Computational Physics Final Report
Simulation of the Tippe Top

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Abstract

The main focus of this project is to analysis the motion of tippe top, using numerical method. Spinning tippe top will flip it's position because of the friction force acting on the contact point between the top and table. By using python programme, we figure out why the top will flip. Moreover, we find that the large friction force and the lower center of mass compared with the geometric center mainly result the different phenomena compared with a regular top.

1 Introduction

Top, a spin like toy, can spin on the ground and keep balance for a while. It is very popular not only among children, but also in scientific world. The research on top has a great impact on engineering and astronomy. Nevertheless, there is a kind of top called tippe top, which is a combination of a concaved ball and a handle stick in the concave. Not like the regular top, which can only spin standing on one point, tippe top can standing on both point, the bottom of the ball and the pin of the handle. The difference between a pin touch and a spherical touch is the former one can treated as a fixed point problem while the latter one is a changing contact point problem. When a tippe top is spinning at a high angular velocity, its handle slowly tilts downward more and more until it lifts the body of the top off the ground with stem pointing downward. As the angular velocity decrease, it loses stability and eventually topples over.

2 Analytical Derivation

2.1 Mass distribution

Since the bottom part of tippe top is a spherical surface, the main reason of the flip should relate to the ball rather than the handle. In order to simply our model, we consider the top is a ball with non-uniform mass distribution.

Denote R as the radius of the ball, O as the center of the ball, C as the center of mass, αR as the length of $\overline{OC}$, A as the contact point.

2.2 Euler angle

Compared with using Cartesian coordinates, we use Euler angle to represent the direction of the top.

Here, we simply introduce the Euler angle. $(\hat{x}, \hat{y}, \hat{z})$ represent the coordinates fixed in our laboratory, while $(\hat{n}, \hat{n}', \hat{3})$ represent the coordinates fixed on the top. Initially, $(\hat{n}, \hat{n}', \hat{3})$ overlap $(\hat{x}, \hat{y}, \hat{z})$. There are three rotations afterwards. Firstly, the coordinates rotates ϕ about $\hat{3}$; secondly, it rotates θ about $\hat{n}$, thirdly, it rotates ψ about $\hat{3}$.

The transformation relation between the Cartesian coordinates and Euler angle are

$$\hat{e}_n = \cos\phi \hat{e}_x + \sin\phi \hat{e}_y \tag{1}$$
$$\hat{e}_{n'} = -\cos\theta \sin\phi \hat{e}_x + \cos\theta \cos\phi \hat{e}_y + \sin\theta \hat{e}_z \tag{2}$$
$$\hat{e}_3 = \sin\theta \sin\phi \hat{e}_x - \sin\theta \cos\phi \hat{e}_y + \cos\theta \hat{e}_z \tag{3}$$

$$\hat{e}_x = \cos\phi \hat{e}_n - \cos\theta \sin\phi \hat{e}_{n'} + \sin\theta \sin\phi \hat{e}_3 \tag{4}$$
$$\hat{e}_y = \sin\phi \hat{e}_n + \cos\theta \cos\phi \hat{e}_{n'} - \sin\theta \cos\phi \hat{e}_3 \tag{5}$$
$$\hat{e}_z = \sin\theta \hat{e}_{n'} + \cos\theta \hat{e}_3 \tag{6}$$

The angular velocity in the laboratory frame is $\vec{\omega}$, and that of coordinates system is $\vec{\alpha}$

$$\begin{aligned}\vec{\omega} =& \dot{\theta}\hat{e}_n + \dot{\phi}\hat{e}_z + \dot{\psi}\hat{e}_3 \\ =& \dot{\theta}\hat{e}_n + \dot{\phi}\sin\theta \hat{e}_{n'} + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{e}_3\end{aligned} \tag{7}$$
$$\begin{aligned}\vec{\alpha} =& \dot{\theta}\hat{e}_n + \dot{\phi}\hat{e}_z \\ =& \dot{\theta}\hat{e}_n + \dot{\phi}\sin\theta \hat{e}_{n'} + \dot{\phi}\cos\theta \hat{e}_3\end{aligned} \tag{8}$$

which has to be emphasised that $\dot{\psi} + \dot{\phi}\cos\theta$ is the angular velocity we observed in the laboratory frame.

2.3 Friction force

Since friction force is the main reason leads to the flip phenomena. We deal with the friction force first. According to the reference, various authors have different opining in treating the friction force. To sum up, there are three big branches:

In 1950s, people used traditional sliding friction law, $\vec{F}_f = \mu F_N$, which is the first branch. However, using that method often will not meet the instable condition, the flip.

The second branch, which is used by most of authors as well as me, is to use viscous friction law, $\vec{F}_f = \mu \vec{v}_A F_N$. Here, I have to explain why it is reasonable to use viscous friction law. In recent paper, authors often use velocity dependent friction force. On the one hand, the simulation outcomes when using this expression match better with the experiments. On the other hand, using this expression can avoid some singularity.

The third branch is to consider both sliding friction and rolling friction (two different coefficients are difficult to choose properly). Moreover, when calculating, it has to figure out the motion is pure rolling or sliding. This means it has to compare that the friction the top need is larger than the maximum static friction force or not. This method is far difficult to solve properly, so I choose the second method.

It will meet some unphysical phenomena, which non of the friction expression can overcome, but it will not have deadly negative effect on the goal of our project. This will be discussed later.

2.4 Force analysis

The normal force, the total force and the acceleration of the center of mass can be expressed like below:

$$|\vec{F}_N| - mg = m\ddot{z}_c \tag{9}$$

$$\ddot{z}_c = \alpha R(\dot{\theta}^2 \cos\theta + \ddot{\theta}\sin\theta) \tag{10}$$

$$\vec{F}_T = \vec{F}_N + \vec{F}_f \tag{11}$$

Due to the theory of angular momentum, we can write the relation between torque and angular acceleration.

$$\vec{N} = \vec{r} \times \vec{F}_T \tag{12}$$

$$\vec{N} = \frac{d\vec{L}}{dt} = \frac{D\vec{L}}{Dt} + \vec{\alpha} \times \vec{L} \tag{13}$$

$$N_3 = I_3(\ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta + \ddot{\psi}) \tag{14}$$

$$N_n = I\ddot{\theta} + I_3\dot{\phi}\dot{\psi}\sin\theta + (I_3 - I)\dot{\phi}^2\sin\theta\cos\theta \tag{15}$$

$$N_{n'} = I\ddot{\phi}\sin\theta + (2I - I_3)\dot{\theta}\dot{\phi}\cos\theta - I_3\dot{\theta}\dot{\psi} \tag{16}$$

Simplify these equations:

$$\ddot{\theta} = \frac{\sin\theta}{I_1}\left(I_1\dot{\phi}^2\cos\theta - I_3\omega_3\dot{\phi} - R\alpha g_n\right) + \frac{R\mu F_N v_x}{I_1}(1-\alpha\cos\theta) \tag{17}$$

$$\ddot{\phi} = \frac{I_3\dot{\theta}\omega_3 - 2I_1\dot{\theta}\dot{\phi}\cos\theta - \mu F_N v_y R(\alpha - \cos\theta)}{I_1\sin\theta} \tag{18}$$

$$\dot{\omega_3} = -\frac{\mu F_N v_y R\sin\theta}{I_3} \tag{19}$$

$$\begin{aligned}\dot{v}_x =& \frac{R\sin\theta}{I_1}\left\{\dot{\phi}\omega_3\left[I_3\left(1-\alpha\cos\theta\right) - I_1\right] + F_N R\alpha(1-\alpha\cos\theta) - I_1\alpha\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)\right\} \\ &- \frac{\mu F_n v_x}{m I_1}\left[I_1 + mR^2(1-\alpha\cos\theta)^2\right] + \dot{\phi}v_y\end{aligned} \tag{20}$$

$$\begin{aligned}\dot{v}_y =& -\frac{\mu F_N v_y}{m I_1 I_3}\left[I_1 I_3 + mR^2 I_3(\alpha-\cos\theta)^2 + mR^2 I_1\sin^2\theta\right] \\ &+ \frac{\omega_3\dot{\theta}R}{I_1}[I_3(\alpha-\cos\theta) + I_1\cos\theta] - \dot{\phi}v_x\end{aligned} \tag{21}$$

where F_N is

$$F_N = \frac{mgI_1 + mR\alpha[\cos\theta(I_1\dot{\phi}^2\sin^2\theta + I_1\dot{\theta}^2) - I_3\dot{\phi}\omega_3\sin^2\theta]}{I_1 + mR^2\alpha^2\sin^2\theta - mR^2\alpha\sin\theta(1-\alpha\cos\theta)\mu v_x} \tag{22}$$

where v_x and v_y are the velocity of the contact point A, not the velocity of the center of mass, and $\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$, which is the total angular velocity along axis $\hat{3}$ that can be observed in laboratory frame.

3 Numerical Method

We have simplify the complicate problem into solving a coupled ordinary equations. The method I choose is the forth order Runge-Kutta method with adaptive time steps.

As we learnt from lecture, firstly, setting a maximum error per second σ. Then, with a set of given initial condition, doing the same problem with two methods, one big step and two half steps. And then, comparing the difference of two solutions. Using a index ρ, if

$$\rho = \frac{30h\sigma}{\sum_i (x_1^i - x_2^i)^2} > 1 \tag{23}$$

we think the large time step will not exceed the error that we set. Then, we continue to solve next time step with step size

$$h' = h\rho^{1/4} \tag{24}$$

However, if $\rho < 1$, we half both time steps repeat the above steps until $\rho > 1$ and then we continue to next time step.

4 Solution and Discussion

4.1 Standard parameter and initial condition

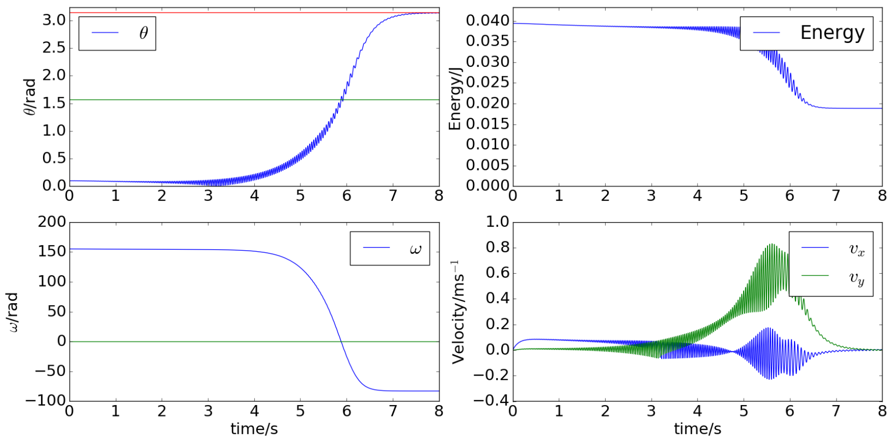

Figure 1: Top parameter: R=0.2cm, g=9.8m/s^2, m=25g, α=0.3, μ=0.3s/m, I=1.25I_3=$\frac{2}{5}mR^2$. Initial condition: $\omega = 155rad/s$, $\theta_0 = 0.1rad$.

From figure 1, we can find that our numerical method successfully simulate the flip motion of a tippe top. The top-left sub-figure shows the reverse of polar angle. It starts from $\theta_0 = 0.1rad$, after about 8 seconds, it increases to π, which means the top reversed.

Moreover, we can split the whole motion into three part. The first part is from 0 to 3 seconds. We can find θ doesn't increase with time. It just nutates, and increase its nutation angle, until it reach 0. When it reaches 0, it comes into the second part. Since polar angle is always positive, when it goes further, it becomes larger again, but it changes the phase of azimuthal angle ($+\pi$). The stable motion is broken, so the polar angle start increase periodically. When it totally reversed (about 7 seconds), the polar angle remains π and won't decrease, which is the third part.

However, it is not physical, since we all know that the top will reverse again and stop ultimately. Since the model here, we consider the contact area is an ideal point, while it should be a small area due to the deformation of the table. In our model, there is no friction force here. No matter what kind of friction expression, there is no friction force as well, unless if we take the deformation into consideration.

The bottom-left figure shows the angular velocity we that we can observe from laboratory frame. We can find when $\theta = \frac{\pi}{2}$, the sign of ω changes as well. It is because the direction of total angular momentum always the same.

The right two figure show the change of total energy and the velocity of contact point. The energy will decrease with the reversion. At end, the total energy not change is because of the ignorance of the deformation.

4.2 Different parameter

1. Change α

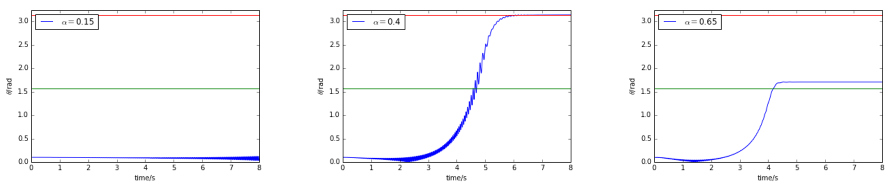

Figure 2: Change the center of mass

Figure 2 shows the motion of different α, which is the ratio of distance between mass center and geometric center. When $\alpha = 0$, two centers overlapped; when $\alpha = 1$, mass center is at the margin of the ball.

We can find that, when α is small, it will not flip. With the increasing α, top will flip. However, when α becomes large enough, the polar angle will not increase to π, since the initial envergy given is not large enough to raise the center of mass. This shows that the eccentric is one of the key point of the flip.

2. Change μ

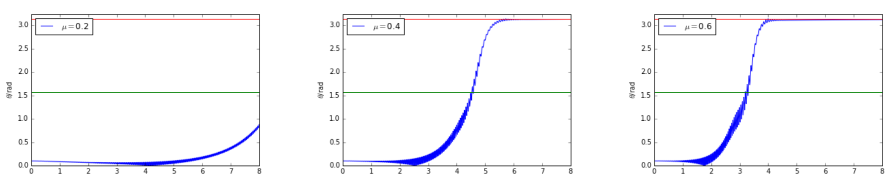

Figure 3: Change friction coefficient

Form figure 3, we can find that the increasing of friction coefficient, will shorten the flip time. The larger the friction, the sooner the flip happens. This shows that the friction is another key point of the flip.

3. Change I/I_3

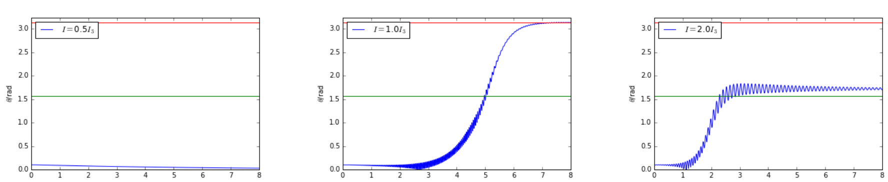

Figure 4: Change moment of inertia ratio

In figure 4, we change the ratio of I/I_3. We can find when $I \approx I_3$, the top will flip. If the difference between I and I_3 are large, the top will have a tendency to rotate to the situation where the larger moment of inertia axis close vertical axis. This is also an important feature of tippe top.

4.3 Different initial condition

1. Change ω

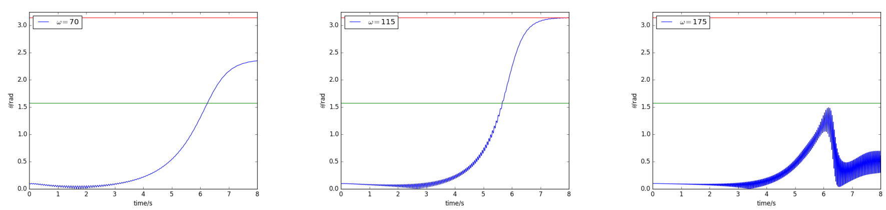

Figure 5: Change initial angular velocity

The initial angular velocity will give top energy to let it raise the center of mass, and flip. When initial angular velocity is small, it will not have enough energy to flip. However, when the initial angular velocity is too large, it will not flip as well. Instead, it will becomes unstable, which is very strange, and I can't explain the phenomena.

2. Change θ_0

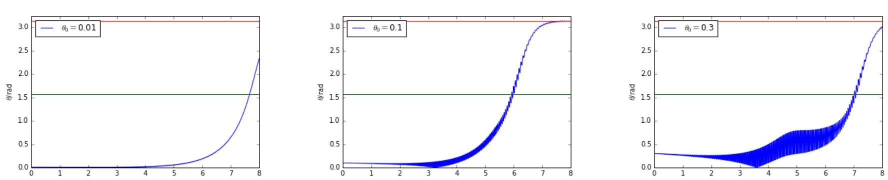

Figure 6: Change initial tilt angle

Since it is numerical simulation, there is no random perturbation compared with the real experiment. We must give the top an initial angle, otherwise, it will not flip. In this unstable equilibrium situation, a small initial angle is necessary.

From figure 6, we can find even the initial angle is large, it will flip as well, while the larger the initial angle, the longer time it takes to flip. As we mentioned above, the first part of the motion is increasing the amplitude of nutation, until $\theta = 0$.

4.4 Compared with regular top

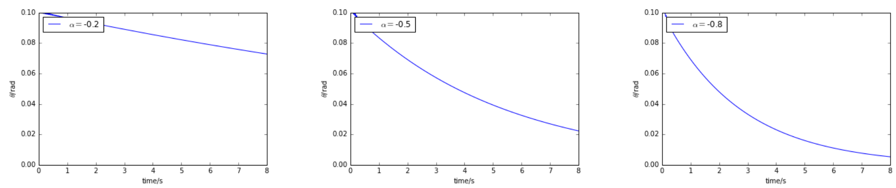

Figure 7: Compare with regular top

As we mentioned in the introduction part, the different between tippe top and regular top is the position of center of mass. The bottom of regular top is always a point. We can consider it is a "ball" (like a ballpen) with small radius. The center of mass is obviously above the geometric center of the "ball". In our model, we just need to change the parameter α from positive (below) to negative (above).

Form figure 7, it is easy to find that the more negative α, the faster polar angle will turn to 0.

5 Next step

1. Taking rolling friction into consideration, and find suitable coefficient.
2. Taking deformation of the table into consideration. Only in this way, can let the top stop ultimately.

6 Acknowledgement

I will thank Professor Andrew MacFadyen for his lecture. He shows many details of derivation of the fundamental knowledge which gives us a deeper learning of computational physics.

Moreover, I would like to thank TA Geoffrey Ryan. Since I have never used python, latex and some other software, he teaches me very patiently. During his recitation, he teaches us a lot, like some useful software, the good habit of programing that we cannot learn from textbook.

7 Reference

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