0.3.14 We will prove it by induction.

For $n=1$, the claim trivially holds.

Assume the claim holds for n i.e.

$$1^3+2^3+\dots+n^3=\left(\frac{n(n+1)}{2}\right)^2 . \quad (I)$$

We will prove that it holds for $n+1$ i.e.

$$1^3+2^3+\dots+(n+1)^3=\left(\frac{(n+1)(n+2)}{2}\right)^2 .$$

Indeed by (I) we have.

$$1^3+2^3+\dots+n^3+(n+1)^3=\left(\frac{n(n+1)}{2}\right)^2+(n+1)^3=$$

$$=\frac{n^2(n+1)^2}{4}+(n+1)^3=$$

$$=\frac{n^2(n+1)^2+4(n+1)(n+1)^2}{4}=$$

$$= \frac{(n+1)^2(n^2+4n+4)}{4} = \frac{(n+1)^2(n+2)^2}{4}.$$

The claim follows by induction $\boxtimes$.

1.1.4 Let $A \subset B$. Since $\sup A$ is upper bound and $\inf A$ is lower bound we have $\inf A \leq x \leq \sup A \quad \forall x \in A$

so $\inf A \leq \sup A \quad (1)$.

Now since $A \subset B$ we have that $x \leq \sup B \quad \forall x \in A$, so $\sup B$ is an upper bound of A. So $\sup A \leq \sup B$ (2).

Similarly we take $\inf B \leq \inf A$ (3).

Combining (1), (2), (3) the claim follows.

1.1.5 Let b be an upper bound of A with $b \in A$. Then b has to be the least upper bound. Indeed if $b' < b$, then b' is not an upper bound since $b \in A$. So $b = \sup A$.

1.2.1. Use the Archimedean property with 1 and $\sqrt{t} > 0$. Then $\exists n \in \mathbb{N}$ s.t.

$$1 < n\sqrt{t} \Rightarrow 1 < n^2 t \Leftrightarrow \frac{1}{n^2} < t.$$

1.2.2. Define the set

$$A = \{ m \in \mathbb{N} \text{ s.t. } t < m \}.$$

Clearly $A \neq \emptyset$ since the natural numbers are unbounded (or Archimedean property). Now by well-ordering of $\mathbb{N}$, the set A

has a minimum element. Let $n = \min A$.

Then $n \in A$, so $t < n$.

If $n > 1$, then $n-1 \notin A$ so $n-1 \leq t$.

If $n=1$ then $n-1 = 0$ so $n-1 \leq t$ again.

In any case $n-1 \leq t < n$.

~~1.2.7~~ we have $(\sqrt{x} - \sqrt{y})^2 \geq 0 \Leftrightarrow$

$\Leftrightarrow x + y - 2\sqrt{xy} \geq 0 \Leftrightarrow$

$\Leftrightarrow \sqrt{xy} \leq \dfrac{x+y}{2}$.

Equality occurs i.f.f. $(\sqrt{x} - \sqrt{y})^2 = 0 \Leftrightarrow$

$\Leftrightarrow \sqrt{x} = \sqrt{y} \Rightarrow \boxed{x = y.}$

1.2.9. WLOG (without loss of generality) we show the claim for the sup.

Let $c \in C \Rightarrow c = a + b$ for some $a \in A$ and $b \in B$.

So $c = a + b \leq \sup A + \sup B$.

Since c is arbitrary we conclude that $\sup A + \sup B$ is an upper bound of C.

Now, given $\varepsilon > 0$, we will find $c \in C$ with $\sup A + \sup B - \varepsilon < c$.

By definition of sup, $\exists\, a \in A$ with $\sup A - \frac{\varepsilon}{2} < a$ (1)

and $b \in B$ with $\sup B - \frac{\varepsilon}{2} < b$ (2).

(1) + (2): $\sup A + \sup B - \varepsilon < a + b$.

Defining $c = a + b \in C$ the conclusion follows.

1.2.13 We will show it by induction.

For $n=1$, the claim trivially holds as an equality.

Assume it holds for n i.e.

$$(1+x)^n \geq 1+nx \quad (I).$$

We show it holds for $n+1$ i.e.

$$(1+x)^{n+1} \geq 1+(n+1)x.$$

Indeed, since $1+x>0$

$$(1+x)^{n+1} = (1+x)(1+x)^{n+1} \overset{(I)}{\geq}$$

$$\geq (1+x)(1+nx) =$$

$$= 1+x+nx+nx^2$$

$$\geq 1+(n+1)x.$$

1.3.5. a) $f(x) \le g(y) \quad \forall x, y \in D$.

Let $y \in D$ Then $f(x) \le g(y) \quad \forall x \in D$

so $\sup_{x \in D} f(x) \le g(y)$. (1)

Since $y \in D$ was arbitrary, we take

$$\sup_{x \in D} f(x) \le \inf_{y \in D} g(y)$$

b). Take $D = [0, 1]$ and

$f(x) = x$

$g(x) = x + \frac{1}{2}$. Then clearly $f(x) \le g(x) \ \forall x$

But $\sup_{x \in D} f(x) = 1$ and $\inf_{x \in D} g(x) = 1/2$.

1.3.9. We take for granted that if f, g is bounded, then $f+g$ is bounded and af is bounded for $a \in \mathbb{R}$ see Ex. 1.38. which was not assigned.

a) Let $h = f+g$. Then

$f = f+g-g = h-g$ so it is

bounded since h and g are.

b) $f(x) = x$, $g(x) = -x$, $D = \mathbb{R}$.

Clearly $f+g = 0 \rightarrow$ bounded.

c) Let f bounded $\Rightarrow \exists M > 0$ s.t. $|f(x)| \leq M$ $\forall x \in D$.

Let $M' > 0$ arbitrary. We will find $x \in D$

with $|f(x)+g(x)| > M'$.

we have $|f(x)| \leq M \Rightarrow -M \leq -|f(x)|$.

Now g is unbounded so $\exists x$

with $|g(x)| > M' + M$.

Then by reverse triangle inequality

$|f(x)+g(x)| \geq |g(x)| - |f(x)| > M' + M - M = M'$.

d) Take $\alpha = 0$.