Solving Integrable Boltzmann Equation for Coscattering Dark Matter

(Final Presentation for PHYS-GA-2000)

Xucheng Gan

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$$\frac{dY_\chi}{dx} = -\frac{45^{3/2}}{2^{7/2}\pi^8}\frac{1}{g_{\star S}(m_\chi/x)\sqrt{g_\star(m_\chi/x)}}\left(f(r)(1+\Delta)^{3/2}\Delta^{1/2}y^4\delta^2\right)\left(\frac{M_{pl}}{m_\chi}\right)\left(\frac{e^{-\Delta x}}{x^2}\right)$$

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$$H(x) = \frac{H(m_\chi)}{x^2} \qquad H(m_\chi) = \sqrt{\frac{\pi^2}{90} \frac{g_*(T)}{{M_{pl}}^2} {m_\chi}^4}$$

$$S(x) = \frac{S(m_\chi)}{x^3} \qquad S(m_\chi) = \frac{2\pi^2}{45} g_{*S}(T) {m_\chi}^3$$

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Exponentially suppressed in low temperature

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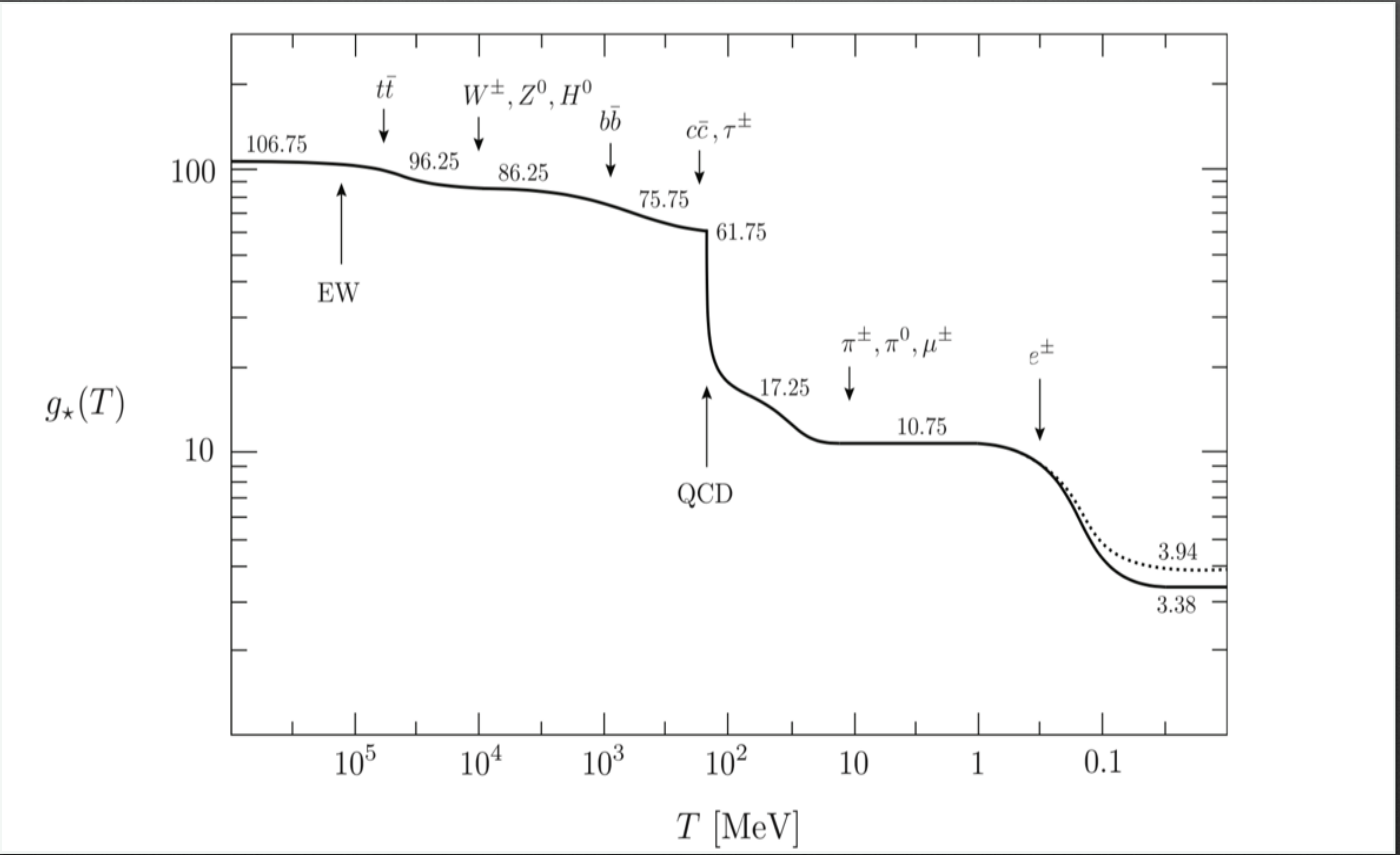

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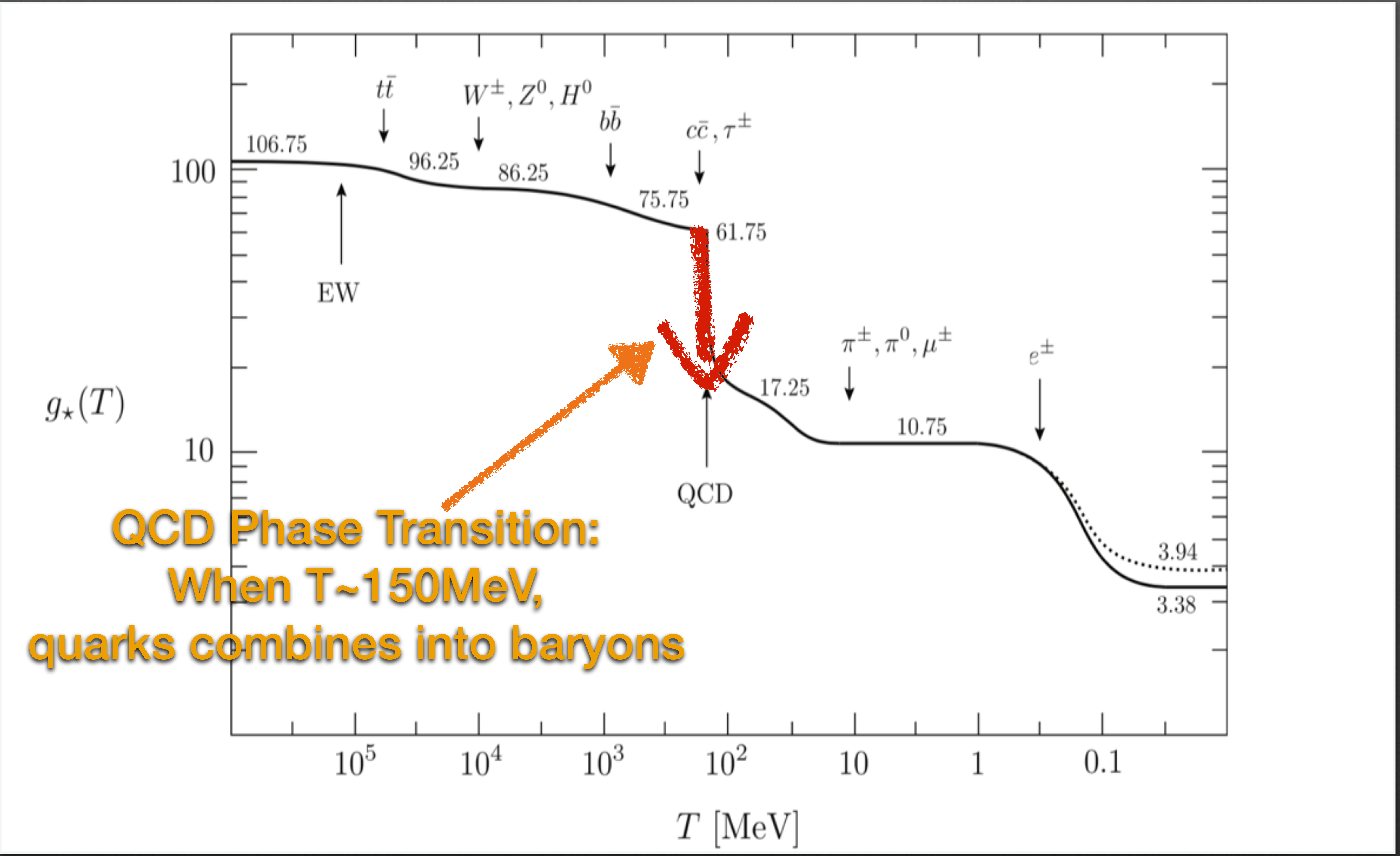

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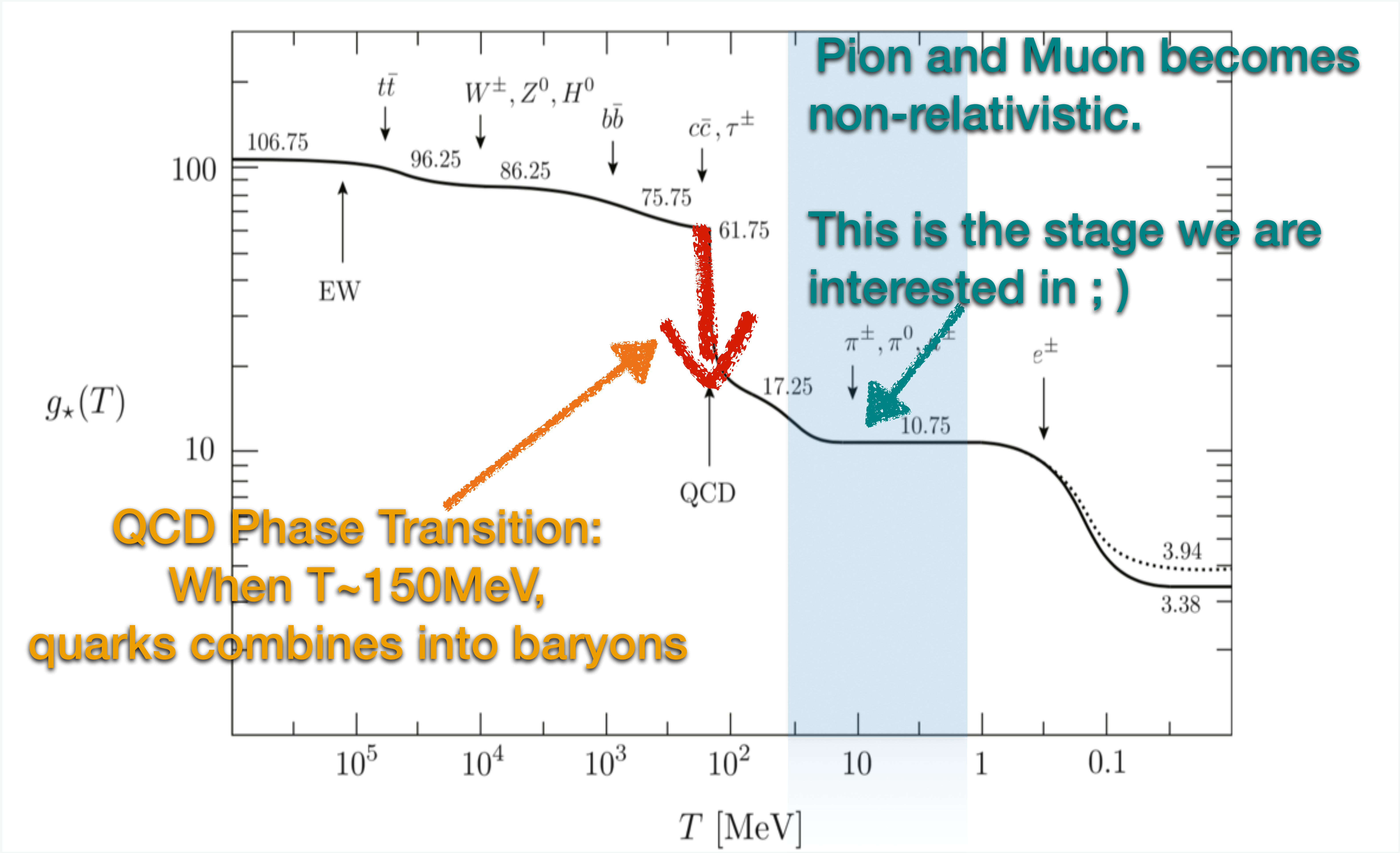

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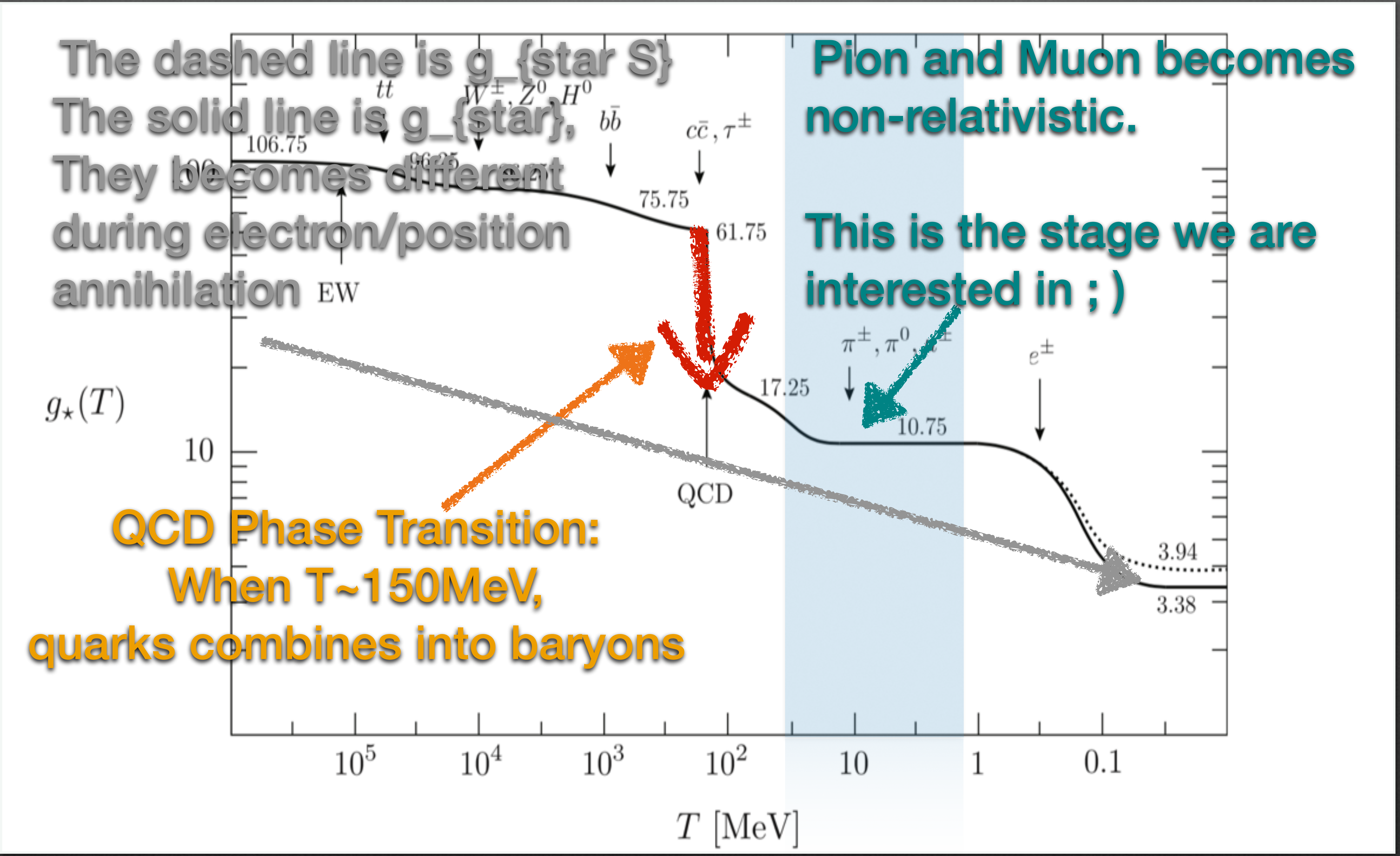

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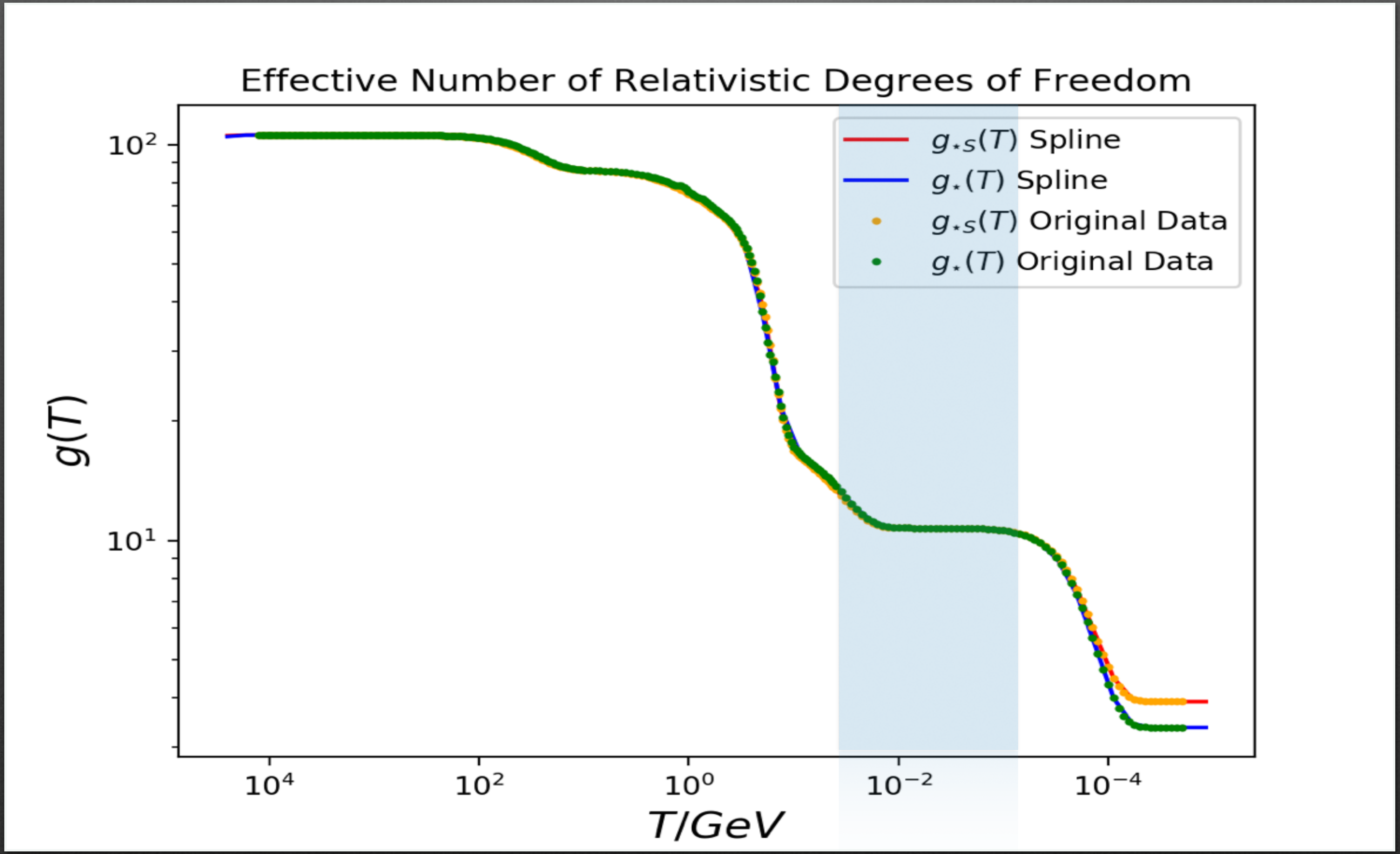

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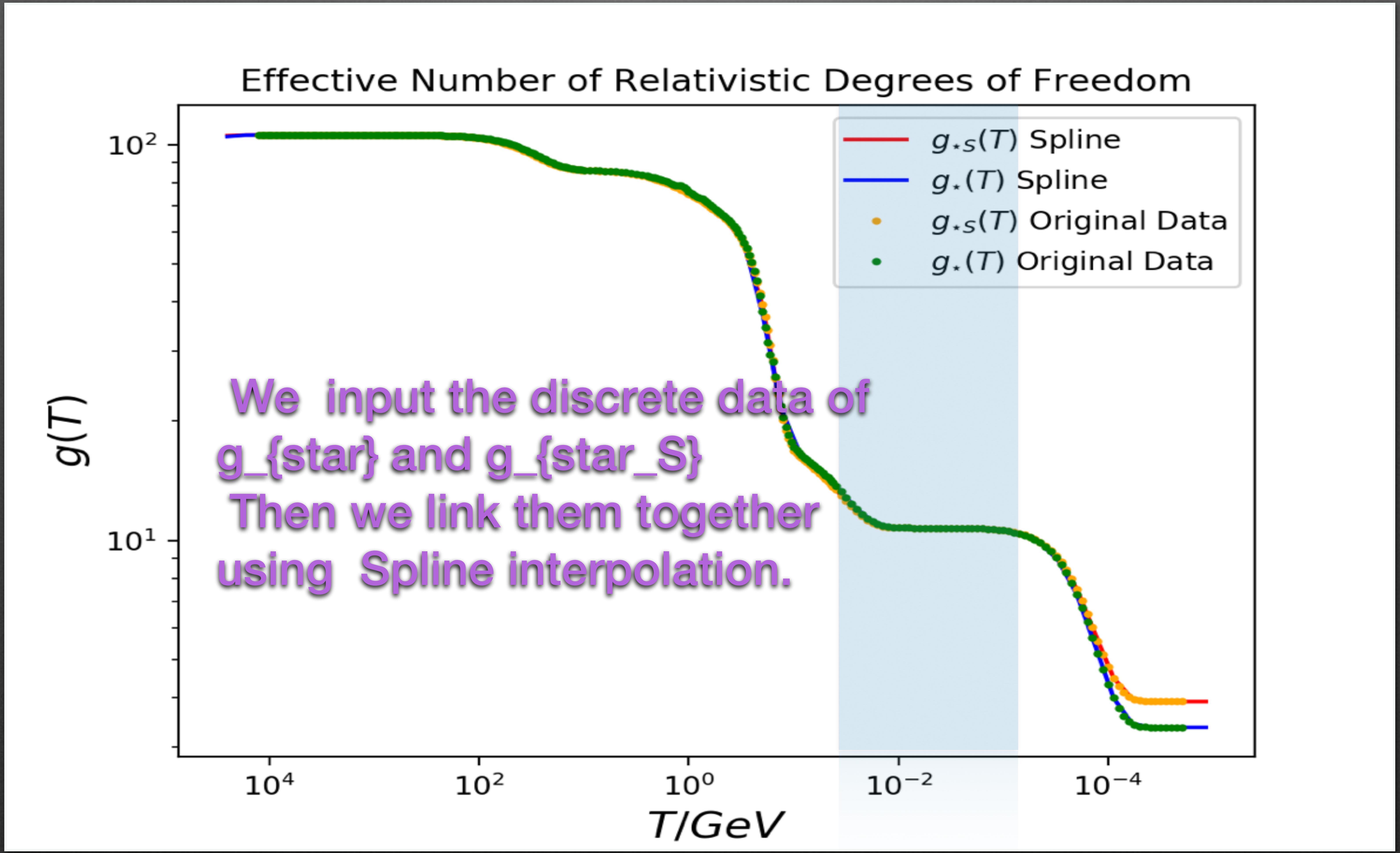

Some Tricky Things About Doing Integration using Python

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Only when this term=10^30,
my code(RK4) works :-(

Dark Matter Freeze Out

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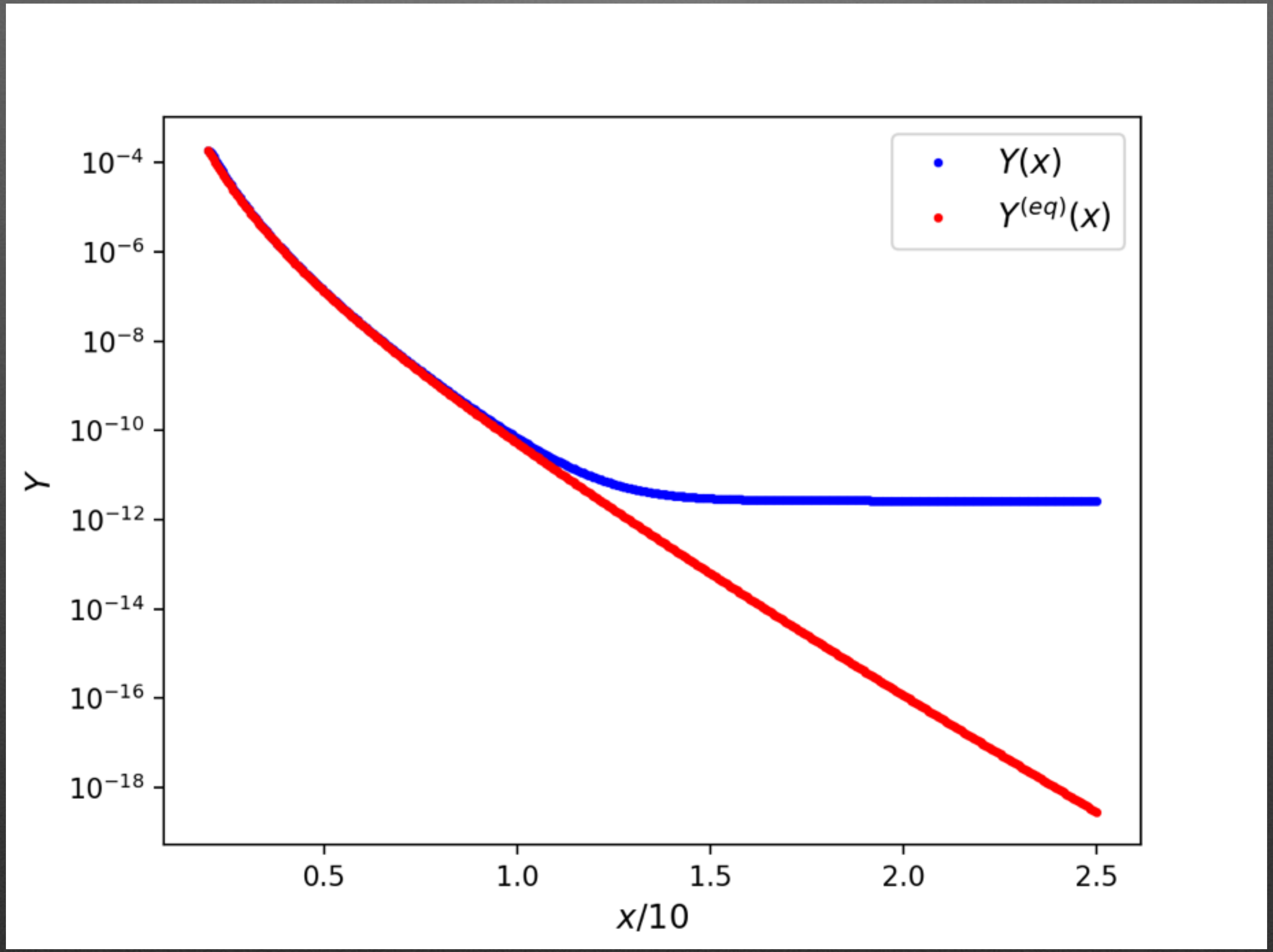

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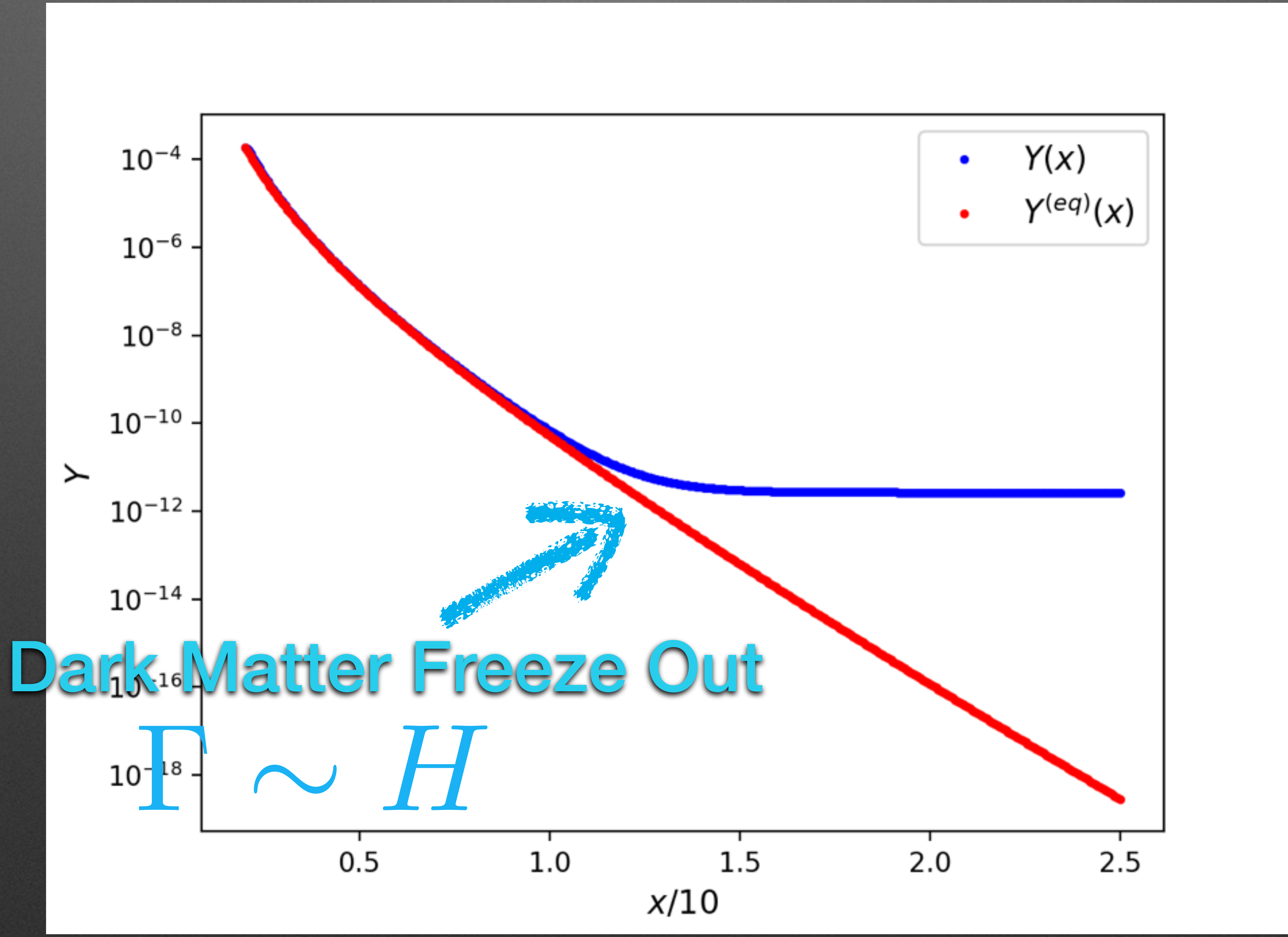
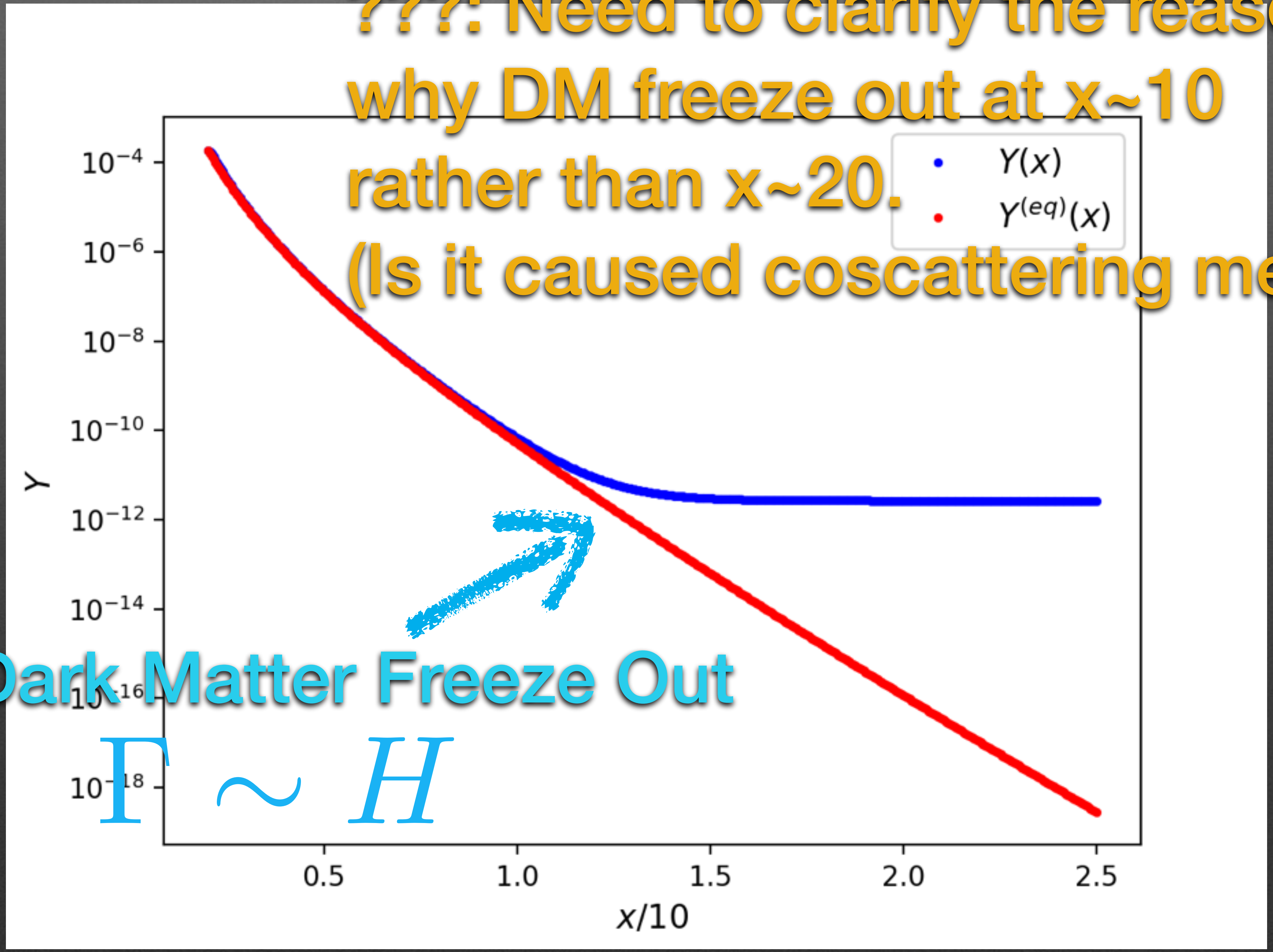


Dark Matter Freeze Out

???: Need to clarify the reason why DM freeze out at x~10 rather than x~20. (Is it caused coscattering mechanism?)



Dark Matter Freeze Out

$$\Gamma \sim H$$

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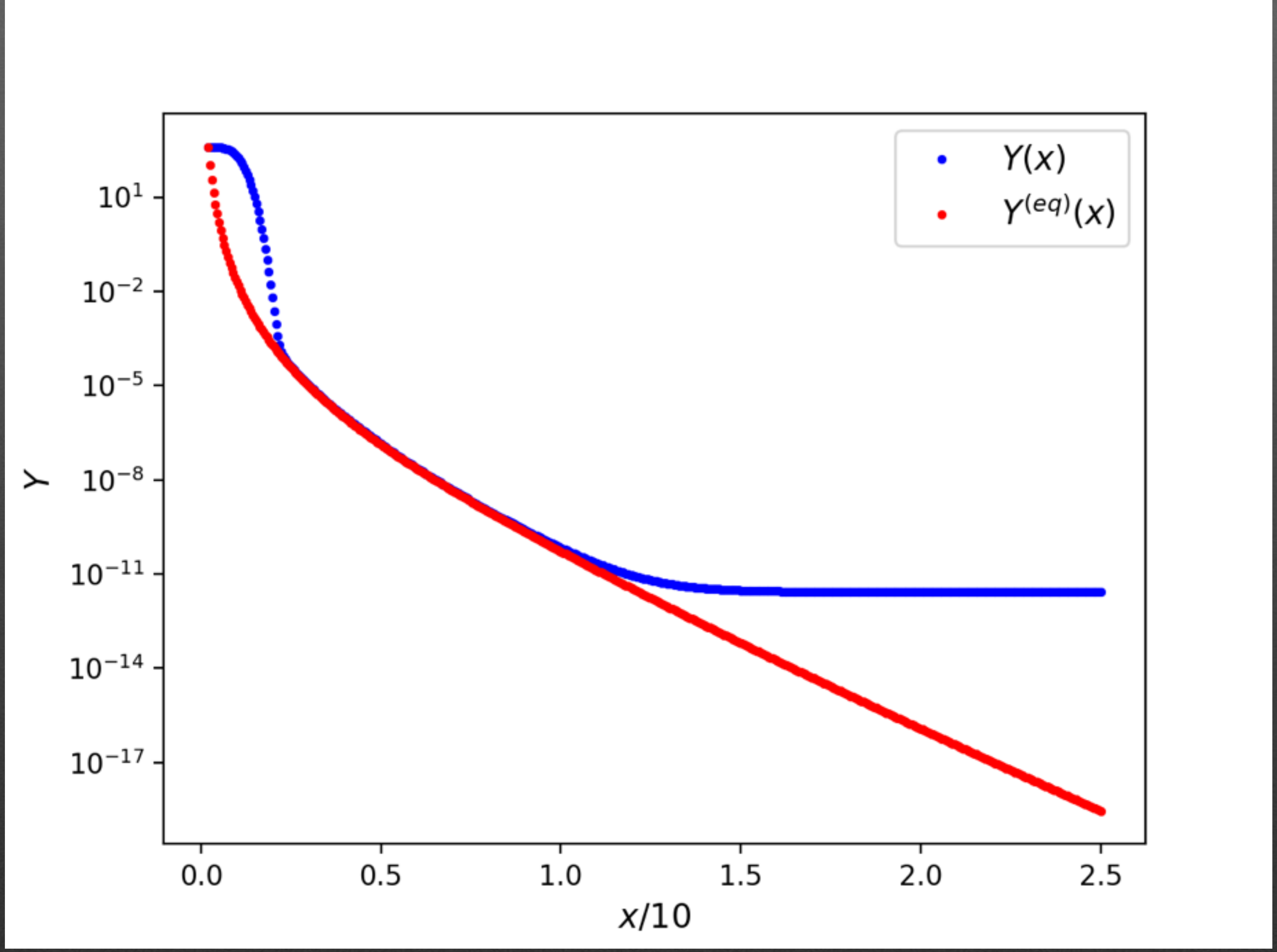

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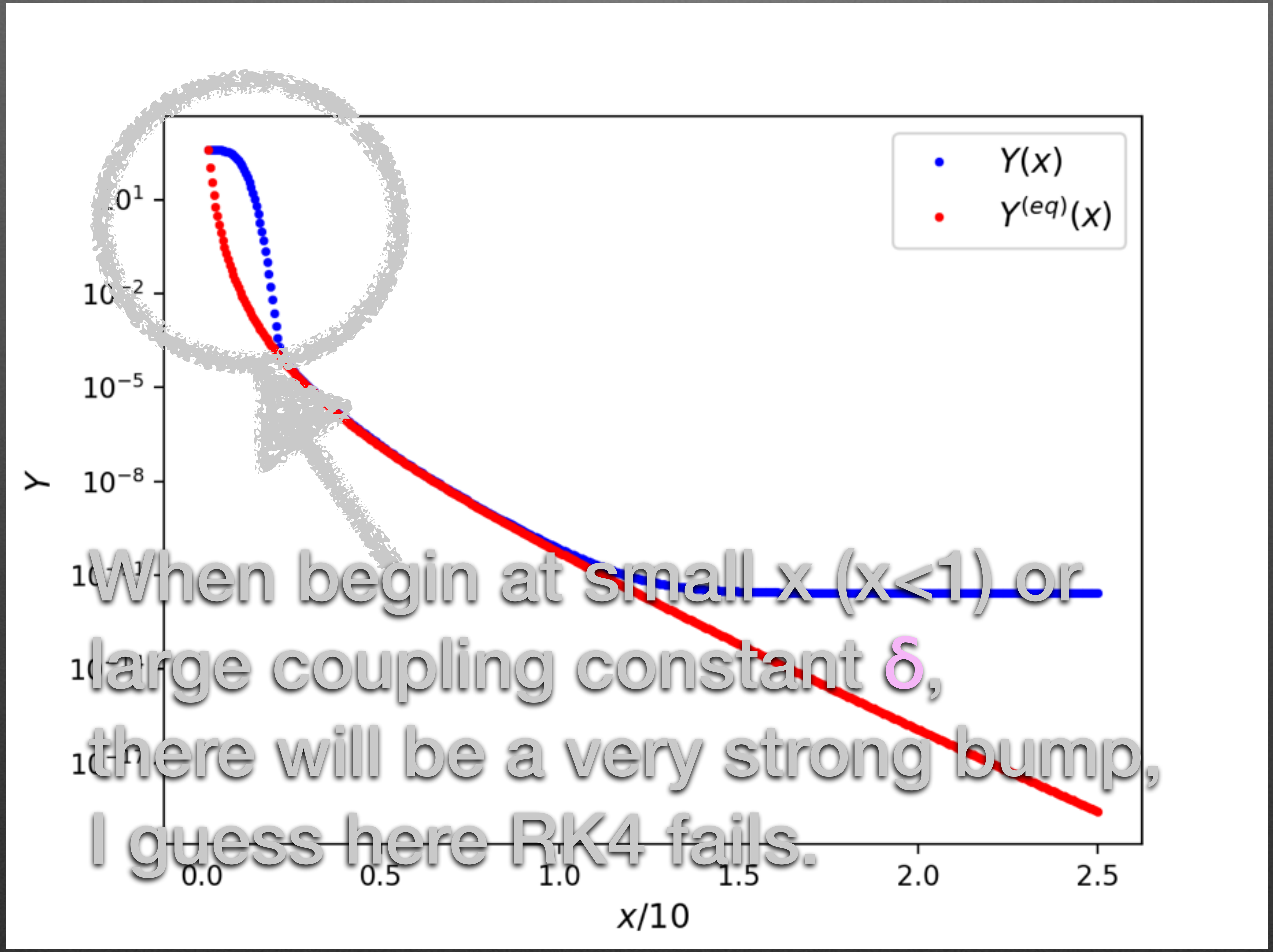

When begin at small x (x<1) or
large coupling constant δ,
there will be a very strong bump,
I guess here RK4 fails.

The Things I want to do later

- Make the calculation works for arbitrary value of δ.
- Reproduce the consequence of WIMP miracle as benchmark.
- Plot the deviation of $I_b^{(eq)}(x, R)$ using my python code and

  exact value (Using Mathematica).
- Plot the freeze-out number density when parameters r and Δ .
- Solve the non-integrable Boltzmann and compare with integrable one. (If TIME permitts)