VaR Subadditivity Failure

Empirical VaR Subadditivity Test

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What is Value-at-Risk?

- **Definition:** $\mathrm{VaR}_\alpha(L_t) = -\inf\{x \in \mathbb{R} : F_{L_t|\mathcal{I}_{t-1}}(x) > \alpha\}$; we use daily horizon with $\alpha \in \{0.95, 0.99\}$.
- **Threshold view:** at level α, VaR_α is the loss we only exceed with probability $1-\alpha$ (e.g., 5% of days for $\alpha = 0.95$).
- **Advantages:** simple, communicable, regulator-friendly; quick to compute from history or scenarios.
- **Limits:** VaR is not subadditive (therefore not coherent), it does not show the severity of loss, and it jumps when regime changes fast.

VaR Illustration

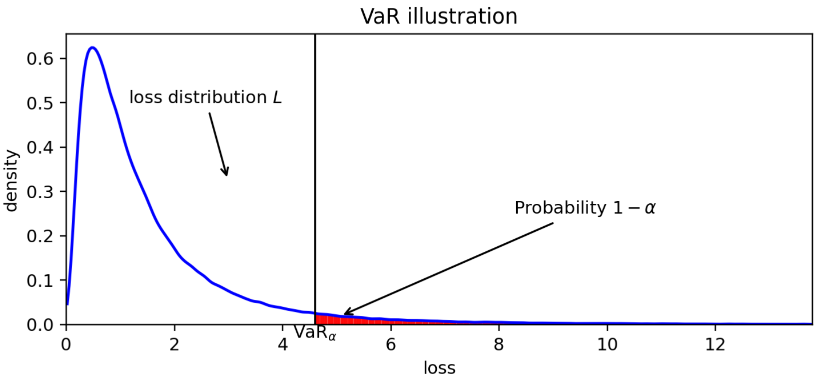


Figure: Blue curve is loss density; red bars are tail mass $1-\alpha$ beyond VaR_α.

Why Subadditivity Is Important

- VaR is not subadditive (therefore not coherent).
- Under VaR-based capital rules, diversification can be penalized: the VaR of the combined portfolio may exceed the sum of stand-alone VaRs. This creates an incentive to keep positions in separate books or entities to reduce reported capital requirements.
- Interpretation: reflects the idea of diversification; "merger does not create extra risk," yet nonsubadditive VaR can push firms to split legally to lower reported capital.

VaR Fails Subadditivity

Setup. Take two independent defaultable zero-coupon bonds, each notional one:

$$L_i = \begin{cases} 0 & \text{w.p. } 0.96, \\ 0.7 & \text{w.p. } 0.04, \end{cases} \qquad i = 1, 2.$$

$\mathrm{VaR}_{0.95}(L_i) = 0$ because $0.96 > 0.95$ keeps 95% quantile at zero.
Portfolio. Mix 50/50 so $L = \frac{1}{2}L_1 + \frac{1}{2}L_2$. Then

$$L = \begin{cases} 0 & \text{w.p. } 0.9216, \\ 0.35 & \text{w.p. } 0.0768, \\ 0.7 & \text{w.p. } 0.0016. \end{cases}$$

Violation. Positive homogeneity gives $\mathrm{VaR}_{0.95}(L_1 + L_2) = 2\mathrm{VaR}_{0.95}(L) = 0.7$, but $\mathrm{VaR}_{0.95}(L_1) + \mathrm{VaR}_{0.95}(L_2) = 0$. Simple mix shows VaR not subadditive (so not coherent).

Our Goal

- Benchmark HS / QR / GJR on the same return data to see which adapts fastest and stays closest to the target breach rate (QR leads in our results).
- Test subadditivity directly by comparing portfolio-level VaR against the sum of stand-alone VaRs to flag where VaR fails.
- Look for patterns in how often and how severely subadditivity breaks as we change portfolio style and confidence level (0.90 to 0.99).

VaR Formula & Inputs

- **Loss driver:** $L_t = -w^\top r_t$, r_t is aligned daily log return vector and w is weights (single asset just $w = 1$).
- We use stock SPY, QQQ, XOM, IWM, TSLA data from Stooq and use close price from 2014-01-02 to 2025-11-11.
- Model inputs:
  - HS: uses the rolling return window $r_{t-252:t-1}$ directly (no extra features).
  - QR: uses `build_feature_frame`$(r) \to$ target $y = r_t$, plus features lag1 $= r_{t-1}$, vol21, vol63 (21d/63d rolling std, all shifted one day for OOS).
  - GJR: uses the same return series but fits conditional variance/Student-t shape via its recursion (no explicit lag/vol features beyond its AR(1)/GJR states).
  - Tail level $p \in \{0.95, 0.99\}$ we check for each name and mix.

Portfolio Test for Subadditivity Failure Rate

- Source: Stooq daily Close for SPY, QQQ, IWM, XOM, TSLA (2014-01-02 to 2025-11-11), converted to daily log returns.
- Goal: measure when and where subadditivity fails more often across mixes and confidence levels.
- SPY/QQQ/IWM are index ETFs even though we treat them as stock exposures.

Historical Simulation Model

- Keep rolling window last 252 daily log returns $r_{t-252:t-1}$ for each asset or weighted portfolio.
- Forecast is $\mathrm{VaR}^{\mathrm{HS}}_{p,t} = \text{quantile}_p(r_{t-252:t-1})$, no parameter except window length.
- It drops one old point each new day, so it adapts slowly.
- Easy to explain: pure percentile of past 252 days—no parameters to explain or tune.
- Run HS for $p = 0.95, 0.99$ so team sees how same history gives different capital ask.

Quantile Regression Model

- Build feature row $x_t = (\text{lag1}, \text{vol}_{21}, \text{vol}_{63})$, target next-day return y_t.
- For each $p \in [0.90, 0.99]$, refit a linear quantile regression every 21 trading days on the expanding return history using these features.
- Every 21 trading days refit $\hat{y}_{p,t} = \beta_p^\top x_t$ using `QuantileRegressor`; a fixed tiny ℓ_1 penalty ($\alpha = 10^{-6}$) is just for stability (no tuning, minimal shrinkage).
- Predictions go directly to $\mathrm{VaR}_{p,t}^{\mathrm{QR}}$, sensitivities come from β_p so governance is clear.

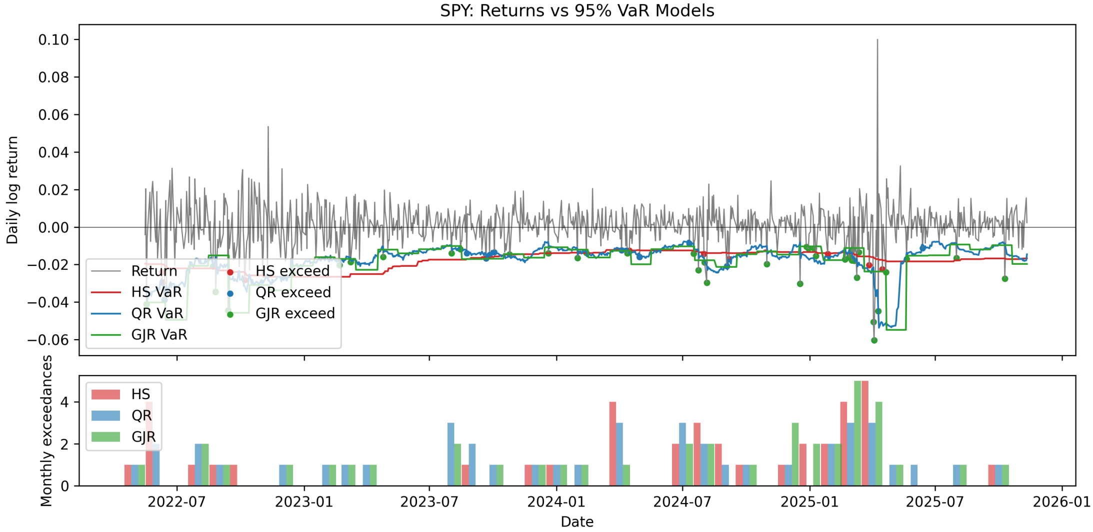
GJR-GARCH Model

- Fit GJR(1,1,1) with Student-t innovations: standardized $z_t = \epsilon_t/\sigma_t$ follow Student-t with fitted dof ν; variance recursion $\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \gamma 1_{\{\epsilon_{t-1}<0\}}\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2$.
- Residual $\epsilon_t = r_t - \mu_t$ uses simple AR(1) mean; refresh whole model monthly so parameters chase new regime.
- Conditional VaR is $\mathrm{VaR}_{p,t}^{\mathrm{GJR}} = \mu_t + \sigma_t F_{t,\nu}^{-1}(p)$, $F_{t,\nu}^{-1}$ is Student-t quantile with fitted dof ν.
- Reacts faster than HS when volatility spikes, keeps tail thickness explicit, but can over-allocate capital when market calm.

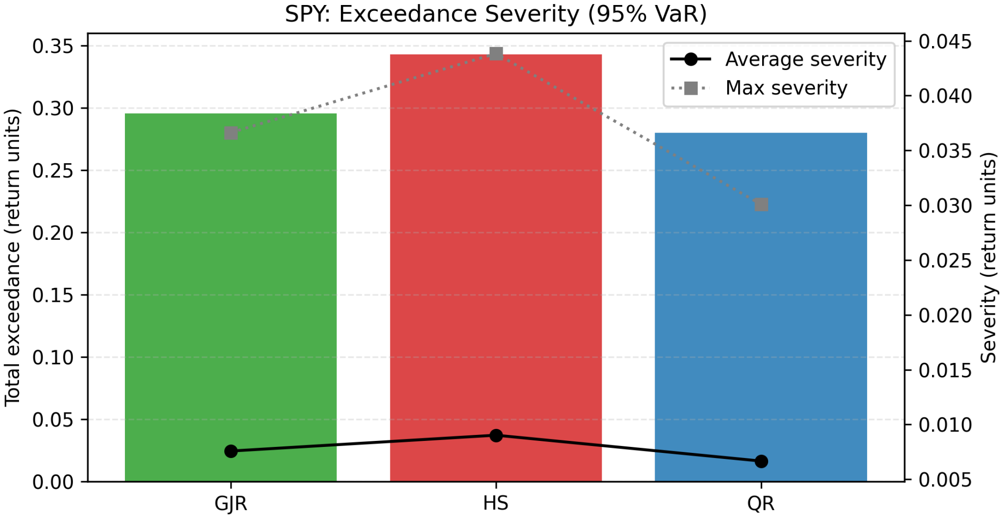
Capital Shortfall Diagnostic

- For each day a return breaches VaR, we record shortfall depth (VaR minus return). Summing depths gives total capital needed as reserve.
- We also track average and max shortfall per model and per p; that shows frequency vs severity on one chart.
- This data is used to construct "Capital Shortfall" plots: each bar is total shortfall; we show average and max loss so treasury sees cash needed when VaR fails.

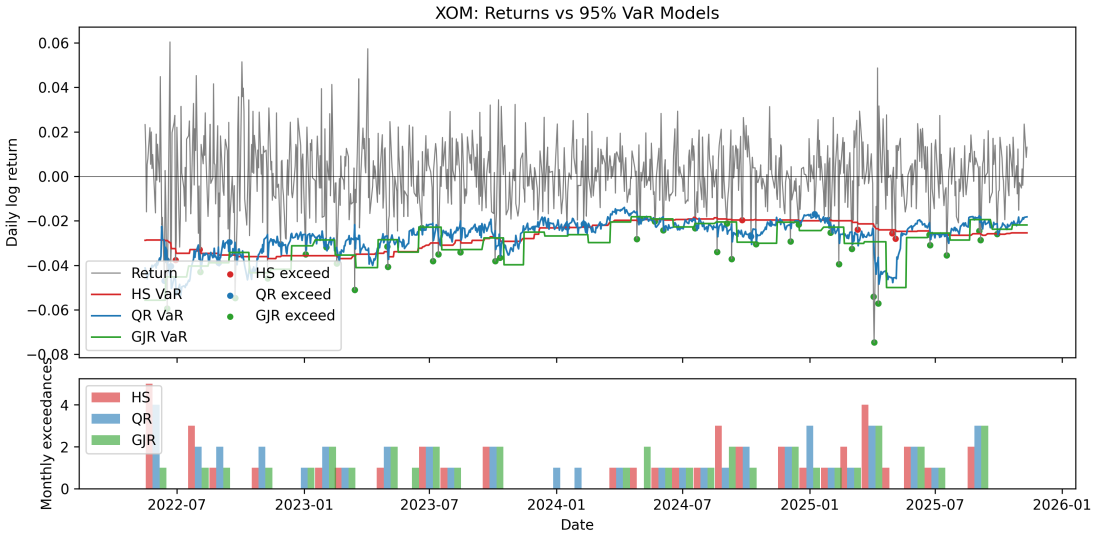
SPY: Returns vs VaR



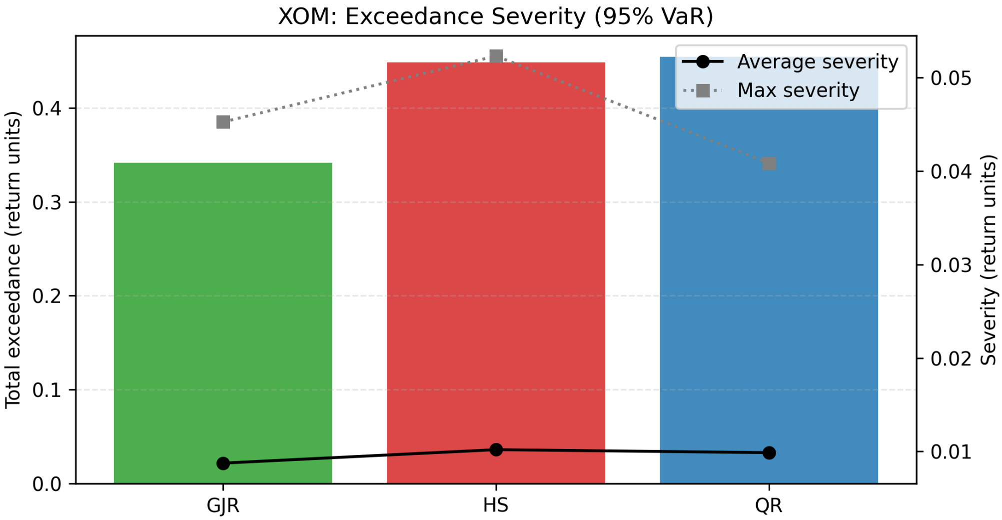
SPY: Exceedance Severity



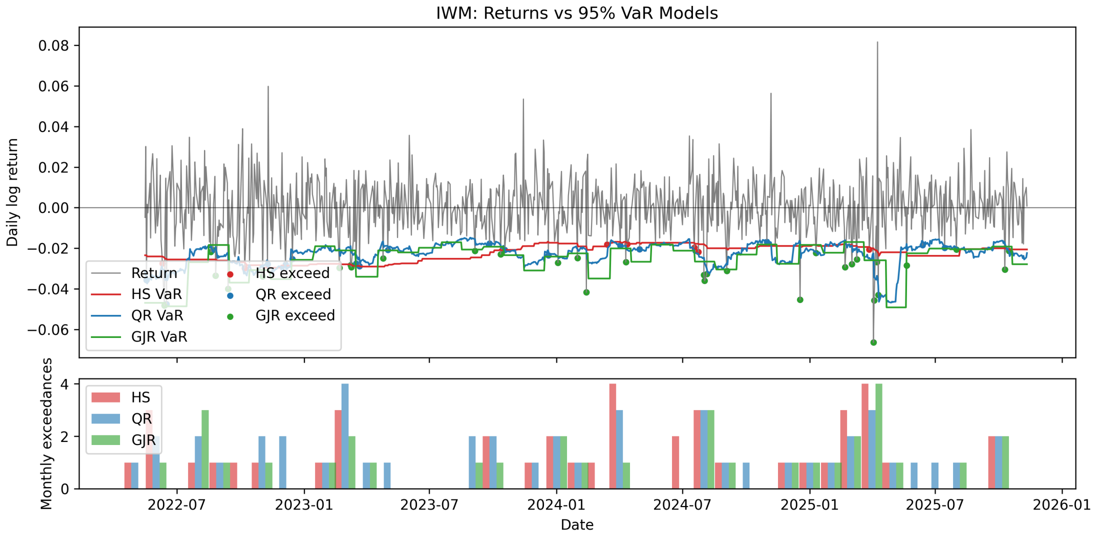
XOM: Returns vs VaR



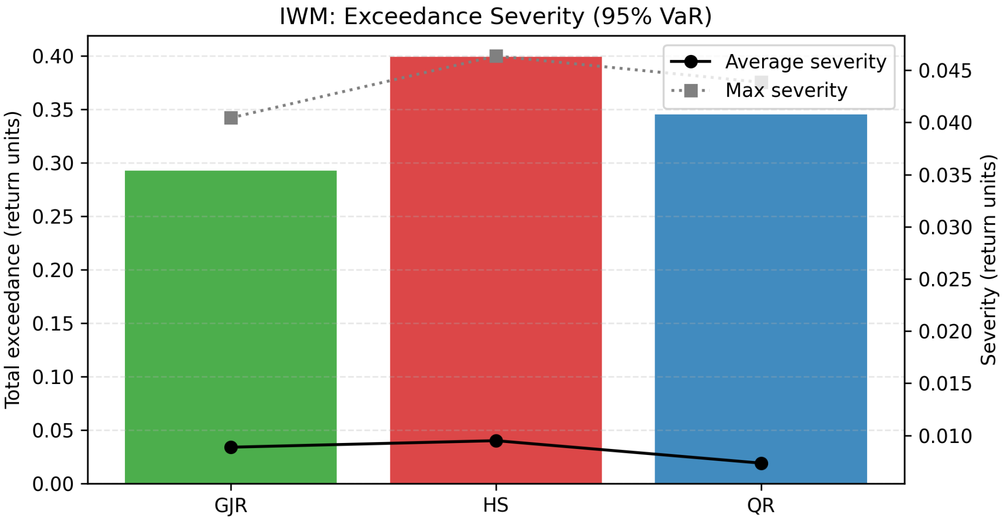
XOM: Exceedance Severity



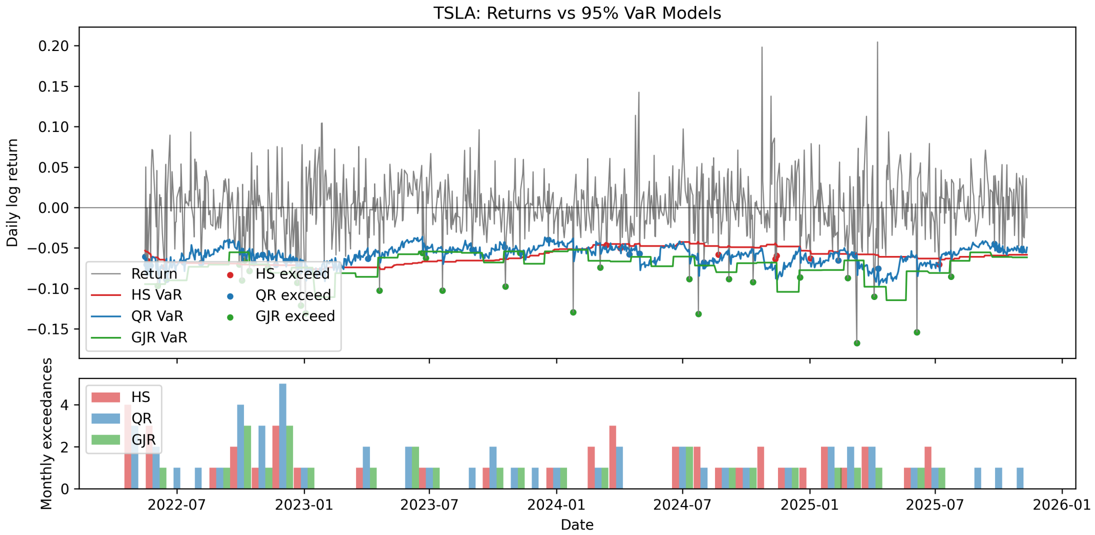
IWM: Returns vs VaR



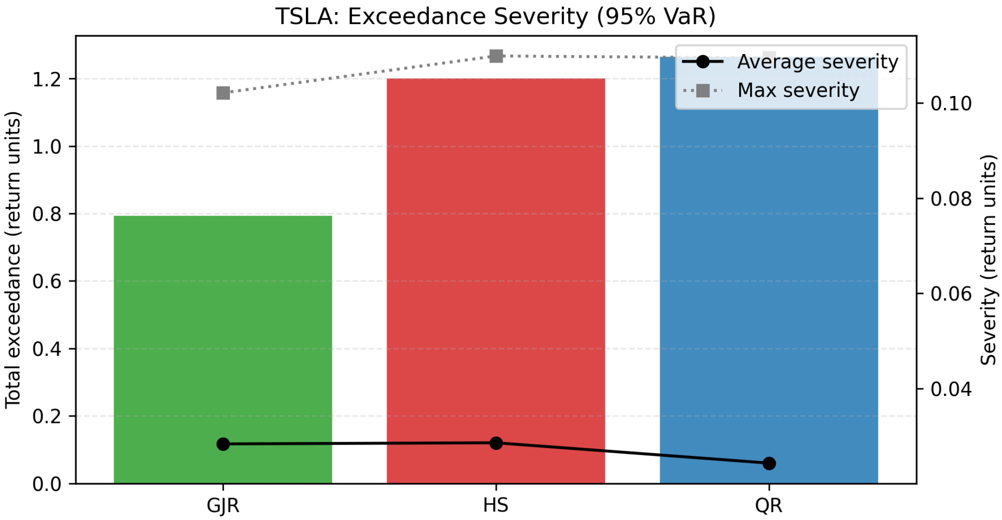
IWM: Exceedance Severity



TSLA: Returns vs VaR



TSLA: Exceedance Severity



Method Comparison Across Stocks

Method	Viol. rate (%)	Avg shortfall (bp)	Max shortfall (bp)	Severity share (%)
GJR	3.96	133.9	1,021.6	26.7
HS	4.73	143.3	1,099.1	37.0
QR	5.33	120.5	1,095.1	36.3

Severity share tells us what percent of total capital shortfall each model gives across SPY/XOM/IWM/TSLA.

Backtest Findings

- **Quantile Regression (QR)** sits nearest 5% target (5.33% breaches) and gives smallest average shortfall (~120 bp) with 36% of severity.
- **GJR-GARCH** is most conservative (3.96% breaches) yet still owns 27% severity while parking extra capital in calm time.
- **Historical Simulation** adapts slow, so it gives 37% severity when volatility jumps fast.
- We lean on QR: keeps compliance close and avoids over-funding the book.

Portfolio Setup

- Baseline mix is simple: 50% SPY (large-cap beta), 30% QQQ (growth tilt), 20% IWM (small-cap flavor) to mimic a diversified equity portfolio.
- Portfolio return is weighted sum of daily log returns; this series feeds direct HS/QR/GJR estimators.
- Portfolio weights used:
  - Value: 34% BRK-B / 33% JPM / 33% WMT.
  - Growth / high ROE: 34% AAPL / 33% MSFT / 33% GOOGL.
  - Stable earnings: 34% JNJ / 33% PG / 33% KO.
  - High beta: 40% TSLA / 35% QQQ / 25% IWM.
  - Small size: 34% INMD / 33% CELH / 33% PLUG.

Portfolio Styles

- **Balanced (SPY/QQQ/IWM):** diversified mix of large-cap, tech growth, and small-cap equities; mirrors broad market risk without one dominant factor.
- **Value (BRK-B/JPM/WMT):** priced cheaply versus fundamentals (low P/E, strong cash flow); steady earners the market may undervalue.
- **Growth / high ROE (AAPL/MSFT/GOOGL):** firms with strong earnings momentum and high profitability (high ROE), examples are mega-cap tech.
- **Stable earnings (JNJ/PG/KO):** defensive consumer and healthcare firms with predictable revenues, low earnings volatility, and consistent cash flows/dividends.
- **High beta (TSLA/QQQ/IWM):** stocks with greater sensitivity to market moves, offering amplified volatility for momentum exposure and VaR stress.
- **Small size (INMD/CELH/PLUG):** small-cap firms with higher volatility and return dispersion, capturing the size premium tied to lower market cap.

Portfolio VaR Backtesting

- We use simple Kupiec coverage tests to check whether portfolio VaR and the sum of individual-stock VaRs hit their target breach rates at $p = 0.95$ and 0.99.
- Every breach gives capital shortfall data (total, average, and max loss) so we can link how often VaR fails with how much money is missing.
- We compare breach dates for the portfolio and for single names to see if diversification really helps or if everything breaks on the same days.
- These checks show whether QR's faster adaptation is enough at portfolio level or if we should lean more on Expected Shortfall or add explicit buffers.

Portfolio VaR (p=0.95)

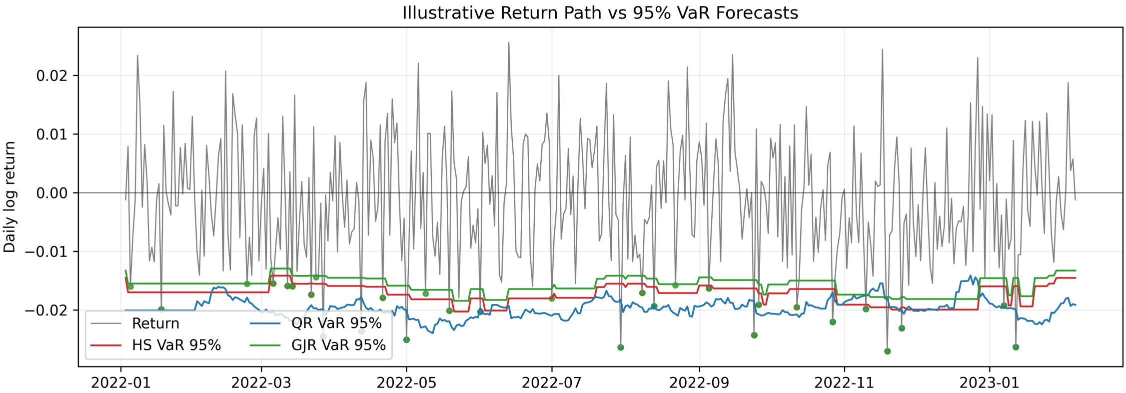


Figure: Direct HS / QR / GJR forecasts versus portfolio returns at 95%; we use this view to see who guards daily capital.

Portfolio VaR (p=0.99)

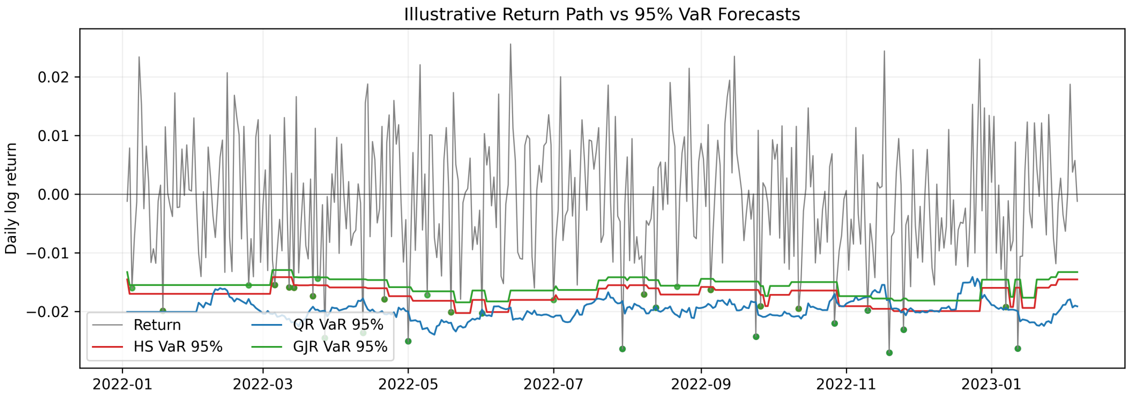


Figure: Same test at 99%; QR adapts quickest while HS trails big jumps, so capital ask widens fast.

Portfolio VaR Breach/Shortfall ($p = 0.95$)

Portfolio	QR direct (rate / shortfall)	HS direct (rate / shortfall)	GJR direct (rate / shortfall)
Balanced	0.057 / 0.187	0.047 / 0.552	0.044 / 0.142
Value	0.039 / 0.170	0.047 / 0.474	0.037 / 0.132
Growth	0.031 / 0.147	0.049 / 0.589	0.029 / 0.116
Stable	0.048 / 0.090	0.051 / 0.376	0.031 / 0.048
High beta	0.042 / 0.264	0.050 / 0.842	0.029 / 0.175
Small size	0.042 / 0.229	0.051 / 1.116	0.018 / 0.088

Breach rate = observed exceedances / observations; total shortfall sums (VaR – return) on breach days.

Portfolio VaR Breach/Shortfall ($p = 0.99$)

Portfolio	QR direct (rate / shortfall)	HS direct (rate / shortfall)	GJR direct (rate / shortfall)
Balanced	0.015 / 0.052	0.019 / 0.173	0.011 / 0.045
Value	0.015 / 0.047	0.014 / 0.140	0.009 / 0.037
Growth	0.013 / 0.031	0.014 / 0.162	0.009 / 0.019
Stable	0.009 / 0.021	0.009 / 0.094	0.002 / 0.007
High beta	0.009 / 0.062	0.011 / 0.195	0.007 / 0.029
Small size	0.007 / 0.039	0.009 / 0.282	0.002 / 0.010

Breach rate = observed exceedances / observations; total shortfall sums (VaR – return) on breach days.

Subadditivity Failure vs Confidence (Value Stocks)

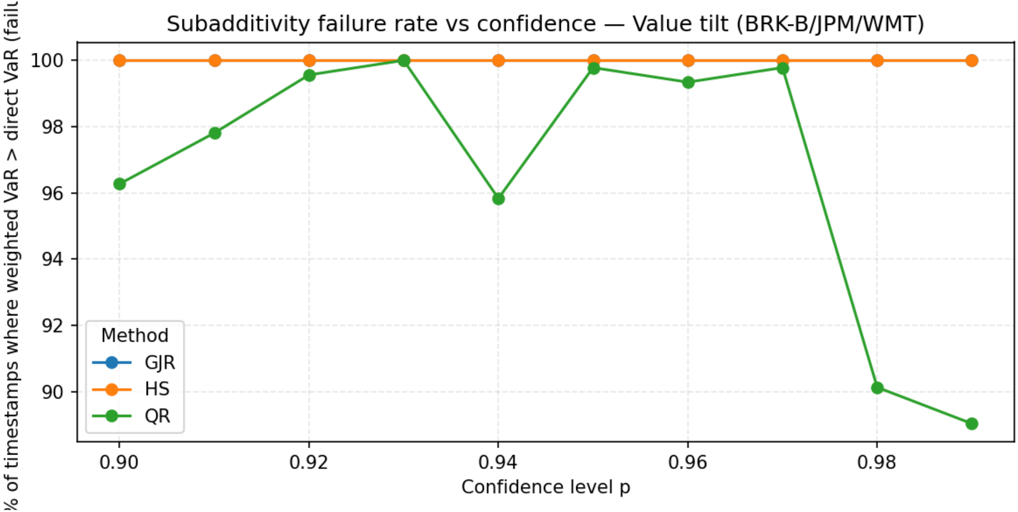


Figure: Value tilt (BRK-B / JPM / WMT): share of days where direct VaR exceeds weighted sum; here

Subadditivity Failure vs Confidence (Small Size)

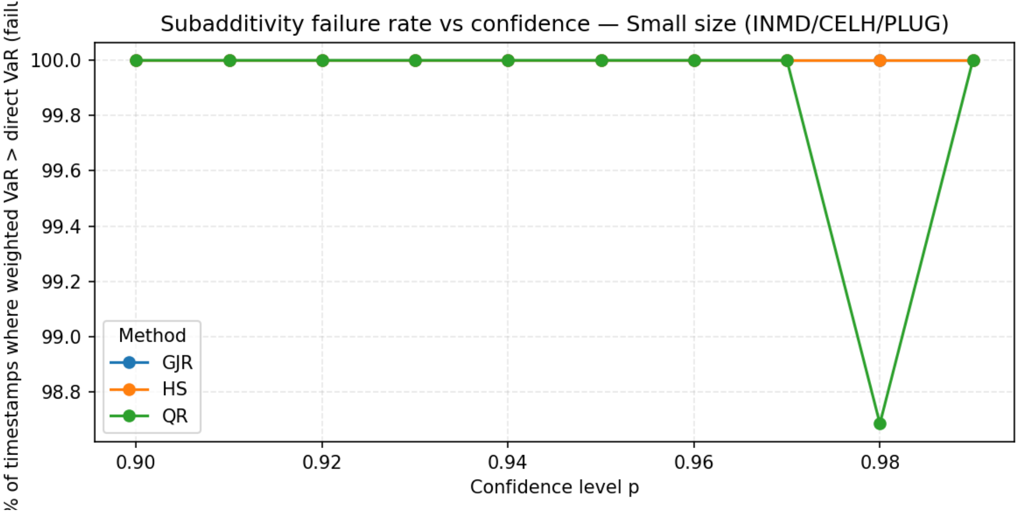


Figure: Small size tilt (INMD / CELH / PLUG): failure rate sits near 100% across p, showing VaR

Subadditivity Failure vs Confidence (Growth / High ROE)

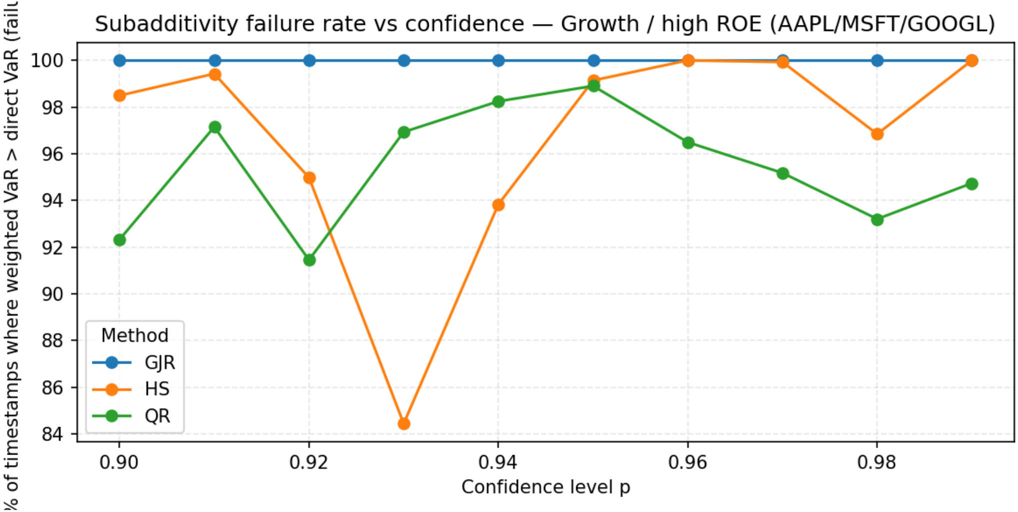


Figure: Growth / high ROE (AAPL / MSFT / GOOGL): failure rate by confidence level for the

Subadditivity Failure vs Confidence (Stable Earnings)

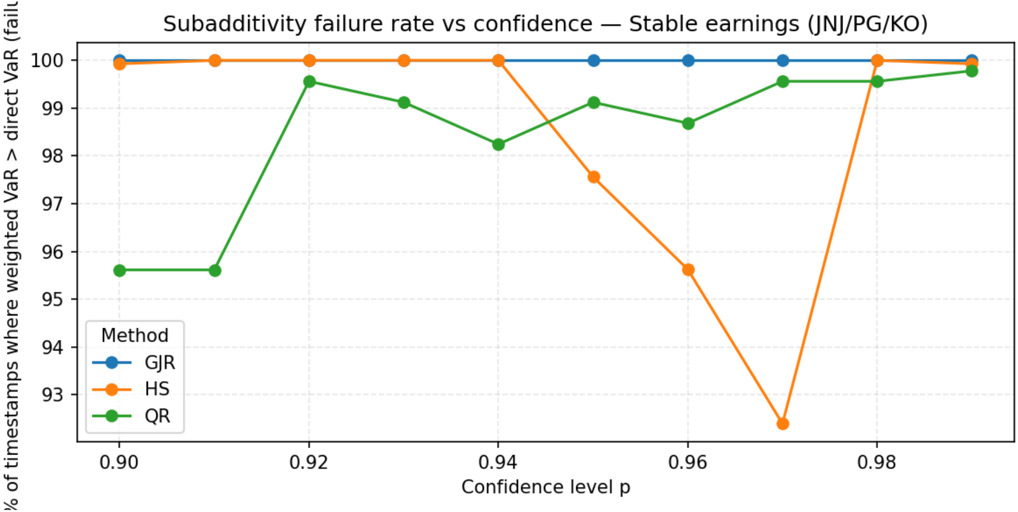


Figure: Stable earnings (JNJ / PG / KO): defensive cash-flow names show high failure share across p.

Subadditivity Failure vs Confidence (High Beta)

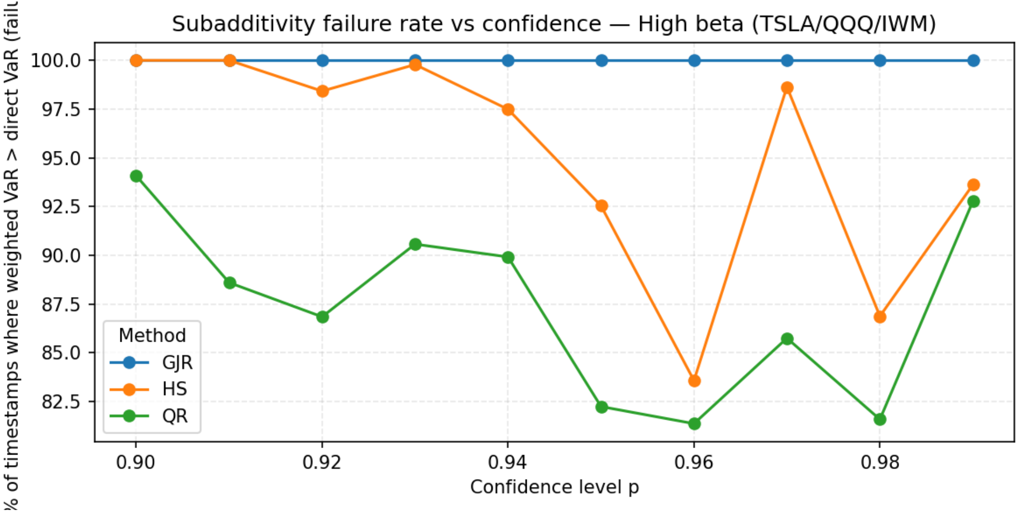


Figure: High beta (TSLA / QQQ / IWM): jumpy mix keeps failure rate elevated across confidence

Failure Trend Takeaways

- Failure stays high across confidence levels; sometimes drifts down near $p = 0.95$ but often bounces back up by $p = 0.99$.
- No clean pattern across styles: value, small-cap, and high-beta all do not show clear patterns; failures vary by mix.
- Bottom line: VaR often fails to diversify; treat results cautiously and keep an ES or buffer plan handy.

Conclusion

- Subadditivity failures show up in every portfolio style; no consistent "safe" mix emerges from value, growth, small-cap, or high-beta baskets.
- Failure rates do not trend cleanly as p rises: they dip near 0.95 for some mixes, then bounce back toward 0.99, so higher confidence does not guarantee coherence.
- Quantile Regression (QR) is the best-behaved VaR: closest to target breach rates with the shallowest shortfalls; HS reacts slowest, GJR is most conservative.
- Overall, we cannot find a clear, stable pattern linking portfolio style or confidence level to how badly VaR fails subadditivity, so VaR should be treated as one diagnostic rather than a fully reliable risk measure.

Thank You

Thank you for listening.

Stocks We Use

Ticker	Quick note	Ticker	Quick note
SPY	Fund owning 500 biggest US firms.	XOM	Oil and gas major.
QQQ	Tech/growth fund (Apple, MSFT heavy).	TSLA	EV and battery maker; volatile.
IWM	Fund holding many small caps.	PLUG	Hydrogen fuel cell / clean energy.
JPM	Large US bank.	BRK-B	Buffett holding company; many busines
AAPL	iPhone/Mac/devices + services.	WMT	Big-box discount retailer.
GOOGL	Search, YouTube, ads, Android.	JNJ	Medicines, devices, health products.
MSFT	Windows, Office, Azure cloud.	PG	Everyday consumer products.
KO	Global soft drinks.	INMD	Aesthetic medical devices.
CELH	Fast-growing energy drinks.		

SPY/QQQ/IWM are ETFs; we treat them as stock exposures in single-asset and portfolio tests.