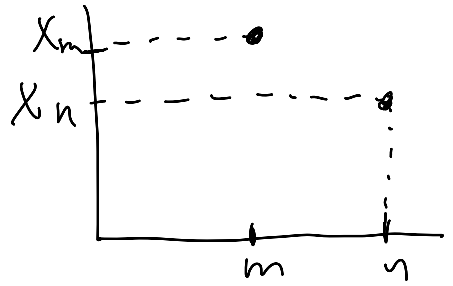
<u>Lemma</u> (Monotone subsequence lemma).
Every sequence (x_n) has a monotone subsequence.

<u>Proof.</u> We construct the set of peaks of (x_n) as follows:

$$A = \{ m \in \mathbb{N} \text{ s.t. } x_m \geq x_n \quad \forall n \geq m \}.$$

So if $m \in A$, we have the following picture



For the set A we have the following cases:

1) $A = \emptyset$. Define $n_1 = 1$. Since $n_1 \notin A$ $\exists\, n_2 > n_1$ with $x_{n_1} < x_{n_2}$.

Since $n_2 \notin A$, $\exists\, n_3 > n_2$ with

$x_{n_2} < x_{n_3}$.

Inductively we construct $1 = n_1 < n_2 < \dots <$
$< n_k < n_{k+1} < \dots$

with $x_{n_k} < x_{n_{k+1}}$ $\forall k \in \mathbb{N}$.

2) A is finite. Let $N = \max A$.
Define $n_1 = N+1$. Then $N+1 \notin A$, so
we follow the same process as before.

3) A is infinite. Let $A = \{n_1, \dots, n_k, \dots\}$.
Then by definition of A the subsequence
$(x_{n_k})_{k \in \mathbb{N}}$ is decreasing.

Theorem (Bolzano - Weierstrass).

Every bounded sequence has a convergent subsequence.

Proof The sequence has a monotone subsequence by the previous Lemma. Since it is bounded, it will converge.