1\. (a) $\lim_{x \to c} \sqrt{x} = \sqrt{c}$ for $c \geqslant 0$

w.t.s. $|\sqrt{x} - \sqrt{c}| < \varepsilon$ if $|x-c| < \delta$

Let $\delta = \varepsilon^2$, we have $|\sqrt{x-c}| < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$

$c \geqslant 0$, so $x-c \geqslant x - 2\sqrt{xc} - c$ $\sqrt{x-c} \geqslant \sqrt{x} - \sqrt{c}$

So $|\sqrt{x} - \sqrt{c}| \leq |\sqrt{x-c}| < \varepsilon$

(b) $\lim_{x \to c} x^2 + x + 1 = c^2 + c + 1$

w.t.s. $|x^2 + x + 1 - (c^2 + c + 1)| < \varepsilon$ if $|x-c| < \delta$

$$\leq |x^2 - c^2| + |x-c| \qquad |x+c| \leq |x-c| + 2|c|$$

$$= |x+c||x-c| + |x-c|$$

$$= |x-c| \cdot (|x+c| + 1)$$

$$\leq |x-c| \cdot (|x-c| + 2|c| + 1)$$

$$= \delta \cdot (\delta + 2|c| + 1)$$

So pick $\delta \cdot (\delta + 2|c| + 1) < \varepsilon$

(c) $\lim_{x \to 0} x^2 \cos(\frac{1}{x}) = 0$

w.t.s. $\left|x^2 \cos(\frac{1}{x}) - 0\right| = x^2 \left|\cos(\frac{1}{x})\right| < \varepsilon$ if $|x-0| = |x| < \delta$

$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$ $\quad \left|\cos\left(\frac{1}{x}\right)\right| \leq 1$ $\quad x^2\left|\cos\left(\frac{1}{x}\right)\right| \leq x^2$

Since $x \to 0$, $|x-0| = |x| < \delta < 1$, so $x^2\left|\cos\left(\frac{1}{x}\right)\right| \leq |x|$

Pick $\delta = \min(1, \varepsilon)$ so $\left|x^2\cos\left(\frac{1}{x}\right) - 0\right| \leq x^2 < \delta \leq \varepsilon$

Using squeeze theorem: $-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$

$-x^2 \leq x^2\cos\left(\frac{1}{x}\right) \leq x^2$ $\quad \lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$

so $\lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right) = \lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$

(d) $\lim_{x\to c} f(x) = L \iff \forall (x_n)_n,\ x_n \xrightarrow{n\to\infty} c \Rightarrow f(x_n) \xrightarrow{x\to\infty} L$

Let $(x_n)_1 := \frac{1}{\frac{\pi}{4}+2n\pi} \xrightarrow{n\to\infty} 0$ $\quad$ but $\quad f(x_{n_1}) \to \frac{1}{2}$,

$(x_n)_2 := \frac{1}{\frac{3\pi}{4}+2n\pi} \xrightarrow{n\to\infty} 0$ $\quad f(x_{n_2}) \to -\frac{1}{2}$

$f(x_{n_1}) \neq f(x_{n_2})$,

so $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)$ does not exist.

(e) $\lim_{x\to 0} \sin(x)\cos(1/x)$

(e) By squeeze theorem: $-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$ $\quad -\sin(x) \leq \sin(x)\cos\left(\frac{1}{x}\right) \leq \sin(x)$

$\lim_{x\to 0} -\sin(x) = \lim_{x\to 0} \sin(x) = 0$, so $\lim_{x\to 0} \sin(x)\cos\left(\frac{1}{x}\right) = 0$

2. Let $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$

$\forall x_n$ s.t. $x_n \in S \setminus \{c\} \ \forall n$

$\lim_{n \to \infty} x_n = L$

and $\{f(x_n)\} \to L$, $\{h(x_n)\} \to L$

So we have $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n > \max\{N_1, N_2\}$

$|f(x_n) - L| < \varepsilon$ and $|h(x_n) - L| < \varepsilon$

$-\varepsilon < f(x_n) - L < \varepsilon$, $-\varepsilon < h(x_n) - L < \varepsilon$

$-\varepsilon < f(x_n) - L \le g(x_n) - L \le h(x_n) - L < \varepsilon$

So $|g(x_n) - L| < \varepsilon$ $\quad \{g(x_n)\} \to L$

Therefore $\lim_{x \to c} g(x)$ exists and

$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \lim_{x \to c} h(x)$

3. $\lim_{x \to c}(f(x)+g(x)) = L_1 \Rightarrow \{c_n := a_n + b_n\} \to L_1$ ①

(a) $\lim_{x\to c} f(x) = L_2$ $\Rightarrow$ $\{a_n\} \to L_2$ ②

$\lim_{x\to c} g(x) = L_3$ $\{b_n\} \to L_3$ ③

w.t.s. $L_1 = L_2 + L_3$

From ② and ③, $\forall \varepsilon > 0$, $\exists N_2, N_3 \in \mathbb{N}$ s.t.
$\forall n > N_2$, $|a_n - a| < \frac{\varepsilon}{2}$ $\forall n > N_3$, $|b_n - b| < \frac{\varepsilon}{2}$

Let $N = \max\{N_2, N_3\}$, we have $\forall n > N$,

$$\left|(a_n + b_n) - (a+b)\right| = \left|(a_n - a) + (b_n - b)\right|$$
$$\leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So $L_1 = L_2 + L_3$. $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

(b) Similarly, $\left|(a_n - b_n) - (a-b)\right| = \left|(a_n - a) - (b_n - b)\right| \leq |a_n - a| + |b_n - b|$
So $\lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$ $< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

(c) $\lim_{x\to c} f(x)g(x) = L_1 \Rightarrow \{a_n b_n\} \to L_1$ ①

$$|a_n b_n - ab| = |a_n b_n + ab_n - ab_n - ab| = |(a_n - a)b_n + a(b_n - b)|$$
$$\leq |(a_n - a)b_n| + |a(b_n - b)| = |a_n - a||b_n| + |a||b_n - b|$$

Since $\{b_n\} \to L_3$, $\exists M > 0$ s.t. $|b_n| < M$, so

$|a_n - a||b_n| + |a||b_n - b| \leq M|a_n - a| + |a||b_n - b|$

Since $|a_n - a| < \frac{\varepsilon}{2M}$, $|b_n - b| < \frac{\varepsilon}{2(|a|+1)}$,

$$|a_n b_n - ab| < \frac{M\varepsilon}{2M} + \frac{|a|\varepsilon}{2(|a|+1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So $L_1 = L_2 \cdot L_3$, $\lim\limits_{x \to c} f(x)g(x) = \lim\limits_{x \to c} f(x) \cdot \lim\limits_{x \to c} g(x)$

(d) Let $b = L_3$, $\forall \frac{|b|}{2} > 0$ $\exists N_1 \in \mathbb{N}$ s.t. $n > N_1$, $|b_n - b| < \frac{|b|}{2}$

So $|b_n| = |b + (b_n - b)| \geq |b| - |b_n - b| > |b| - \frac{|b|}{2} = \frac{|b|}{2} > 0$

Since, $b_n \to L$, $\forall \varepsilon > 0$, $\exists N_3 \in \mathbb{N}$ s.t. $\forall n > N_2$,

Let $N = \max\{N_2, N_3\}$, we have $|b_n - b| < \frac{b^2\varepsilon}{2}$

$\forall n > N$, $\left|\frac{1}{b_n} - \frac{1}{b}\right| \leq \frac{2}{b^2}|b_n - b| < \frac{2}{b^2} \cdot \frac{b^2\varepsilon}{2} = \varepsilon$

So $\{\frac{1}{b_n}\} \to \frac{1}{b} = \frac{1}{L_3}$ From (c), $\{\frac{a_n}{b_n}\} \to \frac{L_2}{L_3} = \frac{a}{b}$

So $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}$

4. w.t.s. $\forall x_n, y_n$ with $x_n, y_n \to c$, $\lim_{x \to \infty} f(x_n) = \lim_{x \to \infty} f(y_n)$

Assume $f(x_n) \to L_1$, $f(y_n) \to L_2$

$z := x_1, y_1, x_2, y_2, \ldots$ $z_n = \begin{cases} x_k, & n = 2k+1 \\ y_k, & n = k+1 \end{cases}$

So $f(z_n)$ is convegent

$f(z_n) \to L$ so x_n and y_n are convergent as subsequences of z_n

$L_1 = L_2 = L$

By the transfer principle, $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} f(x) = L$

5. f at 1: $\forall \varepsilon > 0, \exists \delta$ s.t.

$|f(x) - f(1)| = |x-1| < \delta \leq \varepsilon$ if $|x-1| < \delta$

f is continuous at 1

f at 2: w.t.s. $\exists \varepsilon > 0, \forall \delta$ s.t.

$|f(x) - f(2)| = |x^2 - 2| > \varepsilon$ if $|x-2| < \delta$

When $\varepsilon = 1$, $x = 2 + \frac{\delta}{\sqrt{2}}$,

$|x^2 - 2| = |4 + 2\sqrt{2}\delta + \frac{\delta^2}{2} - 2| = |2 + 2\sqrt{2}\delta + \frac{\delta^2}{2}|$

Since $\delta > 0$, $|x^2 - 2| > 2 > 1 = \varepsilon$

So f is discontinuous at 2

6. $\lim_{x\to c} f(x) = L \iff \forall (X_n)_n,\ X_n \xrightarrow{n\to\infty} c \Rightarrow f(X_n) \xrightarrow{n\to\infty} L$

Let $(X_n)_1 := \frac{1}{\frac{\pi}{2}+2n\pi} \xrightarrow{n\to\infty} 0$ but $f(X_{n_1}) \to 1$

$(X_n)_2 := \frac{1}{\frac{3\pi}{2}+2n\pi} \xrightarrow{n\to\infty} 0$ $f(X_{n_2}) \to -1$

$f(X_{n_1}) \neq f(X_{n_2})$,

so $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist.

so f is not continuous

7, $\forall \varepsilon > 0, \exists \delta = \varepsilon$ s.t. if $|x| < \delta$

$$|f(x) - f(0)| = \begin{cases} |x\sin(\frac{1}{x})| \leq |x| < \delta = \varepsilon \\ 0 < \varepsilon \end{cases}$$

Therefore, f is continuous at 0

f is continuous because composition of

Continuous functions is continuous

8. w.t.s. $|g(c) - f(c)| < \varepsilon \quad \forall \varepsilon > 0$

$$g(x) = f(x) \quad \forall x$$

$$|x - c| < \delta_1 , \quad |f(x) - f(c)| < \frac{\varepsilon}{2}$$

$$|x - c| < \delta_2 , \quad |g(x) - g(c)| < \frac{\varepsilon}{2}$$

$$\delta = \min\{\delta_1, \delta_2\}$$

$\exists\, \delta \in (c - \delta, c + \delta)$ by density of $\mathbb{Q}$

$$f(c) - g(c) = f(c) - f(r) + f(r) - g(r) + g(r) - g(c)$$

$$= f(c) - f(r) + g(r) - g(c)$$

So

$$|f(c) - g(c)| \le |f(c) - f(r)| + |g(r) - g(c)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

9. Take $\varepsilon = \frac{f(c)}{2}$, $\exists \alpha > 0$ s.t.

$\forall x \in (c-\alpha, c+\alpha)$ $\quad |f(x) - f(c)| < \varepsilon = \frac{f(c)}{2}$

Since $f(c) > 0$, $-\frac{f(c)}{2} < f(x) - f(c) < \frac{f(c)}{2}$

$f(x) > \frac{f(x)}{2} = \varepsilon > 0$

10. $\forall \varepsilon > 0$, $\exists \delta$ s.t. if $|x-y| < \delta$, $|g(x) - g(y)| < \varepsilon$

So if $|x-y-0| = |x-y| < \delta$, $|g(x-y) - g(0)| < \varepsilon$

Since $g(0) = 0$, $|g(x-y)| < \varepsilon$

Also if $|x-y| < \delta$, $|f(x) - f(y)| \leq g(x-y) < \varepsilon$

So f is continuous.