1. $\lim_{k\to\infty} U(P_k, f) = \lim_{k\to\infty} L(P_k, f)$

$\lim_{k\to\infty} U(P_k, f) = \sum_{k=1}^{\infty} M_k \Delta x_k$ $\quad \Delta x_k = x_k - x_{k-1}$

$\lim_{k\to\infty} L(P_k, f) = \sum_{k=1}^{\infty} m_k \Delta x_k$

$m_k = \inf\{f(x) : x_{k-1} \leq x \leq x_k\}$ $\quad \underset{k\to\infty}{m_k = M_k}$

$M_k = \sup\{f(x) : x_{k-1} \leq x \leq x_k\}$ $\quad \inf\{f(x)\} = \sup\{f(x)\}$

$\underline{\int_a^b} f = \sup\{L(P_k, f)\} \geq L(P_k, f)$

$||$

$\overline{\int_a^b} f = \inf\{U(P_k, f)\} \leq U(P_k, f)$ $\quad$ So $\underline{\int_a^b} f = \overline{\int_a^b} f$

f is Riemann integrable because

$U(P_k, f) - L(P_k, f) = 0 < \varepsilon \quad \forall \varepsilon > 0$

and $\int_a^b f = \lim_{k\to\infty} U(P_k, f) = \lim_{k\to\infty} L(P_k, f)$

2. $\sup(f+g) \leq \sup f + \sup g$ $\quad \inf(f+g) \geq \inf f + \inf g$

$m_i = \sup\{f+g : x \in [x_{i-1}, x_i]\}$

$U(P, f+g) = \sum_{i=1}^{n} m_i \Delta x_i \leq U(P,f) + U(P,g)$

Similarly, $L(P, f+g) \geq L(P,f) + L(P,g)$

$U(P, f+g) - L(P, f+g) \leq U(P,f) + U(P,g) - L(P,f) - L(P,g) = 0$

$U(P, f+g) - L(P, f+g) \geq 0$ So $U(P, f+g) = L(P, f+g)$

$\int_a^b f+g = \int_a^b f + \int_a^b g$

3, For the sake of contradiction, assume $\exists c \in [a,b]$ s.t. $f(c) \neq 0$

Since $f(x) \geq 0 \ \forall x \in [a,b]$, $f(c) < 0$

Since f is continuous on $[a,b]$, $\exists \varepsilon > 0$ s.t. $f(x) < 0 \ \forall x \in S$,

let $S := \{(c-\varepsilon, c+\varepsilon) \cap [a,b]\}$

Since $f(x) = 0 \ \forall x \in [a,b] \setminus S$, $\int_S f < 0$, which contradicts

$\int_a^b f = 0$. Therefore, $f(x) = 0 \ \forall x \in [a,b]$

4. For the sake of contradiction, assume $f(x) \neq 0 \ \forall x \in [a,b]$

WLOG, assume $f(x) > 0 \ \forall x \in [a,b]$, $\int_a^b f > 0$, contradiction

(if $f(x) < 0 \ \forall x \in [a,b]$, $\int_a^b f < 0$, contradiction)

Therefore, $\exists c \in [a,b]$ s.t. $f(c) = 0$

5. $\int_a^b f = \int_a^b g \quad \int_a^b f - g = 0$ By MVT, $\exists c \in (a,b)$ s.t.

$\int_a^b f - g = (f(c) - g(c))(b-a)$, $f(c) - g(c) = \dfrac{\int_a^b f - g}{b-a} = 0$

Therefore, $\exists c \in [a,b]$ s.t. $f(c) = g(c)$

6. ① If $\alpha \le \beta \le \gamma$, $\int_\alpha^\gamma f = \int_\alpha^\beta f + \int_\beta^\gamma f$

② $\alpha \le \gamma \le \beta$, $\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f$, $\int_\alpha^\gamma f = \int_\alpha^\beta f - \int_\gamma^\beta f = \int_\alpha^\beta f + \int_\beta^\gamma f$

③ $\beta \le \alpha \le \gamma$, $\int_\beta^\gamma f = \int_\beta^\alpha f + \int_\alpha^\gamma f$, $\int_\alpha^\gamma f = \int_\beta^\gamma f - \int_\beta^\alpha f = \int_\alpha^\beta f + \int_\beta^\gamma f$

④ $\beta \le \gamma \le \alpha$, $\int_\beta^\alpha f = \int_\beta^\gamma f + \int_\gamma^\alpha f$, $\int_\alpha^\gamma f = \int_\beta^\gamma f - \int_\beta^\alpha f = \int_\alpha^\beta f + \int_\beta^\gamma f$

⑤ $\gamma \le \alpha \le \beta$, $\int_\gamma^\beta f = \int_\gamma^\alpha f + \int_\alpha^\beta f$, $\int_\alpha^\gamma f = \int_\alpha^\beta f - \int_\gamma^\beta f = \int_\alpha^\beta f + \int_\beta^\gamma f$

⑥ $\gamma \le \beta \le \alpha$, $\int_\gamma^\alpha f = \int_\gamma^\beta f + \int_\beta^\alpha f$, $\int_\alpha^\gamma f = \int_\alpha^\beta f + \int_\beta^\gamma f$

Therefore, no matter in what order, $\forall \alpha, \beta, \gamma \in [a,b]$,

$$\int_\alpha^\gamma f = \int_\alpha^\beta f + \int_\beta^\gamma f$$

7. By linearity, $|h(x) - h(y)| = \left| \int_a^b g(t-x) f(t) dt - \int_a^b g(t-y) f(t) dt \right|$

g is Lip $\Rightarrow$ $|g(x) - g(y)| \le M|x-y|$

$|f(t) \cdot (g(t-x) - g(t-y))| \le |M \cdot (x-y) \cdot f(t)|$

By monotonicity,

f cont. on $[a,b] \Rightarrow f$ bounded

$|f(x)| \le N$, by monotonicity,

Let $K = M \cdot N \cdot (b-a)$

is a constant

$$\begin{aligned}
&= \left| \int_a^b g(t-x) f(t) - g(t-y) f(t) \, dt \right| \\
&= \left| \int_a^b f(t) \cdot (g(t-x) - g(t-y)) \, dt \right| \\
&\le \left| \int_a^b M \cdot (x-y) \cdot f(t) \, dt \right| \\
&= M \cdot |x-y| \cdot \left| \int_a^b f(t) \, dt \right| \\
&\le M \cdot |x-y| \cdot N \cdot \left| \int_a^b dt \right| \\
&= M \cdot N \cdot |x-y| \cdot (b-a) \\
&= K \cdot |x-y|
\end{aligned}$$

So $|h(x) - h(y)| \le K|x-y|$

h is Lipschitz continuous