# Solutions to Relevant Past Practice Exam Questions

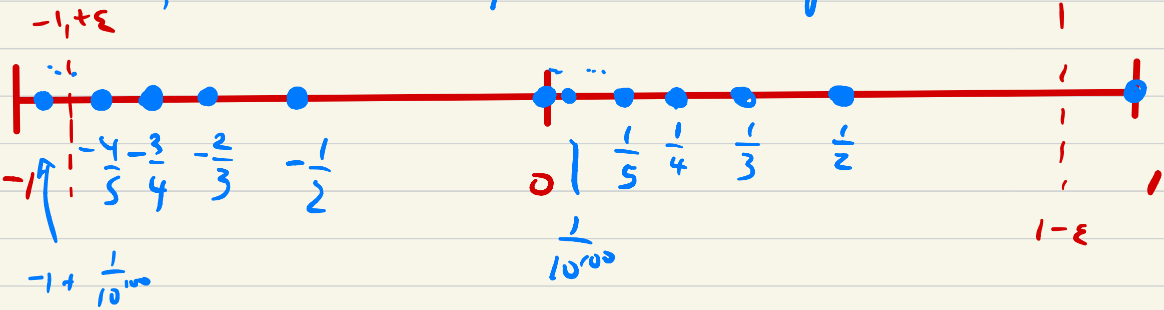
## • Practice Midterm #1

2. Let
$$S = \left\{ \frac{1-k}{k} : k \in \mathbb{N} \right\} \cup \left\{ \frac{1}{k} : k \in \mathbb{N} \right\}.$$

Find $\sup S$ + $\inf S$.

Pf. First observe that
$$\frac{1-k}{k} = -1 + \frac{1}{k}$$
for any $k \in \mathbb{N}$, so we can actually write
$$S = \left\{ -1 + \frac{1}{k} : k \in \mathbb{N} \right\} \cup \left\{ \frac{1}{k} : k \in \mathbb{N} \right\}.$$

With this, we can see the picture more easily:




It is thus clearer that $\sup S = 1$ and $\inf S = -1$.

Let us first prove that $\sup S = 1$. We will use the $\varepsilon$-characterization of supremum.

First, $1$ is an upper bound. If $s \in S$, then $S = -1 + \frac{1}{k}$ or $s = \frac{1}{k}$ for some $k \in \mathbb{N}$. Since $\frac{1}{k} \leq 1$, in either case we have $s \leq 1$ & so $1$ is an upper bound.

Next, it is the least upper bound (intuitively this is clear since $1 \in S$). Let $\varepsilon > 0$ be arbitrary. Since $1 \in S$, by setting $s = 1$ we have some $s \in S$ s.t.

$$1 - \varepsilon < s \leq 1.$$

Thus, $\sup(S) = 1$.

It is a tad trickier to see that $\inf(S) = -1$. First, $-1$ is a lower bound. If $s \in S$, then $S = -1 + \frac{1}{k}$ or $s = \frac{1}{k}$ for some $k \in \mathbb{N}$. Since $\frac{1}{k} > 0$, in either case $s \geq -1 + 0 = -1$. Thus, $-1$ is a lower bound.

To see that it is the greatest lower bound, let $\varepsilon > 0$ be arbitrary. By the Archimedean property, $\exists\, k \in \mathbb{N}$ s.t. $k > \frac{1}{\varepsilon}$, or in other words, $\frac{1}{k} < \varepsilon$. Now, $-1 + \frac{1}{k} \in S$ and observe that

$$-1 < -1 + \frac{1}{k} = s < -1 + \varepsilon.$$

Thus, $\inf(S) = -1$, as desired. $\blacksquare$

4. Let $\{x_n\}$ be a sequence such that $x_n \neq 0$ and $\lim_{n\to\infty} x_n = +\infty$.

Then,

$$\lim_{n\to\infty} \frac{1}{x_n} = 0.$$

Pf. The idea is that since $x_n$ gets large with $n$, $\frac{1}{x_n}$ should get small. Indeed, let $\varepsilon > 0$ be arbitrary. Then, $\exists$ $N$ s.t. $\forall$ $n \geq N$, $x_n > \frac{1}{\varepsilon}$. Observe that since $x_n$ is a large positive number, $|x_n| = x_n$. Thus, $\forall$ $n \geq N$ we have

$$\left| \frac{1}{x_n} - 0 \right| = \frac{1}{|x_n|} = \frac{1}{x_n} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon.$$

$\varepsilon > 0$ was arbitrary, so we conclude that

$$\lim_{n\to\infty} \frac{1}{x_n} = 0. \quad \blacksquare$$

## • Practice Midterm #2

3. Let $\{x_n\}$ be recursively defined as

$$\begin{cases} x_{n+1} := x_n - x_n^2 & n \geq 1 \\ x_1 := \frac{1}{2} \end{cases}.$$

a) Prove that $\{x_n\}$ is a decreasing sequence.

Pf. Observe that $x_n \in \mathbb{R}$, so $x_n^2 \geq 0$. Thus,

$$x_{n+1} = x_n - x_n^2 \leq x_n - 0 = x_n,$$

and so $\{x_n\}$ is monotone decreasing. ☒

b) Prove by induction that $x_n \geq 0$ for all $n \in \mathbb{N}$.

Pf. For our base case, observe that

$$x_1 = \frac{1}{2} \geq 0.$$

Next, for our inductive hypothesis, suppose $x_k \geq 0$. Since we saw in pt. (a) that $\{x_n\}$ is decreasing, we know that $x_k \leq x_1 = \frac{1}{2}$ too (this is technically an inductive argument itself; however, we've seen this enough times that I think we're safe to exclude it :) )

Thus, via the recursion, we have

$$\begin{aligned} x_{k+1} &= x_k - x_k^2 \\ &= x_k(1 - x_k) \\ &\geq 0 \end{aligned}$$

since $x_k, 1 - x_k \geq 0$. So, $x_{k+1} \geq 0$ & we conclude by induction that $x_n \geq 0$ for all $n \in \mathbb{N}$. $\blacksquare$

c) Prove that $\lim\limits_{n\to\infty} x_n$ exists.

Pf. From parts (a) + (b), we see that $\{x_n\}$ is a monotone decreasing sequence that is bounded from below. Thus, by the Monotone Convergence theorem, $\lim\limits_{n\to\infty} x_n := L$ exists and is a real number. $\blacksquare$

d) Find $\lim\limits_{n\to\infty} x_n$.

Pf. Since $x_n \to L$ and $\{x_{n+1}\}$ is a tail sequence, we have

$$\lim_{n\to\infty} x_{n+1} = L$$

too. Using the recursion relation

$$x_{n+1} = x_n - x_n^2$$

and the algebraic properties of limits, we have the equation

$$L = L - L^2.$$

Rearranging, we have $L^2 = 0 \Rightarrow L = 0$. Thus,

$$L = \lim_{n\to\infty} x_n = 0. \quad \blacksquare$$

4. Prove or disprove the following statements:

(i) Let $\{x_n\}$ + $\{y_n\}$ be two sequences. If $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} y_n = 0$$

then

$$\lim_{n\to\infty} x_n y_n = 0.$$

Pf. The idea is just that since $\{x_n\}$ is bounded, any bad behavior that prevents it from converging should be killed by $\{y_n\}$.

Since $x_n$ is bounded, $\exists\ B > 0$ s.t. $|x_n| \leq B$ for all $n \in \mathbb{N}$. Thus, we have the inequalities

$$0 \leq |x_n y_n| \leq B|y_n|$$

for all $n \in \mathbb{N}$. Now, since $y_n \to 0$, $|y_n| \to 0$ too by a HW exercise. By the algebraic properties of limits then,

$$\lim_{n\to\infty} B|y_n| = B \lim_{n\to\infty} |y_n| = B \cdot 0 = 0.$$

Hence, by the Squeeze Theorem, we have

$$\lim_{n\to\infty} |x_n y_n| = 0.$$

By the same HW exercise referenced previously,

$$\lim_{n\to\infty} x_n y_n = 0$$

too. ▨

(ii) Let $\{x_n\}$ and $\{y_n\}$ be two sequences. If $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} y_n = +\infty,$$

then

$$\lim_{n\to\infty} x_n y_n = +\infty.$$

Pf. This statement is false. Intuitively, if we let $x_n \searrow 0$, then $x_n$ should be able to kill off the growth of $\{y_n\}$.

Sure enough, let $x_n := \frac{1}{n}$ and $y_n := n$. Then $\lim_{n\to\infty} y_n = +\infty$ by the Archimedean property ($\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t. $N > \varepsilon$, and so $\forall\, n \geq N$ we have $n \geq N > \varepsilon$), and $0 \leq x_n \leq 1$ for all $n \in \mathbb{N}$. However,

$$x_n y_n = \frac{1}{n} \cdot n = 1$$

for all $n \in \mathbb{N}$, which converges to 1, not $+\infty$. ∎

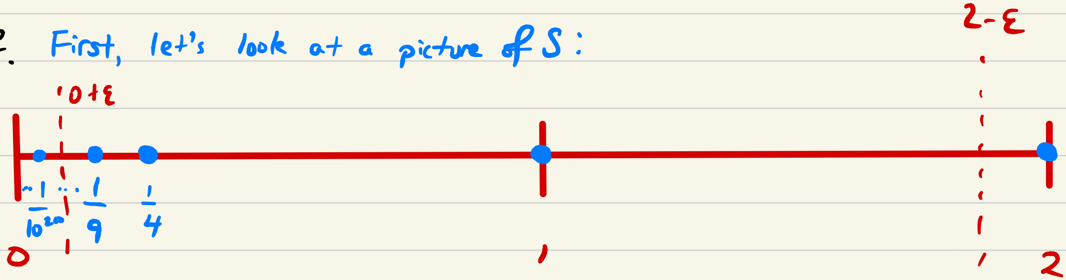
· Practice Midterm #3

2. Let

$$S := \{2\} \cup \left\{ \frac{1}{k^2} : k \in \mathbb{N} \right\}.$$

Find inf $S$ and sup $S$.

Pf. First, let's look at a picture of $S$:




We claim that $\sup S = 2$ and $\inf S = 0$.

To see that $\sup S = 2$, first observe that $2$ is an upper bound for $S$: $2 \le 2$, and $\frac{1}{k^2} \le 2$ for all $k \in \mathbb{N}$. Since all $s \in S$ take one of these forms, $s \le 2$ for all $s \in S$.

$2$ is also the least upper bound. $\forall \varepsilon > 0$, observe that $2 \in S$ and $2 - \varepsilon < 2$. Hence, setting $s = 2$, we have an element $s \in S$ s.t.

$$2 - \varepsilon < 2 = s \le 2,$$

showing that $\sup S = 2$.

For $\inf S = 0$, first observe that $0$ is a lower bound. If $s \in S$, then $s = 2$ or $s = 1/k^2$ for some $k \in \mathbb{N}$, both of which are positive. To see that $0$ is the greatest lower bound, let $\varepsilon > 0$ be arbitrary. Then, by the

Archimedean property, $\exists\, k \in \mathbb{N}$ s.t. $k > \frac{1}{\sqrt{\varepsilon}}$. So, setting $s = \frac{1}{k^2} \in S$, we find

$$0 < \frac{1}{k^2} = s < \frac{1}{\left(\frac{1}{\sqrt{\varepsilon}}\right)^2} = \varepsilon = 0 + \varepsilon$$

and so $\inf S = 0$. ▨

3. See #3 on Practice Midterm #2.

4. Prove or disprove the following statement:
Let $\{x_n\}$ + $\{y_n\}$ be two sequences of strictly positive numbers such that $\{x_n\}$ converges and $\{x_n y_n\}$ converges. Then, $\{y_n\}$ converges.

Pf. We claim that the statement is false. Heuristically, $\{y_n\}$ should be able to blow up in such a way that gets killed by the decay of $\{x_n\}$.

Sure enough, let $x = \frac{1}{n}$ and $y_n = n$. Then, $\lim_{n\to\infty} x_n = 0$ (shown in class) and $x_n y_n = 1$, which converges to $1$. However, $\{y_n\}$ does not converge (in fact it diverges to $+\infty$).

For the sake of completeness, suppose for the sake of a contradiction that $y_n = n$ did converge to a real $L$. Then, $\exists\, N$ st. $\forall\, n \geq N$,

$$|y_n - L| < \frac{1}{2}.$$

However, $\forall\, n \in \mathbb{N}$, $y_{n+1} - y_n = (n+1) - n = 1$. Thus, if $n \geq N$, we have

$$1 = |y_{n+1} - y_n| = |y_{n+1} - L + L - y_n| \leq |y_{n+1} - L| + |y_n - L|$$
$$< \frac{1}{2} + \frac{1}{2} = 1,$$

a contradiction. Hence, $\{y_n\}$ doesn't converge. ▨

## • Midterm 2020 Spring

2. See #2 on Practice Midterm #1.

4. See #4 on Practice Midterm #1.

## • Midterm 2020 Fall

2. Let $\{x_n\}$ be defined recursively as

$$\begin{cases} x_{n+1} = \dfrac{1-x_n}{4} & n \geq 1 \\ x_1 = \dfrac{1}{2} \end{cases}$$

(a) Prove that

$$|x_{n+1} - x_n| \leq \frac{1}{4} |x_n - x_{n-1}|.$$

Pf. Let $n \geq 2$ ($x_{n-1}$ doesn't make sense for $n=1$). Then, via the recursion, we have

$$\begin{aligned} |x_{n+1} - x_n| &= \left| \frac{1-x_n}{4} - x_n \right| \\ &= \left| \frac{1-x_n}{4} - \frac{1-x_{n-1}}{4} \right| \\ &= \left| \frac{1 - x_n - 1 + x_{n-1}}{4} \right| \\ &= \left| \frac{x_{n-1} - x_n}{4} \right| \\ &\leq \frac{1}{4} |x_n - x_{n-1}|. \end{aligned}$$

Technically we have equality in the last line, but this was not asked for. $\boxtimes$

(b) Use (a) to prove that $\{x_n\}$ is a Cauchy sequence.

Pf. This is a bit beyond what we've practiced, but only uses tools we know so I'll include it for completeness.

First, we claim by induction that $|x_{n+1} - x_n| \le \left(\frac{1}{4}\right)^{n-1} |x_2 - x_1|$.

The base case $n=1$ is just $|x_2 - x_1| = 1 \cdot |x_2 - x_1| = \left(\frac{1}{4}\right)^{1-1} |x_2 - x_1|$.

For the inductive step, suppose $|x_{k+1} - x_k| \le \left(\frac{1}{4}\right)^{k-1} |x_2 - x_1|$.
Then, by (a),

$$\begin{aligned} |x_{(k+1)+1} - x_{k+1}| &\le \frac{1}{4} |x_{k+1} - x_k| \\ &\le \frac{1}{4}\left(\frac{1}{4}\right)^{k-1} |x_2 - x_1| \\ &= \left(\frac{1}{4}\right)^{(k+1)-1} |x_2 - x_1|. \end{aligned}$$

Thus, $\forall n \in \mathbb{N}$, $|x_{n+1} - x_n| \le \left(\frac{1}{4}\right)^{n-1} |x_2 - x_1|$.

Now, let $\varepsilon > 0$ be arbitrary & choose $N$ large enough so that $\forall n \ge N$,

$$\frac{4}{3}\left(\frac{1}{4}\right)^{n-1} |x_2 - x_1| < \varepsilon.$$

This is possible since $\lim_{n\to\infty} \left(\frac{1}{4}\right)^n = 0$, as shown in class, and so $\lim_{n\to\infty} \frac{4}{3}\left(\frac{1}{4}\right)^{n-1} |x_2 - x_1| = 0$ too by the algebraic properties of limits.

Now, let $m, n \geq N$. Without loss of generality, let $m > n$ (otherwise, switch indices since $|x_m - x_n| = |x_n - x_m|$; if $m = n$, then $|x_m - x_n| = 0$). Then,

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \cdots - x_{n+1} + x_{n+1} - x_n|$$

$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$

$$\leq \left(\tfrac{1}{4}\right)^{m-2} |x_2 - x_1| + \left(\tfrac{1}{4}\right)^{m-3} |x_2 - x_1| + \cdots + \left(\tfrac{1}{4}\right)^{n-1} |x_2 - x_1|$$

$$= \left( \left(\tfrac{1}{4}\right)^{m-2} + \cdots + \left(\tfrac{1}{4}\right)^{n-1} \right) |x_2 - x_1|$$

$$= \frac{\left(\tfrac{1}{4}\right)^{n-1} - \left(\tfrac{1}{4}\right)^{m-1}}{1 - \tfrac{1}{4}} |x_2 - x_1|$$

$$\leq \tfrac{4}{3} \left(\tfrac{1}{4}\right)^{n-1} |x_2 - x_1| < \varepsilon,$$

where the penultimate line follows from the summation formula for geometric series. $\varepsilon > 0$ was arbitrary, so we conclude that $\{x_n\}$ is Cauchy. $\blacksquare$

(c) Compute $\lim_{n \to \infty} x_n$.

Pf. This is also a bit beyond what we've practiced, but I'll include it for completeness since it only uses things we've seen in class.

By (b), $\{x_n\}$ is Cauchy; thus, as proven in class, $\{x_n\}$ converges to some real number $L$. The tail sequence $\{x_{n+1}\}$ also converges to $L$, + so by the recursion

$$x_{n+1} = \frac{1-x_n}{4}$$

+ the algebraic properties of limits, we have

$$L = \frac{1-L}{4}.$$

Rearranging, we have

$$4L = 1-L$$

$$5L = 1$$

$$L = \frac{1}{5}.$$

Thus,

$$L = \lim_{n\to\infty} x_n = \frac{1}{5}. \qquad \blacksquare$$

4. Prove or disprove the following statements.

(i) Let $\{x_n\}$ be a sequence s.t.

$$\lim_{n\to\infty} x_n = 0.$$

Then,

$$\lim_{n\to\infty} \frac{1}{|x_n|} = +\infty.$$

Pf. We claim that this is true, with the added assumption $x_n \neq 0$ to make the problem well-defined. With this, let $B>0$. Then, since $\lim_{n\to\infty} x_n = 0$, $\exists$ N s.t. $\forall n \geq N$,

$$|x_n - 0| = |x_n| < \frac{1}{B}.$$

In particular, $\forall n \geq N$,

$$\frac{1}{|x_n|} > \frac{1}{\frac{1}{B}} = B.$$

$B>0$ was arbitrary, so $\lim_{n\to\infty} \frac{1}{|x_n|} = +\infty$. ∎

(ii) Let $\{y_n\}$ be a sequence s.t.

$$\lim_{n\to\infty} y_n = 0.$$

Then, either

$$\lim_{n\to\infty} \frac{1}{y_n} = +\infty \quad \underline{\text{OR}} \quad \lim_{n\to\infty} \frac{1}{y_n} = -\infty.$$

Pf. We claim that this statement is false. Intuitively, $y_n$ could have small oscillations that become arbitrarily large when we look at $\frac{1}{y_n}$.

Indeed look at

$$y_n = \frac{(-1)^n}{n},$$

which we've seen converges to zero. However,

$$\frac{1}{y_n} = (-1)^n n = \begin{cases} n & \text{if } n \text{ even;} \\ -n & \text{if } n \text{ odd.} \end{cases}$$

Notice that $\lim_{n\to\infty} \frac{1}{y_n} \neq +\infty$. Suppose for the sake of a contradiction that it did. Then, $\forall B > 0, \exists N$ s.t. $\forall n \geq N$, $\frac{1}{y_n} > B$. However, take $B = 1$; $\exists N$ s.t. $\forall n \geq N$, $\frac{1}{y_n} > B > 0$. However, for any odd $k > N$, $\frac{1}{y_k} < 0$. This is a contradiction, so $\lim_{n\to\infty} \frac{1}{y_n} \neq +\infty$.

Similarly, $\lim_{n\to\infty} \frac{1}{y_n} \neq -\infty$. Suppose for the sake of a contradiction that it did. Then, $\forall B > 0, \exists N$ s.t. $\forall n \geq N$, $\frac{1}{y_n} < -B$. However, take $B = 1$; $\exists N$ s.t. $\forall n \geq N$, $\frac{1}{y_n} < -B < 0$. However, for any even $k > N$, $\frac{1}{y_k} > 0$. This is a contradiction, so $\lim_{n\to\infty} \frac{1}{y_n} \neq -\infty$. ▨