Problem 1. *Prove the mean value theorem for integrals: If $f : [a,b] \to \mathbb{R}$ is continuous, then there exists a $c \in [a,b]$ such that $\int_a^b f = f(c)(b-a)$.*

Take $F(x) = \int_a^x f(t)\,dt$. By FTC, $F'(x) = f(x)$.

$F(b) - F(a) = \int_a^b f(t)\,dt$. Since f is continuous on $[a,b]$, F is continuous on $[a,b]$ and differentiable on (a,b). By MVT,

$\exists\, c \in (a,b)$ s.t. $\dfrac{F(b)-F(a)}{b-a} = F'(c)$,

which is $\int_a^b f = f(c)(b-a)$

Problem 2. *Compute*

$$\frac{d}{dx}\left(\int_{-x}^{x} e^{s^2}\,ds\right).$$

$e^{s^2} \geq 0 \quad \forall s$

Let $F(x) = \int_{-x}^{x} e^{s^2} ds$

$= \int_{-x}^{0} e^{s^2} ds + \int_0^x e^{s^2} ds$

$= 2\int_0^x e^{s^2} ds$

$\frac{d}{dx} F(x) = f(x) = \frac{d}{dx}\left(2\int_0^x e^{s^2} ds\right)$

by FTC, $= 2e^{x^2}$

Problem 3. *Compute*

$$\frac{d}{dx}\left(\int_0^{x^2} \sin(s^2)\,ds\right).$$

$\int_a^b f(g(x))\, g'(x)\,dx = \int_{g(a)}^{g(b)} f(s)\,ds$

So $f(x) = \sin x^2$, $g(x) = x^2$, $g'(x) = 2x$

$f(g(x)) = \sin(x^2)^2 = \sin x^4$

$a = 0$, $g(a) = 0$, $b = x$, $g(b) = x^2$

$\frac{d}{dx}\left(\int_0^{x^2} \sin(s^2)\,ds\right) = \frac{d}{dx}\left(\int_0^x \sin x^4 \cdot 2x\,dx\right)$

$= 2x \cdot \sin x^4$

Problem 4. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let $c \in [a, b]$ be arbitrary. Define*

$$F(x) := \int_c^x f. \; = F(x) - F(c)$$

Prove that F is differentiable and that $F'(x) = f(x)$ for all $x \in [a, b]$.

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\int_x^{x+h} f}{h} = \lim_{h \to 0} f(d)$$

By FTC, $F(x+h) = \int_c^{x+h} f = \int_c^x f + \int_x^{x+h} f = F(x) + \int_x^{x+h} f$

By MVT, $\exists\, d \in [x, x+h]$ s.t. $\int_x^{x+h} f = f(d) \cdot (x+h-x) = f(d) \cdot h$

Since $d \in [x, x+h]$ and $h \to 0$, so $F'(x) = \lim_{d \to x} f(d) = f(x) \quad \forall x \in [a, b]$

Problem 5. *Prove integration by parts. That is, suppose F and G are continuously differentiable functions on $[a, b]$. Then prove*

$$\int_a^b F(x)G'(x) = F(b)G(b) - F(a)G(a) - \int_a^b F'(x)G(x)\, dx.$$

$(F(x)\, G(x))' = F'(x)\, G(x) + F(x)\, G'(x)$

$F(x)\, G'(x) = (F(x)\, G(x))' - F'(x)\, G(x)$

$\int_a^b F(x)\, G'(x) = \int_a^b (F(x)\, G(x))' - \int_a^b F'(x)\, G(x)$

by FTC, $= F(b)\, G(b) - F(a)\, G(a) - \int_a^b F'(x)\, G(x)$

Problem 6. *Suppose F and G are continuously differentiable functions defined on $[a, b]$ such that $F'(x) = G'(x)$ for all $x \in [a, b]$. Using the fundamental theorem of calculus, show that F and G differ by a constant. That is, show that there exists a $C \in \mathbb{R}$ such that $F(x) - G(x) = C$.*

$F'(x) = G'(x)$ $\forall x \in [a, b]$. Take the integral on both sides,

$\int F'(x) = \int G'(x)$ $\forall x \in [a, b]$. Since by FTC,

$\int f = F + C$, so $F(x) + C_1 = G(x) + C_2$, $F(x) - G(x) = C$

Problem 7. *Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $\int_a^x f = \int_x^b f$ for all $x \in [a, b]$. Show that $f(x) = 0$ for all $x \in [a, b]$.*

$\int_a^x f = \int_x^b f$, $\int_a^x f = -\int_b^x f$, $\frac{d}{dx} \int_a^x f = \frac{d}{dx}\left(-\int_b^x f\right)$

$f(x) = -f(x)$, $2f(x) = 0$, $f(x) = 0$ $\forall x \in [a, b]$

Problem 8. *A function f is an odd function if $f(x) = -f(-x)$, and f is an even function if $f(x) = f(-x)$. Let $a > 0$. Assume f is continuous. Prove:*

(a) If f is odd, then $\int_{-a}^{a} f = 0$.

(b) If f is even, then $\int_{-a}^{a} f = 2\int_{0}^{a} f$.

(a) $\int_{-a}^{a} f = \int_{-a}^{0} f + \int_{0}^{a} f$

Since $f(x) = -f(-x)$, $f(x) \geq 0 \Rightarrow f(-x) \leq 0$ or $f(x) \leq 0 \Rightarrow f(-x) \geq 0$.

And since $\forall x \in [0,a]$, $f(x) = -f(-x)$. For any $x_1 \in [0,a]$, $\exists x_2 \in [-a,0]$ s.t. $f(x_1) + f(x_2) = f(x_1) - f(x_1) = 0$

$f(x_2) = -f(x_1)$, $x_2 = -x_1$

So $\int_{-a}^{0} f = -\int_{0}^{a} f$, $\int_{-a}^{a} f = 0$

(b) $\int_{-a}^{a} f = \int_{-a}^{0} f + \int_{0}^{a} f$

Since $f(x) = f(-x)$, $f(x) \geq 0 \Rightarrow f(-x) \geq 0$ or $f(x) \leq 0 \Rightarrow f(-x) \leq 0$

For any $x_1 \in [0,a]$, $\exists x_2 \in [-a,0]$ s.t. $f(x_2) = f(x_1)$, $x_2 = -x_1$

So $\int_{-a}^{0} f = \int_{0}^{a} f$, $\int_{-a}^{a} f = 2\int_{0}^{a} f$