***Exercise* 0.3.14:** *Prove* $1^3+2^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2$ *for all* $n\in\mathbb{N}$.

Proof by induction

Let $P(n)$ be the statement that

$1^3+2^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2$ is true $\forall n\in\mathbb{N}$

- $P(1)$ is true by plugging in $n=1$: $\quad 1^3=1=\left(\frac{1\cdot(1+1)}{2}\right)^2$

- Assume $P(n)$ is true, that is,

  $1^3+2^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2$ holds

  Add $(n+1)^3$ to both sides we have

$$1^3+2^3+\cdots+n^3+(n+1)^3=\left(\frac{n(n+1)}{2}\right)^2+(n+1)^3=\frac{n^2(n+1)^2}{4}+\frac{4(n+1)^3}{4}$$

$$=\frac{(n+1)^2}{4}\cdot(n^2+4n+4)=\frac{(n+1)^2(n+2)^2}{4}=\left(\frac{(n+1)(n+2)}{2}\right)^2$$

  and hence $P(n+1)$ is true. ✓

- By principle of induction, $P(n)$ is true for all $n\in\mathbb{N}$.

  $1^3+2^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2 \quad \forall n\in\mathbb{N}$ $\square$

Exercise 1.1.4: *Let S be an ordered set. Let $B \subset S$ be bounded (above and below). Let $A \subset B$ be a nonempty subset. Suppose all the* infs *and* sups *exist. Show that*

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

Since $A \subset B$, an upper bound of B is also an upper bound of A. Therefore, $\sup B$ is an upper bound of B and is also an upper bound of A.

By definition, $\sup A$ satisfies that for any upper bound a of A, $\sup A \leq a$. Therefore, $\sup A \leq \underbrace{\sup B}_{\text{upper bound of } A}$ (1)

Similarly, $\inf B \leq \inf A$. (2)

By definition, if $\inf A$ and $\sup A$ exists,

$\inf A \leq x \leq \sup A \quad \forall x \in A$. (3)

From (1), (2), (3), we have $\inf B \leq \inf A \leq \sup A \leq \sup B$ $\square$

Exercise 1.1.5: *Let S be an ordered set. Let $A \subset S$ and suppose b is an upper bound for A. Suppose $b \in A$. Show that $b = \sup A$.*

For any $\varepsilon > 0$, by definition, since $b \in A$ and $b > b - \varepsilon$,

$b - \varepsilon$ is not an upper bound for A. Therefore,

b would be the least upper bound for A

as any $b - \varepsilon$ is not an upper bound for A.

$b = \sup A$. $\square$

Exercise 1.2.1: *Prove that if $t > 0$ ($t \in \mathbb{R}$), then there exists an $n \in \mathbb{N}$ such that $\frac{1}{n^2} < t$.*

$t > 0$ ($t \in \mathbb{R}$), so $\sqrt{t} > 0$ ($\sqrt{t} \in \mathbb{R}$)

The Archimedean property claims that:

Let $x, y \in \mathbb{R}$ with $x > 0$. Then, $\exists\, n \in \mathbb{N}$ with $nx > y$

In this case, $\sqrt{t}, 1 \in \mathbb{R}$ with $\sqrt{t} > 0$.

Therefore, $\exists\, n \in \mathbb{N}$ s.t. $n \cdot \sqrt{t} > 1$

Square both sides: $n^2 \cdot t > 1$

Divide both sides by n^2: $t > \frac{1}{n^2}$, $\frac{1}{n^2} < t$ $\square$

($n > 0$ so $n^2 > 0$)

Exercise 1.2.2: *Prove that if $t \geq 0$ ($t \in \mathbb{R}$), then there exists an $n \in \mathbb{N}$ such that $n - 1 \leq t < n$.*

The Archimedean property claims that:

Let $x, y \in \mathbb{R}$ with $x > 0$. Then, $\exists\, n \in \mathbb{N}$ with $nx > y$

Define $A = \{k \in \mathbb{N} \text{ s.t. } k > t\} \neq \emptyset$ by AP.

By well-ordering, $\exists\, n = \min A$ s.t. $n \leq k \ \forall k \in A$

Since $n = \min A \in A$, we also have $n \in \mathbb{N}, n \in A, n > t$. (1)

If $n = 1$, $n - 1 = 0$. Given that $t \geq 0$, substitute

and combine with (1), we have $n - 1 \leq t < n$ ✓ (2)

If $n > 1$, $n - 1 > 0$. $n \in \mathbb{N}$ so $n - 1 \in \mathbb{N}$ too,

but $n - 1 \notin A$ because $n = \min A$. Therefore, while $n - 1$

satisfies that $n - 1 \in \mathbb{N}$, it does not satisfy the

statement that $n - 1 > t$, which means $n - 1 \leq t$.

Combine with (1) we have $n - 1 \leq t < n$. ✓ (3)

From (2) and (3), we conclude that if $t \geq 0$ ($t \in \mathbb{R}$),

$\exists\, n \in \mathbb{N}$ s.t. $n - 1 \leq t < n$

$\square$

Exercise 1.2.7: *Prove the* arithmetic-geometric mean inequality. *That is, for two positive real numbers x, y, we have*

$$\sqrt{xy} \leq \frac{x+y}{2}.$$

Furthermore, equality occurs if and only if $x = y$.

Since $x \in \mathbb{R}$ and $y \in \mathbb{R}$, we have $x - y \in \mathbb{R}$.

Therefore, $(x-y)^2 = x^2 - 2xy + y^2 \geqslant 0 \quad x^2 + 2xy + y^2 \geqslant 4xy \quad (1)$

$xy > 0$ because $x > 0$ and $y > 0$, so take the square root of both sides of (1):

$(x+y)^2 \geqslant 4xy \Rightarrow x+y \geqslant 2\sqrt{xy}$, $\sqrt{xy} \leq \frac{x+y}{2}$

Furthermore, if $x = y$, $\sqrt{xy} = \sqrt{x^2} = x$ because $x > 0$

$\frac{x+y}{2} = \frac{2x}{2} = x$, $x = x$, so $\sqrt{xy} = \frac{x+y}{2}$. On the other hand,

if $\sqrt{xy} = \frac{x+y}{2}$, square both sides to get $xy = \frac{x^2+2xy+y^2}{2}$

$2xy = x^2 + 2xy + y^2 \Rightarrow x^2 + y^2 = 0 \Rightarrow x = y = 0$

Therefore, we can conclude that $\sqrt{xy} = \frac{x+y}{2}$ if and only if $x = y$. $(x, y > 0,\ x, y \in \mathbb{R})$

$\square$

Exercise 1.2.9: *Let A and B be two nonempty bounded sets of real numbers. Let* $C := \{a+b : a \in A, b \in B\}$*. Show that C is a bounded set and that*

$$\sup C = \sup A + \sup B \quad \text{and} \quad \inf C = \inf A + \inf B.$$

Since A and B are two nonempty bounded sets of real numbers,

$\exists \sup A \geq a \ \forall a \in A$ and $\exists \sup B \geq b \ \forall b \in B$

$\sup A + \sup B \geq a+b \ \forall a \in A \ \forall b \in B$

Therefore, $\sup A + \sup B$ is an upper bound for $C := \{a+b : a \in A, b \in B\}$

Similarly, $\exists \inf A \leq a \ \forall a \in A$, $\exists \inf B \leq b \ \forall b \in B$

$\inf A + \inf B \leq a+b \ \forall a \in A \ \forall b \in B \Rightarrow \inf A + \inf B$ lower bound for C

Therefore, C is bounded both below and above.

So $\sup C$ and $\inf C$ exists.

Furthermore, $\sup C \leq \sup A + \sup B$ because $\sup A + \sup B$ is an upper bound for C.

If $\sup C < \sup A + \sup B$, $\sup A + \sup B - \sup C > 0$

From $\sup A$ and $\sup B$ we have

$\forall \varepsilon_1 > 0, \exists a \in A$ s.t. $\sup A - \varepsilon_1 < a$

$\forall \varepsilon_2 > 0, \exists b \in B$ s.t. $\sup B - \varepsilon_2 < b$

Add them together, $\sup A + \sup B - (\varepsilon_1 + \varepsilon_2) < a+b \quad (1)$

Since ε_1 and ε_2 are any arbitrary positive numbers, Let $\varepsilon_1 + \varepsilon_2 = \sup A + \sup B - \sup C > 0$

Plug in (1): $\sup A + \sup B - \sup A - \sup B + \sup C = \sup C < a+b$

However, $\sup C \geq a+b \quad \forall a \in A \ \forall b \in B$, contradiction.

Therefore, we eliminate the possibility that

$\sup C < \sup A + \sup B$

Therefore, $\sup C = \sup A + \sup B$

Similarly, $\inf C \geq \inf A + \inf B$

By contradiction, $\inf C > \inf A + \inf B$ is false.

Therefore, $\inf C = \inf A + \inf B$ $\square$

Exercise 1.2.13: *Prove the so-called* Bernoulli's inequality*: *If* $1+x>0$, *then for all* $n \in \mathbb{N}$, *we have* $(1+x)^n \geq 1+nx$.

Proof by induction.

Let $P(n)$ be the statement that

$(1+x)^n \geqslant 1+nx$ is true $\forall n \in \mathbb{N}$ if $1+x>0$

- $P(1)$ is true by plugging in $n=1$: $(1+x)^1 \geqslant 1+1 \cdot x$ ✓

- Assume $P(n)$ is true, that is, $(1+x)^n \geqslant 1+nx$ holds

  Since $1+x>0$, $(1+x)^n \cdot (1+x) \geqslant (1+nx)(1+x) = 1+nx+x+nx^2$ (1)

  Since $n \in \mathbb{N}$, $nx^2 \geqslant 0$, so $1+nx+x+nx^2 \geqslant 1+nx+x = 1+(n+1)x$ (2)

  Combining (1) and (2), $(1+x)^{n+1} \geqslant 1+(n+1)x$ ✓

  and hence $P(n+1)$ is true.

- By principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$.

  $(1+x)^n \geqslant 1+nx$ is true $\forall n \in \mathbb{N}$ if $1+x>0$

$\square$

Exercise 1.3.5: *Let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be functions (D nonempty).*

a) Suppose $f(x) \le g(y)$ for all $x \in D$ and $y \in D$. Show that

$$\sup_{x\in D} f(x) \le \inf_{x\in D} g(x).$$

b) Find a specific D, f, and g, such that $f(x) \le g(x)$ for all $x \in D$, but

$$\sup_{x\in D} f(x) > \inf_{x\in D} g(x).$$

a) $f(x) \le g(y) \ \forall x \in D \ \forall y \in D$

So $\sup\limits_{x\in D} f(x)$ is lower bound of $g(y)$

$\sup\limits_{x\in D} f(x) \le g(y)$. Similarly, $\sup\limits_{x\in D} f(x) \le \inf\limits_{x\in D} g(y)$

Therefore, $\sup\limits_{x\in D} f(x) \le \inf\limits_{x\in D} g(x)$ $\square$

b) $D = (-1, 1)$

$f = -x^2$

$g = -x^2 + 1$

$\sup\limits_{x\in D} f(x) = 0 > \inf\limits_{x\in D} g(x) = -1$

Exercise 1.3.9: *Let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be functions, $\alpha \in \mathbb{R}$, and recall what $f+g$ and αf means from the previous exercise.*

a) *Prove that if $f+g$ and g are bounded, then f is bounded.*

b) *Find an example where f and g are both unbounded, but $f+g$ is bounded.*

c) *Prove that if f is bounded but g is unbounded, then $f+g$ is unbounded.*

d) *Find an example where f is unbounded but αf is bounded.*

a) $f+g$ and g are bounded, so let $|f+g| \le C$ $(C>0)$

and $|g| \le B$ $(B>0)$. By the triangle inequality,

$$|f| = |f+g-g| \le |f+g| + |-g| = |f+g| + |g|$$
$$\le C + B$$

Therefore, if $f+g$ and g are bounded, f is also bounded $\square$

b) $f = x$, $g = -x$, $f+g = x + (-x) = 0$

c) Prove by contradiction. Assume $f+g$ is bounded. From a),

Since $f+g$ and f are bounded, then g must be bounded.

But given that g is unbounded, contradiction.

Therefore, if f is bounded and g is unbounded,

$f+g$ is unbounded. $\square$

d) $f = x$, $\alpha = 0$