A Case of Numerical Relativity: Scalar field propagation in AdS_3

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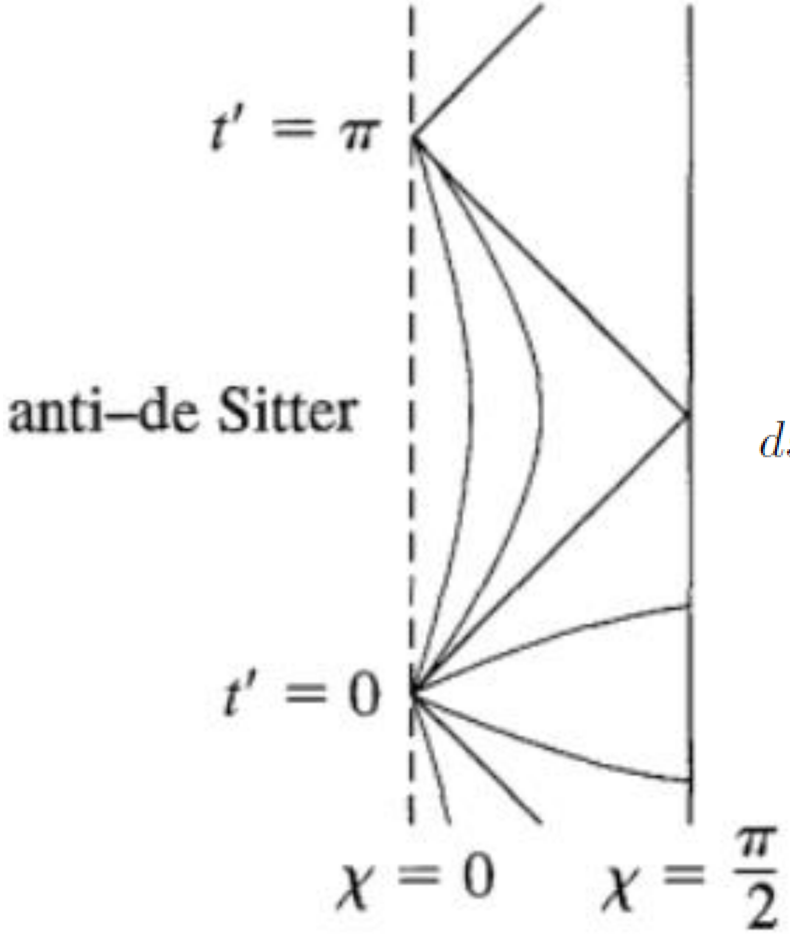
Introduction to the background

Numerical Relativity

- Einstein's field equations are nonlinear 2nd order partial differential equations, which can only be solved analytically in few cases.
- Due to the nonlinearity in the equations, we have to introduce new techniques for solving partial differential equations.

What’s special for Anti de-Sitter (AdS) spacetime?



- The null infinity I is timelike: we cannot compactify time direction even in Penrose diagram. This means that light rays from the origin can travel across AdS and reflect back in finite proper time.
- We need to use compactified coordinates and specify corresponding boundary conditions on null infinity

$$ds^2 = \frac{e^{2A(r,t)}}{\cos^2(r/\ell)}\left(dr^2 - dt^2\right) + \ell^2 \tan^2(r/\ell) e^{2B(r,t)} d\theta^2$$

- The metric for AdS_3 with matter content.

The equations we need to solve

- Einstein's field equation (we can rescale away Newton's constant):

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- Klein-Gordon equation for massless scalar field:

$$\nabla_\mu \nabla^\mu \phi = 0$$

$$\Phi(r,t) \equiv \partial_r \phi(r,t), \quad \Pi(r,t) \equiv \partial_t \phi(r,t)$$

$$\partial_t A(r,t) \equiv A_t(r,t), \quad \partial_t B(r,t) \equiv B_t(r,t)$$

$$\partial_t A_t = \partial_r^2 A + \left(\frac{\pi}{2}\right)^2 \frac{1-e^{2A}}{cos^2(\pi r/2)} + 2\pi(\Phi^2 - \Pi^2)$$

$$\partial_t B_t = \partial_r^2 B + \partial_r B\left(\partial_r B + \frac{2\pi}{sin(\pi r)}\right) - B_t^2 + \frac{\pi^2}{2}\frac{1-e^{2A}}{cos^2(\pi r/2)}$$

$$\partial_r^2 B + \partial_r B\left(\partial_r B - \partial_r A + \frac{\pi(1+cos^2(\pi r/2))}{sin(\pi r)}\right) - \frac{\pi \partial_r A}{sin(\pi r)}$$

$$-A_t B_t + \frac{\pi^2(1-e^{2A})}{4cos^2(\pi r/2)} + 2\pi(\Phi^2 + \Pi^2) = 0$$

$$\partial_r B_t + B_t\left(\partial_r B - \partial_r A + \frac{\pi}{2tan(\pi r/2)}\right) - A_t\left(\partial_r B + \frac{\pi}{sin(\pi r/2)}\right) + 4\pi\Phi\Pi = 0$$

Constraint equations

$$\partial_t \Phi = \partial_r \Pi$$

$$\partial_t \Pi = \Phi \partial_r B + \partial_r \Phi + \frac{\pi \Phi}{sin(\pi r)} - B_t \Pi$$

Here I rescale AdS radius l so that $r \in [0,1]$.

Boundary conditions at the origin and on null infinities

1. Regularity at r = 0.
2. Asymptotically AdS at spatial infinity.
3. Conservation of scalar matter at the boundary during propagation.

$$\partial_r A(0,t) = 0, \quad \partial_r B(0,t) = 0, \quad \Phi(0,t) = 0, \quad \partial_r \Pi(0,t) = 0$$
$$A(1,t) = 0, \quad \partial_r B(1,t) = 0, \quad \Phi(1,t) = 0, \quad \Pi(1,t) = 0$$

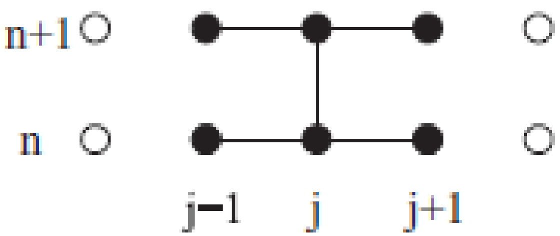
There are extra boundary conditions which we can use as a consistent check during evolution.

The numerical techniques I use.

Iterative Crank-Nicolson scheme

- Normal Crank-Nicolson scheme is implicit:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\frac{1}{2}\left[F_i^{n+1}\left(u,\, x,\, t,\, \frac{\partial u}{\partial x},\, \frac{\partial^2 u}{\partial x^2}\right)+F_i^n\left(u,\, x,\, t,\, \frac{\partial u}{\partial x},\, \frac{\partial^2 u}{\partial x^2}\right)\right]$$



- But for nonlinear systems, it is hard to solve such a simultaneous equations.

Iterative Crank-Nicolson scheme

- An alternative is the iterative Crank-Nicolson scheme. It is a explicit scheme that converges to Crank-Nicolson scheme.

$$
{}^{(1)}u_j^{n+1} = u_j^n + \Delta t F_i^n(u^n, x, t)
$$

$$
{}^{(2)}u_j^{n+1} = u_j^n + \frac{1}{2}\Delta t(F_i^n({}^{(1)}u^{n+1}, x, t) + F_i^n(u^n, x, t))
$$

$$
{}^{(3)}u_j^{n+1} = u_j^n + \frac{1}{2}\Delta t(F_i^n({}^{(2)}u^{n+1}, x, t) + F_i^n(u^n, x, t))
$$

$$
\dots
$$

Iterative Crank-Nicolson scheme

- Theoretically it will converge to implicit scheme.
- Saul A. Teukolsky[1] found out we should carry out exactly two steps or the instabilities will grow.
- With this scheme, I can obtain more stable results than normal 2nd order explicit scheme.

1 Teukolsky, S. A. "On the stability of the iterated Crank-Nicholson method in numerical relativity, 1999." arXiv preprint gr-qc/9909026.

Higher order representation of Spatial derivatives

There will instable growing modes at the boundary due to terms like $\frac{C}{\sin(\pi r)}$ so in order to smear the divergences I use higher-order spatial scheme to discretize the source function.

$$(\partial_x f)_i = \frac{1}{12\Delta x}(f_{i-2} - 8f_{i-1} + 8f_{i+1} - f_{i+2})$$

$$(\partial_x^2 f)_i = \frac{1}{12(\Delta x)^2}(-f_{i-2} + 16f_{i-1} - 30f_i + 16f_{i+1} - f_{i+2})$$

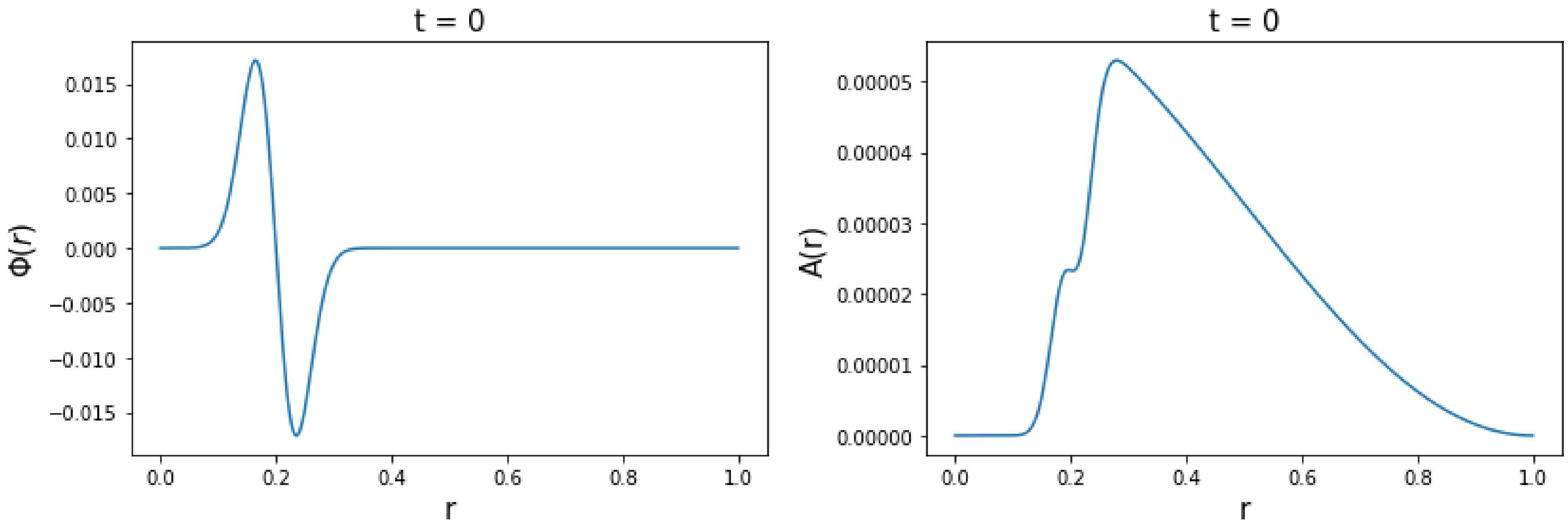
Kreiss-Oliger dissipation

- We can introduce an artificial viscosity: high order derivatives and small amplitude. Then it will damp the very high frequency modes during the evolutions. And it turns out that it is necessary to stabilize the system.
- In this case I introduce the following 3rd order numerical viscosity for each functions:

$$\partial_t u_m \to \partial_t u_m + \frac{\sigma}{64\Delta x}(u_{m+3} - 6u_{m+2} + 15u_{m+1} - 20u_m + 15u_{m-1} - 6u_{m-2} + u_{m-3})$$

Actual Running & Numerical Results

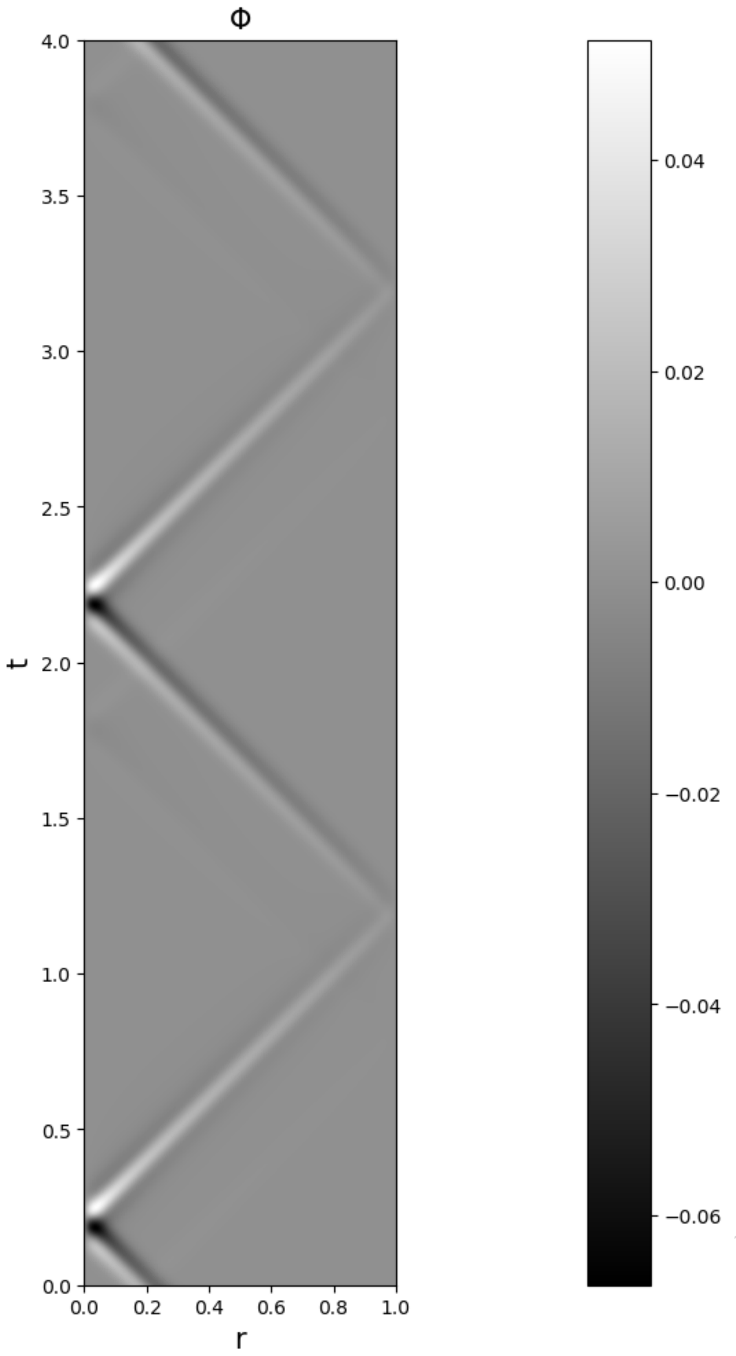
Initial conditions: Gaussian wave



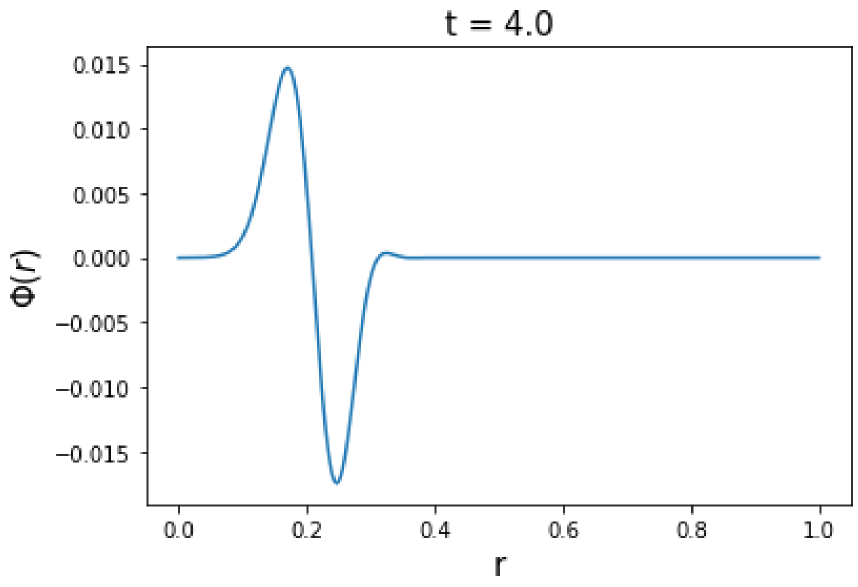
A(r, 0) is solved from constraint equations using RK4.

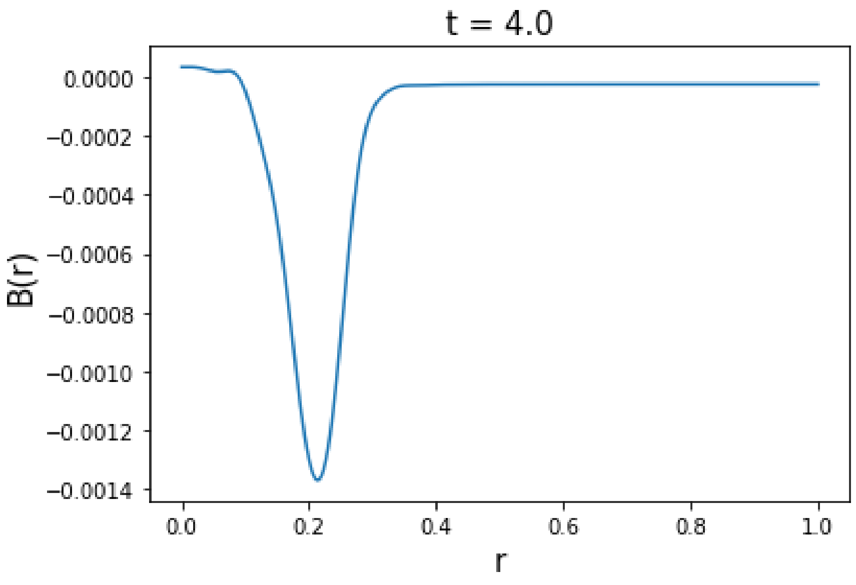
Time evolution of an ingoing Gaussian wave

Setting small amplitude of wave so that we can neglect back-reaction which will introduce instabilities in my code.

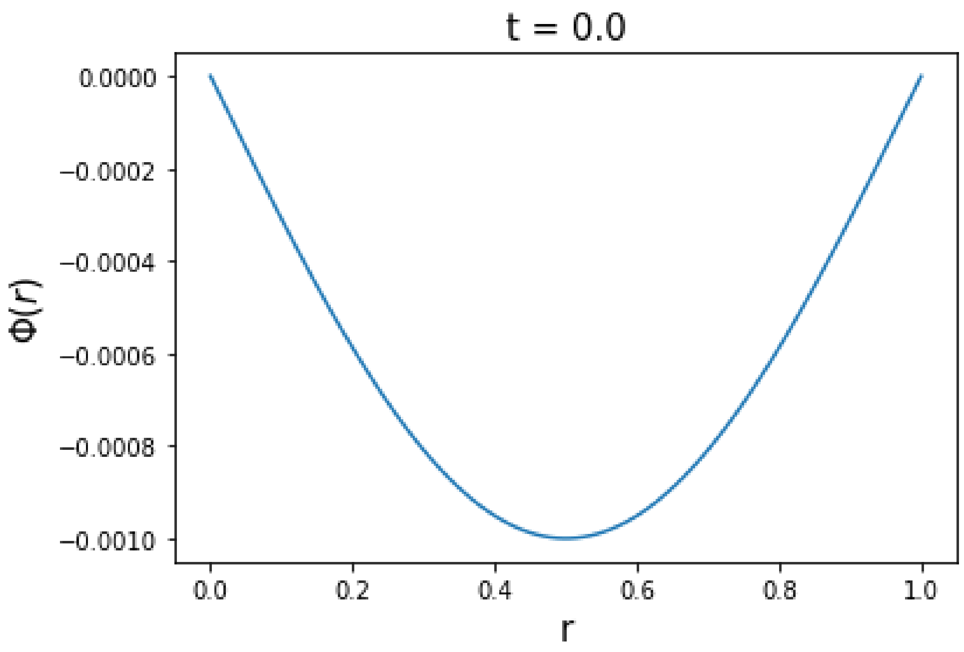


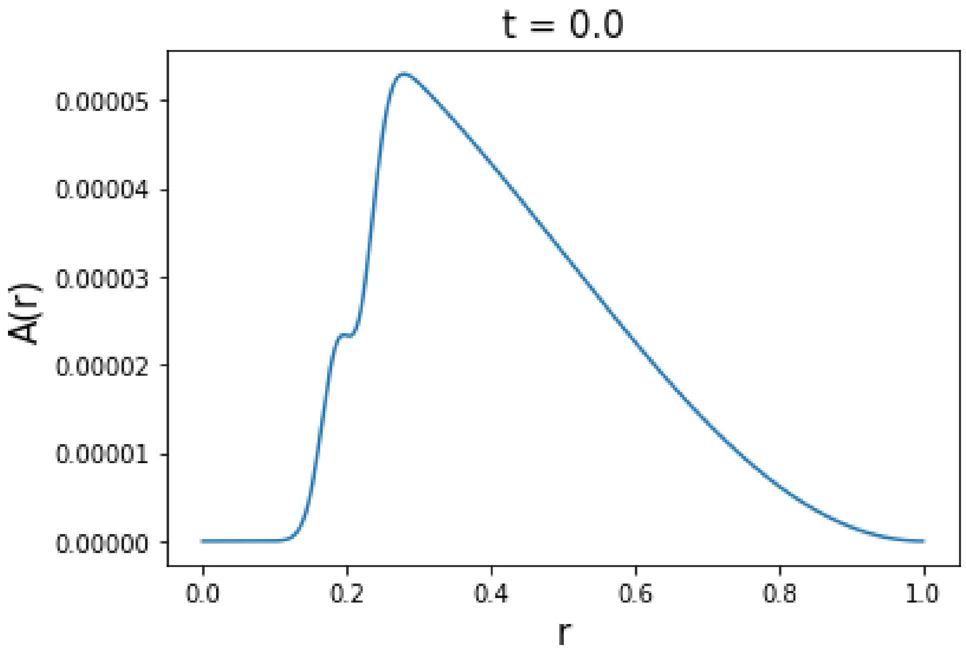
Time evolution of an ingoing Gaussian wave





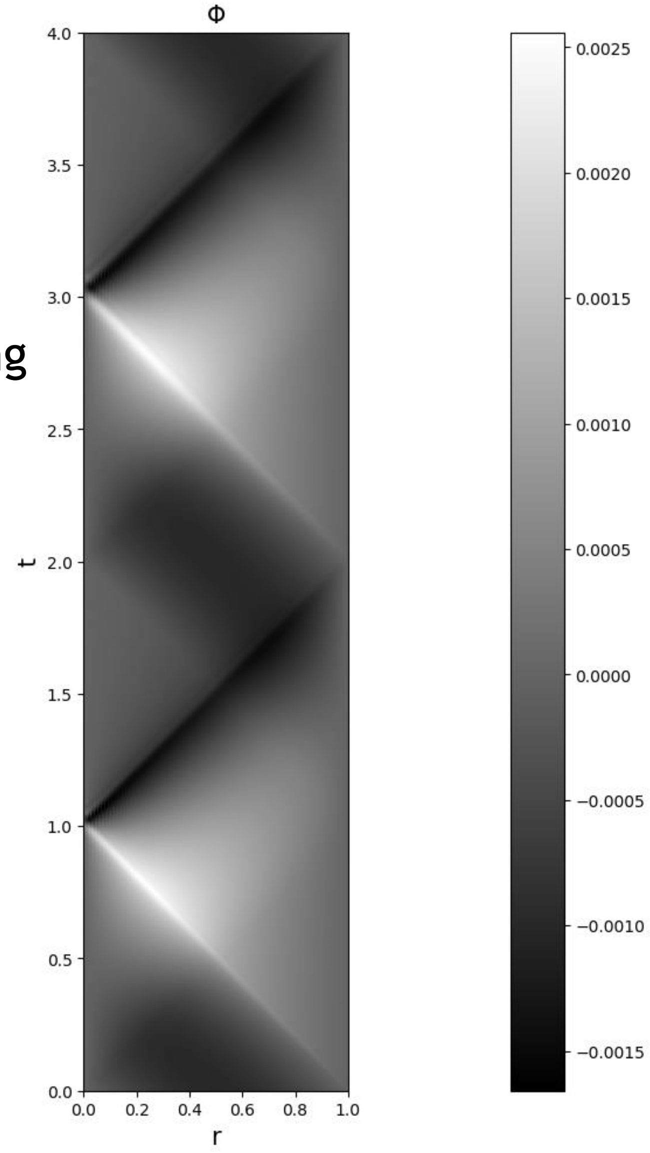
Time evolution of an harmonic wave



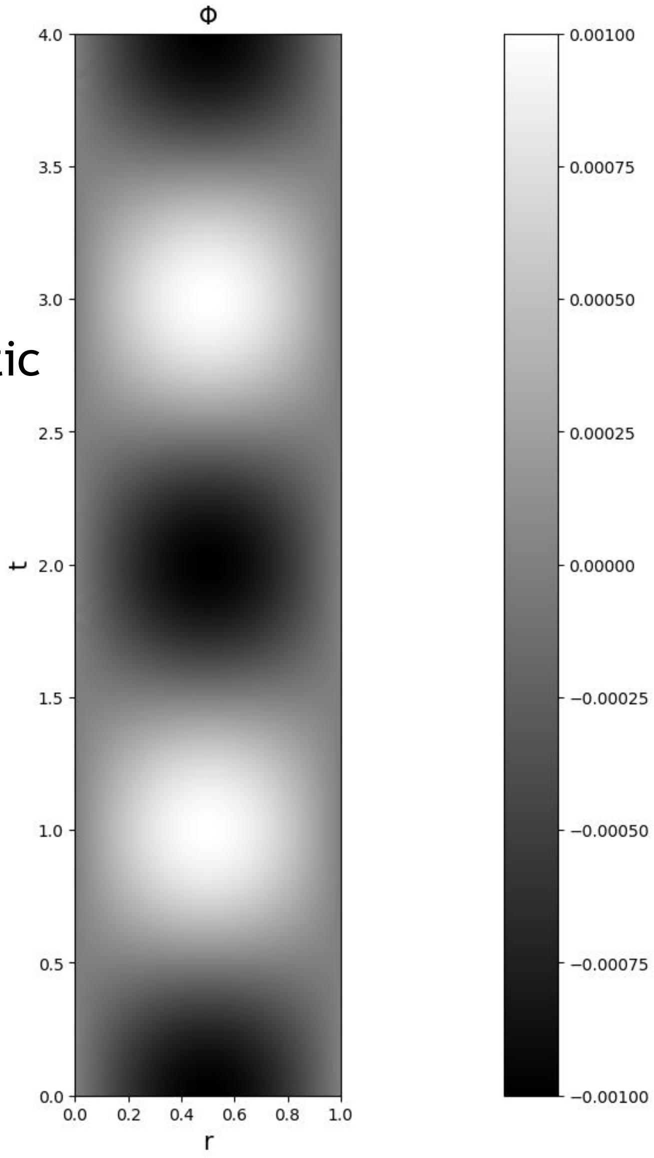


Time evolution of an harmonic wave

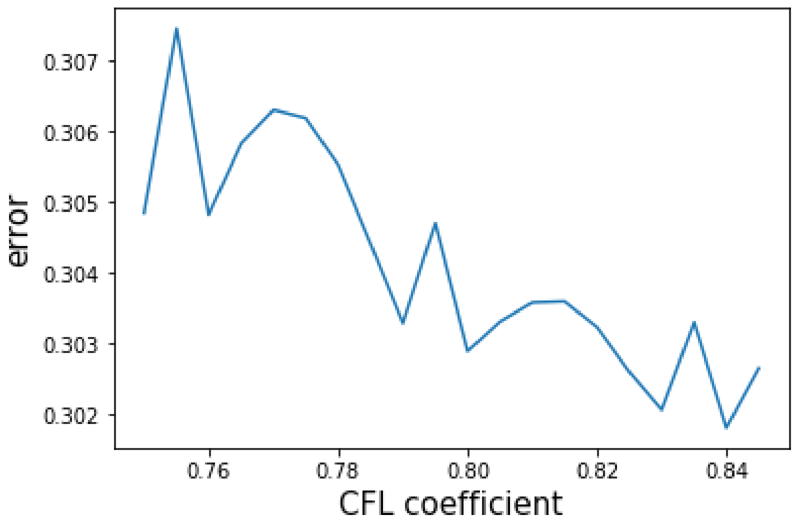
Ingoing

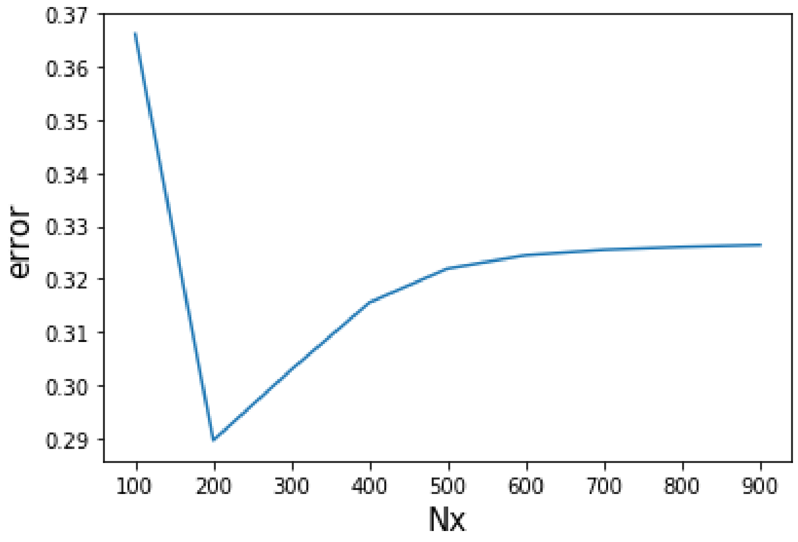


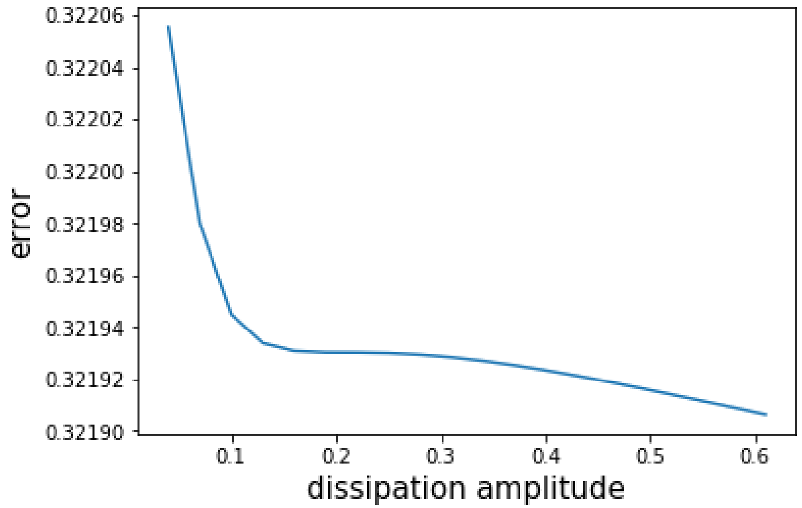
Static



Error analysis







Future plan

- Modify my code for better stability so that it can work for waves with amplitude at the same order with AdS radius.
- If it works, it is interesting to study gravitational collapse cases, which will have critical behaviors.

$$M_{\mathrm{BH}} = C|p - p^*|^\gamma$$

- I can also try them in asymptotically flat spacetime, which may be easier.

Thank you!

Comments? Suggestions? Critiques?