5.2.4 We can use FTC+MVT or Min/max + IVT prove this statement.

<u>FTC + MVT</u>: Define $F(x)=\int_a^x f(t)\,dt$, which is well-defined and differentiable since f is continuous with $F'=f$ by the FCT. Applying MVT on F, $\exists c \in (a,b)$

$$\text{with } F'(c)=\frac{F(b)-F(a)}{b-a} \iff f(c)=\frac{\int_a^b f(t)\,dt}{b-a} \iff$$

$$\iff f(c)(b-a)=\int_a^b f(t)\,dt.$$

<u>Min/max + IVT</u>: Since f is continuous, by the min-Max theorem, $\exists\, x_{min}, x_{max} \in [a,b]$ with

$$f(x_{min}) \le f(x) \le f(x_{max}) \quad \forall x \in [a,b].$$

$$\text{Thus } f(x_{min})(b-a) \le \int_a^b f \le f(x_{max})(b-a)$$

$$\Rightarrow f(x_{min}) \le \frac{\int_a^b f}{b-a} \le f(x_{max}).$$

By the IVT, $\exists c$ between x_{min} and x_{max} with

$$f(c)=\frac{\int_a^b f}{b-a} \iff f(c)(b-a)=\int_a^b f.$$

5.3.1 By FTC $\frac{d}{dx}\int_{-x}^{x} e^{s^2}\,ds =$

$= e^{x^2} - (-1)e^{x^2} = 2e^{x^2}$.

5.3.2 $\frac{d}{dx}\int_0^{x^2} \sin(s^2)\,ds \overset{FTC}{=} \sin(x^4)(x^2)' =$

$= 2x\sin(x^4)$.

5.3.4 Let $c \in [a,b]$ and $F(x) = \int_c^x f(t)\,dt$.

Notice that $\int_a^x f(t)\,dt = \int_a^c f(t)\,dt + F(x)$. $\Rightarrow$

$\Rightarrow F(x) = \int_a^x f(t)\,dt - \int_a^c f(t)\,dt$.

By the FTC F is differentiable

and $F'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt - \int_a^c f(t)\,dt\right) = f(x)$.

5.3.5. By the product rule, we have:

$$\left(F(x)G(x)\right)' = F'(x)G(x) + F(x)G'(x)$$

$$\Rightarrow \int_a^b \left[F(x)G(x)\right]' dx = \int_a^b F'(x)G(x)\,dx + \int_a^b F(x)G'(x)\,dx$$

$$\overset{FTC}{\Rightarrow} \quad F(b)G(b) - F(a)G(a) = \int_a^b F'(x)G(x)\,dx + \int_a^b F(x)G'(x)\,dx$$

$$\Leftrightarrow \int_a^b F(x)G'(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b F'(x)G(x)\,dx$$

5.3.6. Since F, G are continuously differentiable, F', G' are continuous thus integrable.

So $F'(t) = G'(t) \ \forall t \in [a,b] \Rightarrow$

$$\Rightarrow \int_a^x F'(t)\,dt = \int_a^x G'(t)\,dt \quad \overset{FTC}{\Rightarrow}$$

$$\Rightarrow F(x) - F(a) = G(x) - G(a) \Rightarrow$$

$$\Rightarrow F(x) - G(x) = F(a) - G(a) := C.$$

~~5.3.8.~~ We have $\int_a^x f(t)dt = \int_x^b f(t)dt \quad \forall x \; (1)$.

Since f is continuous we can differentiate (1) and use FTC_1 to obtain

$$f(x) = -f(x) \quad \forall x \Rightarrow f(x) = 0 \quad \forall x.$$

~~5.3.10.~~ a) We have $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx.$ (1)

But $\int_{-a}^{0} f(x)dx \overset{odd}{=} \int_{-a}^{0} -f(-x)dx = -\int_{-a}^{0} f(-x)dx =$

$\overset{u=-x}{=} -\int_0^a f(u)du$.

Thus by (1) $\int_{-a}^{a} f(x)dx = 0.$

b) Expression (1) still holds. But now:

But now $\int_{-a}^{0} f(x)dx \overset{even}{=} \int_{-a}^{0} f(-x)dx \overset{u=-x}{=} \int_0^a f(u)du$

This (1) implies $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.