Exercise 2.1.3: *Is the sequence* $\left\{\frac{(-1)^n}{2n}\right\}$ *convergent? If so, what is the limit?*

$\lim_{n\to\infty} \frac{(-1)^n}{2n} = 0$, so the limit of the sequence $\left\{\frac{(-1)^n}{2n}\right\}$ is 0.

(oscillates between $\frac{-1}{\infty}$ and $\frac{1}{\infty}$)

Let ε be an arbitrary positive number.

W.T.S. $\exists N$ s.t. $\forall n \geq N$, $\left|\frac{(-1)^n}{2n} - 0\right| < \varepsilon \quad \forall \varepsilon > 0$

$$\left|\frac{(-1)^n}{2n} - 0\right| = \frac{1}{2n}$$

$\exists N = \lceil \frac{1}{2\varepsilon} \rceil$ s.t. $\frac{1}{2N} < \varepsilon$. $\forall \varepsilon > 0$

$\forall n \geq N$, $\left|\frac{(-1)^n}{2n} - 0\right| = \frac{1}{2n} \leq \frac{1}{2N} < \varepsilon$

Therefore, by definition, $\left\{\frac{(-1)^n}{2n}\right\}$ is convergent $\square$

Exercise 2.1.6: *Is the sequence* $\left\{\frac{n}{n^2+1}\right\}$ *convergent? If so, what is the limit?*

$\lim_{n\to\infty} \frac{n}{n^2+1} = 0$, so the limit of the sequence $\left\{\frac{n}{n^2+1}\right\}$ is 0.

Let ε be an arbitrary positive number.

W.T.S. $\exists N$ s.t. $\forall n \geq N$, $\left|\frac{n}{n^2+1} - 0\right| < \varepsilon$

$$\left|\frac{n}{n^2+1} - 0\right| = \frac{n}{n^2+1}$$

$\exists N = \lceil \frac{1}{\varepsilon} \rceil$ s.t. $\frac{1}{N} < \varepsilon$.

$\forall n \geq N$, $\left|\frac{n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$

Therefore, by definition, $\left\{\frac{n}{n^2+1}\right\}$ is convergent

$\square$

Exercise 2.1.7: *Let $\{x_n\}$ be a sequence.*

a) Show that $\lim x_n = 0$ *(that is, the limit exists and is zero) if and only if* $\lim |x_n| = 0$.

b) Find an example such that $\{|x_n|\}$ *converges and* $\{x_n\}$ *diverges.*

a) $\lim x_n = 0 \Rightarrow \lim |x_n| = 0$ $\qquad x = \lim |x_n| = 0$

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon \quad \forall n \geq N$ $\qquad ||x_n| - \quad < \varepsilon$

$x = \lim x_n = 0$, so $|x_n| < \varepsilon$

So $|x_n| - 0 < \varepsilon$, $||x_n| - 0| < \varepsilon$, $||x_n| - x| < \varepsilon$, which satisfies that

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $||x_n| - x| < \varepsilon \quad \forall n \geq N$

So $\lim |x_n| = 0$

$\lim |x_n| = 0 \Rightarrow \lim x_n = 0$

Similarly, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $||x_n| - x| < \varepsilon \quad \forall n \geq N$

$x = \lim |x_n| = 0$, so $||x_n|| < \varepsilon \qquad |x_n| < \varepsilon$

So $|x_n - 0| < \varepsilon$, $|x_n - x| < \varepsilon$, which satisfies that

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon \quad \forall n \geq N$

So $\lim x_n = 0$ $\qquad \square$

b) $\{(-1)^n\}$ diverges

$\{|(-1)^n|\}$ converges

Exercise 2.1.13: *Let $\{x_n\}$ be a convergent monotone sequence. Suppose there exists a $k \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} x_n = x_k.$$

Show that $x_n = x_k$ for all $n \geq k$.

We know $\{X_n\}$ is convergent monotone sequence.

Assume it's monotone increasing, then

$X_k = \lim_{n\to\infty} X_n = \sup\{X_n : n \in \mathbb{N}\}$.

X_k $(k \in \mathbb{N})$ is within $\{X_n : n \in \mathbb{N}\}$

So $X_k \leq X_n \quad \forall n \geq k$

Since $\{X_n\}$ is monotone increasing, $X_k = X_n \ \forall n \geq k$

Similarly, if $\{X_n\}$ is monotone decreasing,

$X_k = \lim_{n\to\infty} X_n = \inf\{X_n : n \in \mathbb{N}\} \Rightarrow X_k \geq X_n \Rightarrow X_k = X_n \ \forall n \geq k$

$\square$

Exercise 2.1.16: *Let $\{x_n\}$ be a sequence. Suppose there are two convergent subsequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$. Suppose*

$$\lim_{i\to\infty} x_{n_i} = a \qquad \textit{and} \qquad \lim_{i\to\infty} x_{m_i} = b,$$

where $a \neq b$. Prove that $\{x_n\}$ is not convergent, without using Proposition 2.1.17.

Assume $\{x_n\}$ is convergent. Let $\lim x_n = L$, such that

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|x_n - L| < \varepsilon$ $\forall n \geq N$ (1)

We know that $\lim_{i\to\infty} x_{n_i} = a$, $\lim_{i\to\infty} x_{m_i} = b$, $a \neq b$

So either $L \neq a \neq b$, or $L \neq a$, or $L \neq b$

Assume $L \neq a$, let $k = |L - a| > 0$ (2)

Since $\{x_{n_i}\}$ is convergent,

$\forall \varepsilon > 0$ $\exists M \in \mathbb{N}$ s.t. $\forall n_i \geq M$,

$|x_{n_i} - a| < \varepsilon$

Choose $\varepsilon = \frac{k}{2} > 0$, so $|x_{n_i} - a| < \frac{k}{2}$ (3)

From (2), $k = |L - a| = |L - x_{n_i} + x_{n_i} - a| \leq |L - x_{n_i}| + |x_{n_i} - a|$

From (3), $k < |L - x_{n_i}| + \frac{k}{2}$, So $|L - x_{n_i}| > \frac{k}{2} = \varepsilon$ (4)

(1) and (4) contradiction.

Similarly, if we otherwise assume $L \neq b$, $\varepsilon = \frac{k}{2} = \frac{|L-b|}{2} > 0$

(4) will become $|L - x_{m_i}| > \frac{k}{2} = \varepsilon$ (5)

(1) and (5) contradiction.

Therefore, $\{x_n\}$ is not convergent. $\square$

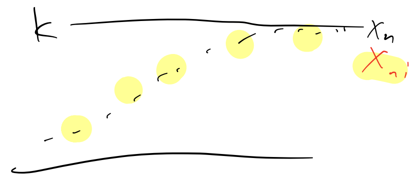
Exercise 2.1.23: *Suppose that $\{x_n\}$ is a monotone increasing sequence that has a convergent subsequence. Show that $\{x_n\}$ is convergent. Note: So Proposition 2.1.17 is an "if and only if" for monotone sequences.*

$\{x_n\}$ is monotone increasing, so its subsequences are also monotone increasing. Let $\{x_{n_i}\}$ be the given convergent subsequence.

Since $\{x_{n_i}\}$ is convergent and monotone increasing,

$\lim\limits_{i\to\infty} x_{n_i} = \sup\{x_{n_i} : i \in \mathbb{N}\} = k$, so $x_n \leq k \ \forall n \in \mathbb{N}$

By Proposition 2.1.10, since $\{x_n\}$ is upper bounded, it is convergent



$\square$

Exercise 2.2.4: *Suppose* $x_1 := \frac{1}{2}$ *and* $x_{n+1} := x_n^2$. *Show that* $\{x_n\}$ *converges and find* $\lim x_n$. *Hint: You cannot divide by zero!*

$x_1 = \frac{1}{2} < 1$ $\quad 0 < x_2 = \frac{1}{4} < x_1 = \frac{1}{2}$ $\quad$ so $\quad x_{n+1} = x_n^2 < x_n \quad \forall n \in \mathbb{N}$

$0 < x_n < \frac{1}{2} \quad \forall n \in \mathbb{N}$

$\{x_n\}$ is decreasing and lower bounded, so it converges

$\lim\limits_{n\to\infty} x_n = L$ $\quad$ Let $n \to \infty$, $L = L^2$, $L^2 - L = 0$, $L = 0$ or 1

Since $\{x_n\}$ is monotone decreasing, $x_n < \frac{1}{2}$ $\forall n \in \mathbb{N}$,

$L \neq 1$. Therefore, $\lim x_n = 0$

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Exercise 2.2.5: *Let* $x_n := \frac{n-\cos(n)}{n}$. *Use the squeeze lemma to show that* $\{x_n\}$ *converges and find the limit.*

$$-1 \leq \cos(n) \leq 1$$

$$-\frac{1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n}$$

$$-\frac{1}{n} \leq -\frac{\cos(n)}{n} \leq \frac{1}{n}$$

$$1-\frac{1}{n} \leq \underbrace{1-\frac{\cos(n)}{n}}_{= \frac{n-\cos(n)}{n}} \leq 1+\frac{1}{n}$$

Since $\lim_{n\to\infty} \{1-\frac{1}{n}\} = 1 = \lim_{n\to\infty} \{1+\frac{1}{n}\}$,

by squeeze lemma, $\lim_{n\to\infty} \{x_n\} = 1$

So $\{x_n\}$ converges to 1

Exercise 2.2.12:

a) Suppose $\{a_n\}$ is a bounded sequence and $\{b_n\}$ is a sequence converging to 0. Show that $\{a_n b_n\}$ converges to 0.

b) Find an example where $\{a_n\}$ is unbounded, $\{b_n\}$ converges to 0, and $\{a_n b_n\}$ is not convergent.

c) Find an example where $\{a_n\}$ is bounded, $\{b_n\}$ converges to some $x \neq 0$, and $\{a_n b_n\}$ is not convergent.

a) $\exists B \in \mathbb{R}$ s.t. $|a_n| \leq B \quad \forall n \in \mathbb{N}$

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $|b_n - 0| = |b_n| < \varepsilon \quad \forall n \geq N$

Therefore, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $|a_n b_n - 0| < B \cdot \varepsilon = \varepsilon_0 \quad \forall \varepsilon_0 > 0$

b). $a_n := n^2 \qquad b_n := \frac{1}{n} \qquad a_n b_n = n$ not convergent

c) $a_n := (-1)^n \qquad b_n := 1 \qquad a_n b_n = (-1)^n$ not convergent

Exercise 2.2.14: *Suppose $x_1 := c$ and $x_{n+1} := x_n^2 + x_n$. Show that $\{x_n\}$ converges if and only if $-1 \le c \le 0$, in which case it converges to 0.*

$-1 \le c \le 0 \Rightarrow \{x_n\}$ converges to 0

$x_{n+1} = x_n^2 + x_n > x_n$, so $\{x_n\}$ is monotonic increasing.

$\lim x_n = L \qquad L = L^2 + L \qquad L = 0$

$\{x_n\}$ converges to $0 \Rightarrow -1 \le c \le 0$

If $c > 0$, $\{x_n\}$ is monotone increasing, $\{x_n\}$ is bounded below by c, so $\lim x_n \geq c > 0 \Rightarrow \Leftarrow$

If $c < -1$, $\{x_n\}$ is monotone increasing with no upper bound.

So $-1 \le c \le 0$