3.3.6. Since g is a monic polynomial of even degree, we can find $M>0$ large enough with $g(M)>0$ and $g(-M)>0$.

Now since $g(0)<0$, we may apply Bolzano's theorem to each of the intervals $[-M,0]$, $[0,M]$ and obtain at least a root in each of them. Thus, such a polynomial has at least two roots.

3.3.8. Restrict first f in a period, say $[0,P]$. By the min/max theorem

$\exists\, X_{max} \in [0,P]$ s.t. $f(x) \leq f(X_{max})$ $\forall x \in [0,P]$

$\exists\, X_{min} \in [0,P]$ s.t. $f(x) \geq f(X_{min})$ $\forall x \in [0,P]$.

Now take an arbitrary $x \in \mathbb{R}$. Then $\exists\, \tilde{x} \in [0,P]$ and $k \in \mathbb{Z}$ s.t. $x = \tilde{x} + kP$

Then $f(x) = f(\tilde{x} + kP) = f(\tilde{x}) \leq f(x_{max})$

and similarly $f(x) = f(\tilde{x}) \geq f(x_{min})$.

So f attains min and max.

3.3.10. $f : [0,1] \to [0,1]$.

If $f(0) = 0$ or $f(1) = 1$, there is clearly a fixed point.

So we may assume $f(0) > 0$ and $f(1) < 1$.

Define the function $g(x) = f(x) - x$ which is continuous. Then $g(0) = f(0) - 0 > 0$

and $g(1) = f(1) - 1 < 0$.

so by Bolzano's theorem $\exists c \in (0,1)$

with $g(c) = 0 \Leftrightarrow f(c) = c$.

In any case $\exists c \in [0,1]$ s.t. $f(c) = c$.

3.3.12 We will show that $f(\mathbb{R}) = \mathbb{R}$.

In othere words $\forall y \in \mathbb{R}$, $\exists c \in \mathbb{R}$ with $y = f(c)$.

Fix $y \in \mathbb{R}$. Then $y \leq f(y+1)$ by assumption. Similarly $f(y-1) \leq y$.

so $f(y-1) \leq y \leq f(y+1)$.

Thus by the IVT, $\exists c \in [y-1, y+1]$ with $f(c) = y$. So $f(\mathbb{R}) = \mathbb{R}$.

3.3.3 Define $\tilde{f}(x) = \begin{cases} f(x), & x \in (0,1) \\ 0, & x = 0, 1 \end{cases}$

Then $\tilde{f} : [0,1] \to \mathbb{R}$ is continuous on $[0,1]$ since $\lim_{x \to 0^+} \tilde{f}(x) = 0 = \tilde{f}(0)$

and $\lim_{x \to 1^-} \tilde{f}(x) = 0 = \tilde{f}(1)$.

So by the min/max theorem $\tilde{f}$ achieves a max at a point $X_{max} \in [0,1]$ and a min at a point $X_{min} \in [0,1]$

In particular
$\tilde{f}(X_{min}) \leq f(x) \leq \tilde{f}(X_{max}) \; \forall x \in (0,1)$.

We have the following cases.

- $X_{min}, X_{max} \in (0,1)$. Then f achieves both min and max.

- $X_{min} \in (0,1)$, $X_{max} \in \{0,1\}$. Then f achieves a min at X_{min} for sure.

- $x_{max} \in (0,1)$, $x_{min} \in \{0,1\}$.

Same b/4 with maximum.

- $x_{min}, x_{max} \in \{0,1\}$

Then $\tilde{f}(x_{min}) = f(x_{max})$

$= 0$.

Thus $f(x) = 0 \quad \forall x \in (0,1)$

so it achivs both min and max trivially.

3.4.3. Let $x, y \in (c, \infty)$ where $c > 0$. Then

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| \leq \frac{1}{c^2}|x-y|.$$

so $f(x) = \frac{1}{x}$ is Lipschitz in (c, ∞) with constant $\frac{1}{c^2}$.

3.4.4. Assume f is Lipschitz continuous in $(0, \infty)$ with constant L. I.e.

$$\left|\frac{1}{x} - \frac{1}{y}\right| \leq L|x-y| \quad \forall x, y > 0.$$

$\circledast$

Fix $x \in (0,\infty)$ and set $y = 2x$.

Then ⊛ implies.

$$\frac{1}{2x} \leq Lx \Rightarrow x^2 \geq \frac{1}{2L} \Rightarrow$$

$$\Rightarrow x \geq \sqrt{\frac{1}{2L}}$$

which contradicts the validity of ⊛.

3.4.7. Define $\tilde{g} : [0,1] \to \mathbb{R}$

~~By~~ by $\tilde{g}(x) = \begin{cases} g(x), & x \in (0,1) \\ 0, & x = 0,1. \end{cases}$

Since f is continuous in $[0,1)$ then $g(x)=xf(x)$ is continuous in $(0,1)$, so $\tilde{g}$ is continuous in $(0,1)$.

Moreover for any $x\in(0,1)$, we have

$$|\tilde{g}(x)| = |x(1-x)||f(x)| \leq$$

$$\leq |x(1-x)| M$$

But $\lim_{x\to 0^+} |x(1-x)| M = 0$.

So by squeezing we take

$$\lim_{x\to 0^+} \tilde{g}(x) = 0 = \tilde{g}(0)$$

So $\tilde{g}$ is continuous at $x=0$.
Similarly we can show that $\tilde{g}$ is continuous at $x=1$.
Thus $\tilde{g}$ is continuous at $[0,1]$. By Theorem on uniform continuity, $\tilde{g}$ is U.C. on $[0,1]$. Thus $\tilde{g}$ is U.C. in $(0,1) \subset [0,1]$. But $\tilde{g}$ coincides with g in $(0,1)$.
So g is U.C. in $(0,1)$.

3.4.10

a) Let $x_n = \frac{1}{2n} \in (0,1)$ $\forall n$.

$x_n \to 0$ so it is Cauchy.

Take $f(x) = \frac{1}{x}$, $x \in (0,1)$ which is continuous.

Then $f(x_n) = \frac{1}{x_n} = 2n$ which is unbounded so not Cauchy.

b) Let $f: \mathbb{R} \to \mathbb{R}$ continuous.

Let (x_n) be Cauchy. So

(x_n) is bounded i.e. $\exists M > 0$

s.t. $x_n \in [-M, M]$ $\forall n \in \mathbb{N}$.

Now if we restrict f in

$[-M, M]$ it becomes U.C.

since it is closed + bounded

interval.

So $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. for

all $x, y \in [-M, M]$ with

$|x-y|<\delta$, we have

$|f(\cancel{x_n})-f(y)|<\varepsilon$.

Now (x_n) is Cauchy so

$\exists N\in\mathbb{N}$ s.t. $|x_n-x_m|<\delta$

Moreover $x_n\in[-M,M]$ $\forall n\in\mathbb{N}$.

Thus for any $m>n>N$

we have $|f(x_n)-f(x_m)|<\varepsilon$

so $(f(x_n))$ is Cauchy.

3.4.16. Let $\varepsilon > 0$. Since $g(0) = \lim_{z \to 0^+} g(z)$, $\exists \delta > 0$ s.t. for all $0 < z < \delta$ we have $g(z) < \varepsilon$.

Now take x, y with $|x - y| < \delta$.

Then $|f(x) - f(y)| \leq g(|x-y|)$

$< \varepsilon$.

so f is U.C.