RELATIVISTIC HYDRO NOTES

MARC WILLIAMSON

The equation of state for a relativistic, ideal fluid is...

$$p(\rho, \epsilon) = \rho\epsilon(\gamma - 1)$$

where ρ is the rest mass, ϵ is the specific internal energy, and γ is the adiabatic index. The ideal fluid equation of state is related to another type of fluid, called a Polytropic fluid, which has equation of state...

$$p(\rho) = K\rho^{\Gamma}$$

where K is a constant across the entire fluid, and Γ is the polytropic exponent. The polytropic equation of state is more general than the ideal fluid, but they coincide when the fluid transforms isentropically. We can see this by differentiating the ideal fluid EOS...

$$\begin{aligned} dp &= (\gamma - 1)\left(\frac{p}{\rho} + \epsilon\right) d\rho \\ dp &= \gamma\left(\frac{p}{\rho}\right) d\rho \end{aligned}$$

where we have used the differential form of the first law of thermodynamics...

$$d\epsilon = tdS - pd\left(\frac{1}{\rho}\right)$$

with $dS = 0$ for isentropic transformations. Thus we can integrate...

$$\begin{aligned} dp &= \gamma\left(\frac{p}{\rho}\right) d\rho \\ \frac{1}{p}dp &= \gamma\left(\frac{1}{\rho}\right) d\rho \\ \int \frac{1}{p}dp &= \int \gamma\left(\frac{1}{\rho}\right) d\rho \\ \ln(p) &= \gamma\ln(\rho) + k \\ \ln(p) &= \ln\left(\rho^{\gamma}\right) + k \\ p &= K\rho^{\gamma} \end{aligned}$$

In numerical simulations, we always choose $\Gamma = \gamma$ so that we can calculate the internal energy using the first law of thermodynamics...

$$\begin{aligned} d\epsilon &= -pd\left(\rho^{-1}\right) \\ d\epsilon &= \frac{1}{\rho^2}pd\rho \\ d\epsilon &= \frac{K\rho^{\Gamma}}{\rho^2}d\rho \\ d\epsilon &= K\rho^{\Gamma-2}d\rho \\ \epsilon &= \frac{K\rho^{\Gamma-1}}{\Gamma - 1} \end{aligned}$$

You should now be able to calculate the internal energy using the above relation!

We can now calculate the sound speed. In general for an arbitrary equation of state, we have...

$$c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

which we can re express through some algebra as...

$$\begin{aligned} c_s^2 &= \frac{1}{h}\left(\frac{dp}{d\rho}\right)_s \\ &= \frac{1}{h}\left[\left(\frac{\partial p}{\partial \rho}\right)_s + \frac{p}{\rho^2}\left(\frac{\partial p}{\partial \epsilon}\right)_\rho\right] \end{aligned}$$

In our case, the polytropic equation of state does not explicitly depend on the internal energy, so we have...

$$\begin{aligned} c_s^2 &= \frac{1}{h}\left(\frac{\partial p}{\partial \rho}\right) \\ c_s^2 &= \frac{1}{h}\left(\frac{\partial}{\partial \rho}\left[K\rho^\Gamma\right]\right) \\ c_s^2 &= \frac{1}{h}\left(\Gamma K \rho^{\Gamma-1}\right) \\ c_s^2 &= \frac{\Gamma p}{h\rho} \end{aligned}$$

where $h = 1 + \epsilon + \frac{p}{\rho}$ is the specific enthalpy. Some algebra gives us...

$$\begin{aligned} c_s^2 &= \frac{\Gamma p}{h\rho} \\ c_s^2 &= \frac{\Gamma p}{\rho + \rho\epsilon + p} \\ c_s^2 &= \frac{\Gamma p}{\rho + \rho\left(\frac{K\rho^{\Gamma-1}}{\Gamma-1}\right) + K\rho^\Gamma} \\ c_s^2 &= \frac{\Gamma p}{\rho + \rho\left(\frac{K\rho^{\Gamma-1}}{\Gamma-1}\right) + K\rho^\Gamma} \\ c_s^2 &= \frac{\Gamma p}{\rho + K\rho^\Gamma\left[1 + \frac{1}{\Gamma-1}\right]} \\ c_s^2 &= \frac{\Gamma p}{\rho + p\left[1 + \frac{1}{\Gamma-1}\right]} \\ c_s^2 &= \frac{\Gamma\left(\Gamma-1\right)p}{\rho\left(\Gamma-1\right) + p\left[\left(\Gamma-1\right)+1\right]} \\ c_s^2 &= \frac{\Gamma\left(\Gamma-1\right)p}{\rho\left(\Gamma-1\right) + p\Gamma} \\ c_s^2 &= \left[\frac{\rho\left(\Gamma-1\right) + p\Gamma}{\Gamma\left(\Gamma-1\right)p}\right]^{-1} \\ c_s^2 &= \left[\frac{\rho}{\Gamma p} + \frac{1}{\left(\Gamma-1\right)}\right]^{-1} \\ c_s^2 &= \left[\frac{1}{\Gamma K\rho^{\Gamma-1}} + \frac{1}{\left(\Gamma-1\right)}\right]^{-1} \end{aligned}$$

You should now be able to easily calculate the sound speed using the above equation!

I highly recommend reading chapter 2.4, and all of chapter 2, of Relativistic Hydrodynamics by Luciano Rezzolla and Olindo Zanotti. These derivations can be found there in more detail.