4.1.2. We first show that $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$, $x \neq 0$.

Indeed, let $x \neq 0$. Take y close enough to x so that $y \neq 0$. We have

$$\frac{\frac{1}{y} - \frac{1}{x}}{y-x} = \frac{\frac{x-y}{xy}}{y-x} = -\frac{1}{xy}.$$

so as $y \to x$, we obtain

$$\lim_{y \to x} \frac{\frac{1}{y} - \frac{1}{x}}{y-x} = \lim_{y \to x} \frac{-1}{xy} = -\frac{1}{x^2}.$$

Thus $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

Now if we call $h(x) = \frac{f(x)}{g(x)}$ and $K(x) = \frac{1}{g(x)}$

we have $h(x) = f(x) K(x)$.

Notice that by the chain rule, $K(x)$ is differentiable with $K'(x) = -\frac{g'(x)}{g^2(x)}$.

so by the product rule, we have

$$h'(x) = \left(f(x)\,k(x)\right)' = f'(x)\,k(x) + f(x)\,k'(x)$$

$$= \frac{f'(x)}{g(x)} + \frac{-g'(x)}{g^2(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}.$$

4.1.6. Assume the inequality $|x - \sin x| \leq x^2$

Let's compute for $h \neq 0$

$$\frac{\sin(x+h) - \sin x}{h} = \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} =$$

$$= \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\,\frac{\sin h}{h}. \quad (*)$$

Now we have $|h - \sin h| \leq h^2 \Rightarrow$

$$\Rightarrow \left|1 - \frac{\sin h}{h}\right| \leq |h| \quad \forall\, h \neq 0$$

so by the squeeze theorem $\lim_{h \to 0}\left|1 - \frac{\sin h}{h}\right| = 0$

$$\Leftrightarrow \lim_{h \to 0} \frac{\sin h}{h} = 1$$

Now $\cos h - 1 = \cos h - \cos^2 h - \sin^2 h =$

$$= \cos h\,(1 - \cos h) - \sin^2 h$$

$$\Rightarrow \sin^2 h = \cos h\,(1 - \cos h) + 1 - \cos h =$$

$$= (\cos h + 1)(1 - \cos h)$$

$$\Rightarrow 1 - \cos h = \frac{\sin^2 h}{1 + \cos h} \Rightarrow$$

$$\Rightarrow \frac{1 - \cos h}{h} = \frac{\sin h}{h} \cdot \frac{\sin h}{1 + \cos h}.$$

so $\displaystyle \lim_{h\to 0} \frac{1 - \cos h}{h} = \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{1} \; \underbrace{\lim_{h\to 0} \frac{\sin h}{1 + \cos h}}_{0} =$

$$= 0.$$

so by (*).

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \sin x \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{0} + \cos x \underbrace{\lim \frac{\sin h}{h}}_{1}$$

$$= \cos x.$$

so $\sin x$ is differentiable and $(\sin x)' = \cos x$.

4.1.8. Assume first $n \in \mathbb{N}$. We prove the claim by induction.

For $n=1$ it is trivially true.

Assume the claim holds for n i.e.

$$(f^n)' = n f^{n-1} f'$$

We compute $(f^{n+1})'$.

We have $(f^{n+1})' = (f f^n)' =$

$$= f' f^n + f (f^n)' =$$

$$= f' f^n + f n f^{n-1} f' =$$

$$= f' f^n + n f^n f' =$$

$$= (n+1) f^n f'.$$

By induction the claim follows.

Now if $n<0$, let us write $k = -n > 0$

Write $g = \frac{1}{f}$. Then $f^n = f^{-k} = (f^{-1})^k =$
$= g^k$.

We use what we have proved to obtain:

$$(f^n)' = (g^k)' = k g^{k-1} g' = k f^{1-k} \left(\frac{1}{f}\right)' =$$
$$= k f^{1-k} \left(-\frac{1}{f^2}\right) f' =$$
$$= -k f^{-1-k} f' = n f^{n-1} f'.$$

~~4.1.9.~~ f is assumed Lipschitz so $\exists L > 0$
s.t. $|f(x) - f(y)| \leq L|x-y| \quad \forall x, y \in I$.

$$\Rightarrow \left| \frac{f(x) - f(y)}{x - y} \right| \leq L \quad \forall x, y \in I,\ x \neq y.$$

So for ~~all~~ for arbitrary $c \in I$, we have.

$$\left|\frac{f(x)-f(c)}{x-c}\right| \leq L \quad \forall x \neq c.$$

Letting $x \to c$ we take.

$$|f'(c)| = \left|\lim_{x \to c} \frac{f(x)-f(c)}{x-c}\right| =$$

$$= \lim_{x \to c} \left|\frac{f(x)-f(c)}{x-c}\right| \leq L.$$

4.1.11. We cannot apply the product rule because we merely know that f is bounded. So we need to go through the definition

Since f is bounded, there is $M>0$ s.t.

$$|f(x)| \leq M \quad \forall x \in I.$$

Also we know that $g(c)=0$ and

$$g'(c) = \lim_{x \to c} \frac{g(x)-\cancel{g(c)}}{x-c} = \lim_{x \to c} \frac{g(x)}{x-c} = 0. \quad (1)$$

$$\text{so} \quad \left|\frac{h(x)-h(c)}{x-c}\right| = \left|\frac{f(x)g(x) - f(c)\cancel{g(c)}}{x-c}\right| =$$

$$= \left| \frac{f(x)g(x)}{x-c} \right| \leq M \left| \frac{g(x)}{x-c} \right|.$$

But by ① $\lim_{x \to c} \left| \frac{g(x)}{x-c} \right| = 0.$

so by the squeeze theorem

$\lim_{x \to c} \frac{h(x)-h(c)}{x-c} = 0.$ Thus $h'(c) = 0.$

~~4.1.13~~ Let $M > 0$ s.t. $|h(x)| \leq M \ \forall x \in (-1,1).$

Let us compute

$$g(x) - g(0) = x^2 h^2(x) - 0^2 \cdot h^2(0) = x^2 h^2(x).$$

so $\frac{g(x)-g(0)}{x} = x h^2(x)$

so $\left| \frac{g(x)-g(0)}{x} \right| \leq M^2 x.$

since $\lim_{x \to 0} M^2 x = 0$, we take that (by squeeze).

$$\lim_{x \to 0} \left| \frac{g(x) - g(0)}{x} \right| = 0.$$

so $\lim_{x \to 0} \frac{g(x) - g(0)}{x} = 0$. so g is diff. at 0 and $g'(0) = 0$.

(b) Take $f(x) = \sqrt{|x|}$. Then $f(0) = 0$, it is continuous but $f^2(x) = |x|$ which is not differentiable at the origin