Computational Physics Homework #1: Mandelbrot Set and Millikan Experiment

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ABSTRACT

In this assignment, we solve two problems. The first is to compute and plot the Mandelbrot set, which has a fractal structure. We find that iterating over the equation and plotting the values in the set shows the expected fractal pattern, and that the points outside the set vary somewhat continuously based on iteration number. The second problem is to implement the least squares fit and apply it to data from the Millikan experiment determining the photoelectric effect. We perform this fit, which fits very well to the data, and use it to calculate the value of Planck's constant, obtaining a value of $h = 6.549 \times 10^{-34} m^2 kg/s$.

1 MANDELBROT SET

1.1 Introduction

The Mandelbrot set was discovered by Benoît Mandelbrot, a French mathematician and self-described "fractalist." It uses an equation involving complex numbers over which we iterate recursively. If the output for a given complex input never passes a certain value, that input is in the Mandelbrot set. The real and imaginary parts of the values in the set create a fractal pattern when plotted on a grid, which is self-similar and has recursive detail along the boundary.

1.2 Theory

Consider a set of values c, where all of the values in c are complex, of the form $c = x + iy$. Take x and y to range from -2 to 2, with N x-y pairs. Then, input each value of c into the equation

$$z' = z^2 + c$$

where z begins at 0. This outputs a value z', which we then feed back into the equation as z. We continue iterating until either *a*) z' hits the maximum value of 2 or *b*) a maximum number of iterations is reached (in our case, 100).

Case *a*) means that the complex number is not in the Mandelbrot set, while case *b*) means that the value is in the set. We note that, in theory, a point is in the set if it never reaches the maximum over infinite iterations, but we choose a maximum for obvious reasons; the results are not significantly dependent on this value.

1.3 Results

We plot our results in Figure 1. We choose $N = 1000$, giving 10^6 complex numbers. The plot shows x vs. y values where those are the components of the complex numbers that are input into the function above. Black points show values in the Mandelbrot set, while the white area is outside of the set. We find that we successfully obtain the beetle-like shape characteristic of the set.

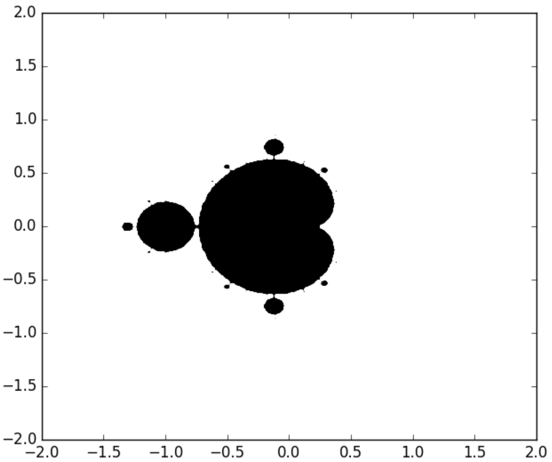

Figure 1. x vs. y values of the complex numbers $c = x + iy$ in the range $-2 \leq x, y \leq 2$. Black points indicate values in the Mandelbrot set.

We also plot the set color-coded by the number of iterations until the maximum value is reached (or black if they never reach it after 100 iterations). The results are shown in Figure 2.

Finally, we show the points outside of the set color-coded by the logarithm of the number of iterations. The results are shown in Figure 3.

2 DISCUSSION & CONCLUSIONS

We successfully implement the equation and processes that give rise to the Mandelbrot set. We find that the values inside

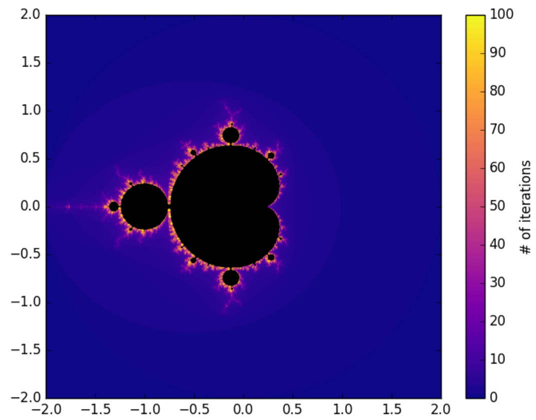

Figure 2. As in Figure 1, but color-coded by the number of iterations before reaching the maximum value of 2. Points in the Mandelbrot set are still shown in black.

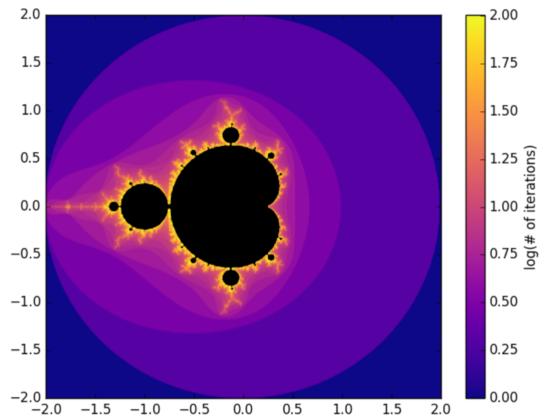

Figure 3. As in Figure 1, but color-coded by the logarithm of the number of iterations before reaching the maximum value of 2. Points in the Mandelbrot set are still shown in black.

the set, when plotted in terms of the real and imaginary parts of the complex numbers that give rise to the values, display a characteristic fractal pattern.

Moreover, we find that color-coding the points outside the set by the number of iterations it took before reaching the maximum value shows a fractal pattern shadowing the one of the set itself. The gradient is fairly continuous, with many iterations near the edge of the Mandelbrot set and quickly fading to very few iterations before the maximum is reached.

When plotting the logarithm of the iterations, we see a very discrete distribution further from the center of the set. This is due to there being many values with low iteration numbers, but only a few at higher iterations, which skews the log-scaled values.

Overall, we successfully implement and plot the values of the Mandelbrot set.

3 LEAST-SQUARES FIT TO THE MILLIKAN EXPERIMENT

3.1 Introduction

In 1914, American physicist Robert Millikan conducted a Nobel Prize-winning experiment to measure the photoelectric effect. This effect occurs when light shines onto a metal, occasionally causing an electron to be ejected. The energy of this electron is proportional to the frequency of the light beam, given one subtracts the amount of energy it took to remove the electron from the metal, a quantity known as the work function. The constant of proportionality involves Planck's constant h, so Millikan used the measurement of this effect to calculate a value for h.

In order to measure this constant, it is necessary to fit the experimental data to a line. In this part of the assignment, we calculate and implement the method of least-squares and use it to fit Millikan's data and calculate h.

3.2 Theory

Given a set of points on the x-y plane, we want to calculate the straight line that best fits these points. The equation for a line is

$$y = mx + c$$

where m is the slope and c is the y-intercept. For some line, the vertical distance between the line at (x, y) and the point at (x_i, y_i) is

$$y - y_i = mx_i + c - y_i$$

. To calculate the squares of these distances for N data points, we use

$$\chi^2 = \sum_{i=1}^{N}(mx_i + c - y_i)^2$$

. We want to minimize this value in order to find the line that best fits all of the points.

To perform this minimization, we take the derivative with respect to m and then with respect to c. After rearranging to solve for these quantities, we find

$$m = \frac{E_{xy} - E_x E_y}{E_{xx} - E_x^2}$$

$$c = \frac{E_{xx}E_y - E_x E_x y}{E_{xx} - E_x^2}$$

where we have defined

$$E_x = \frac{1}{N}\sum_{i=1}^{N} x_i, \;\; E_y = \frac{1}{N}\sum_{i=1}^{N} x y_i, \;\; E_x x = \frac{1}{N}\sum_{i=1}^{N} x_i^2, \;\; E_x y = \frac{1}{N}\sum_{i=1}^{N} x_i y_i$$

The photoelectric effect is described by the equation

$$V = \frac{h}{e}\nu - \phi$$

where V is the voltage of the ejected electron, h is Planck's constant, e is the charge of an electron, ν is the frequency of the light used, and ϕ is the work function.

Millikan's experiment measured the frequencies and voltages, or the x and y components respectively. Then,

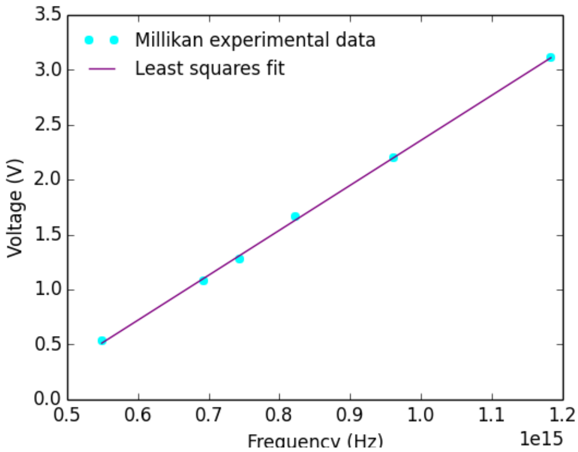

Figure 4. Voltages of ejected electrons as a function of the frequency of the light beam used. Blue circles show the results of Millikan's experiment. The purple line shows the least-squares fit to the data.

$m = \frac{h}{e}$ (and $c = -\phi$). Thus, calculating m using the method described above allows us to find Planck's constant using

$$h = me$$

where $e = 1.602 \times 10^{-19}$ C.

3.3 Results

We show our results in Figure 4. The plot shows the voltage of each electron ejected from the metal as a function of the light that was shone. The blue circles are the data points from Millikan's experiment, and the purple line is the least squares fit calculated using the method described above.

We find a fit very close to the data points, showing a tight relation between the voltage and frequency. The slope and intercept of the fit are

$$m = 4.088 \times 10^{-15}$$

$$c = -1.731$$

Using these values and the equation described above, we obtain a value of Planck's constant of

$$h = 6.549 \times 10^{-34}$$

The accepted value is $h = 6.626 \times 10^{-34}$, so our value is only 1.16% lower.

3.4 Discussion and Conclusions

We successfully implement the method of least squares to find a first-order fit to the data. This allows us to take the data used in the Millikan experiment on the photoelectric effect and apply our method to obtain fit parameters.

We use the slope of our fit line to calculate the value of Planck's constant, and find a value only 1.16% lower than the accepted value.

REFERENCES

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