1. Let $g(x) = \begin{cases} f(x) & \text{for } x \in (0,1) \\ 0 & \text{for } x = 0, 1 \end{cases}$ Since f is continuous on $(0,1)$, and $\lim_{x \to 0} f(x) = 0 = g(0)$, $\lim_{x \to 1} f(x) = 0 = g(1)$, g is continuous on $[0,1]$. By the extreme value theorem, g has both an absolute maximum and an absolute minimum on $[0,1]$. If $g(0) = g(1) = 0$ happens to be the absolute minimum on $[0,1]$, f would still have the absolute maximum on $(0,1)$. Vice versa. Therefore, we Can conclude that f has either an absoulte maximum or an absolute minimum on $(0,1)$.

2. W.t.s. $g(n) > 0$ for a large $n \in \mathbb{N}$

$$\left|\frac{b_{d-1}n^{d-1} + \cdots + b_1 n + b_0}{n^d}\right| \leq \frac{|b_{d-1}|n^{d-1} + \cdots + |b_1|n + |b_0|}{n^d}$$

$$\leq \frac{n^{d-1}(|b_{d-1}| + \cdots + |b_1| + |b_0|)}{n^d} = \frac{1}{n}(|b_{d-1}| + \cdots + |b_1| + |b_0|)$$

Therefore, $\lim_{n\to\infty} \frac{b_{d-1}n^{d-1} + \cdots + b_1 n + b_0}{n^d} = 0$ ①

So $\exists M \in \mathbb{N}$ s.t. $\left|\frac{b_{d-1}M^{d-1} + \cdots + b_1 M + b_0}{M^d}\right| < 1$,

which implies $-(b_{d-1}M^{d-1} + \cdots + b_1 M + b_0) < M^d$

Therefore, $g(M) > 0$. By the intermediate value theorem,

$g(0) < 0 < g(M) \Rightarrow \exists\, c_1 \in (0, M)$ s.t. $g(c_1) = 0$

Similarly, consider $g(-n)$ for a large $n \in \mathbb{N}$,

① still holds as $(-n)^d = n^d$ since d is even

So $\exists K \in \mathbb{N}$ s.t. $g(-K) > 0 \Rightarrow \exists\, c_2 \in (-K, 0)$ s.t. $g(c_2) = 0$

Therefore, g at least have 2 roots (c_1, c_2)

3. f is continuous on $[C, C+P]$, $C \in \mathbb{R}$ since $P > 0$

Let $g(x) = f(x)$ $\forall x \in [C, C+P]$. by the extreme value theorem, g has both absolute maximum and minimum on $[C, C+P]$. So f has both as well.

4. Let $g(x) = f(x) - x$ $\quad [0,1] \to [0,1] \quad a=0, b=1$

$g(a) = g(0) = f(0) - x = -x \quad$ Since $0 \le x \le 1$, $\quad -x \le 0 \le 1-x$

$g(b) = g(1) = f(1) - x = 1-x \quad g(a) \le 0 \le g(b)$

By the intermediate value theorem, $\exists\, x \in [0,1]$ s.t. $g(x) = 0$

which means $\exists\, x\, [0,1]$ s.t. $f(x) = x$

5. $f(\mathbb{R}) = \mathbb{R}$

W.t.s, $\forall r \in \mathbb{R}\ \exists\, c \in \mathbb{R}$ s.t. $f(c) = r$

Since $r \in \mathbb{R}$, $r \le f(r) \le r+1$, by intermediate value theorem,

$\exists\, c \in [r, r+1]$ s.t. $f(c) = r$

6. Take $x, y \in (c, \infty)$. Then $|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| = \frac{|x-y|}{xy}$

For $x, y > c > 0$, $|f(x) - f(y)| < \frac{1}{c^2}|x-y|$

Therefore, f is Lipschitz continuous on (c, ∞)

7. Assume $\exists L > 0$ s.t. $|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{xy} \leq L \cdot |x-y|$

$\frac{1}{xy} \leq L \quad \forall x, y > 0$. Let $y = 2x$, $\frac{1}{2x^2} \leq L \quad \frac{1}{L} \leq 2x^2$,

which cannot be true $\forall x > 0$, contradiction.

Therefore, f is not Lipschitz continuous on $(0, \infty)$

8. $0 < x, c < 1$ $\qquad |x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

$$|x(1-x)f(x) - c(1-c)f(c)| = |xf(x) - x^2 f(x) - cf(c) + c^2 f(c)|$$

$$= |xf(x) - xf(c) + xf(c) - cf(c) - x^2 f(x) + x^2 f(c) - x^2 f(c) + c^2 f(c)|$$

$$= |x(f(x) - f(c)) + f(c)(x-c) - x^2(f(x) - f(c)) - f(c)(x^2 - c^2)|$$

$$\leq |f(x) - f(c)| \cdot |x(x-1)| + |f(c)| \cdot |x-c| \underbrace{|x+c-1|}_{\leq |x|+|c|+1}$$

$$\leq (1-x)\varepsilon + f(c) \cdot 3\delta$$

So g is continuous on $(0,1) \to \mathbb{R}$ since f is

Therefore, g is uniformly continuous

9. (a) $f(x)=\frac{1}{x}$ $\{x_n\}=\frac{1}{n}$ $f(x_n)=n$, not Cauchy

(b) f is continuous $\Rightarrow \exists \delta>0$ s.t. $|f(x)-f(y)|<\varepsilon$ if $|x-y|<\delta$

So $|f(x_m)-f(x_n)|<\varepsilon$ if $|x_m-x_n|<\delta$

$\{x_n\}$ is Cauchy $\Rightarrow$ $|x_m-x_n|<\varepsilon$ $\forall \varepsilon>0\ \exists N\in\mathbb{N}$ s.t. $\forall m,n\geq N$

We can choose a $M\in\mathbb{N}$ s.t. $|x_m-x_n|<\delta$ $\forall$ $m,n\geq M$

Then $\forall$ $m,n\geq M$ $|f(x_m)-f(x_n)|<\varepsilon$ $f(x_n)$ is Cauchy

10. g is continuous at 0 and $g(0)=0$

$||x-y|-0|=|x-y|<\delta \Rightarrow |g(|x-y|)-g(0)|=|g(|x-y|)|<\varepsilon$

$|f(x)-f(y)|\leq g(|x-y|)<\varepsilon$ $\forall$ $|x-y|<\delta$

So f is uniformly continuous.