2.1.3 $\lim_{n\to\infty} \frac{(-1)^n}{2n} = 0$.

Indeed let $\varepsilon > 0$. Then

$\left| \frac{(-1)^n}{2n} \right| = \frac{1}{2n} < \varepsilon$, as long as

$n > \frac{\varepsilon}{2} =: N(\varepsilon)$.

2.1.6. $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$. Indeed, let $N \in \mathbb{N}$. Then for $n > N$ we have $\left| \frac{n}{n^2+1} \right| = \frac{n}{n^2+1} = \frac{1}{n+\frac{1}{n}} \leq \frac{1}{n} \leq$

$\leq \frac{1}{N}$. So given $\varepsilon > 0$, if we pick $\frac{1}{N} < \varepsilon \iff N > \frac{1}{\varepsilon}$ we have

$\left|\frac{n}{n^2+1}\right| < \varepsilon \quad \forall n \geq N$.

2.1.7

a) Let $a_n = |x_n|$.

Then $x_n \to 0$ means that

$\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n \geq N$ we

have $|x_n| < \varepsilon$.

But notice that $|x_n| = a_n = |a_n|$

so $x_n \to 0$ is equivalent to

$a_n = |x_n| \to 0$.

b) Take $x_n = (-1)^n$. Then (x_n) diverges but $|x_n| = 1 \ \forall n \in \mathbb{N}$ which obviously converges.

2.1.13 Assume WLOG that (x_n) is increasing.

Arguing by contradiction, assume the claim is not true.

i.e. $\exists N > K$ with $x_N \neq x_K$.

Since (x_n) is increasing and $N > K$

we have $x_N > x_K$ so

Now consider $\varepsilon := x_N - x_K > 0$.

Since $\lim_{n \to \infty} x_n = x_K$ $\exists N^* > N$

s.t. $|x_n - x_K| < \varepsilon = x_N - x_K$

$\forall n > N^* \geq N \geq K$

Pick such an n. Then $x_n \geq x_N > x_K$

so $x_n - \cancel{x_K} = |x_n - x_K| < x_N - \cancel{x_K}$.

$\Rightarrow x_n < x_N$ which is a contradiction since $n > N$.

2.1.16. If we used Proposition 2.1.17 the claim would be immediate. Let us not use it.

So let $x_{n_i} \xrightarrow{i\to\infty} a$, $x_{m_i} \xrightarrow{i\to\infty} b$ and $a \neq b$.

Arguing by contradiction, assume (x_n) converges to some x.

Then either $x \neq a$ or $x = a$.

Assume first that $x \neq a$.

define $\varepsilon := \frac{|a-x|}{2}$ Then

$$(a-\varepsilon, a+\varepsilon) \cap (x-\varepsilon, x+\varepsilon) = \emptyset. \quad (1)$$

But since $x_n \to x$, $\exists N$ with

$x_n \in (x-\varepsilon, x+\varepsilon)$. $\forall n > N$. (2)

Since $x_{n_i} \to a$, $\exists K$ with

$x_{n_i} \in (a-\varepsilon, a+\varepsilon)$ $\forall i > K$ (3).

Pick $i > \max\{K, N\}$.

Then $n_i \geq i > \max\{K, N\}$

So both (2)-(3) need to hold.

But this is a contradiction by (1).

2.11.23. Since (x_n) is increasing it either converges, or it diverges to $+\infty$. If it diverges to $+\infty$, every subsequence would go to $+\infty$, which is not the case by assumption. So (x_n) has to converge.

<u>Note</u> You can show it by the ε-definition too.

2.2.5. $x_n = \dfrac{n - \cos n}{n} = 1 - \dfrac{\cos n}{n}$.

But $\left|\dfrac{\cos n}{n}\right| \leq \dfrac{1}{n}$ so by squeeze lemma $\frac{\cos n}{n} \to 0$ as $n \to \infty$.

So $X_n = 1 - \frac{\cos n}{n} \xrightarrow{n \to \infty} 1$.

2.2.12.

a) let $|a_n| \leq M$. Since $b_n \to 0$

$\forall \varepsilon > 0 \; \exists N$ s.t. $|b_n| < \frac{\varepsilon}{M} \; \forall n > N$.

So for $n > N$ we have

$|a_n b_n| < M \cdot \frac{\varepsilon}{M} = \varepsilon$ so

$a_n b_n \to 0$.

b) Take $a_n = n^2$, $b_n = \frac{1}{n}$.

Then $b_n \to 0$ but $a_n b_n = n \to \infty$.

c) Take $a_n = (-1)^n$, $b_n = 1$.

Then $b_n \to 1$ but $a_n b_n = (-1)^n$ which diverges.

~~2.2.14~~ First notice that (x_n) is increasing since

$x_{n+1} = x_n^2 + x_n \geq x_n \quad \forall n \in \mathbb{N}$.

"$\Rightarrow$" Assume $x_n \to L$. We will show that $c \in [-1, 0]$ and that $L = 0$.

Indeed since $x_n \to L$, letting $n \to \infty$ in the inductive scheme we get

$$L = L^2 + L \Rightarrow L^2 = 0 \Rightarrow L = 0,$$

we easily see that $c \leq 0$.

Indeed $c = x_1 \leq x_n \quad \forall n \in \mathbb{N}$.

Since $x_n \to 0$, we let $n \to \infty$

so we take $c \leq 0$.

Now assume that $c < -1$.

Then we have $c^2 > -c \overset{(c \leq 0)}{\Rightarrow}$.

$$\Rightarrow c^2 + c > 0. \quad (1)$$

Now $x_n \geq x_2 = x_1^2 + x_1 = c^2 + c$
$\forall n \geq 2$.

letting $n \to \infty$ we take

$c^2 + c \leq 0$ (2). (1) and (2)

contradict. So $c \geq -1$.

We conclude $c \in [-1, 0]$.

"$\Leftarrow$" Assume $c \in [-1, 0]$. We will

show $x_n \to 0$.

We have already showed that

if the limit exists, it has to

be 0, and that (x_n) is increasing.

So it suffices to show that (x_n) is upper bounded.

In fact we will use induction to show that $-1 \le x_n \le 0 \ \forall n$.

For $\underline{n=1}$ this holds since $c \in [-1, 0]$.

Assume $-1 \le x_n \le 0$ for some n, we will show $-1 \le x_{n+1} \le 0$.

The left hand side is trivial

because by monotonicity we have

$x_{n+1} \geq x_1 = c \geq -1$.

For the right hand side we have $-1 \leq x_n \leq 0 \Rightarrow$

$\Rightarrow \quad -x_n \geq x_n^2 \Rightarrow x_n^2 + x_n \leq 0$

so $\quad x_{n+1} = x_n^2 + x_n \leq 0$.

We conclude that $-1 \leq x_{n+1} \leq 0$ which closes the induction.

2.2.4

$$\begin{cases} x_{n+1} = x_n^2 \\ x_1 = 1/2 \end{cases}$$

We first show by induction that

$0 < x_n \leq 1/2 \quad \forall n \in \mathbb{N}$.

<u>n=1</u>, it is true since $x_1 = 1/2$

assume $0 < x_n \leq \frac{1}{2}$.

Then $x_{n+1} = x_n^2 \in (0, \frac{1}{4}] \subseteq (0, 1/2]$.

so by induction we conclude that

$0 < x_n \leq \frac{1}{2} \quad \forall n \in \mathbb{N}$.

We then notice that (x_n) is decreasing. Indeed since $0 < x_n \leq \frac{1}{2}$ $\forall n$, we have $x_{n+1} = x_n^2 < x_n$.

Thus (x_n) converges. Let $L = \lim x_n$.

Then letting $n \to \infty$ in the inductive scheme we obtain $L = L^2 \Rightarrow$

$\Rightarrow L = 0$ or $L = 1$.

Clearly we cannot have $L = 1$ since $x_n \leq \frac{1}{2}$ $\forall n$.

so $L = 0$.