5.1.3. Assume $(P_k)_k$ is a sequence of partitions of $[a,b]$ with $\lim_{k\to\infty}\left[U(P_k,f)-L(P_k,f)\right]=0$.

Let $\varepsilon>0$. Then $\exists\, k_0\in\mathbb{N}$ s.t.

$$U(P_{k_0},f)-L(P_{k_0},f)<\varepsilon$$

So $0\le\overline{\int_a^b} f-\underline{\int_a^b} f\le U(P_{k_0},f)-L(f,P_{k_0})<\varepsilon.$

Since ε is arbitrary we have

$$\overline{\int_a^b} f-\underline{\int_a^b} f=0 \Rightarrow \overline{\int_a^b} f=\underline{\int_a^b} f \text{ so}$$

f is integrable.

Now, we show that $\lim_{k\to\infty} U(f,P_k)=\int_a^b f$.

For any $k\in\mathbb{N}$ we have $L(f,P_k)\le\int_a^b f\le U(f,P_k)$.

Thus $\int_a^b f\le U(f,P_k)=U(f,P_k)-L(f,P_k)+L(f,P_k)\le$

$$\le U(f,P_k)-L(f,P_k)+\int_a^b f.$$

But $\lim_{k\to\infty}\left(U(f,P_k)-L(f,P_k)\right)=0$ so by the squeeze lemma, the sequence $\left(U(f,P_k)\right)_k$ converges and $\lim_{k\to\infty}U(f,P_k)=\int_a^b f$.

Now $\lim_{k\to\infty}L(f,P_k)=\lim_{k\to\infty}\left(L(f,P_k)-U(f,P_k)+U(f,P_k)\right)$

$=\lim_{k\to\infty}\left[L(f,P_k)-U(f,P_k)\right]+\lim_{k\to\infty}U(f,P_k)=$

$=\int_a^b f$.

5.2.2 Notice that for any interval $I\subset[a,b]$ we have

$$\inf_{x\in I}f(x)+\inf_{x\in I}g(x)\le\inf_{x\in I}\left(f(x)+g(x)\right)\le\sup_{x\in I}\left[f(x)+g(x)\right]\le$$
$$\le\sup_{x\in I}f(x)+\sup_{x\in I}g(x).$$

Thus for any partition P we have

$$L(f,P)+L(g,P)\le L(f+g,P)\le U(f+g,P)\le U(f,P)+U(g,P) \quad (1).$$

Now let $\varepsilon > 0$. We will show that $\exists P$ with

$$U(f+g, P) - L(f+g, P) < \varepsilon.$$

Indeed let $\varepsilon > 0$. Since f is integrable $\exists P_1$ with

$$U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2} \quad (2)$$

Similarly, since g is integrable, $\exists P_2$ with

$$U(g, P_2) - L(g, P_2) < \frac{\varepsilon}{2} \quad (3).$$

Let $P = P_1 \cup P_2$. By (2)-(3), get

$$U(f, P) - L(f, P) < \frac{\varepsilon}{2} \text{ and } U(g, P) - L(g, P) < \frac{\varepsilon}{2}.$$

so adding those, we get

$$U(f, P) + U(g, P) - L(f, P) - L(g, P) < \varepsilon. \quad (4)$$

By (1), (4) we get $U(f+g, P) - L(f+g, P) < \varepsilon$

so $f+g$ is integrable.

Additionally, for any partition P we have:

$$\int_a^b (f+g) \leq U(f+g, P) \overset{(1)}{\leq} U(f, P) + U(g, P).$$

Since the left hand side is independent of P we can take the inf on the right hand side.

So $$\int_a^b (f+g) \le \inf_P \{ U(f,P) + U(g,P) \}$$

$$= \inf_P U(f,P) + \inf_P U(g,P) = \overline{\int_a^b} f + \overline{\int_a^b} g =$$
$$= \int_a^b f + \int_a^b g$$

so $$\int_a^b (f+g) \le \int_a^b f + \int_a^b g.$$

Similarly (using lower sums instead and taking suprema).

we can show that
$$\int_a^b f + \int_a^b g \le \int_a^b (f+g).$$

Thus $$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

5.2.5 By continuity, it suffices to show that $f(x) = 0$ $\forall x \in (a,b)$.

Assume $\exists\, c \in (a,b)$ with $f(c) > 0$. Then since f is continuous and $c \in (a,b)$, $\exists\, \delta > 0$ s.t. $(c-\delta, c+\delta) \subset (a,b)$ and $f(x) > 0$ $\forall x \in (c-\delta, c+\delta)$.

Now by the Min-Max theorem, $\exists\, m > 0$ s.t. $m \leq f(x)$ $\forall x \in [c - \frac{\delta}{2}, c + \frac{\delta}{2}]$. $\Rightarrow$

$$\Rightarrow 0 < m\delta \leq \int_{c-\delta/2}^{c+\delta/2} f \leq \int_a^b f \quad \text{since } f \geq 0.$$

which is a contradiction, since $\int_a^b f = 0$.

5.2.6. If $f(a) = 0$ or $f(b) = 0$, the claim is trivial. So assume $f(a)f(b) \neq 0$.

Case 1: $f(a)f(b) < 0$. Then, by Bolzano's theorem, $\exists\, c \in [a,b]$ with $f(c) = 0$.

Case 2: $f(a)f(b) > 0$. Assume WLOG that $f(a) > 0$, $f(b) > 0$. Assume that $f(x) \neq 0$ $\forall x \in (a,b)$. If $f(x) > 0$ $\forall x \in (a,b)$, that leads to

contradiction because $\int_a^b f = 0$ and Ex. 5.2.5.
If $\exists x_0 \in (a,b)$ with $f(x_0) < 0$, then by Bolzano's theorem $\exists c \in (a, x_0)$ with $f(c) = 0$ which contradicts the assumption $f(x) \neq 0$ $\forall x \in (a,b)$.

<u>Note</u>. The easiest way to do that problem is to use Ex. 5.2.4 which was not assigned in this homework. In that case $\exists c \in [a,b]$ s.t. $f(c) = \frac{\int_a^b f}{b-a} = 0$.

5.2.7. Define $h = f - g$ which is continuous.
Moreover $\int_a^b h = \int_a^b (f-g) = \int_a^b f - \int_a^b g = 0$.

So by Ex. 5.2.6. $\exists c \in [a,b]$ with $h(c) = 0$
$\Leftrightarrow f(c) = g(c)$.

5.2.8. If $\alpha < \beta < \gamma$ we know it.
Assume now that $\alpha < \gamma < \beta$. Then

$$\int_a^{\beta} f = \int_a^{\gamma} f + \int_{\gamma}^{\beta} \Rightarrow \int_a^{\gamma} f = \int_a^{\beta} f - \int_{\gamma}^{\beta} f =$$

$$= \int_a^{\beta} f + \int_{\beta}^{\gamma} f .$$

So we have showed that the formula holds -

no matter what the ordering between γ is:

So assume WLOG that $\beta < \gamma$.

It remains to consider the cases $\beta < \alpha < \gamma$ and $\beta < \gamma < \alpha$.

Assume $\beta < \alpha < \gamma$. Then $\int_{\beta}^{\gamma} f = \int_{\beta}^{\alpha} f + \int_{a}^{\gamma} f \Rightarrow$

$$\Rightarrow \int_a^{\gamma} f = \int_{\beta}^{\gamma} f - \int_{\beta}^{\alpha} f = \int_{\alpha}^{\beta} f + \int_{\beta}^{\gamma} f .$$

Assume now $\beta < \gamma < \alpha$. Then $\int_{\beta}^{a} f = \int_{\beta}^{\gamma} f + \int_{\gamma}^{\alpha} f \Rightarrow$

$$\Rightarrow \int_{\gamma}^{\alpha} f = \int_{\beta}^{a} f - \int_{\beta}^{\gamma} f \Rightarrow$$

$$\Rightarrow \int_{\alpha}^{\gamma} f = \int_{a}^{\beta} f + \int_{\beta}^{\gamma} f .$$

The claim is proved.

5.2.17 g is Lipschitz so $\exists L > 0$ s.t.
$|g(x) - g(y)| \leq L|x-y| \quad \forall x,y \in \mathbb{R}$.

Also f is continuous so it is bounded i.e. $\exists M > 0$ s.t. $|f(x)| \leq M \quad \forall x \in [a,b]$

Thus $|h(x) - h(y)| = \left| \int_a^b g(x-t) f(t)\, dt - \int_a^b g(y-t) f(t)\, dt \right| =$

$$= \left| \int_a^b (g(x-t) - g(y-t)) f(t)\, dt \right| \leq$$

$$\leq \int_a^b |g(x-t) - g(y-t)|\, |f(t)|\, dt \leq$$

$$\leq M \int_a^b |x - \cancel{t} - y + \cancel{t}|\, dt = ML(b-a)\, |x-y|.$$

So h is Lipschitz with constant $ML(b-a)$.