Problem 1

$$|x_{n+1}-x_n| \le C|x_n-x_{n-1}| \le C^2|x_{n-1}-x_{n-2}| \le C^{n-1}|x_2-x_1|$$

w.t.s. $\forall \varepsilon>0, \exists N\in\mathbb{N}$ s.t. $\forall n\ge N\ \forall m\ge N$, $|x_m-x_n|<\varepsilon$

Without loss of generality, assume $m=n+k$ $(k\ge 0)$

$$|x_m-x_n| = |x_{n+k}-x_n| = |x_{n+k}-x_{n+k-1}+x_{n+k-1}-x_{n+k-2}+x_{n+k-2}\cdots+x_{n+1}-x_n|$$

$$=|(x_{n+k}-x_{n+k-1})-(x_{n+k-2}-x_{n+k-1})-(x_{n+k-3}-x_{n+k-2})-\cdots-(x_n-x_{n+1})|$$

$$\le |x_{n+k}-x_{n+k-1}|+|x_{n+k-1}-x_{n+k-2}|+\cdots+|x_{n+1}-x_n|$$ by triangular inequality

$$\le C^{n+k-2}|x_2-x_1|+C^{n+k-3}|x_2-x_1|+\cdots+C^{n-1}|x_2-x_1|$$

$$=C^{n-1}\cdot|x_2-x_1|\cdot(1+C+C^2+\cdots+C^{k-1})=\frac{1-C^k}{1-C}\cdot C^{n-1}\cdot|x_2-x_1|$$

Since $0<C<1$, $0<1-C<1$, $0<1-C^k<1$, so $\frac{1-C^k}{1-C}\cdot C^{n-1} < C^{n-1}$ <1

Therefore, $\forall \varepsilon>0, \exists N\in\mathbb{N}$ s.t. $\forall n\ge N\ \forall m\ge N$,

$|x_m-x_n|<C^{n-1}|x_2-x_1|<\varepsilon$.

$|x_m-x_n|\to 0$ as $n\to\infty$

so $\{x_n\}$ is Cauchy.

□

Problem 2.

$\lim Y_n = 0$, so $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $|Y_n - 0| = |Y_n| < \varepsilon$ ① $\forall n \geq N$

$|X_m - X_k| \leq |Y_k|$ ② $\forall k \in \mathbb{N}$ $\forall m \geq k$

Since ① satisfies $\forall n \geq N \in \mathbb{N}$ and ② satisfies $\forall k \in \mathbb{N}$,

$\exists Y_k = Y_n$. So $|X_m - X_k| \leq |Y_n| < \varepsilon$, $\{X_n\}$ is Cauchy $\square$

Problem 3

$\forall M \in \mathbb{N}$, $\exists k \geq M$ $\exists n \geq M$ s.t. $x_k < 0$, $x_n > 0$

So $x_n - x_k > x_n$

Since $\{x_n\}$ is Cauchy, by definition, $|x_n - x_k| < \varepsilon$

$|x_n - x_k| \leq |x_n| + |x_k| = x_n - x_k < \varepsilon$

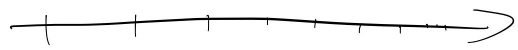
So $x_n < \varepsilon$ $\quad |x_n - 0| = x_n < \varepsilon$

Therefore $\{x_n\}$ converges to 0 $\square$

Problem 4

$$|x_n - x_{n-1}| \leq \frac{n}{(n+1)^2} \qquad \frac{n}{(n-1)^2} > \frac{n+1}{n^2}$$

$$|x_{n+1} - x_n| \leq \frac{n+1}{n^2}$$



So the distance is getting smaller and smaller,

Using the result from Problem 6,

$\{x_n\}$ is Cauchy

$\square$

Problem 5

$\{x_n\}$ is Cauchy, so $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ $\forall k \geq N$,

$|x_n - x_k| < \varepsilon$. Since this satisfies $\forall k \geq N$, and

$\exists$ infinitely many n s.t. $x_n = c$, $\exists k$ s.t. $x_k = c$.

$|x_n - c| < \varepsilon$. Therefore, $\{x_n\}$ converges to c and $\lim x_n = c$.

$\square$

Problem 6

$\{x_n\}$ is Cauchy, so $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ $\forall m \geq N$

$$|x_n - x_m| < \varepsilon$$

$$|x_{n+1} - x_n| = |(x_{n+1} - x_{n-1}) - (x_n - x_{n-1})| \leq |x_{n+1} - x_{n-1}| + |x_n - x_{n-1}|$$

$|x_{n+1} - x_{n-1}| < \frac{\varepsilon}{2}$ $\forall n \geq N_1$, $|x_n - x_{n-1}| < \frac{\varepsilon}{2}$ $\forall n \geq N_2$

Therefore, $\forall n \begin{cases} > N_1, & \text{if } N_1 \geq N_2 \\ > N_2, & \text{if } N_1 < N_2 \end{cases}$,

$$|x_{n+1} - x_n| \leq |x_{n+1} - x_{n-1}| + |x_n - x_{n-1}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore, the difference between two consecutive terms is getting closer and closer since it's bounded by any positive ε ($|x_{n+1} - x_n| < \varepsilon$). So $|x_{n+1} - x_n| \leq |x_n - x_{n-1}|$.

Problem 7

a) Since $\frac{3}{9n+1} < \frac{1}{n}$ $\forall n \geq 1$, by comparison test,

since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{3}{9n+1}$ also diverges.

b) Since $\frac{1}{2n-1} \leqslant \frac{1}{n}$ $\forall n \geq 1$, by comparison test,

since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ also diverges.

c) $\frac{(-1)^n}{n^2} = (-1)^n \cdot \frac{1}{n^2}$, $\frac{1}{n^2} > 0$ and decreasing

and $\lim_{n\to\infty} \frac{1}{n^2} = 0$, so by alternating test,

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges

(d) $\frac{1}{n(n+1)} = \frac{n+1-n}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

Since $\lim_{n\to\infty} \frac{1}{n+1} = 0$, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1

(e) $\frac{x_{n+1}}{x_n} = \frac{(n+1)e^{-(n+1)^2}}{n e^{-n^2}} = \frac{n+1}{n} \cdot e^{-2n-1}$, so $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot e^{-2n-1}$

$= 1 \cdot 0 = 0 < 1$

Therefore, by ratio test, $\sum_{n=1}^{\infty} ne^{-n^2}$ converges.

Problem 8

a) $\sum |x_n y_n| \leq \sum |x_n| |y_n|$

Since $\sum |x_n|$ and $\sum |y_n|$ converge,

$\sum |x_n| |y_n|$ also converge.

By comparison test, $\sum |x_n y_n|$ also converge

$\sum x_n y_n$ absolutely converges.

b) $x_n = \frac{(-1)^n}{n}$ $\quad$ $y_n = \frac{(-1)^n}{n}$ $\quad$ $x_n y_n = \frac{1}{n^2}$

not converge absolutely $\quad$ not converge absolutely $\quad$ converge absolutely

c) $x_n = \frac{(-1)^n}{\sqrt{n}}$ $\quad$ $y_n = \frac{(-1)^n}{\sqrt{n}}$ $\quad$ $x_n y_n = \frac{1}{n}$

all three converge absolutely, but

$\sum x_n \sum y_n \neq \sum x_n y_n$

Problem 9

$$\left|\sum_{n=1}^{\infty} x_n\right| = \left| x_1 + x_2 + \cdots \right| \leq |x_1| + |x_2| + \cdots = \sum_{n=1}^{\infty} |x_n|$$

$\square$

Problem 10

Since $\sum x_n$ converges, $\exists N > 0$ s.t. $x_n \le N \quad \forall n \geq 1$

So $x_n^2 = x_n \cdot x_n \leq N \cdot x_n \quad \forall n \geq 1$

$\sum x_n$ converges, so $N \cdot x_n$ converges

By comparison test, $\sum x_n^2$ also converges.

$\square$