Research Statement

Yunpeng Niu

I work in symplectic geometry, especially on mirror symmetry and Fukaya categories.

1. Introduction

Mirror symmetry is the mathematical interpretation of certain dualities predicted by string theory, where two seemingly different physical systems are isomorphic in a nontrivial way. This leads to many unexpected duality results in geometry, the most influential of which is **Homological Mirror Symmetry** (HMS), initiated by Kontsevich [Kon95]. HMS predicts an isomorphism between two invariants, the **Fukaya category** $Fuk(X)$ of a symplectic manifold X, a symplectic invariant, and the bounded derived category of coherent sheaves $D^bCoh(\hat{X})$ on a certain "dual" algebraic variety $\hat{X}$, an algebro-geometric invariant. While $D^bCoh(\hat{X})$ is a central object in algebraic geometry known to encode all information of an underlying algebraic variety in many cases [BO95], it is unknown how much symplectic geometry the Fukaya category detects.

The Central Research Question. Motivated by HMS, the central question I am trying to answer is the following: *How much does the Fukaya category detect about the geometry of the symplectic manifold to which it is associated?*

One way to answer this question is to realize the predictions of HMS. Unlike $D^bCoh(\hat{X})$, fewer structures are known on $Fuk(X)$ and it is a longstanding project in symplectic geometry to put "mirror" structures on $Fuk(X)$. For instance, while the Fourier-Mukai transformation has no obvious counterpart in symplectic geometry, groundbreaking work of Wehrheim-Woodward [WW10] associated to **Lagrangian correspondences** a correspondence functor that parallels the algebraic picture. Similarly, while the tensor product provides an obvious **monoidal structure** on $D^bCoh(\hat{X})$, no mirror monoidal structure on $Fuk(X)$, although long conjectured, is known to exist in general.

A crucial testing ground for this conjecture, motivated by **SYZ mirror symmetry** [SYZ96], are symplectic manifolds equipped with a **Lagrangian torus fibration** with section. In 2010, Subotic [Sub10] constructed a mirror monoidal structure on the Fukaya category in the case of a symplectic manifold with a regular Lagrangian fibration. This includes elliptic curves, the complex dimension 1 case of mirror symmetry. The main result of my research thus far is a generalization of Subotic's construction to the first singular case.

> **Main Result.** For symplectic 4-fold that admits a Lagrangian fibration with section, allowing generic type of singular fiber, we can construct geometrically an immersed Lagrangian correspondence, which gives rise to a monoidal structure on the Fukaya category under suitable geometric hypothesis.

This includes the classical setting of complex dimension 2 mirror symmetry: elliptic K3 surfaces. This result is related to functorality in categorical symplectic geometry [AB22], brings more understanding of the mirror symmetry of K3 surfaces, and provides a key example for Abouzaid's proposal of **spectrum mirror symmetry**.

Future Proposals. I expect to apply the methodology of this result to study more general singular fibrations, construct monoidal structures on more Fukaya categories, and construct mirror varieties via HMS.

2. Background

In this section, I state necessary background and summarize Subotic's result. A $2n$-dimensional symplectic manifold (M^{2n}, ω) is said to be equipped with a regular **Lagrangian fibration** if there exists a locally trivial, proper submersion $\pi : M \to B$ onto an n-dimensional manifold B whose fiber are Lagrangian (i.e. $\omega|_{F_b} = 0$). This is a natural structure to consider and examples abound, for instance the cotangent bundle of any smooth manifold equipped with its standard symplectic form. If we require the fibers to be connected then they are tori T^n [Eva23].

Another ingredient we need is the **Fukaya category** $Fuk(M)$ of a symplectic manifold M. Its objects are (admissible) Lagrangian submanifolds L. The morphisms are Floer chain complexes with boundary operators given by counting pseudoholomorphic polygons with boundary on these Lagrangians, satisfying a particular algebraic relation called the A_∞ relations. This algebraic restriction leads to geometric hypotheses on M and L. Especially, we need L to be **unobstructed** (i.e. it does not bound any pseudoholomorphic disks).

After choosing a local trivialization on the base, a regular Lagrangian fibration becomes a completely integrable system. This means it is equipped with a maximal set of Poission-commuting Hamiltonian functions, which generate flows that periodically circle each fiber F_b, generating $H_1(F_b; \mathbb{Z})$. The time coordinates parametrizing the flow define a group structure on each fiber that assembles the natural addition on tori $T^n = \mathbb{R}^n/\mathbb{Z}^n$ after choosing a Lagrangian section $\sigma : B \to M$ to serve as the origin. This fibrewise addition has an associated addition graph:

$$\Gamma := \{(x, y, z) \in M^3 \ | \ z = x + y\}.$$

For regular Lagrangian fibrations, Subotic [Sub10] proved that this addition graph Γ is a smooth Lagrangian correspondence from $M \times M$ to M; that is, a smooth Lagrangian submanifold of $(M^3, -\omega \oplus -\omega \oplus \omega)$. Moreover, using the work of [WW10], he showed that it induces a correspondence functor from $Don^{\#}(M \times M)$ to $Don^{\#}(M)$ under natural geometric hypotheses, inducing a monoidal structure on $Don^{\#}(M)$. Here $Don^{\#}(M)$ is a certain variant of the Fukaya category called the (extended) Donaldson-Fukaya category. Unlike the Fukaya category, the morphisms in $Don^{\#}(M)$ are Floer homologies, making it a genuine category.

3. Research to Date

In this section, I summarize my past work generalizing Subotic's construction to the first case of singular Lagrangian fibrations.

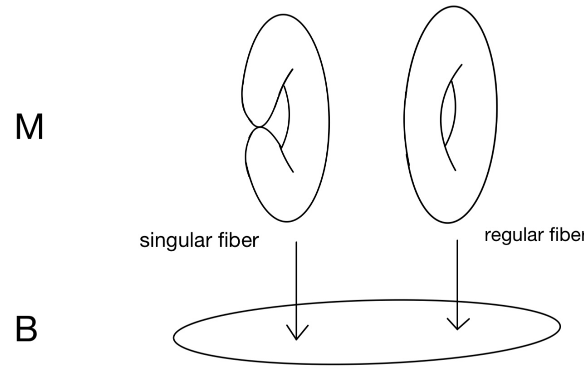
3.1. An Immersed Lagrangian Correspondence.

A natural class of singularities to consider are *focus-focus singularities*. In dimension 4, these singularities have a local neighborhood that is fibered-symplectomorphic to the following standard model:

$$\begin{aligned}(\mathbb{C}^2, \omega_0 = dp_1 \wedge dq_1 + dp_2 \wedge dq_2) &\longrightarrow \mathbb{C} \\ (p, q) &\longrightarrow -\bar{p}q\end{aligned}$$

If a singular fiber has k focus-focus singularities and no other singular points, the fiber will look like a torus pinched at k circles. A singular fiber with one focus-focus singularity and no other singular points is called a *symplectic A_1 singular fiber*.

In my collaboration with M. Abouzaid and N. Bottman [ABN24], we study symplectic 4-folds (M, ω) equipped with a Lagrangian fibration with section, having singular fibers of symplectic A_1 type (see the figure below). In this case, we find that for each singular fiber F_s, the group structure on the regular torus fibers induces a well-defined group structure on its regular part, isomorphic to $(\mathbb{C}^*, \times)$. Under this identification, the singular point s is identified with 0 and



∞. As such, the addition on the singular fiber F_s can only be defined on pairs of points in $F_s \times F_s \backslash (s, s)$. This is consistent with Kodaira's observations [Kod63] regarding elliptic fibrations with (complex) A_1-type singular fibres. Thus, we can define the addition graph Γ, as in the case of regular fibrations, on those pairs of points where addition is well-defined. Our main result is:

Theorem 3.1 (Niu, with Abouzaid and Bottman). *The closure $\bar{\Gamma}$ of the addition graph Γ is an immersed Lagrangian correspondence from $M \times M$ to M, with ordinary double points at each (s, s, s), for s a critical point.*

The proof of this theorem relies on a result of Ngọc [Ngọ03] that enables us to work with certain model fibrations. In these models, we build charts that cover $\bar{\Gamma}$ and use them to explicitly construct a smooth manifold $\widetilde{\Gamma}$ with an immersion onto $\bar{\Gamma}$, from which one can see the transversal double point singularities clearly.

We also analyze the geometry of this Lagrangian correspondence. On the pair of points where addition is well-defined, the map $(x, y) \in M \times_B M \to (x, y, x + y) \in \widetilde{\Gamma}$ is a diffeomorphism onto its image. The complement of its image is diffeomorphic to separate copies of S^2, each corresponding to a pair of singular points (s, s). The normal bundle of $S^2 \subset \widetilde{\Gamma}$ is isomorphic to the underlying real bundle of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{P}^1$, via an explicit isomorphism we construct using our local charts for $\widetilde{\Gamma}$. In this way, $\widetilde{\Gamma}$ can be interpreted as the minimal resolution of $M \times_B M$: the singularities of $M \times_B M$ are the points (s, s), with local models of the form $\{xy = zw\} \subset \mathbb{C}^4$. Indeed, any minimal resolution would resolve the origin with an exceptional $\mathbb{P}^1$, and such an exceptional curve always has normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ [Ati58].

3.2. **The Generic Elliptic K3 case.** The singular Lagrangian fibration we considered in [ABN24] includes a classical setup for mirror symmetry in complex dimension 2: generic elliptic K3 surfaces. An elliptic K3 is a K3 surface (M, J) equipped with an elliptic fibration $\pi : M \to \mathbb{P}^1$. Among those K3 that admit such fibrations, generically they have 24 (complex) A_1-type singular fibers [Huy16]. Each singular fiber has a fibered neighborhood smoothly equivalent to that of a symplectic A_1 singular fiber. The hyperKähler structure on K3 enables us to rotate the complex structure (thus the compatible symplectic structure), turning holomorphic submanifolds into special Lagrangian submanifolds, to get an S^1-family of Lagrangian fibrations.

In my work [Niu], I consider this geometric setup. Consider a generic elliptic K3 with (holomorphic) section σ and denote the corresponding S^1-family of Lagrangian fibrations (after hyperKähler rotation) on it by $\pi : (M, \omega_\theta) \to S^2$ for $\theta \in S^1$, where ω_θ is the S^1-family of Kähler forms. I find that the addition stays the same for all $\theta \in S^1$, thus giving the same addition graph. Denote by Γ this common addition graph. I then prove the following key result:

Theorem 3.2 (Niu). *$\bar{\Gamma}$ is unobstructed for generic ω_θ in the S^1-family.*

That is to say, $\bar{\Gamma}$ does not bound any nonconstant pseudo-holomorphic disks for the complex structure compatible with $\widetilde{\omega}_\theta = -\omega_\theta \oplus -\omega_\theta \oplus \omega_\theta$. Thus, together with a choice of spin structure on $\widetilde{\Gamma}$ that is used to orient the corresponding moduli spaces, the previous theorem implies that $\bar{\Gamma}$ is an admissible object in the Fukaya category $Fuk((M \times M)^- \times M, \widetilde{\omega}_\theta)$. By using the framework developed by Fukaya [Fuk23], where objects are allowed to be Lagrangians with clean self-intersections, I further show the following.

Theorem 3.3 (Niu). *$\bar{\Gamma}$ induces an A_∞ bi-functor on $Fuk(M, \omega_\theta)$ for generic $\theta \in S^1$, which gives a symmetric monoidal structure with unit object $\sigma(B)$.*

This A_∞ bi-functor is constructed from the composition of the Künneth functor $Fuk(M) \times Fuk(M) \to Fuk(M \times M)$ and the correspondence functor $Fuk(M \times M) \to Fuk(M)$ induced by $\bar{\Gamma}$. Then, under a natural definition of monoidal structure on an A_∞ category, I show this A_∞ bi-functor induces a symmetric monoidal structure, upgrading Subotic's result to the A_∞ category. This confirms that the Fukaya category "sees," through its own algebraic structure, the Lagrangian fibration structure on the underlying symplectic manifold.

I also discover in this case the following unexpected geometric feature: while the elliptic fibration itself carries a fiberwise addition induced from the elliptic curve, which gives a complex addition graph whose closure has the same geometric features as in [ABN24], this does not in general coincide with the above symplectic addition graph. In [Niu], I provide an equivalent condition for the two addition graphs to coincide at every pair of points. In particular, this condition forces the hyperKhäler metric to be semi-flat (that is, flat when restricted to each fiber), which is not the case in general for elliptic K3 surfaces.

4. Future Work

In the following I will describe my research plan for the period of the fellowship.

4.1. **Monoidal Structures on Fukaya Categories.**

Given our work in [ABN24], it is natural to ask the following question: when can the immersed Lagrangian correspondence from [ABN24] induce a monoidal structure on the Fukaya category?

In an ongoing project with M. Abouzaid and N. Bottman [ABN], we expect to answer this question in the case of exact symplectic manifolds; that is, when the symplectic form satisfies $\omega = d\lambda$ for $\lambda \in \Omega^1(M)$ and the Lagrangians L considered in the Fukaya category are exact with respect to λ. This is a classical setup for instance for the wrapped Fukaya category in the noncompact case and includes an important example formulated by Auroux [Aur07].

Expectation 4.1. *$\bar{\Gamma}$ induces a monoidal structure on $Fuk(M)$ in a suitable exact setup.*

In this case, we can show that there is a canonical grading for M that grades both fibers and sections, generalizing Seidel's construction in [Sei00]. This grading induces a canonical grading on $(M \times M)^- \times M$, under which the correspondence is graded and can be shown to be unobstructed. We then expect to use a similar setup to that of symplectic cluster manifolds in [Gro22], where one can put restrictions on how the fibration behaves near infinity to give geometric boundness conditions on M so that a suitable exact Fukaya category is defined. From there, we expect to use the construction of an SYZ mirror in [GV24] to prove that the resulting monoidal structure is in fact mirror to the tensor product on D^bCoh.

4.2. **Geometry of Symplectic Addition.** I also plan to include more singularity types to study the geometry of the addition graph, starting from symplectic A_n-type singular fibers (i.e., a torus pinched at multiple points with each singular point of focus-focus type) for $n \geq 2$. These also arise naturally for elliptic K3 surfaces. I expect the group structure on the regular part of the singular fiber is isomorphic to $(\mathbb{Z}_n, +) \times (\mathbb{C}^*, \times)$. I also expect the closure of the addition graph will include extra pieces from the summation of pairs of singular points that each give an extra S^2, corresponding to one component in the multi-pinched torus. Thus, I aim to prove:

Conjecture 4.2. *$\bar{\Gamma}$ is an embedded Lagrangian correspondence from $M \times M$ to M, when M has singular fibers of symplectic A_n-type ($n \geq 2$).*

To prove this, we can no longer use a complete classification result as in the symplectic A_1 case [ABN24], but I expect a similar local-to-global analysis to hold by connecting the standard focus-focus chart at each singular point with local Hamiltonian flow. Similarly, the extra S^2 should correspond to resolution of the singularity at each pair of singular points in $M \times_B M$.

I also plan to include more types of singularities: either from the symplectic viewpoint, other nondegenerate singularities for Hamiltonian systems like elliptic and hyperbolic type singularities; or, from the complex viewpoint, other types of singular fibers that could arise for elliptic K3 surfaces, for instance those of D_n and E_n type. In this way, I expect to study more concrete examples in the elliptic K3 case. For instance, there is a natural torus fibration arising from the construction of Kummer K3s, in which case the singular fibers are of D_4 type.

4.3. **Reconstructing Mirror Variety.** I expect to use the symmetric monoidal strucuture in [Niu] to study mirror symmetry of K3 surfaces. Through the construction of twisted complexes, the A_∞ category $Fuk(X)$ becomes a triangulated category $DFuk(X)$. I expect the A_∞ bi-functor in [Niu] can be upgraded to a tensor product $\otimes$ on $DFuk(X)$ compatible with exact triangles. By predictions of HMS, $D^bCoh(\hat{X})$ for the mirror variety $\hat{X}$ is equivalent to $DFuk(X)$. I expect this equivalence to be an equivalence of tensor triangulated categories [Bal02]:

Conjecture 4.3. *$(DFuk(X), \otimes)$ and $(D^bCoh(\hat{X}), \hat{\otimes})$ are equivalent as tensor triangulated categories.*

As for an elliptic K3 surface with section, its D^bCoh determines the original variety [Huy16], it is reasonable to expect we can reconstruct the mirror variety $\hat{X}$ from $DFuk(X)$.

Balmer's spectrum construction [Bal02] suggests one can obtain a locally ringed space $Spec((D^bCoh(\hat{X}), \hat{\otimes}))$ from the tensor triangulated category $(D^bCoh(\hat{X}), \hat{\otimes})$ and $\hat{X} \cong Spec((D^bCoh(\hat{X}), \hat{\otimes}))$. I expect to reconstruct $\hat{X}$ by $Spec((DFuk(X), \otimes))$ under nice assumptions. On the other hand, suggested by the SYZ picture, it should also be possible to reconstruct $\hat{X}$ geometrically. It is conjectured by Fukaya that under the assumption that the mirror variety $\hat{X}$ is projective, the SYZ mirror can be constructed explicitly as a projective scheme over a certain graded commutative ring:

Conjecture 4.4 (Fukaya)**.** *$\hat{X} \cong Proj(\oplus_{i=0}^{\infty} HF(L_0, L_i))$ as SYZ mirror.*

Here L_0 denotes the Lagrangian section we use as the origin for the group structure and L_i is the ith iterated sum of another Lagrangian section L_1. Here L_0 is considered to be mirror to the structure sheaf on $\hat{X}$ and L_1 is considered to be mirror to an ample line bundle. I expect to realize the construction of $\hat{X}$ in this way and, via the above equivalence, to show that the upgraded monoidal structure is in fact mirror to the derived tensor product on $D^bCoh(\hat{X})$: by showing $Spec((DFuk(X), \otimes) \cong Proj(\oplus_{i=0}^{\infty} HF(L_0, L_i))$ as locally ringed space, this shows the SYZ mirror is in fact the homological mirror of X.

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