2.4.2. It suffices to show that for any $n, p \in \mathbb{N}$ we have

$|x_{n+p} - x_n| \leq a_n$, for some sequence $(a_n)$ with $a_n \to 0$. (see Ex. 2.4.4).

We have

$$|x_n - x_{n+p}| = |x_n - x_{n+1} + x_{n+1} - x_{n+2} + \dots + x_{n+p-1} - x_{n+p}|$$

$$\leq \sum_{i=0}^{p-1} |x_{n+i} - x_{n+i+1}| \quad (1)$$

But for any $k \in \mathbb{N}$, $k > 1$, we have

$$|x_{k+1} - x_k| \leq C|x_k - x_{k-1}| \leq$$

$$\leq \dots \leq C^{k-1}|x_2 - x_1|$$

So for $k = n+i$ we take

$$|x_{n+i+1} - x_{n+i}| \leq C^{n+i-1}|x_2 - x_1|. \quad (2)$$

so by (1)-(2) we have

$$|x_{n+p}-x_n| \leq \sum_{i=0}^{p-1} c^{n+i-1} |x_2-x_1| =$$

$$= c^{n-1} |x_2-x_1| \sum_{i=0}^{p-1} c^i \leq$$

$$\leq c^{n-1} |x_2-x_1| \sum_{i=0}^{\infty} c^i =$$

$$= \frac{c^{n-1}}{1-c} |x_2-x_1|$$

since $c<1$.

Moreover, since $c<1$ we take $c^{n-1} \to 0$.
so the sequence is Cauchy.

~~2.4.4.~~ Let $\varepsilon>0$. Since $y_k \to 0$,
we can find $N$ with $y_k < \varepsilon \ \forall k>N$.
so for $m>k>N$, we have

$|x_m - x_k| \leq y_k < \varepsilon$ so $(x_n)$ is Cauchy.

2.4.6. We will apply the result of Ex. 2.4.4.

Let $n, k \in \mathbb{N}$ with $k > n$. Then

$$|x_k - x_n| \leq \frac{n}{k^2} \leq \frac{n}{n^2} = \frac{1}{n}.$$

Since $\frac{1}{n} \to 0$, the claim follows from Ex. 2.4.4.

2.4.5. $(x_n)$ is Cauchy so it converges to some limit $x$. We will show that $x = 0$.

We will construct a positive and a negative subsequence.

Indeed let $M = 1$. Then $\exists\, n_1 \geq 1$ with $x_{n_1} > 0$.

Now let $M = n_1$. Then $\exists\, n_2 > n_1$ with $x_{n_2} > 0$.

Keep doing this process we construct a subsequence $(x_{n_k})$ with $x_{n_k} > 0\ \forall k$.

Similarly we can construct another subsequence $(x_{m_k})$ with $x_{m_k} < 0\ \forall k$.

Since $(x_n)$ converges to $x$, then $x_{n_k} \to x$ and $x_{m_k} \to x$ as $k \to \infty$.

But $x_{n_k} > 0\ \forall k \Rightarrow x = \lim\limits_{k\to\infty} x_{n_k} \geq 0$ since $x_{n_k} > 0\ \forall k$.

Similarly we take that $x \leq 0$.

So the only possibility is $\boxed{x = 0.}$.

2.4.7. Let $\varepsilon > 0$. Then $\exists N$ s.t.

$|x_m - x_n| < \varepsilon\ \forall m > n > N$. (1).

Since there infinitely many $n$'s with $x_n = c$, we can find $N' > N$ with $x_{N'} = c$

Then for $m > N'$, (1) implies

$$|x_m - c| = |x_m - x_{N'}| < \varepsilon$$

so $x_m \to c$ as $m \to \infty$.

2.4.8. False

Consider the sequence:

$$1, 0, 0, \frac{1}{2}, 0, 0, \ldots, \frac{1}{n}, 0, 0 \ldots$$

This sequence clearly goes to 0 but does not satisfy the condition.

2.5.3.

a) $\sum_{n=1}^{\infty} \frac{3}{9n+1} = \infty$. Indeed take $n$ large so that $9n+1 < 10n \Leftrightarrow$

$$\Leftrightarrow \frac{1}{10n} < \frac{1}{9n+1}.$$

$\sum_{n=1}^{\infty} \frac{1}{10n} = \infty$ so by comparison it diverges

b) Same

c) It converges absolutely since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

d) It converges since $\frac{1}{n(n+1)} < \frac{1}{n^2}$.

e) It converges since for $n$ large

we have $ne^{-n^2} < \frac{1}{n^2}$ and $\sum \frac{1}{n^2} < \infty$.

alternatively, one can use ratio test.

~~2.5.9~~ Since $\sum x_n$ converges absolutely, $x_n \to 0$

so it is bounded. Let $|x_n| \leq M$.

Then $\sum_{n=1}^{\infty} |x_n y_n| \leq M \sum_{n=1}^{\infty} |y_n| < \infty$

since $\sum_{n=1}^{\infty} |y_n| < \infty$

b) Take $x_n = n$, $y_n = \frac{1}{n^3}$.

$\sum x_n y_n = \sum \frac{1}{n^2} < \infty$ but $\sum x_n = \infty$.

c) $x_n = y_n = \left(\frac{1}{2}\right)^n$

Then $\sum_{n=0}^{\infty} x_n = \sum_{n=0}^{\infty} y_n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2$.

and ~~so~~ $\sum_{n=0}^{\infty} x_n y_n = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$.

2.5.10. By triangle inequality we have (1)

$$\left| \sum_{k=1}^{n} x_k \right| \le \sum_{k=1}^{n} |x_k| \le \sum_{k=1}^{\infty} |x_k| < \infty.$$

Since $\sum x_k$ absolutely converges we have that $\sum_{k=1}^{\infty} x_k$ converges. By (1) and properties of the limits (in particular that the

absolute value commutes with limits).
we obtain the result.

2.5.14. $\sum x_n$ converges so $x_n \to 0$.

So $\exists N$ s.t. $x_n < 1 \quad \forall n > N$.

so $x_n^2 < x_n$ (since $x_n$ positive).

Since $\sum x_n$ converges, by comparison we take that $\sum x_n^2$ converges.