3.1.1 a) Let $\delta > 0$ and assume $|x-c| < \delta$. Then

$$\left|\sqrt{x}-\sqrt{c}\right| = \frac{|x-c|}{\sqrt{x}+\sqrt{c}} < \frac{\delta}{\sqrt{x}+\sqrt{c}} \leq \frac{\delta}{\sqrt{c}} \quad (1).$$

So given $\varepsilon > 0$, choose δ s.t. $\varepsilon = \frac{\delta}{\sqrt{c}} \Leftrightarrow \delta = \sqrt{c}\,\varepsilon$.

Then by (1) we have

$$\left|\sqrt{x}-\sqrt{c}\right| < \varepsilon, \text{ whenever } |x-c| < \delta.$$

b) One can prove that using the definition directly. But we have shown in class that $\lim_{x\to c} x^2 = c^2$. Moreover $\lim_{x\to c} x = c$ and

$\lim_{x\to c} 1 = 1$.

$$\text{So } \lim_{x\to c}(x^2+x+1) = \lim_{x\to c} x^2 + \lim_{x\to c} x + \lim_{x\to c} 1 =$$

$$= c^2 + c + 1.$$

c) For any $x \neq 0$, we have $|\cos \frac{1}{x}| \leq 1$

$\Rightarrow |x^2 \cos \frac{1}{x}| \leq x^2 \Leftrightarrow -x^2 \leq x^2 \cos \frac{1}{x} \leq x^2 \quad \forall x \neq 0$

But $\lim\limits_{x \to 0} x^2 = 0$, so by the squeeze theorem (for functions) we get that

$\lim\limits_{x \to 0} x^2 \cos \frac{1}{x} = 0$.

d) Notice that for $x \neq 0$, we have

$\sin \frac{1}{x} \cos \frac{1}{x} = \frac{1}{2} \sin \frac{2}{x}$.

We claim that $\lim\limits_{x \to 0} \frac{1}{2} \sin \frac{2}{x}$ does not exist.

Indeed, take the sequence $x_n = \frac{4}{n\pi}$

clearly $x_n \xrightarrow{n \to \infty} 0$ but

$\frac{1}{2} \sin \frac{2}{x_n} = \frac{1}{2} \sin \frac{2}{\frac{4}{n\pi}} = \frac{1}{2} \sin \frac{n\pi}{2}$, which

clearly diverges since it takes the values

$\frac{1}{2}, 0, -\frac{1}{2}, 0, \frac{1}{2}, 0, \ldots$

so by the transfer principle the limit does not exist.

e) For $x \neq 0$, we have $\left|\cos\frac{1}{x}\right| \leq 1 \Rightarrow$

$\Rightarrow \left|\sin x \cos\frac{1}{x}\right| \leq |\sin x| \Leftrightarrow$

$\Leftrightarrow -|\sin x| \leq \sin x \cos\frac{1}{x} \leq |\sin x| \quad \forall x \neq 0.$

Since $\lim_{x\to 0} |\sin x| = 0$, we have

that $\lim_{x \to 0} \sin x \cos\frac{1}{x} = 0$ by the

squeeze theorem.

3.1.3 This is the squeeze theorem for functions. Let $L = \lim_{x\to c} f(x) = \lim_{x\to c} h(x)$ (*)

We will show that $\lim_{x\to c} g(x) = L$.

By the TP, it suffices to show that for any sequence $x_n \to c$, we have

$g(x_n) \to L$. So take a sequence $x_n \to c$.

By $(*)$ and the TP, we have that

$f(x_n) \to L$ and $h(x_n) \to L$.

Also by assumption $f(x_n) \leq g(x_n) \leq h(x_n)$.

So by the squeeze theorem for sequences

we take that $g(x_n) \to L$. Thus

$\lim_{x \to c} g(x) = L$.

3.1.4. All the claims follow using the T.P. and the corresponding properties of limits of sequences. So WLOG we will show only the first claim $\lim_{x \to c} (f(x)+g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$.

Denote $F = \lim_{x \to c} f(x)$, $G = \lim_{x \to c} g(x)$.

Consider a sequence (x_n) with $x_n \to c$.

It suffices to show that $f(x_n)+g(x_n) \to F+G$.

But $\lim_{x \to c} f(x) = F \overset{T.P.}{\Longrightarrow} f(x_n) \xrightarrow{n\to\infty} F$

similarly $g(x_n) \xrightarrow{n\to\infty} G$

So $f(x_n)+g(x_n) \to F+G$ by properties of limits of sequences.

Consequently we have showed that

$$\lim_{x\to c}\left(f(x)+g(x)\right) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x).$$

The rest of the claims follow by similar arguments.

~~3.1.11 We did this problem in class.~~ Although it is similar to the TP, it is a weaker assumption. Essentially we need to show that for any sequences $(x_n), (y_n)$ converging to c, and

$L_1 = \lim_{n \to \infty} f(x_n)$, $L_2 = \lim_{n \to \infty} f(y_n)$ we have $L_1 = L_2$. Then the claim will follow by the T.P.

Define the sequence $z: (x_1, y_1, x_2, y_2, \dots)$

In otherwords define

$$z_n = \begin{cases} x_k, & n = 2k-1, \ k \geq 1 \\ y_k, & n = 2k, \ k \geq 1. \end{cases}$$

Now $\lim_{k \to \infty} z_{2k-1} = \lim_{k \to \infty} x_k = c$

and $\lim_{k \to \infty} z_{2k} = \lim_{k \to \infty} y_k = c$.

So $z_n \to c$ since it can be decomposed to subsequences converging to c.

Thus by assumption $f(z_n) \to L$ for some $L \in \mathbb{R}$.

But $f(z_{2k-1}) = f(x_k) \to L_1$

and $f(z_{2k}) = f(y_k) \to L_2$.

Since $f(z_{2k-1})$, $f(z_{2k})$ are subsequences of $f(z_n)$ we get that $L_1 = L_2$ and the claim is proved.

3.2.3 $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ x^2, & x \notin \mathbb{Q} \end{cases}$

We can prove it by the definition of continuity as the problem suggests or by the T.P. I will exhibit both approaches.

<u>Direct proof.</u>

- Continuity at $x = 1$.

Let $\delta > 0$ and assume $|x - 1| < \delta$.

Now if $x \in \mathbb{Q}$ we have $|f(x) - f(1)| = |x - 1| < \delta$ (1).

if $x \notin \mathbb{Q}$ we have $|f(x) - f(1)| = |x^2 - 1| =$

$= |x - 1|\,|x + 1| < \delta\,(|x| + 1)$.

but $\cancel{|x-2|}$ $|x| - 1 \leq |x-1| < \delta \Rightarrow$

$\Rightarrow |x| < 1 + \delta$.

So for $x \notin \mathbb{Q}$ we have $|f(x) - f(1)| <$

$< \delta(|x|+1) < \delta(\delta + 2)$

assuming $\delta < 1$ we get

$|f(x) - f(1)| < 3\delta$. (2).

By (1)-(2), given $\varepsilon > 0$, we pick

δ such that

$$\begin{cases} \delta < \varepsilon \\ \delta < 1 \\ 3\delta < \varepsilon \Leftrightarrow \delta < \frac{\varepsilon}{3} \end{cases}$$

So for $\delta = \min\{1, \frac{\varepsilon}{3}\}$ we have

$|f(x) - f(1)| < \varepsilon$, whenever $|x-1| < \delta$.

- Discontinuity at $x=2$. It suffices to show that $\exists \varepsilon > 0$ s.t. $\forall\ 0 < \delta < 1/2$.

$\exists\, x_\delta$ with $|x_\delta - 2| < \delta$ and
$|f(x_\delta) - f(2)| \geq \varepsilon$.

Take $\varepsilon = 1/8$ and arbitrary $0 < \delta < 1/2$

Consider $|x_\delta - 2| < \delta \Leftrightarrow 2 - \delta < x_\delta < 2 + \delta$.

Then since $\delta < 0$ we get $x_\delta > 2 - \frac{1}{2} > \frac{3}{2}$.

so $x_\delta^2 > \frac{9}{4}$.

Now $|f(x_\delta) - f(2)| = |x_\delta^2 - 2| \geq x_\delta^2 - 2 >$
$> \frac{9}{4} - 2 = \frac{1}{8} = \varepsilon$.

Proof by T.P.

Take any sequences $(r_n)_n, (i_n)_n$ of rational and irrational numbers with $r_n \to 1$ and $i_n \to 1$.

Then $f(r_n) = r_n \to 1 = f(r)$.
and $f(i_n) = i_n^2 \to 1 = f(r)$. since $i_n \to 1$.

Now an arbitrary sequence $X_n \to 1$
has three possibility.

- contains finitely many irrationals
- contains finitely many rationals
- contains infinitely many rationals and irrationals

In the first case it behaves like a rational sequence in the limit, so $f(X_n) = X_n \to 1$.
similarly in the second case, $f(X_n) = X_n^2 \to 1$.
In the third case (X_n) can be uniquely split in two rational and irrational subsequences.
(r_n) and (i_n) with $i_n, r_n \to 1$.

We have shown that $f(r_n), f(i_n) \to 1$.
So $f(X_n) \to 1 = f(r)$.
Thus f is continuous at 1.

To prove discontinuity at 2 consider an irrational sequence $i_n \to 2$.

Then $f(i_n) = i_n^2 \to 4 \neq 2 = f(2)$.

So by the T.P. f is discontinuous at 2.

3.2.4 f is not continuous because continuity fails at 0, since

$\lim_{x \to 0} \sin \frac{1}{x}$ DNE. (We have seen that in class and above for $\sin \frac{2}{x}$).

3.2.5. $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

is continuous.

First, f is continuous at any $c \neq 0$ as a composition of continuous functions.

(do not forget that!!).

Now it remains to check 0. For any $x \neq 0$, we have $\left|\sin\frac{1}{x}\right| \leq 1 \Rightarrow$

$$\Rightarrow -|x| \leq x\sin\frac{1}{x} \leq |x|.$$

Since $\lim_{x\to 0}|x| = 0$, the squeeze theorem implies $\lim_{x\to 0} x\sin\frac{1}{x} = 0 = f(0)$.

So f is continuous at 0 as well, so it is overall continuous.

3.2.10. Let $f(r) = g(r)$ $\forall r \in \mathbb{Q}$.

Let $x \in \mathbb{R}$. Take a rational sequence (r_n) with $r_n \to x$. By continuity of f and g we take that $f(r_n) \to f(x)$ and $g(r_n) \to g(x)$.

Since $r_n \in \mathbb{Q}$ we have $f(r_n) = g(r_n)$ $\forall n \Rightarrow$

$\underset{n \to \infty}{\Rightarrow} f(x) = g(x)$.

3.2.11 Take any $\varepsilon > 0$ with $\varepsilon < f(c)$.
That's possible because $f(c) > 0$.

Now since f is continuous at c $\exists \alpha > 0$
s.t. $|f(x) - f(c)| < \varepsilon \quad \forall |x-c| < \alpha$.

$$f(c) - f(x) \le |f(c) - f(x)| < \varepsilon \Rightarrow$$

$$\Rightarrow f(c) - \varepsilon \le f(x) \quad \forall |x-c| < \alpha.$$

but $\varepsilon < f(c)$ so $f(x) > 0$.

3.2.15 In fact with what we know now
f will in fact be uniformly continuous.
To start, g is continuous at 0 and $g(0) = 0$
so $\lim_{z \to 0} g(z) = g(0) = 0$.

So $\forall \varepsilon > 0$, $\exists \delta$ s.t. if $|z| < \delta$ we have

$$|g(z)| < \varepsilon.$$

Now take $|x-y| < \delta$. Then $|g(x-y)| < \varepsilon$.

so $|f(x) - f(y)| \leq g(|x-y|) < \varepsilon$

so f is U.C.