4.2.1. We prove that f is decreasing i.f.f $f'(x) \leq 0$.

Let f be decreasing. Then for any $x \neq y$ we have $$\frac{f(x)-f(y)}{x-y} \leq 0.$$

Taking $x \to y$ and using the fact that f is differentiable at y we take $f'(y) \leq 0$.

Let $f'(x) \leq 0 \quad \forall x \in I$.

Let $x, y \in I$ with $x < y$. We will show $f(x) \geq f(y)$. Apply MVT in $[x, y]$. Then $\exists\, \xi \in (x, y)$ s.t. $$f'(\xi) = \frac{f(y)-f(x)}{y-x}.$$

But $f'(\xi) \leq 0$ so $$\frac{f(y)-f(x)}{y-x} \leq 0.$$

Since $x < y$, we take $f(x) \geq f(y)$ so f is decreasing.

4.2.3. Since f' is bounded, $\exists M > 0$ s.t. $|f'(x)| \leq M \quad \forall x \in \mathbb{R}$.

Let $x, y \in \mathbb{R}$. Assume WLOG that $x < y$.

Then by MVT $\exists \xi \in (x, y)$ s.t.

$$\frac{f(x) - f(y)}{x - y} = f'(\xi) \Rightarrow$$

$$\Rightarrow |f(x) - f(y)| = |f'(\xi)| \, |x - y| \leq M |x - y|.$$

so f is Lipschitz.

4.2.5. Let $x \neq y$ Then $|f(x) - f(y)| \leq |x - y|^2 \Rightarrow$

$$\Rightarrow \left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|$$

so by squeeze theorem $\lim\limits_{x \to y} \frac{f(x) - f(y)}{x - y} = 0$.

so f is differentiable at y and $f'(y) = 0$.

thus f is constant.

4.2.6. Assume $f'(x) > 0$. Let $x<y$. We will show $f(x) < f(y)$.

By MVT $\exists\, \xi \in (x,y)$ with $\frac{f(x)-f(y)}{x-y} = f'(\xi)$.

since $f' > 0$ we take $\frac{f(x)-f(y)}{x-y} > 0$

since $x<y$, we take $f(x) < f(y)$. Thus f is strictly increasing.

4.2.7. I accidentally assigned this problem since it requires the use of Darboux theorem which was not covered in class. You will all receive full credit for this problem.

Darboux theorem is an IVT for the derivative of a differentiable function, without requiring that f' is continuous. Feel free to read it in the book (Theorem 4.2.11). We will apply it to solve this problem.

Arguing by contradiction assume $\exists\, \tilde{c} \in (a,b)$ with $f'(\tilde{c}) \leq 0$.

If $f'(\tilde{c}) = 0$ we reach contradiction immediately since $f'(x) \neq 0 \quad \forall x \in (a,b)$.

so assume $f'(\tilde{c}) < 0$.

since $f'(c) > 0$ and $f'(\tilde{c}) < 0$

by Darboux's theorem there exists a ξ between c and $\tilde{c}$ with $f'(\xi) = 0$.

This is a contradiction since $f'(x) \neq 0 \ \forall x$.

Thus $f'(x) > 0 \ \forall x$.

4.2.8 Define $h = f - g$. Then $h' = f' - g' = 0$. So h is constant: $h(x) = C \Rightarrow f(x) - g(x) = C$

$\Rightarrow f(x) = g(x) + C \ \forall x$.

4.2.10. Assume f' is bounded. Then by previous problem f is Lipschitz i.e.

$\exists M>0$ s.t. $|f(x)-f(y)| \leq M|x-y| \quad \forall x,y \in (a,b)$.

Since $x,y \in (a,b)$ we have $|x-y| < b-a$.

so $|f(x)-f(y)| \leq M(b-a) \quad \forall x \in (a,b)$

Take $y = \frac{a+b}{2} \in (a,b)$.

Then we have

$$|f(x)| - \left|f\left(\frac{a+b}{2}\right)\right| \leq \left|f(x) - f\left(\frac{a+b}{2}\right)\right| \leq$$

$$\leq M(b-a) \Rightarrow$$

$$\Rightarrow |f(x)| \leq M(b-a) + \left|f\left(\frac{a+b}{2}\right)\right| \quad \forall x \in (a,b).$$

Thus f is bounded, which is a contradiction.

4.2.12. Define $h(x) = f(x) - ax - b$.

Then $h'(x) = f'(x) - a = a - a = 0$
$\forall x$.

so h is constant.

Thus $h(x) = h(0) = f(0) - b = b - b = 0$.

so $h(x) = 0 \quad \forall x \in (a,b)$.

so $f(x) = ax + b \quad \forall x \in (a,b)$.

4.2.15. To prove Cauchy's MVT (which generalizes the regular MVT choosing $\phi(x) = x$), we mimic the proof of the regular MVT. Assume first that $\boxed{\phi(a) \neq \phi(b)}$.

Define $g(x) = f(x) - f(b) - \dfrac{f(b)-f(a)}{\phi(b)-\phi(a)}(\phi(x) - \phi(b))$.

This function is well defined since $\phi(b) \neq \phi(a)$ and differentiable since f, ϕ are. The derivative is $g'(x) = f'(x) - \dfrac{f(b)-f(a)}{\phi(b)-\phi(a)}\phi'(x)$.

Now, $g(a) = f(a) - f(b) - \dfrac{f(b)-f(a)}{\phi(b)-\phi(a)}(\phi(a) - \phi(b))$

$= f(a) - f(b) + f(b) - f(a) = 0$.

$g(b) = 0$.

So by Rolle's theorem $\exists\, \xi \in (a,b)$ with

$$g'(\xi) = 0 \iff f'(\xi) - \frac{f(b)-f(a)}{\phi(b)-\phi(a)}\,\phi'(\xi) = 0$$

$$\Rightarrow \big(\phi(b)-\phi(a)\big) f'(\xi) = \big(f(b)-f(a)\big)\,\phi'(\xi).$$

Now if $\phi(a) = \phi(b)$, by Rolle's theorem $\exists\, \xi \in (a,b)$ with $\phi'(\xi) = 0$. For this ξ the conclusion of the theorem clearly holds.

4.2.9. First, we prove that $g(x) \neq 0$ for $x \neq c$. Let $x \neq c$. Apply MVT between x and c to find a ξ between x and c with

$$g'(\xi) = \frac{g(x) - \cancel{g(c)}}{x-c} \Rightarrow g(x) = g'(\xi)(x-c).$$

since $g'(\xi) \neq 0$ and $x - c \neq 0$ we take $g(x) \neq 0$.

So now $\dfrac{f(x)}{g(x)}$ is perfectly defined when $x \neq c$.

Now call $L = \lim_{x \to c} \dfrac{f'(x)}{g'(x)}$.

Then $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. if $0 < |\xi - c| < \delta$ we have $\left| \dfrac{f'(\xi)}{g'(\xi)} - L \right| < \varepsilon$. $(*)$

Let $x \in (a,b)$ with $0 < |\cancel{\xi} - c| < \delta$.

By Cauchy's MVT $\exists \xi$ between x and c

with $\big(f(x) - f(c)\big) g'(\xi) = \big(g(x) - g(c)\big) f'(\xi)$

$$\overset{f(c)=g(c)=0}{\Longleftrightarrow} \quad f(x)\, g'(\xi) = g(x)\, f'(\xi) \quad \xRightarrow{\substack{g'(\xi) \neq 0 \\ g(x) \neq 0}}$$

$$\Rightarrow \quad \frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}.$$

In particular $|\xi - c| < |x - c| < \delta$ so $(*)$ holds.

Thus we have.

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f'(\xi)}{g'(\xi)} - L \right| < \varepsilon .$$

we conclude that $\displaystyle\lim_{x \to c} \frac{f(x)}{g(x)} = L$.