{1}{\theta} \log(1 + \exp(-t)(\exp(-\theta) - 1))$
Gumbel	$\exp \left[-((-\log(u))^\theta + (-\log(v))^\theta)^{1/\theta} \right]$	$\theta \in [1, \infty)$	$(-\log(t))^\theta$	$\exp(-t^{1/\theta})$
Independence	uv		$-\log(t)$	$\exp(-t)$
Joe	$1 - [(1-u)^\theta + (1-v)^\theta - (1-u)^\theta(1-v)^\theta]^{1/\theta}$	$\theta \in [1, \infty)$	$-\log(1 - (1-t)^\theta)$	$1 - (1 - \exp(-t))^{1/\theta}$

Table 3: Archimedean Generator

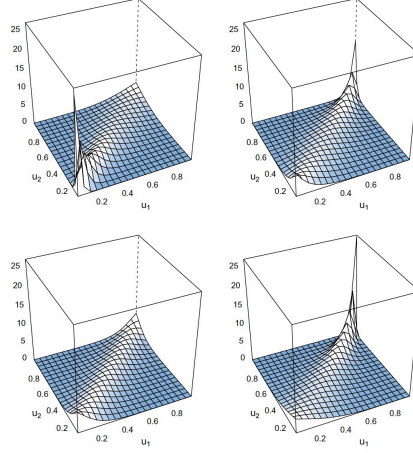


Figure 2: Top left: Clayton, top right: Gumbel, bottom left: Frank, bottom right: Joe.

5.1 Simple Archimedean Copula

5.2 Laplace-Lebesgue-Stieltjes Transform



5.3 Nested Archimedean Copula

5.4 Simulations

Algorithm 5.3 (*Marshall and Olkin*):

1. Sample V from $F = LS^{-1}(\psi^{-1})$
 2. Sample i.i.d E_1, \dots, E_p from $Exp(1)$, independent of V .
 3. Return $U = (\psi^{-1}(\frac{E_1}{V}), \dots, \psi^{-1}(\frac{E_p}{V}))$
-

6 test

7 Applications in Assurance

8 Applications in Derivatives