Fokker-Planck Equation and Heat Equation with Crank-Nicolson Method

$$\frac{\partial p(v,t)}{\partial t} = -\mu \frac{\partial p(v,t)}{\partial v} + D \frac{\partial^2 p(v,t)}{\partial v^2}$$

$$\frac{\partial u(x,t)}{\partial t} = D \frac{\partial u^2(x,t)}{\partial x^2}$$

(& Alternating Direction Implicit (ADI) Method)

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Computational Physics
Thursday, December 15, 2016

Fokker-Planck Equation

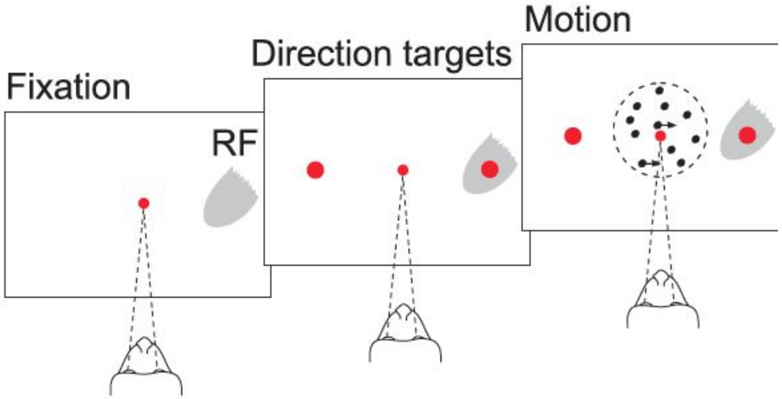
$$\frac{\partial p(v,t)}{\partial t} = \left[-\frac{\partial \mu(v,t)}{\partial v} + \frac{\partial^2}{\partial v^2} D(v,t) \right] p(v,t)$$

p(v,t): probability density function

D: diffusion coeff. μ: advection coeff.

- A PDE for the distribution function describing Brownian motion.
- The time evolution of the probability density function of the velocity of a particle under the influence of drag forces and random forces.

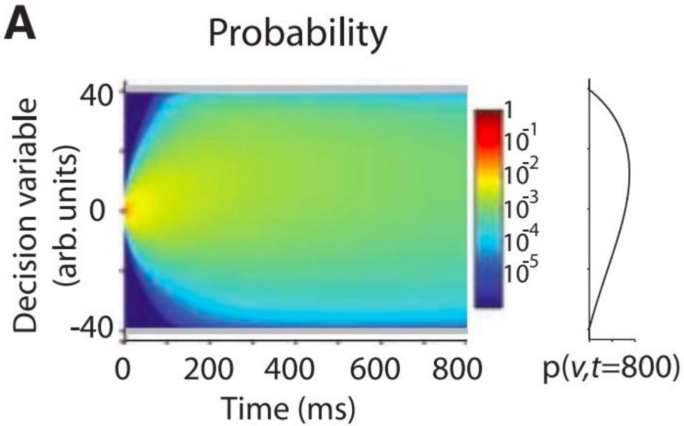
Fokker-Planck Equation



RF: Response Field

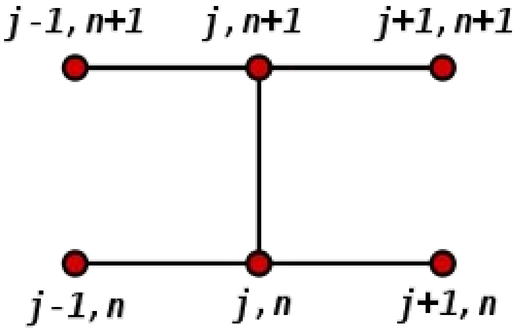
Bounded Accumulation Model

$p(v,t)$: the propagation of the probability density of decision variable.
(v: decision variable, μ: strength of momentary evidence)



R. Kiani and M. N. Shadlen, *Science* 324, 759 (2009)

Crank-Nicolson Method (1D)



$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = F_i^n\left(u,\, x,\, t,\, \frac{\partial u}{\partial x},\, \frac{\partial^2 u}{\partial x^2}\right) \qquad \text{(forward Euler)}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = F_i^{n+1}\left(u,\, x,\, t,\, \frac{\partial u}{\partial x},\, \frac{\partial^2 u}{\partial x^2}\right) \qquad \text{(backward Euler)}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2}\left[F_i^{n+1}\left(u,\, x,\, t,\, \frac{\partial u}{\partial x},\, \frac{\partial^2 u}{\partial x^2}\right) + F_i^n\left(u,\, x,\, t,\, \frac{\partial u}{\partial x},\, \frac{\partial^2 u}{\partial x^2}\right)\right] \qquad \text{(Crank--Nicolson).}$$

For 1D diffusion, $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{a}{2(\Delta x)^2}\left((u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) + (u_{i+1}^n - 2u_i^n + u_{i-1}^n)\right) \qquad r = \frac{a\Delta t}{2(\Delta x)^2}$$

$$-ru_{i+1}^{n+1} + (1+2r)u_i^{n+1} - ru_{i-1}^{n+1} = ru_{i+1}^n + (1-2r)u_i^n + ru_{i-1}^n$$

Crank-Nicolson Method (1D)

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}$$

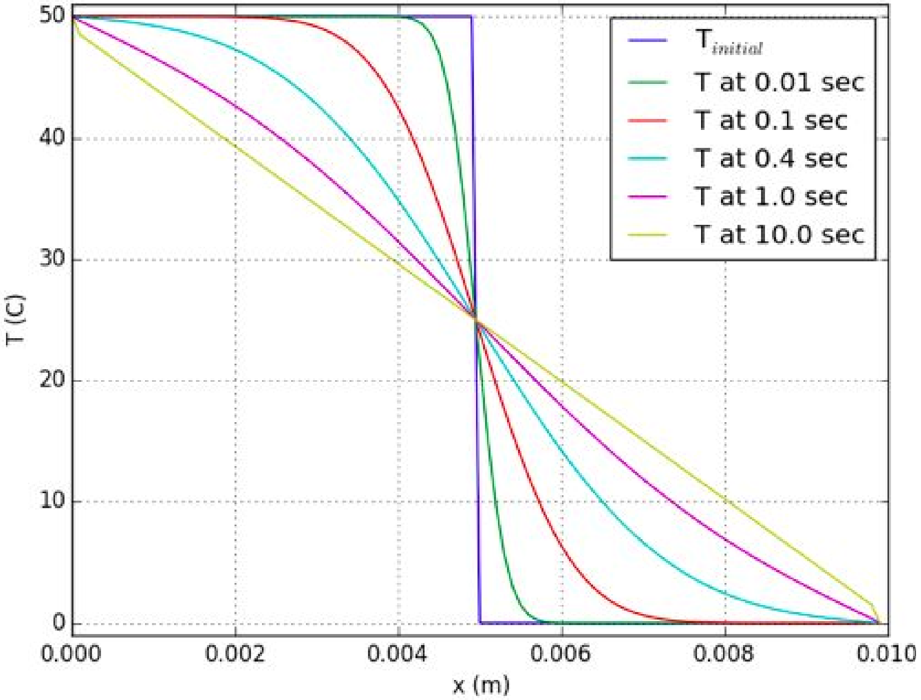
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{a}{2(\Delta x)^2}\left((u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) + (u_{i+1}^n - 2u_i^n + u_{i-1}^n)\right)$$

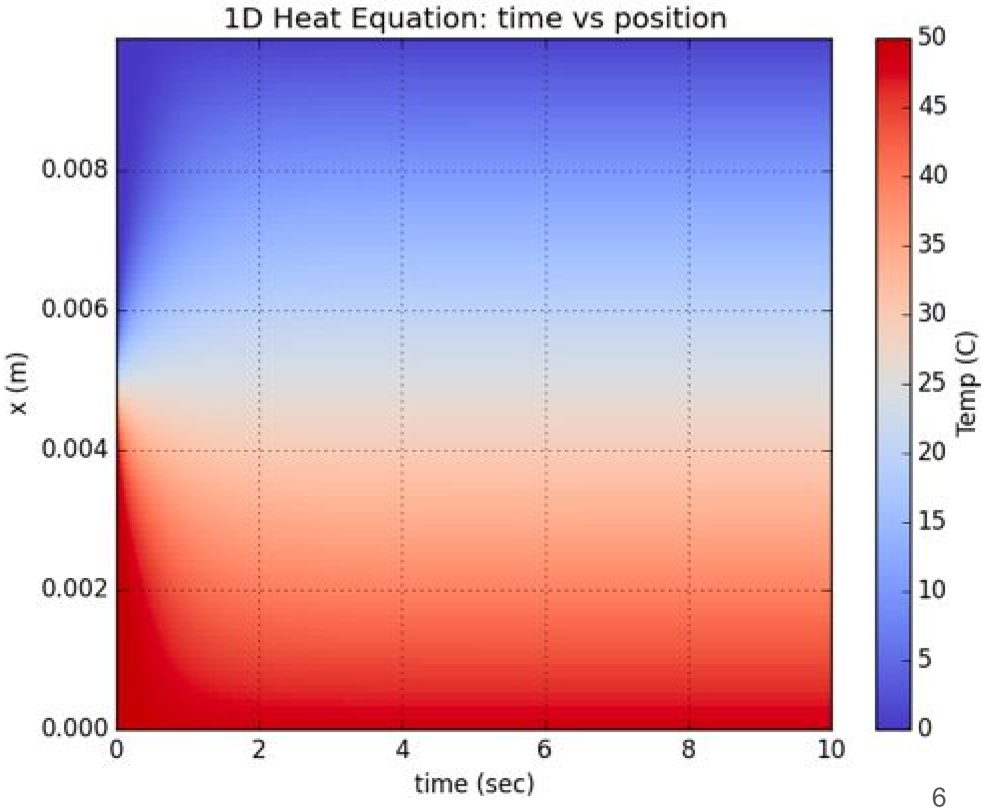
$$r = \frac{a\Delta t}{2(\Delta x)^2}$$

$$-ru_{i+1}^{n+1} + (1+2r)u_i^{n+1} - ru_{i-1}^{n+1} = ru_{i+1}^n + (1-2r)u_i^n + ru_{i-1}^n$$

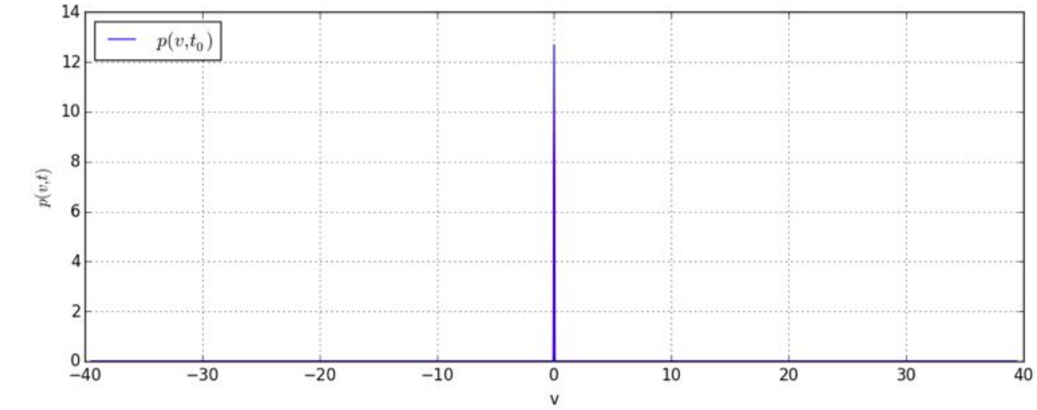
$$\boxed{\begin{gathered}\mathbf{A}\mathbf{U}^{n+1} = \mathbf{B}\mathbf{U}^n \longrightarrow \mathbf{U}^{n+1} = \mathbf{A}^{-1}\mathbf{B}\mathbf{U}^n \\[2em] \mathbf{U}^{n+1} = \begin{bmatrix} U_0^{n+1} \\ U_1^{n+1} \\ \vdots \\ U_{J-1}^{n+1} \end{bmatrix}, \mathbf{U}^n = \begin{bmatrix} U_0^n \\ U_1^n \\ \vdots \\ U_{J-1}^n \end{bmatrix}\end{gathered}}$$

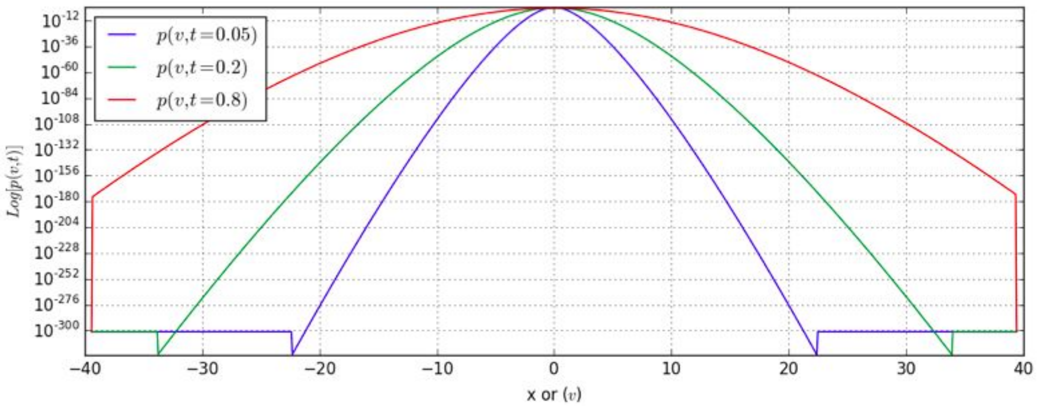
[Results] 1D Heat Equation

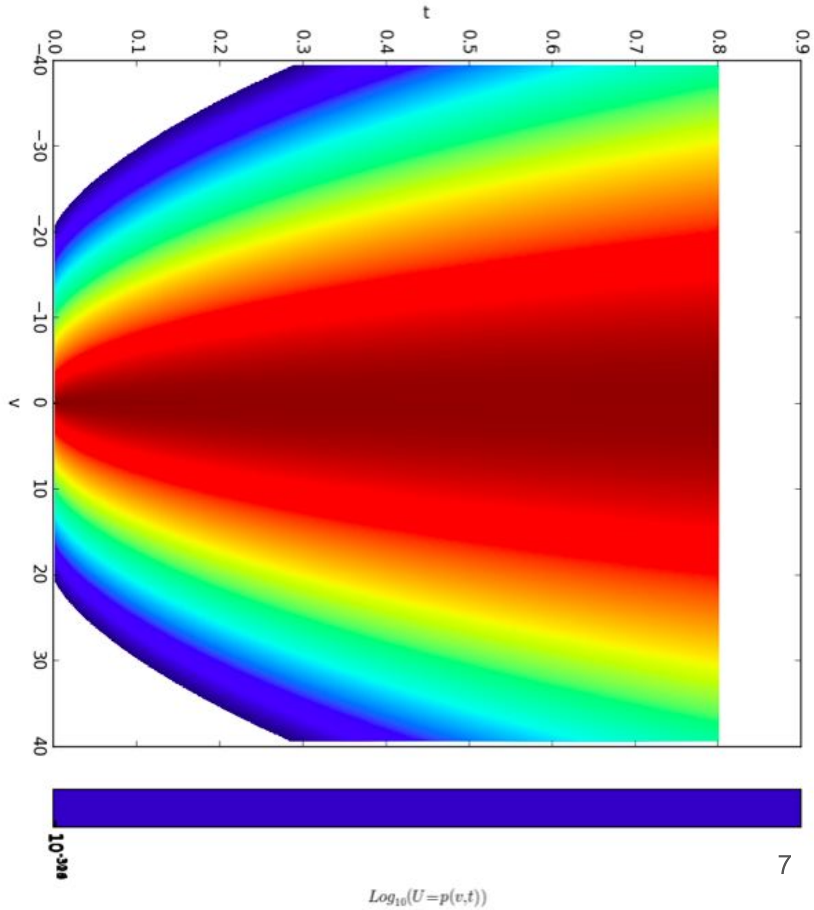




[Results] 1D Fokker-Planck Equation







Crank-Nicolson Method (2D)

For 2D diffusion,

$$\frac{\partial u(x,y,t)}{\partial t} = D\left[\frac{\partial u^2(x,y,t)}{\partial x^2} + \frac{\partial u^2(x,y,t)}{\partial y^2}\right]$$

Generalization of CN method:

$$\frac{U_{j,k}^{n+1} - U_{j,k}^{n}}{\Delta t} = \frac{D}{2\Delta x^2}\left(U_{j+1,k}^{n} - 2U_{j,k}^{n} + U_{j-1,k}^{n+1} + U_{j,k}^{n+1} - 2U_{j,k}^{n+1} + U_{j-1,k}^{n+1}\right)$$
$$+ \frac{D}{2\Delta y^2}\left(U_{j,k+1}^{n} - 2U_{j}^{n} + U_{j,k-1}^{n+1} + U_{j,k}^{n+1} - 2U_{j,k}^{n+1} + U_{j,k-1}^{n+1}\right)$$

$$\mathbf{A}\mathbf{U}^{n+1} = \mathbf{B}\mathbf{U}^{n} \longrightarrow \mathbf{U}^{n+1} = \mathbf{A}^{-1}\mathbf{B}\mathbf{U}^{n}$$

$$\mathbf{U}_{;,k}^{n+1} = \begin{bmatrix} U_{0,k}^{n+1} \\ U_{1,;}^{n+1} \\ \vdots \\ U_{J-1,k}^{n+1} \end{bmatrix}, \mathbf{U}_{;,k}^{n} = \begin{bmatrix} U_{0,k}^{n} \\ U_{1,k}^{n} \\ \vdots \\ U_{J-1,k}^{n} \end{bmatrix}$$

$\Delta x = \Delta y \qquad r = \frac{D\Delta t}{2\Delta x^2}$

Crank-Nicolson Method (2D)

For 2D diffusion,

$$\frac{\partial u(x,y,t)}{\partial t} = D\left[\frac{\partial u^2(x,y,t)}{\partial x^2} + \frac{\partial u^2(x,y,t)}{\partial y^2}\right]$$

Generalization of CN method:

$$\frac{U^{n+1}_{j,k} - U^n_{j,k}}{\Delta t} = \frac{D}{2\Delta x^2}\left(U^n_{j+1,k} - 2U^n_{j,k} + U^{n+1}_{j-1,k} + U^{n+1}_{j,k} - 2U^{n+1}_{j,k} + U^{n+1}_{j-1,k}\right)$$
$$+ \frac{D}{2\Delta y^2}\left(U^n_{j,k+1} - 2U^n_j + U^{n+1}_{j,k-1} + U^{n+1}_{j,k} - 2U^{n+1}_{j,k} + U^{n+1}_{j,k-1}\right)$$

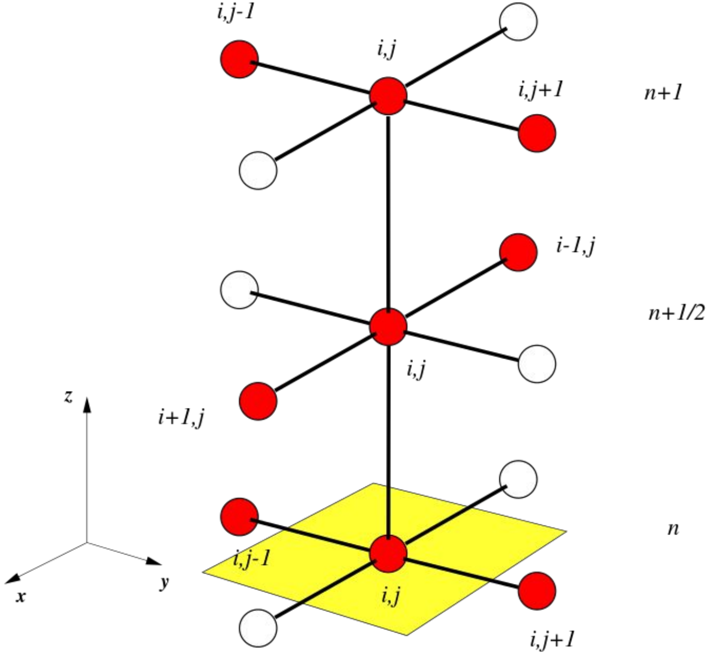
$$\mathbf{A}\mathbf{U}^{n+1} = \mathbf{B}\mathbf{U}^n - \quad r = \frac{D\Delta t}{2\Delta x^2}$$

Computationally very inefficient

$$\mathbf{U}_{;,k}$$

$$\left[U^{n+1}_{J-1,k}\right] \qquad \left[U_{J-1,k}\right]$$

Alternating Direction Implicit Method



Two Steps:

1. n → n + ½ (in *x* direction)
2. n + ½ → n + 1 (in *y* direction)

$$\frac{U_{j,i}^{n+1/2} - U_{j,i}^{n}}{\Delta t/2} = \frac{D}{2\Delta x^2}\left(U_{j+1,i}^{n+1/2} - 2U_{j,i}^{n+1/2} + U_{j-1,i}^{n+1/2}\right) + \frac{D}{2\Delta y^2}\left(U_{j,i+1}^{n} - 2U_{j,i}^{n} + U_{j,i-1}^{n}\right)$$

$$\frac{U_{j,i}^{n+1} - U_{j,i}^{n+1/2}}{\Delta t/2} = \frac{D}{2\Delta x^2}\left(U_{j+1,i}^{n+1/2} - 2U_{j,i}^{n+1/2} + U_{j-1,i}^{n+1/2}\right) + \frac{D}{2\Delta y^2}\left(U_{j,i+1}^{n+1} - 2U_{j,i}^{n+1} + U_{j,i-1}^{n+1}\right)$$

Explicit Method

$$\frac{U_{j,k}^{n+1} - U_{j,k}^{n}}{\Delta t} = \frac{U_{j+1,k}^{n} - 2U_{j,k}^{n} + U_{j-1,k}^{n}}{\Delta x^2} + \frac{U_{j,k+1}^{n} - 2U_{j,k}^{n} + U_{j,k-1}^{n}}{\Delta y^2}$$

$$\downarrow$$

$$\Delta x = \Delta y \qquad r = \frac{\Delta t}{\Delta x^2}$$

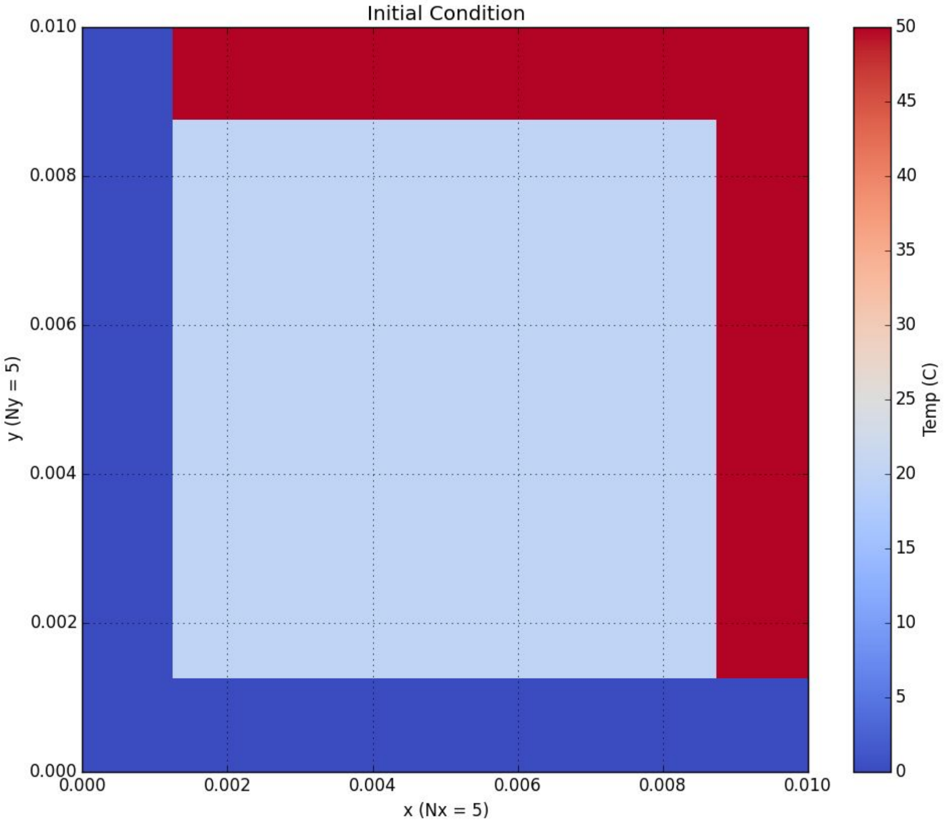
$$U_{j,k}^{n+1} = (1-4r)U_{j,k}^{n} + r\left(U_{j+1,k}^{n} + U_{j-1,k}^{n} + U_{j,k+1}^{n} + U_{j,k-1}^{n}\right)$$

Stability Condition: $r \leq \frac{1}{2}$ $\left(r = \frac{\Delta t}{\Delta x^2} \right)$

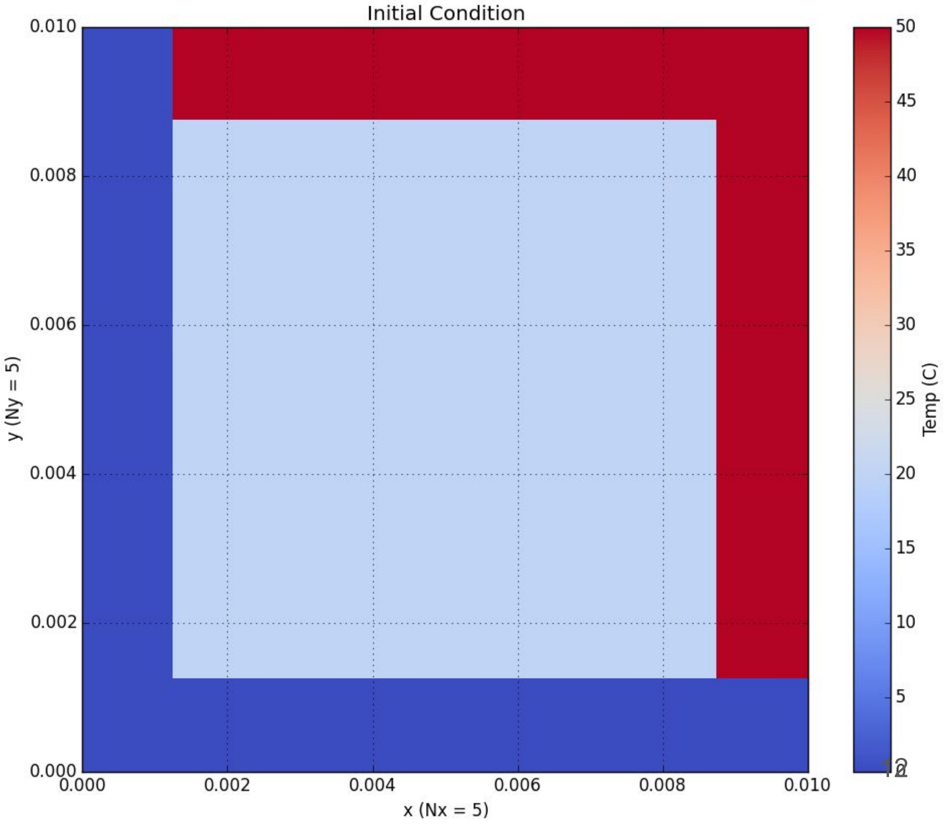
Inefficient for a large grid (Nx ~ 100).

[Results] 2D Heat Equation

(Explicit Method)

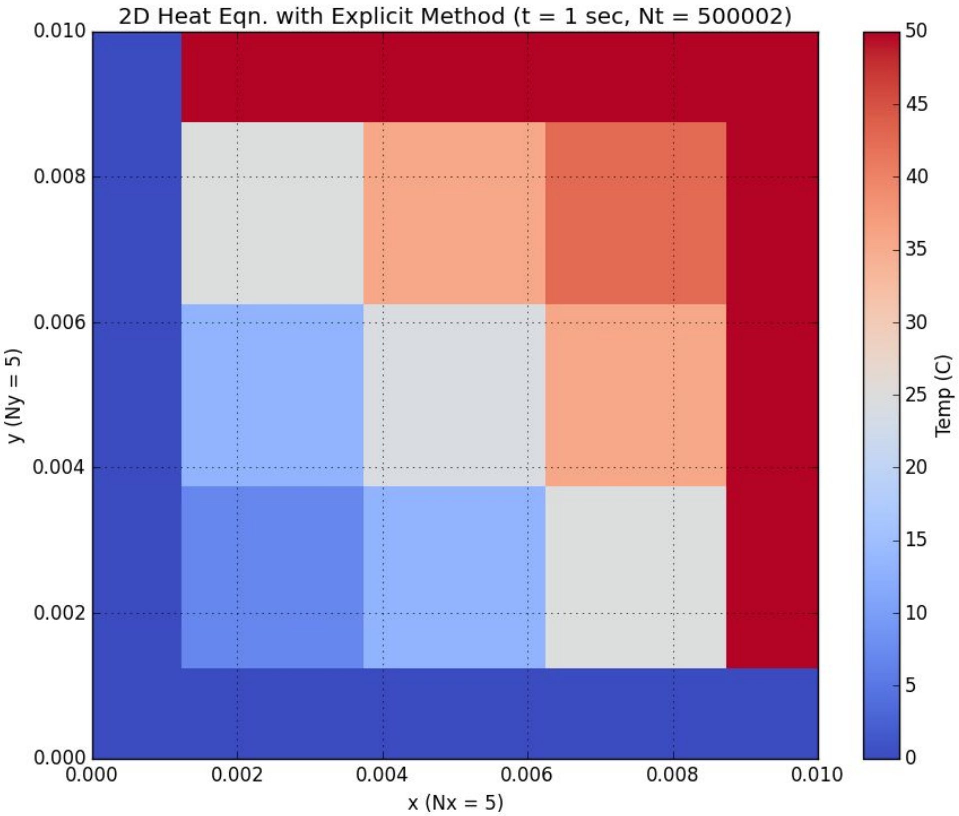


(ADI Method)

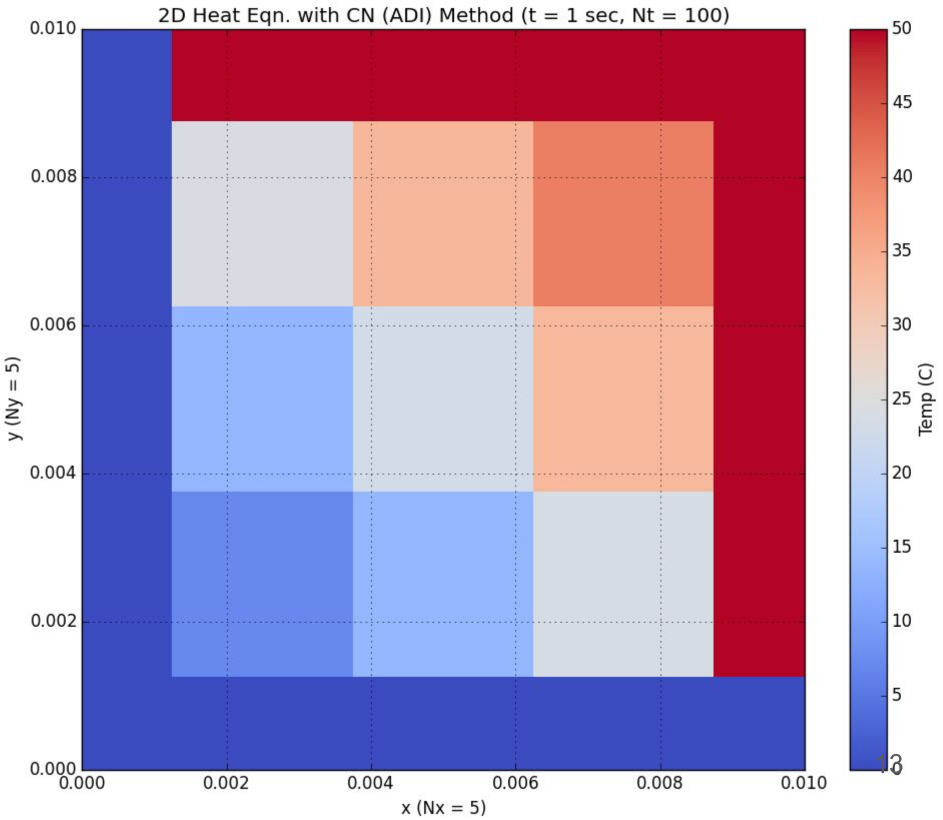


[Results] 2D Heat Equation

(Explicit Method)

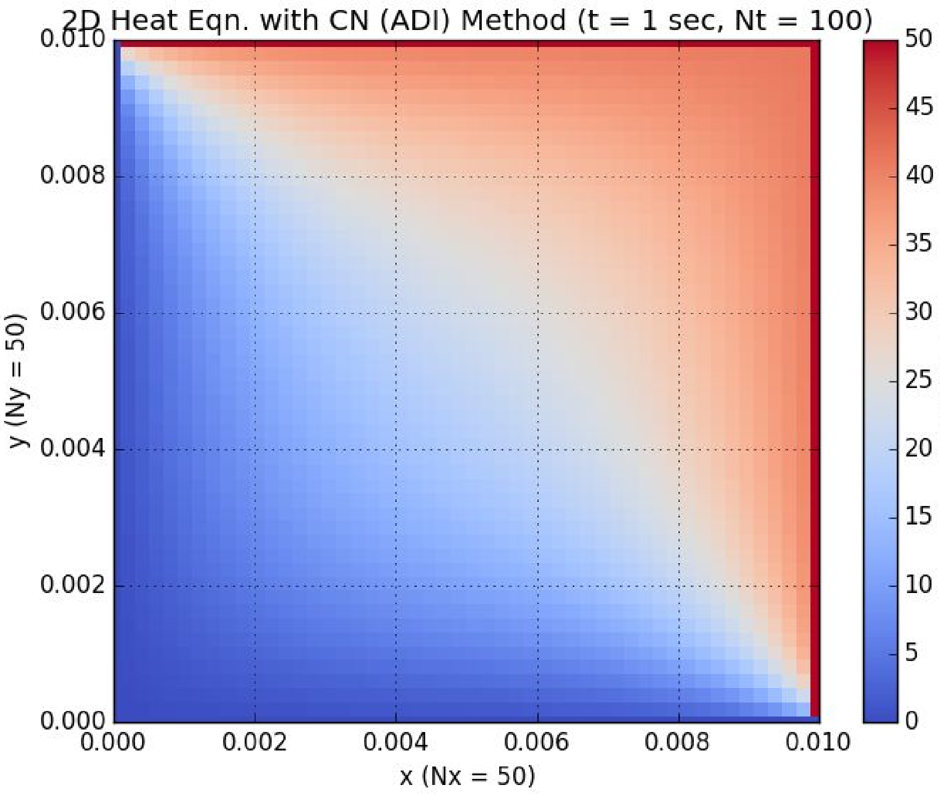


(ADI Method)

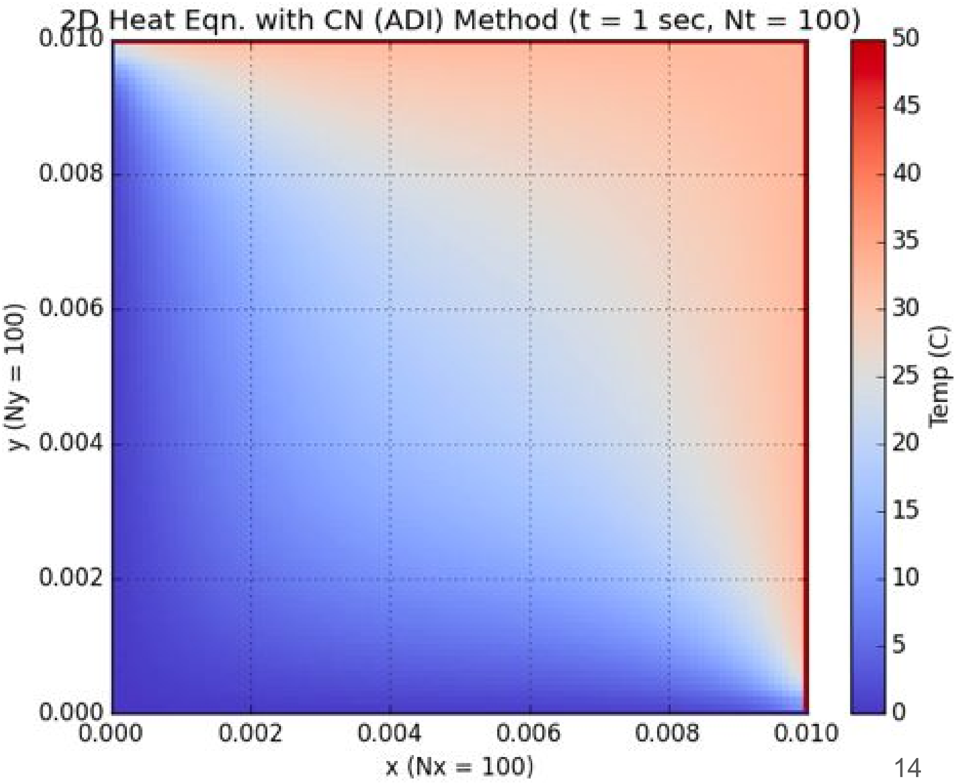


[Results] 2D Heat Equation

(ADI Method, N x N = 50 x 50)



(ADI Method, N x N = 100 x 100)



Discussion & Further Developments

- 1D Heat Equation & Fokker-Planck Equation
- 2D Heat Equation with Explicit method and Crank-Nicolson (ADI) method

- Error analysis (1D & 2D heat equation)
- 2D Fokker-Planck Equation

Acknowledgements

Professor Xiao-Jing Wang

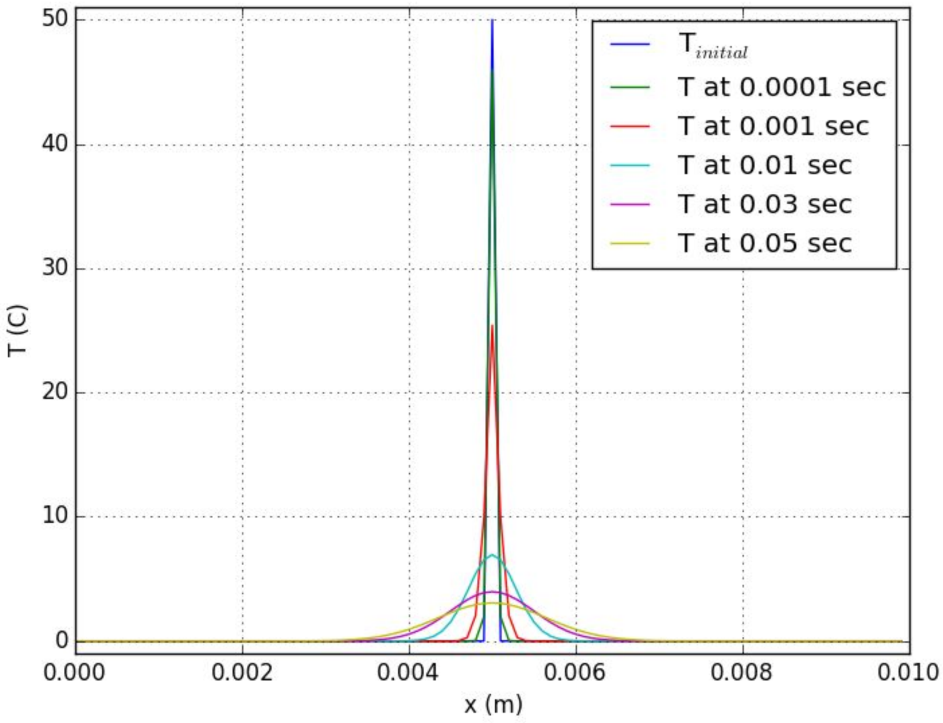
Dr. Francis Song

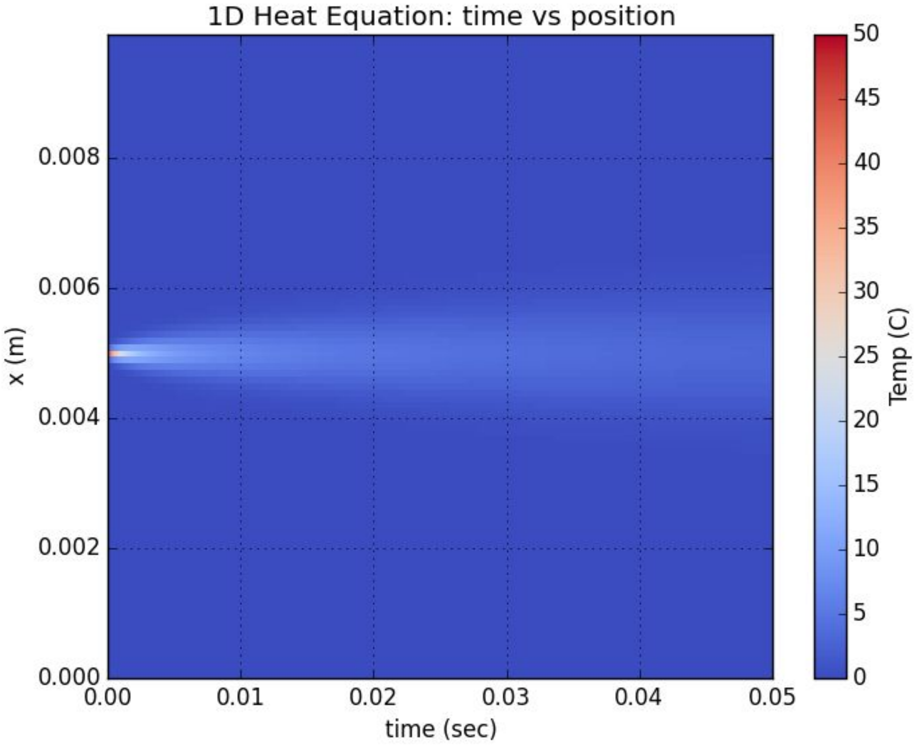
References

1. R. Kiani and M. N. Shadlen, *Science* 324, 759 (2009).
2. R. Kiani, L. Corthell, and M. N. Shadlen, *Neuron* 84, 1329 (2014).
3. T. Lakoba, *MATH 337*, The Heat equation in 2 and 3 spatial dimensions (2016).
4. Wikipedia: Crank-Nicolson method
5. Wikipedia: Alternating direction implicit method.

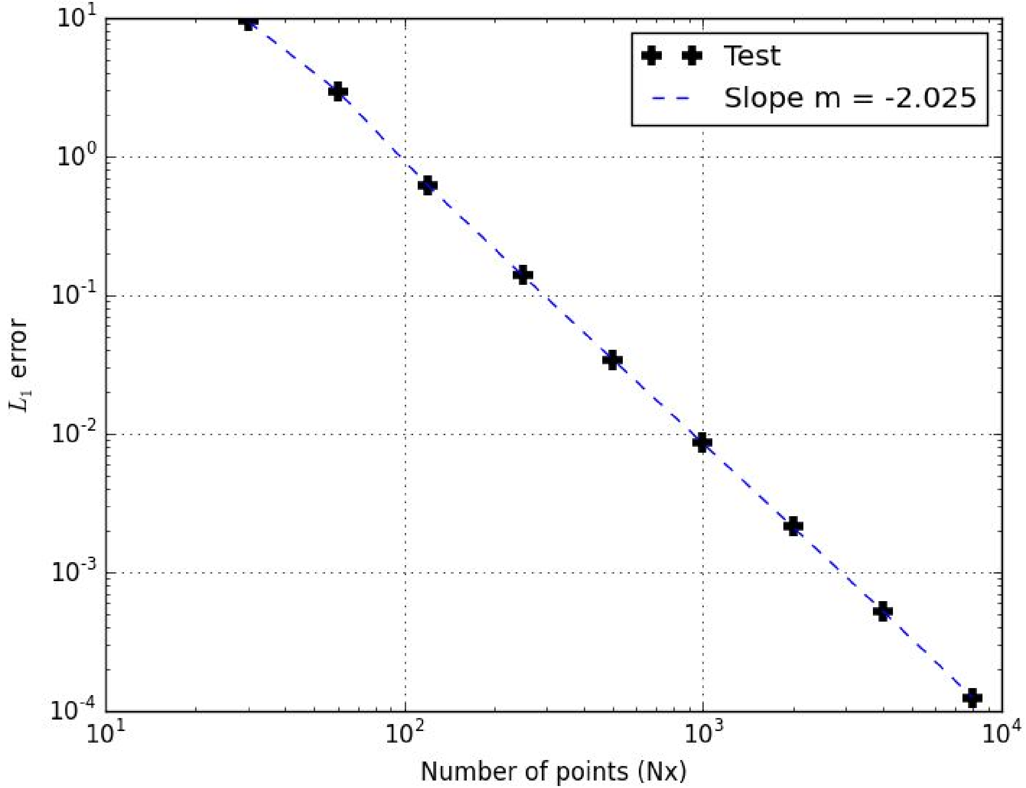
Supplementary Slides

[Results] 1D Heat Equation & Convergence

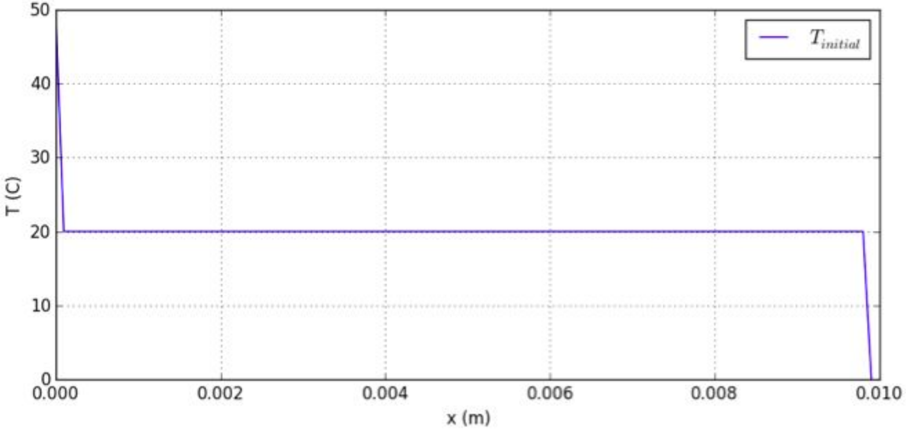


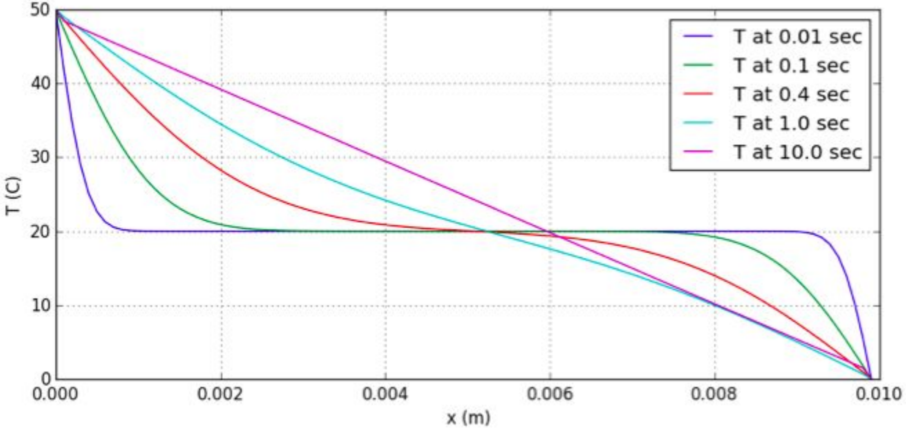


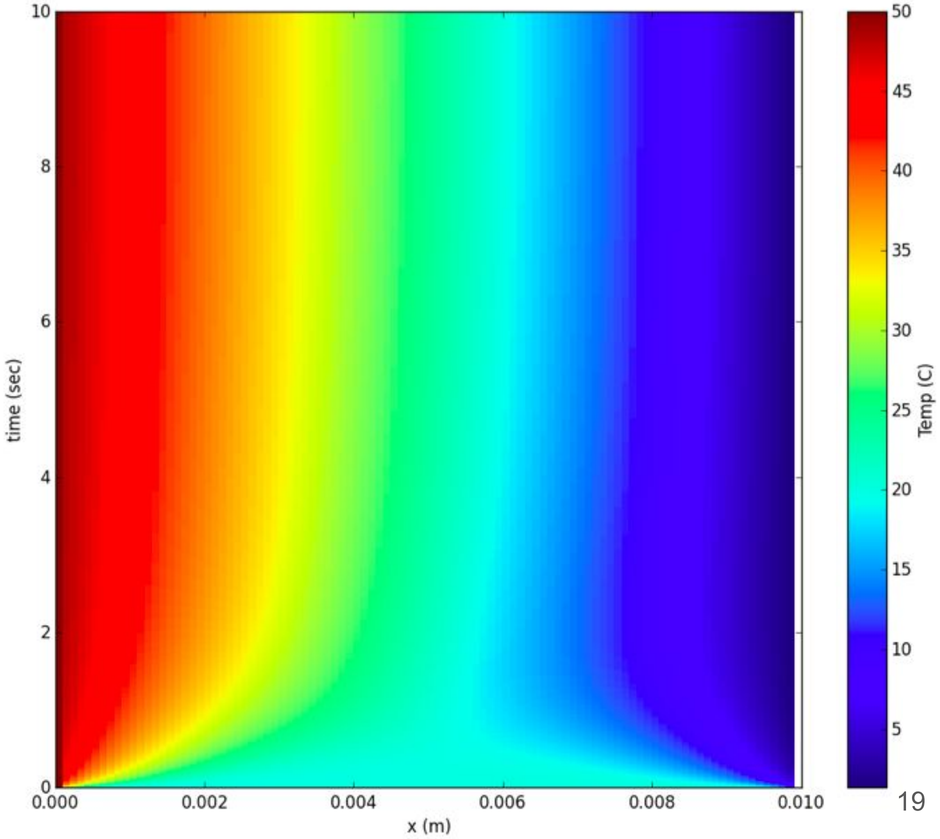
[Results] 1D Heat Equation & Convergence



[Results] 1D Heat Equation







Alternating Direction Implicit Method

$$\frac{U_{j,i}^{n+1/2} - U_{j,i}^{n}}{\Delta t/2} = \frac{D}{2\Delta x^2}\left(U_{j+1,i}^{n+1/2} - 2U_{j,i}^{n+1/2} + U_{j-1,i}^{n+1/2}\right) + \frac{D}{2\Delta y^2}\left(U_{j,i+1}^{n} - 2U_{j,i}^{n} + U_{j,i-1}^{n}\right)$$

$$\frac{U_{j,i}^{n+1} - U_{j,i}^{n+1/2}}{\Delta t/2} = \frac{D}{2\Delta x^2}\left(U_{j+1,i}^{n+1/2} - 2U_{j,i}^{n+1/2} + U_{j-1,i}^{n+1/2}\right) + \frac{D}{2\Delta y^2}\left(U_{j,i+1}^{n+1} - 2U_{j,i}^{n+1} + U_{j,i-1}^{n+1}\right)$$

$$\left(1 - \frac{r}{2}\delta_x^2\right)\vec{\mathbf{U}}_{;,l}^{n+1/2} = \vec{\mathbf{U}}_{;,l}^{n} + \frac{r}{2}\left(\vec{\mathbf{U}}_{;,l+1}^{n} - 2\vec{\mathbf{U}}_{;,l}^{n} + \vec{\mathbf{U}}_{;,l-1}^{n}\right) + \frac{r}{2}\vec{b}'_{;,l}$$

$$\vec{\mathbf{U}}_{;,l}^{n+1/2} = \begin{bmatrix} U_{1,l}^{n+1/2} \\ U_{2,l}^{n+1/2} \\ \vdots \\ U_{M-2,l}^{n+1/2} \\ U_{M-1,l}^{n+1/2} \end{bmatrix}, \vec{\mathbf{U}}_{;,l}^{n} = \begin{bmatrix} U_{1,l}^{n} \\ U_{2,l}^{n} \\ \vdots \\ U_{M-2,l}^{n} \\ U_{M-1,l}^{n} \end{bmatrix}, \vec{\mathbf{b}}'_{;,l} = \begin{bmatrix} g_{top,l} \\ 0 \\ \vdots \\ 0 \\ g_{bottom,l} \end{bmatrix}$$

$$\left(1 - \frac{r}{2}\delta_y^2\right)\vec{\mathbf{U}}_{m,;}^{n+1} = \vec{\mathbf{U}}_{m,;}^{n+1/2} + \frac{r}{2}\left(\vec{\mathbf{U}}_{m+1,;}^{n+1/2} - 2\vec{\mathbf{U}}_{m,;}^{n+1/2} + \vec{\mathbf{U}}_{m-1,;}^{n+1/2}\right) + \frac{r}{2}\vec{b}_{m,;}$$

$$\vec{\mathbf{U}}_{m,;}^{n+1} = \begin{bmatrix} U_{m,1}^{n+1} \\ U_{m,2}^{n+1} \\ \vdots \\ U_{m,L-2}^{n+1} \\ U_{m,L-1}^{n+1} \end{bmatrix}, \vec{\mathbf{U}}_{m,;}^{n+1/2} = \begin{bmatrix} U_{m,1}^{n+1/2} \\ U_{m,2}^{n+1/2} \\ \vdots \\ U_{m,L-2}^{n+1/2} \\ U_{m,L-1}^{n+1/2} \end{bmatrix}, \vec{\mathbf{b}}_{m,;} = \begin{bmatrix} g_{m,right} \\ 0 \\ \vdots \\ 0 \\ g_{m,left} \end{bmatrix}$$

Goals

1. 1D Heat Equation (with Crank-Nicolson)

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial u^2(x,t)}{\partial x^2}$$

2. 1D Fokker-Planck Equation (with CN)

$$\frac{\partial p(v,t)}{\partial t} = -\mu\frac{\partial p(v,t)}{\partial v} + D\frac{\partial^2 p(v,t)}{\partial v^2}$$

3. 2D Heat Equation
   - Simple Explicit Method
   - Alternating Direction Implicit (ADI) Method (CN)

$$\frac{\partial u(x,y,t)}{\partial t} = D\left[\frac{\partial u^2(x,y,t)}{\partial x^2} + \frac{\partial u^2(x,y,t)}{\partial y^2}\right]$$

D: diffusion coeff.
μ: advection coeff.

4. 2D Fokker-Planck Equation (with CN/ADI)

$$\frac{\partial p(\vec{v},t)}{\partial t} = -\sum_{i=1}^{2}\mu_i\frac{\partial p(\vec{v},t)}{\partial v} + \sum_{i=1}^{2}\sum_{j=1}^{2}D_{ij}\frac{\partial^2 p(\vec{v},t)}{\partial v_i \partial v_j}$$