4.2.1, 3, 5, 6, 7, 8, 9, 10, 12, 15

***Exercise* 4.2.1:** *Finish the proof of Proposition 4.2.7.*

Proposition 4.2.7. *Let I be an interval and let $f: I \to \mathbb{R}$ be a differentiable function.*

(i) f is increasing if and only if $f'(x) \geq 0$ for all $x \in I$.

(ii) f is decreasing if and only if $f'(x) \leq 0$ for all $x \in I$.

(ii) $\Rightarrow$ Assume $f'(x) \leq 0 \ \forall x \in I$. WLOG, take $x, y \in I$ s.t. $x < y$. Since $[x, y] \subset I$, by MVT, $\exists c \ (x, y)$ s.t. $f(y) - f(x) = f'(c)(y - x)$

Since $f'(c) \leq 0$, $y - x > 0$, $f(y) - f(x) \leq 0$

$f(y) \leq f(x) \ \forall [x, y] \subset I$, $x < y$

So f is decreasing

$\Leftarrow$ Assume f is decreasing, $\forall x, c \in I$ with $x \neq c$, $\dfrac{f(x) - f(c)}{x - c} \leq 0$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \leq 0 \quad \forall c \in I$$

Exercise 4.2.3: *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that f' is a bounded function. Prove that f is a Lipschitz continuous function.*

f' is bounded, so $\exists M \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}$

$|f'(x)| \leq M$. Take any $[a, b] \subset \mathbb{R}$,

f is continuous on $[a, b]$ and

differentiable on (a, b), by MVT,

$\exists\, c \in (a, b)$ s.t. $f(b) - f(a) = f'(c)(b-a)$

$|f(b) - f(a)| \leq M(b-a) \quad \forall\, a, b \in \mathbb{R}$

So f is Lipschitz continuous

***Exercise* 4.2.5:** *Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that $|f(x) - f(y)| \leq |x-y|^2$ for all x and y. Show that $f(x) = C$ for some constant C. Hint: Show that f is differentiable at all points and compute the derivative.*

***Exercise* 4.2.6:** *Finish the proof of Proposition 4.2.8. That is, suppose I is an interval and $f\colon I \to \mathbb{R}$ is a differentiable function such that $f'(x) > 0$ for all $x \in I$. Show that f is strictly increasing.*

***Exercise* 4.2.7:** *Suppose $f\colon (a,b) \to \mathbb{R}$ is a differentiable function such that $f'(x) \neq 0$ for all $x \in (a,b)$. Suppose there exists a point $c \in (a,b)$ such that $f'(c) > 0$. Prove $f'(x) > 0$ for all $x \in (a,b)$.*

***Exercise* 4.2.8:** *Suppose $f\colon (a,b) \to \mathbb{R}$ and $g\colon (a,b) \to \mathbb{R}$ are differentiable functions such that $f'(x) = g'(x)$ for all $x \in (a,b)$, then show that there exists a constant C such that $f(x) = g(x) + C$.*

***Exercise* 4.2.9:** *Prove the following version of L'Hôpital's rule. Suppose $f\colon (a,b) \to \mathbb{R}$ and $g\colon (a,b) \to \mathbb{R}$ are differentiable functions and $c \in (a,b)$. Suppose that $f(c) = 0$, $g(c) = 0$, $g'(x) \neq 0$ when $x \neq c$, and that the limit of $f'(x)/g'(x)$ as x goes to c exists. Show that*

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

Compare to Exercise 4.1.15. Note: Before you do anything else, prove that $g(x) \neq 0$ when $x \neq c$.

***Exercise* 4.2.10:** *Let $f\colon (a,b) \to \mathbb{R}$ be an unbounded differentiable function. Show $f'\colon (a,b) \to \mathbb{R}$ is unbounded.*

4.2.5. $|f(x) - f(y)| \leq |x-y|^2$

$$-|x-y| \leq \frac{f(x)-f(y)}{x-y} \leq |x-y|$$

Since $\lim_{x \to y} -|x-y| = \lim_{x \to y} |x-y| = 0$,

$$-0 \leq \lim_{x \to y} \frac{f(x)-f(y)}{x-y} \leq 0$$

Therefore, $f'(x) = 0 \ \forall x$, which means

$f(x) = C \ \forall x$ for some costant $C \in \mathbb{R}$

4.2.6 Assume $f'(x) > 0 \ \forall x \in I$. WLOG, take $x, y \in I$ s.t. $x < y$. Since $[x, y] \subset I$, by MVT, $\exists c \in (x, y)$ s.t.

$$f(y) - f(x) = f'(c) \cdot (y - x)$$

Since $f'(c) > 0$, $y - x > 0$, $f(y) - f(x) > 0$

$f(y) > f(x) \quad \forall [x, y] \subset I$, $x < y$

So f is strictly increasing

4.2.7 Since $f'(x) \neq 0 \ \forall x \in (a,b)$,

Either $f'(x) < 0$ or $f'(x) > 0 \quad \forall x \in (a,b)$

Assume $\exists d \in (a,b)$ s.t. $f'(d) < 0$.

Given that $\exists c \in (a,b)$ s.t. $f'(c) > 0$.

Since $f : (a,b) \to \mathbb{R}$ is differentiable,

$f' : (a,b) \to \mathbb{R}$ is continuous

$f' : [d,c] \to \mathbb{R}$ is continuous.

By IVT, $\exists x \in (d,c)$ s.t. $f'(x) = 0$

Contradiction. Therefore, $\forall x \in (a,b)$,

neither $f'(x) < 0$ or $f'(x) = 0$,

which means $f'(x) > 0 \ \forall x \in (a,b)$.

4.2.8 Define $h:(a,b) \to \mathbb{R}$ by

$h(x) := f(x) - g(x)$, so $h'(x) = f'(x) - g'(x)$

Since $f'(x) = g'(x)$ $\forall x \in (a,b)$,

$h'(x) = 0$ $\forall x \in (a,b)$. Therefore,

$\exists C$ s.t. $h(x) = C$ $\forall x \in (a,b)$

which means there exists a

constant C s.t. $f(x) = g(x) + C$, $\forall x \in (a,b)$

4.2.9. $g'(x) \neq 0 \ \forall x \neq c$, and $g:(a,b) \to \mathbb{R}$ is continuous since $g:(a,b) \to \mathbb{R}$ is differentiable

So $\forall x > c$, $g'(x)$ can't have jump discontinuity from $g'(x) > 0$ to $g'(x) < 0$ or from $g'(x) < 0$ to $g'(x) > 0$. So either $g'(x) > 0$ or $g'(x) < 0 \ \forall x > c$. So $g(x) \neq 0 \ \forall x > c$. Similarly, $g(x) \neq 0 \ \forall x < c$. So $g(x) \neq 0 \ \forall x \neq c$.

$\forall x > c$, since f and g are continuous on $[c, x]$ and differentiable on (c, x), by MVT, $\exists\, k \in (c, x)$ s.t. $\frac{f'(k)}{g'(k)} = \frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f(x)}{g(x)}$

So $\lim_{x \to c} \frac{f'(x)}{g'(x)} = \lim_{x \to c} \frac{f(x)}{g(x)} \ \forall x > c$. Similarly,

$\lim_{x \to c} \frac{f'(x)}{g'(x)} = \lim_{x \to c} \frac{f(x)}{g(x)} \ \forall x < c$

4.2.10 f is unbounded $\forall M \in \mathbb{R}$

$\exists x \in \mathbb{R}$ s.t. $|f(x)| > M$.

Take any $[a, b] \subset \mathbb{R}$, by MVT,

$\exists c \in (a, b)$ s.t. $f(b) - f(a) = f'(c)(b-a)$

Assume f' is bounded. $|f'(c)| \leq N$

$|f(b) - f(a)| \leq N(b-a)$. However this

is impossible because $\forall a \in \mathbb{R}$,

$\exists b \in \mathbb{R}$ s.t. $|f(b) - f(a)| > N(b-a)$

Contradiction.

Therefore, f' is unbounded.

4.2.12. $f(x) = ax+b$

$f'(x) = a \quad \forall x \in \mathbb{R}$

So $f(x_1) - f(x_2) = a(x_1 - x_2) \quad \forall x_1, x_2 \in \mathbb{R}$

It always holds when $x_1 = x_2$, so

when $x_2 = 0$, $f(x_1) - b = a \cdot x_1$

$f(x_1) = ax_1 + b \quad \forall x_1 \neq x_2$

So $f(x) = ax+b \quad \forall x \in \mathbb{R}$

***Exercise* 4.2.15:** *Prove Theorem 4.2.5.*

Theorem 4.2.5 (Cauchy's mean value theorem)**.** *Let $f\colon [a,b] \to \mathbb{R}$ and $\varphi\colon [a,b] \to \mathbb{R}$ be continuous functions differentiable on (a,b). Then there exists a point $c \in (a,b)$ such that*

$$\big(f(b) - f(a)\big)\varphi'(c) = f'(c)\big(\varphi(b) - \varphi(a)\big).$$

4.2.15. Define $g: [a,b] \to \mathbb{R}$ by

$$g(x) := f(x) - f(b) - \frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}(\varphi(x) - \varphi(b))$$

g is differentiable on (a,b), continuous on $[a,b]$, and $g(a) = 0$, $g(b) = 0$. By Rolle's theorem, $\exists\, c \in (a,b)$ s.t. $g'(c) = 0$, which gives

$$g'(c) = f'(c) - \frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}\varphi'(c) = 0$$

$$f'(c) = \frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}\varphi'(c)$$

$$(f(b)-f(a))\varphi'(c) = f'(c)(\varphi(b) - \varphi(a))$$