Due: April 4^{th}, 2023

Analysis Homework 4

Problem 1. *Find the limit (and prove it of course) or prove that the limit does not exist.*

(a) $\lim_{x\to c} \sqrt{x}$, *for* $c \geq 0$

(b) $\lim_{x\to c} x^2 + x + 1$, *for* $c \in \mathbb{R}$

(c) $\lim_{x\to 0} x^2 \cos(1/x)$

(d) $\lim_{x\to 0} \sin(1/x)\cos(1/x)$

(e) $\lim_{x\to 0} \sin(x)\cos(1/x)$

Problem 2. *Let* $S \subset \mathbb{R}$ *and let* c *be a cluster point of* S. *Suppose* $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$, *and* $h : S \to \mathbb{R}$ *are functions such that, for all* $x \in S$,

$$f(x) \leq g(x) \leq h(x).$$

Suppose the limits of $f(x)$ *and* $h(x)$ *as* x *goes to* c *exist, and*

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x).$$

Then the limit of $g(x)$ *as* x *goes to* c *exists and*

$$\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = \lim_{x\to c} h(x).$$

Problem 3. *Let* $S \subset \mathbb{R}$ *and let* c *be a cluster point of* S. *Suppose* $f : S \to \mathbb{R}$ *and* $g : S \to \mathbb{R}$ *are functions such that the limits of* $f(x)$ *and* $g(x)$ *as* x *goes to* c *both exist. Then*

(a) $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.

(b) $\lim_{x\to c}(f(x) - g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$.

(c) $\lim_{x\to c} f(x)g(x) = (\lim_{x\to c} f(x))(\lim_{x\to c} g(x))$.

(d) If $\lim_{x\to c} g(x) \neq 0$ *and* $g(x) \neq 0$ *for all* $x \in S \setminus \{c\}$, *then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

Problem 4. *Let* $S \subset \mathbb{R}$, c *be a cluster point of* S, *and* $f : S \to \mathbb{R}$ *be a function. Suppose that for every sequence* $\{x_n\}$ *in* $S \setminus \{c\}$ *such that* $\lim x_n = c$ *the sequence* $\{f(x_n\}$ *is convergent. Then show that the limit of* $f(x)$ *as* $x \to c$ *exists.*

Problem 5. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) := \begin{cases} x & x \in \mathbb{Q} \\ x^2 & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Using the definition of continuity directly prove that f *is continuous at* 1 *and discontinuous at* 2.

Problem 6. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) := \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Is f *continuous? Prove your assertion.*

Problem 7. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) := \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Is f *continuous? Prove your assertion.*

Problem 8. *Let* $f : \mathbb{R} \to \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$ *be continuous functions. Suppose that for all rational numbers* r, $f(r) = g(r)$. *Show that* $f(x) = g(x)$ *for all* x.

Problem 9. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be continuous. Suppose* $f(c) > 0$. *Show that there exists an* $\alpha > 0$ *such that for all* $x \in (c - \alpha, c + \alpha)$, *we have* $f(x) > 0$.

Problem 10. *Suppose* $g : \mathbb{R} \to \mathbb{R}$ *is a continuous function such that* $g(0) = 0$, *and suppose* $f : \mathbb{R} \to \mathbb{R}$ *is such that* $|f(x) - f(y)| \leq g(x - y)$ *for all* x *and* y. *Show that* f *is continuous.*