Comp Physics Final Project

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Our goal is to solving the relativistic non-integrable Boltzmann equation in early universe(FLRW metric) numerically. Firstly, we shall try to solve Integrable Boltzmann equation and then in later research we can try more complicated non-integrable Boltzmann equation.

1 Formulas in cosmology

The Planck mass

$$M_{pl} = \sqrt{\frac{\hbar c}{8\pi G}} \tag{1}$$

Here(Also in the later calculation) we choose $\hbar = 1$, $c = 1$. So we have

$$8\pi G = \frac{1}{{M_{pl}}^2} \tag{2}$$

Entropy is conserved

$$Sa^3 = const \tag{3}$$

$$(or) \quad S(x) = \frac{S_0}{x^3} \tag{4}$$

$$(or) \quad \frac{dS}{dt} + 3HS = 0 \tag{5}$$

1st Friedmann equation is(From [1] [7] [9])

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a} \tag{6}$$

More abstract version of 1st Friedmann equation is

$$\left(\frac{1}{H_0}\frac{da/dt}{a}\right)^2 = \Omega_{r,0}\left(\frac{1}{a^4}\right) + \Omega_{m,0}\left(\frac{1}{a^3}\right) + \Omega_{k,0}\left(\frac{1}{a^2}\right) + \Omega_{\Lambda,0} \tag{7}$$

In radiation dominated universe ($x = \frac{m_\chi}{T}$)
We have the energy density

$$\rho = \sum_i \rho_i = \frac{\pi^2}{30} g_*(T) T^4 \tag{8}$$

$$H(x) = \frac{H(m_\chi)}{x^2} \tag{9}$$

where

$$H(m_\chi) = \sqrt{\frac{\pi^2}{90}\frac{g_*(T)}{{M_{pl}}^2}{m_\chi}^4} \tag{10}$$

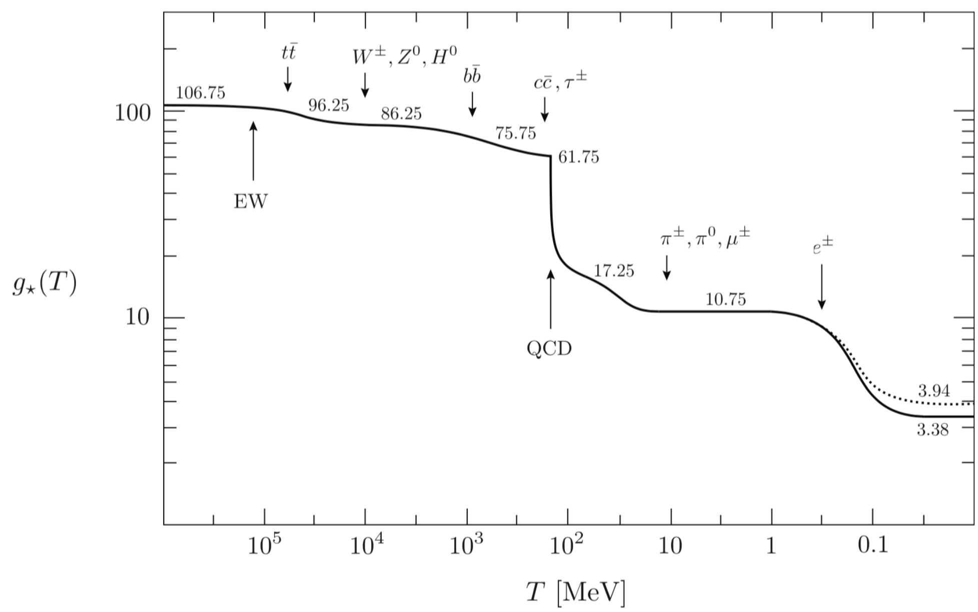

Figure 1: The change of effective massless degree of freedom $g_*(T)$ in terms of temperature T in radiation dominant universe(For standard model $SU(3) \otimes SU(2) \otimes U(1)$). Solid line is $g_*(T)$ and doted line is $g_{*S}(T)$. (From D.Baumann's cosmology note [7])

For the entropy S, we can also have [1] [7]

$$S(T) = \frac{2\pi^2}{45} g_{*,S}(T)T^3 \tag{11}$$

So we can have that

$$S(x) = \frac{2\pi^2}{45} g_{*S}(T){m_\chi}^3 \frac{1}{x^3} \tag{12}$$

Number density in arbitrary distribution

$$n_i(T) = n(T; m_i) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\exp\left(-\frac{\sqrt{k^2+{m_i}^2}-\mu_i}{T}\right) + \eta_i} \tag{13}$$

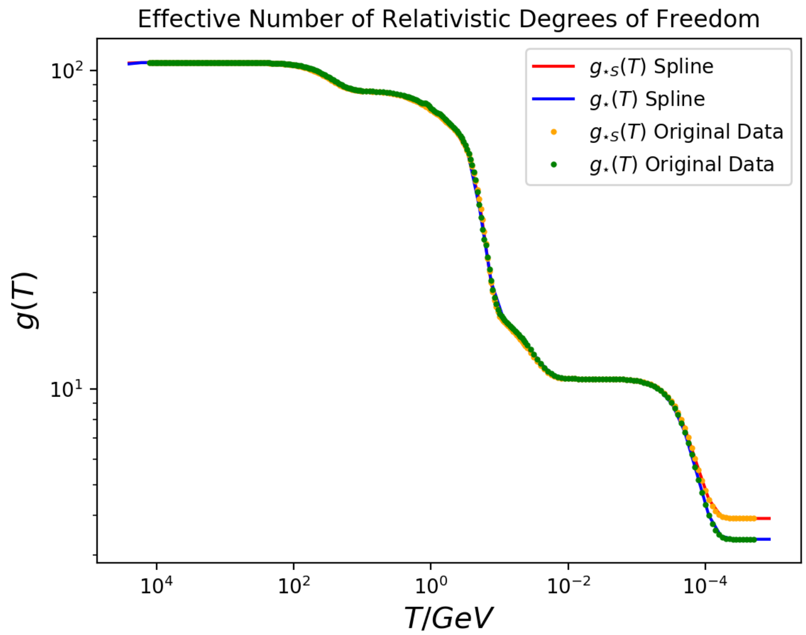

Figure 2: We input the numerical value of $g(T)$ in the scheme of Standard Model $SU(3) \otimes SU(2) \otimes U(1)$. We also use the Spline to do the interpolation.

(Boltzmann $\eta = 0$, Bose-Einstein $\eta = -1$, Fermi-Dirac $\eta = +1$).

In non-relativistic limit $\sqrt{k^2 + m_i{}^2} \approx m_i + \frac{k^2}{2m_i}$, and in low temperature limit we have $exp((E_i - \mu_i)/T) \gg 1$. n_i will follow Boltzmann distribution(Non-relativistic limit)

$$n_i(T) = g_i \Big(\frac{m_i T}{2\pi}\Big)^{3/2} exp\Big(\frac{-(m_i - \mu_i)}{T}\Big) \tag{14}$$

Integrable Boltzmann equation for $1 + 2 \leftrightarrow 3 + 4$

$$\frac{dn_1}{dt} + 3Hn_1 = \frac{dn_2}{dt} + 3Hn_2 = -\langle\sigma_{12\to34}v\rangle n_1 n_2 + \langle\sigma_{34\to12}v\rangle n_3 n_4 \tag{15}$$

$$\frac{dn_3}{dt} + 3Hn_3 = \frac{dn_4}{dt} + 3Hn_4 = -\langle\sigma_{34\to12}v\rangle n_3 n_4 + \langle\sigma_{12\to34}v\rangle n_1 n_2 \tag{16}$$

So we have (Don't need detailed balance)

$$\frac{dY_1}{dx} = \frac{dY_2}{dx} = -\frac{dY_3}{dx} = -\frac{dY_4}{dx} \tag{17}$$

Detailed balance for $1 + 2 \leftrightarrow 3 + 4$ for integrable Boltzmann equation

$$\langle\sigma_{12\to34}\rangle n_1^{(eq)} n_2^{(eq)} = \langle\sigma_{34\to12}\rangle n_3^{(eq)} n_4^{(eq)} \tag{18}$$

2 Boltzmann equation

2.1 Boltzmann Equation in FLRW (Isotropic&Homogeneous)

The general form of Boltzmann equation is

$$\frac{1}{E}L[f_\chi] = \frac{1}{E}C[f_\chi] \tag{19}$$

where the Liouville operator L in FLRW metric behaves like

$$\frac{1}{E}L = \frac{\partial}{\partial t} - \frac{\dot{a}(t)}{a(t)}\frac{|\vec{p}|^2}{E}\frac{\partial}{\partial E} \tag{20}$$

In relativistic situation, we have $E = \sqrt{p^2 + m_\chi^2}$. Using $EdE = pdp$ we have that

$$(\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p})f_\chi(p,t) = \frac{1}{E}C[f_\chi] \tag{21}$$

where $p_\chi^\mu = (E, \vec{p})$.

Here we use the definition $p := |\vec{p}|$ and we assume that the density in phase space doesn't depend on the direction of the momentum(Isotropic) $\vec{p}$ and the location $\vec{x}$ (Homogeneous) , which means $f(p^\mu, x^\mu) = f(p,t)$.

What we will calculate later is coscattering collision process

$$\chi(\vec{p}) + \phi(\vec{k}) \to \psi(\vec{p'}) + \phi(\vec{k'}) \tag{22}$$

in our calculation ϕ and ψ are in thermal equilibrium(We will use this to simplify the collision in later calculation), however χ is treated as out of equilibrium.

2.2 Liouville Term Simplification

To simplify the Liouville term, we do the change of variable using dimensionless quantity

$$x(t) = \frac{m_\chi}{T(t)} \qquad (Time) \tag{23}$$

$$q(t,p) = \frac{p}{T(t)} \qquad (Momentum) \tag{24}$$

We will also make use of the definition of Hubble parameter $H(t) = \frac{\dot{a}(t)}{a(t)}$ and the change of radiation temperature in terms of scale parameter, which is $T(t) \propto \frac{1}{a(t)}$. So we will have that

$$\frac{1}{E}L = \frac{\partial}{\partial t} - H\vec{p}\cdot\frac{\partial}{\partial \vec{p}} = H\big(t(x)\big)x\frac{\partial}{\partial x} \tag{25}$$

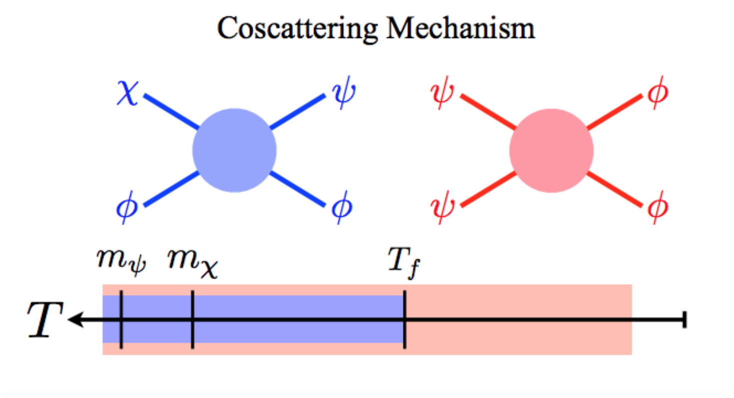

Figure 3: The mechanism of coscattering process and coannihilation process [2]. Here the blue Feynman diagram is related to the coscattering process $\chi + \phi \to \psi + \phi$. And the red Feynman diagram is related to the coannihilation process $\psi + \psi \to \phi + \phi$ (Since the process $\chi + \chi \to \phi + \phi$ and $\chi + \psi \to \phi + \phi$ is negligible, we don't plot their Feynmann diagram.). The mass of darkmatter particle χ is smaller than particle ψ(What is ψ, is it SM particle?).

2.3 Collision Term Simplification

The collision term is

$$C[f_\chi] = \frac{1}{2}\int d\Omega_{\vec{k}} d\Omega_{\vec{k}'} d\Omega_{\vec{p}'} (2\pi)^4 \delta^{(4)}(p + k - p' - k') |\overline{\mathcal{M}(\chi + \phi \leftrightarrow \psi + \phi)}|^2 \\ \times \Big[f_\phi(k', t) f_\psi(p', t) - f_\phi(k, t) f_\chi(p, t) \Big] \tag{26}$$

where $d\Omega_{\vec{p}} = \frac{d^3\vec{p}}{(2\pi)^3 2E(p)}$ is the Lorentz invariant phase space.

We can define the velocity-averaged cross section(Invariant under Lorentz Transformation), which is

$$\langle \sigma v \rangle(s) = \int d\Omega_{\vec{p}'} d\Omega_{\vec{k}'} (2\pi)^4 \delta^{(4)}(p + k - p' - k') \frac{1}{2E(k)} \frac{1}{2E(p)} |\mathcal{M}|^2 \tag{27}$$

Because as we have mentioned before, ϕ and ψ are in thermal equilibrium($T << E_X$), we have that

$$f_X^{(eq)}(p, t) = \frac{1}{exp(\frac{E_{X,\vec{p}}}{T}) + 1} \approx exp(-\frac{\sqrt{|\vec{p}|^2 + m_X^2}}{T}) = exp(-\sqrt{(m_X/m_\chi)^2 q^2 + x^2}) \tag{28}$$

where $X = \phi, \psi, \chi$. Here χ is not in thermal equilibrium but we need $f_\chi^{(eq)}$ when using formula $f_\phi^{(eq)}(k', t) f_\psi^{(eq)}(p', t) = f_\phi^{(eq)}(k, t) f_\chi^{(eq)}(p, t)$ later.

So we have that

$$\frac{1}{E(p)}C[f_\chi] = \int \frac{d^3\vec{k}}{(2\pi)^3}\langle\sigma v\rangle(s)\Big[f_\phi(k',t)f_\psi(p',t) - f_\phi(k,t)f_\chi(p,t)\Big] \tag{29}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3}\langle\sigma v\rangle(s)\Big[f_\phi^{(eq)}(k',t)f_\psi^{(eq)}(p',t) - f_\phi^{(eq)}(k,t)f_\chi(p,t)\Big] \tag{30}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3}\langle\sigma v\rangle(s)\Big[f_\phi^{(eq)}(k,t)f_\chi^{(eq)}(p,t) - f_\phi^{(eq)}(k,t)f_\chi(p,t)\Big] \tag{31}$$

$$= \Big[f_\chi^{(eq)}(p,t) - f_\chi(p,t)\Big]\int \frac{d^3\vec{k}}{(2\pi)^3}\langle\sigma v\rangle(s)f_\phi^{(eq)}(k,t) \tag{32}$$

Here s is one of the Mandelstam variable, which is $s = (k+p)^2$.
Here we define that

$$C(p,t) = \int \frac{d^3\vec{k}}{(2\pi)^3}\langle\sigma v\rangle(s)f_\phi^{(eq)}(k,t) \tag{33}$$

So we have that

$$C[f_\chi] = \Big[f_\chi^{(eq)}(p,t) - f_\chi(p,t)\Big]C(p,t) \tag{34}$$

After combing the Liouville term and collision term together, we have that

$$H(x)x\frac{\partial f_\chi(x,q)}{\partial x} = \Big[exp(-\sqrt{x^2+q^2}) - f_\chi(x,q)\Big]C(x,q) \tag{35}$$

Here $C(x,q)$ is a 3D integration so I plan to use Monte-Carlo method to do the integration.
After getting $f_\chi(x,q)$ numerically, we can calculate the related number density n_χ, which is

$$n_\chi\big(T(t)\big) = \int \frac{d^3\vec{p}}{(2\pi)^3}f_\chi(p,T) = T^3\int \frac{d^3\vec{q}}{(2\pi)^3}f_\chi(q,x) = \Big(\frac{m_\chi}{x}\Big)^3\int \frac{d^3\vec{q}}{(2\pi)^3}f_\chi(q,x) \tag{36}$$

2.4 $\langle\sigma v\rangle$ calculation

Let's follow the calculation of [4] and [5] and extend the consequence to more general case $m_1 \neq m_2$.

In cosmic comoving frame, where gas is assumed to be at rest as a whole part, we have

$$\langle\sigma v\rangle = \frac{\int \sigma v_{Mol}e^{-E_1/T}e^{-E_2/T}d^3\vec{p_1}d^3\vec{p_2}}{\int e^{-E_1/T}e^{-E_2/T}d^3\vec{p_1}d^3\vec{p_2}} \tag{37}$$

Here $v_{Mol} = \sqrt{|\vec{v_1}-\vec{v_2}|^2 + |\vec{v_1}\times\vec{v_2}|^2}$. Even though v_{Mol} itself is not invariant under Lorentz transformation, we can prove that $v_{Mol}E_1E_2$ is invariant under Lorentz transformation. Let's prove it:

$$
\begin{aligned}
F(\tilde{s}) &= \sqrt{(p_1^{\mu} p_{\mu 2})^2 - {m_1}^2 {m_2}^2} \\
&= \frac{\sqrt{\tilde{s}^2 - 2({m_1}^2 + {m_2}^2)\tilde{s} + (m_1^2 - m_2^2)^2}}{2}
\end{aligned} \tag{38}
$$

Finally, we can show that

$$
v_{Mol} = \frac{F(\tilde{s})}{E_1 E_2} \tag{39}
$$

The distribution function is $f(E) = e^{-E/T}$, which is Maxwell-Boltzmann distribution.

Let's simplify $d^3\vec{p_1} d^3\vec{p_2}$.

If function $f = f(\theta_{12}, |\vec{v_1}|, |\vec{v_2}|)$ only depends on the angle θ_{12} between $\vec{p_1}$ and $\vec{p_2}$, $|\vec{p_1}|$ and $|\vec{p_2}|$, we will have that

$$
\begin{aligned}
&\int d^3\vec{p_1} d^3\vec{p_2} \quad y(\theta_{12}, p_1, p_2) \\
&= \int {p_1}^2 dp_1 dcos\theta_1 d\phi_1 \int {p_2}^2 dp_2 dcos\theta_{12} d\phi_{12} \quad y(\theta_{12}, p_1, p_2) \\
&= \int (4\pi)^2 p_1 E_1 dE_1 p_2 E_2 dE_2 \frac{1}{2} dcos\theta_{12} \quad y(\theta_{12}, p_1, p_2)
\end{aligned} \tag{40}
$$

We do the replacement of variable, which is

$$
E_{\pm} = E_1 \pm E_2 \tag{41}
$$

$$
\tilde{s} = (p_1^{\mu} + p_2^{\mu})^2 = ({m_1}^2 + {m_2}^2) + 2E_1E_2 - 2p_1p_2cos\theta_{12} \tag{42}
$$

Here we can know the range of e

So we have that

$$
\begin{aligned}
&\int 2\pi^2 E_1 E_2 dE_+ dE_- d\tilde{s} \quad y(\theta_{12}, p_1, p_2) \\
&= \int 2\pi^2 E_1 E_2 \frac{\partial(E_+, E_-, \tilde{s})}{\partial(E_1, E_2, cos\theta_{12})} dE_1 dE_2 dcos\theta_{12} \quad y(\theta_{12}, p_1, p_2) \\
&= \int (4\pi)^2 p_1 E_1 dE_1 p_2 E_2 dE_2 \frac{1}{2} dcos\theta_{12} \quad y(\theta_{12}, p_1, p_2)
\end{aligned} \tag{43}
$$

So we have that for $y(\theta_{12}, p_1, p_2)$,

$$
d^3\vec{p_1} d^3\vec{p_2} = 2\pi^2 E_1 E_2 dE_+ dE_- d\tilde{s} \tag{44}
$$

When $\frac{p_1}{p_2} = \frac{m_1}{m_2}$ and $cos\theta_{12} = 1$, we can get its minimum of $\tilde{s}$

$$
\tilde{s}_{min} = (m_1 + m_2)^2 \tag{45}
$$

We can have the range

$$|E_-| \le \sqrt{1-\frac{2({m_1}^2+{m_2}^2)}{\tilde{s}}}\sqrt{{E_+}^2-\tilde{s}} \tag{46}$$
$$E_+ \ge \sqrt{\tilde{s}} \tag{47}$$
$$\tilde{s} \ge 2({m_1}^2+{m_2}^2) \tag{48}$$

The integration representation of Modified Bessel Function we will use later is(See [8])

$$K_n(x) = \frac{\sqrt{\pi}(x/2)^n}{\Gamma(n+\frac{1}{2})}\int_0^{+\infty} dt \quad \sinh t^{2n} \exp(-\cosh tx) \tag{49}$$
$$K_n(x) = \int_0^{+\infty} dt \quad \cosh nt \exp(-\cosh tx) \tag{50}$$

So we can have the integration

$$\begin{aligned}
&\int \sigma v_{Mol} e^{-E_1/T} e^{-E_2/T} d^3\vec{p}_1 d^3\vec{p}_2 \\
&= 2\pi^2 \int_{\tilde{s}=2({m_1}^2+{m_2}^2)}^{+\infty} d\tilde{s} \quad \sigma(\tilde{s})F(\tilde{s}) \int_{E_+=\sqrt{\tilde{s}}}^{+\infty} e^{-E_+/T} dE_+ \int_{E_-=-\sqrt{1-\frac{2({m_1}^2+{m_2}^2)}{\tilde{s}}}\sqrt{{E_+}^2-\tilde{s}}}^{E_-=+\sqrt{1-\frac{2({m_1}^2+{m_2}^2)}{\tilde{s}}}\sqrt{{E_+}^2-\tilde{s}}} dE_- \\
&= 4\pi^2 \int_{\tilde{s}=2({m_1}^2+{m_2}^2)}^{+\infty} d\tilde{s} \quad \sigma(\tilde{s})F(\tilde{s})\sqrt{1-\frac{2({m_1}^2+{m_2}^2)}{\tilde{s}}} \int_{E_+=\sqrt{s}}^{+\infty} dE_+ e^{-E_+/T}\sqrt{{E_+}^2-\tilde{s}} \\
&= 4\pi^2 T \int_{\tilde{s}=2({m_1}^2+{m_2}^2)}^{+\infty} d\tilde{s} \quad \sigma(\tilde{s})F(\tilde{s})\sqrt{\tilde{s}-2({m_1}^2+{m_2}^2)} K_1(\frac{\sqrt{\tilde{s}}}{T})
\end{aligned} \tag{51}$$

and

$$\int e^{-E_{1,2}/T} d^3\vec{p}_{1,2} \tag{52}$$
$$= 4\pi \int_{E_{1,2}=m_{1,2}}^{+\infty} \exp(-E_{1,2}/T)\sqrt{{E_{1,2}}^2-{m_{1,2}}^2}E_{1,2}dE_{1,2} \tag{53}$$
$$= 4\pi(m_{1,2})^2 T K_0(\frac{m_{1,2}}{T}) \tag{54}$$

Finally we have

$$\langle \sigma v_{Mol} \rangle = \frac{\int_{\tilde{s}=2({m_1}^2+{m_2}^2)}^{\tilde{s}=+\infty} d\tilde{s} \quad \sigma(\tilde{s})F(\tilde{s})\sqrt{\tilde{s}-2({m_1}^2+{m_2}^2)}K_1(\frac{\sqrt{\tilde{s}}}{T})}{4T\Big({m_1}^2K_0(\frac{m_1}{T})\Big)\Big({m_2}^2K_0(\frac{m_2}{T})\Big)} \tag{55}$$

2.5 Compare with integrable Boltzmann equation

After the calculation using non-integrable Boltzmann equation, we can calculate it using integrable integrable version(To get integrable version, we need to do some approximation on its collision term $\int \frac{d^3\vec{p}}{(2\pi)^3}C[f_\chi]$.

Since we have the detailed balance

$$\langle \sigma_{\chi\to\psi} v\rangle n_\chi^{(eq)} = \langle \sigma_{\psi\to\chi}\rangle n_\psi^{(eq)} \tag{56}$$

$$\langle \sigma_{\psi\psi\to\phi\phi}\rangle n_\psi^{(eq)} n_\psi^{(eq)} = \langle \sigma_{\phi\phi\to\psi\psi}\rangle n_\phi^{(eq)} n_\phi^{(eq)} \tag{57}$$

So we can have the equation(Assuming that $n_\psi = n_\psi^{(eq)}$ and $n_\phi = n_\phi^{(eq)}$)

$$\frac{dn_i}{dt} + 3Hn_i = -\sum_j \left[n_\phi^{(eq)} \langle \sigma_{i\to j} v\rangle \big(n_i - n_i^{(eq)} \frac{n_j}{n_j^{(eq)}} \big) + \langle \sigma_{ij} v\rangle \big(n_i n_j - n_i^{(eq)} n_j^{(eq)} \big) \right] \tag{58}$$

The first term is related to the coannihilation process $\chi + \phi \to \psi + \phi$ and its inverse process $\psi + \phi \to \chi + \phi$. The second term is related to the coscattering process $\psi + \psi \to \phi + \phi$, $\chi + \psi \to \phi + \phi$(Negligible) and $\chi + \psi \to \phi + \phi$(Negligible). The index i, j can be chosen as $i, j = \chi, \psi$.

Assuming that we can have that

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -n_\phi^{(eq)} \langle \sigma_{\chi\to\psi} v\rangle \Big(n_\chi - n_\chi^{(eq)} \Big) \tag{59}$$

$$\frac{dn_\psi}{dt} + 3Hn_\psi = -n_\phi^{(eq)} n_\psi^{(eq)} \langle \sigma_{\psi\to\chi} v\rangle \Big(1 - \frac{n_\chi}{n_\chi^{(eq)}} \Big) \tag{60}$$

We define that

$$Y_i = \frac{n_i}{S} \tag{61}$$

Since we have that

$$\frac{d}{dt} = H(x) x \frac{d}{dx} \tag{62}$$

$$\frac{dS}{dt} + 3H(x)S(x) = 0 \tag{63}$$

So we have that

$$\frac{dn_i}{dt} + 3H(x)n_i = SH(x)x\frac{d}{dx} \tag{64}$$

So in radiation dominated universe

$$\frac{dY_\chi}{dx} = -\frac{S(x)}{H(x)x} \langle \sigma_{\chi\to\psi} v\rangle Y_\phi^{(eq)} Y_\chi^{(eq)} \left(\frac{Y_\chi}{Y_\chi^{(eq)}} - 1 \right) \tag{65}$$

Here let's derive $H(m_\chi)$. We already have the Friedmann equation (6) which is $H^2 = \frac{1}{3{M_{pl}}^2}\rho$ (Here $k \approx 0$). For the radiation dominant universe, we have

$$H(x) = \frac{H(m_\chi)}{x^2} \tag{66}$$

$$S(x) = \frac{S(m_\chi)}{x^3} \tag{67}$$

where

$$H(m_\chi) = \sqrt{\frac{\pi^2}{90}\frac{g_*(T)}{{M_{pl}}^2}{m_\chi}^4} \tag{68}$$

$$S(m_\chi) = \frac{2\pi^2}{45}g_{*S}(T){m_\chi}^3 \tag{69}$$

Don't forget that $x = \frac{m_\chi}{T}$.

This equation can be solved using adaptive RK4(Not stable), Crank-Nicolson(More stable, but still not enough), Backward Euler(Very stable). I will mainly use Backward Euler in later calculation.

We will compare (35) (36) and (60) to see the difference between these two approaches. My another goal is to reproduce the consequences in Joshua's paper [2].

3 Numerical Calculation

3.1 Equation to solve

From [2], we know the example dark sector(Toy model) is

$$\mathcal{L} \supset -\frac{m_\chi}{2}\chi^2 - \frac{m_\psi}{2}\psi^2 - \delta m\chi\psi - \frac{y}{2}\phi\psi^2 + h.c. \tag{70}$$

$$\langle\sigma_{\chi\to\psi}v\rangle(x) = \left(\frac{m_\psi}{m_\chi}\right)^{\frac{3}{2}} e^{-\Delta x}\langle\sigma_{\psi\to\chi}v\rangle \tag{71}$$

$$\approx \left(\frac{m_\psi}{m_\chi}\right)^{\frac{3}{2}} e^{-\Delta x} f(r)\sqrt{\Delta}\frac{y^4\delta^2}{2\pi m_\chi^2} \tag{72}$$

where

$$f(r) = \frac{(r^2+r+2)^2}{\sqrt{2}(r-2)^2 r^{9/2}(r+1)^{7/2}} \tag{73}$$

Also here we use the dimensionless quantity to represent the m_ϕ and m_ψ.

$$r = \frac{m_\phi}{m_\chi} \tag{74}$$

$$\Delta = \frac{m_\psi - m_\chi}{m_\chi} \tag{75}$$

After some algebra, we have the ODE

$$\begin{aligned} \frac{dY_\chi}{dx} = &-\frac{45^{3/2}}{2^{7/2}\pi^8}\frac{1}{g_{\star S}(m_\chi/x)\sqrt{g_\star(m_\chi/x)}}\left(f(r)(1+\Delta)^{3/2}\Delta^{1/2}y^4\delta^2\right)\left(\frac{M_{pl}}{m_\chi}\right)\left(\frac{e^{-\Delta x}}{x^2}\right) \\ &\times I_b^{(eq)}(x, R=r)I_b^{(eq)}(x, R=1)\left(\frac{Y_\chi}{Y_\chi^{(eq)}} - 1\right) \end{aligned} \tag{76}$$

Here

$$I_b^{(eq)}(x,R) = \int_0^{+\infty} d\xi \frac{\xi^2}{\exp\left(\sqrt{\xi^2+(Rx)^2}\right)-1} \tag{77}$$

and

$$Y_i^{(eq)}(x) = \frac{45}{4\pi^4 \cdot g_{\star S}(m_\chi/x)} I_b^{(eq)}(x, R=\frac{m_i}{m_\chi}) \tag{78}$$

3.2 Some tricky treatment for doing integration

There is some trick thing on the calculation of $I_b^{(eq)}(x,R)$ when using Python. It will be very easy to meet the situation when the calculation overflows. When $x \leq 20$, we can set the do such operation:

$$I_b^{(eq)}(x,R) = \int_0^{+\Lambda} d\xi \frac{\xi^2}{\exp\left(\sqrt{\xi^2+(Rx)^2}\right)-1} \tag{79}$$

Here we choose $\Lambda = 700$(If $\Lambda > 700$ the numerical consequence of $I_b^{(eq)}(x,R)$ will diverge.). When $x > 20$, since $\frac{1}{e^{20}} \sim 10^{-9}$, so we can do the approximation

$$I_b^{(eq)})(x,R) \approx \int_0^{+\infty} d\xi \frac{\xi^2}{\exp\left(\sqrt{\xi^2+(Rx)^2}\right)} = (Rx)^2 K_2(Rx) \tag{80}$$

So we technically we make the $I_b^{(eq)}(x,R)$ to be

$$I_b^{(eq)}(x,R) = \begin{cases} \int_0^{+\Lambda} d\xi \frac{\xi^2}{\exp\left(\sqrt{\xi^2+(Rx)^2}\right)-1} & 0 \leq Rx < 20 \\ (Rx)^2 K_2(Rx) & Rx \geq 20 \end{cases} \tag{81}$$

Let's compare different method for the calculation of $I_b^{(eq)}(x,R)$, the first method is (81). The second one is using python recommend "integrate.quad" to calculate $I_b^{(eq),2}(x,R) = \int_0^\infty d\xi \frac{\xi^2}{\exp\left(\sqrt{\xi^2+(Rx)^2}\right)-1}$ by brutal force. We compare the numerical consequence of these two method and compare it with the consequence of Mathematica(15 digits at least)(See Figure 4). We can find blue line(81) behaves much better than green line(brutal force), so we will use the formula (81) to do later calculation.

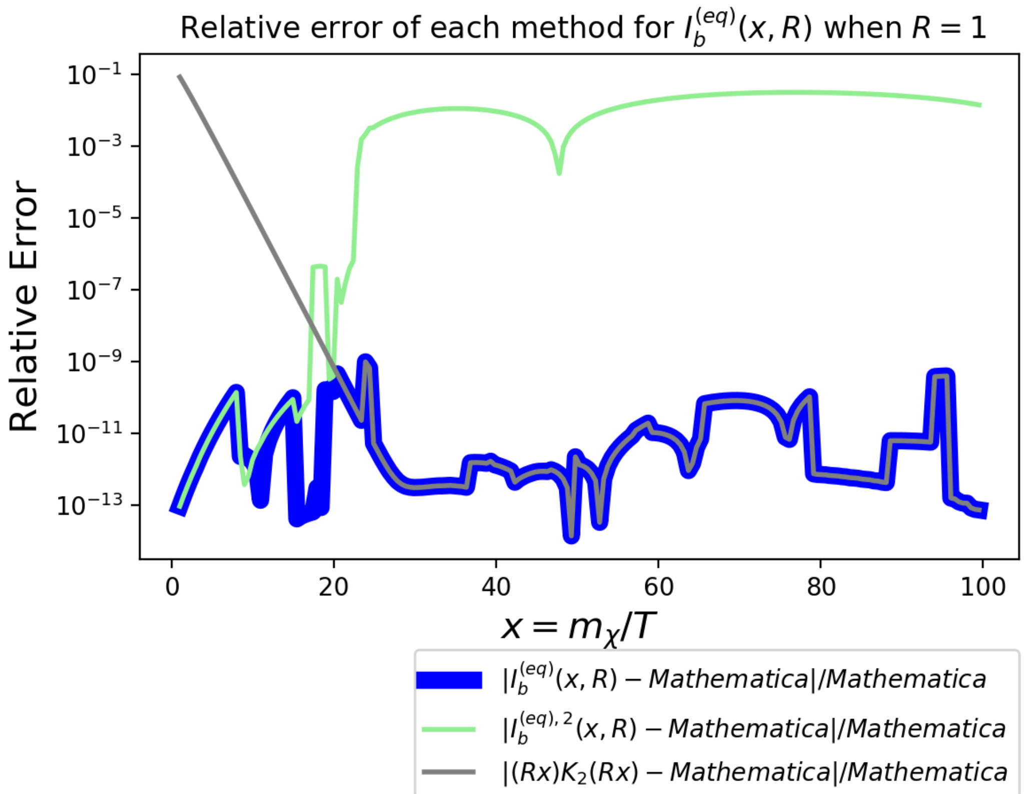

Figure 4: Here the green line is related to the brutal force calculation of $\int_0^\infty d\xi \quad \xi^2/\left[\exp\left(\sqrt{\xi^2+(Rx)^2}\right)-1\right]$ using python. The blue line is our choosed method which calculate $\int_0^\Lambda d\xi \quad \xi^2/\left[\exp\left(\sqrt{\xi^2+(Rx)^2}\right)-1\right]$ when $Rx < 20$(Here we choose $\Lambda = 700$). When $Rx \geq 20$, the whole integration tends to become much more like Boltzmann integration, so we omit -1 and get analytical approximation $RxK_2(Rx)$. We comepare these two consequence with the consequence from Mathematica(as bench mark). We can find blue line behaves much better and $Relative \quad error \sim 10^{-9}$, however the green line behavies badly, when $Rx \geq 20$, $Relative \quad error \sim 10^{-1}$. However the region $Rx \geq 20$ still should be considered so it is not wise to use brutal force calculation(green line).

3.3 Change parameter $r = m_\phi/m_\chi$

Here we varies the

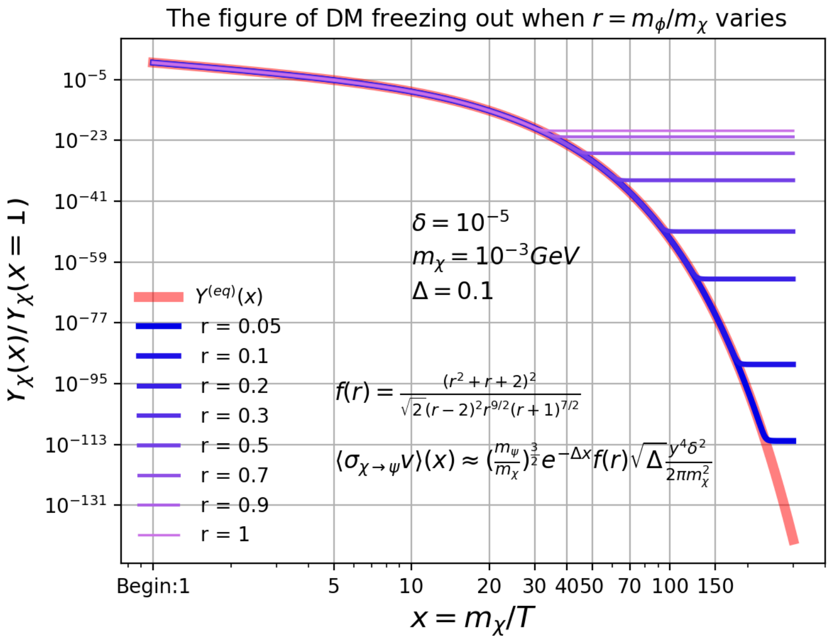

Figure 5: sa

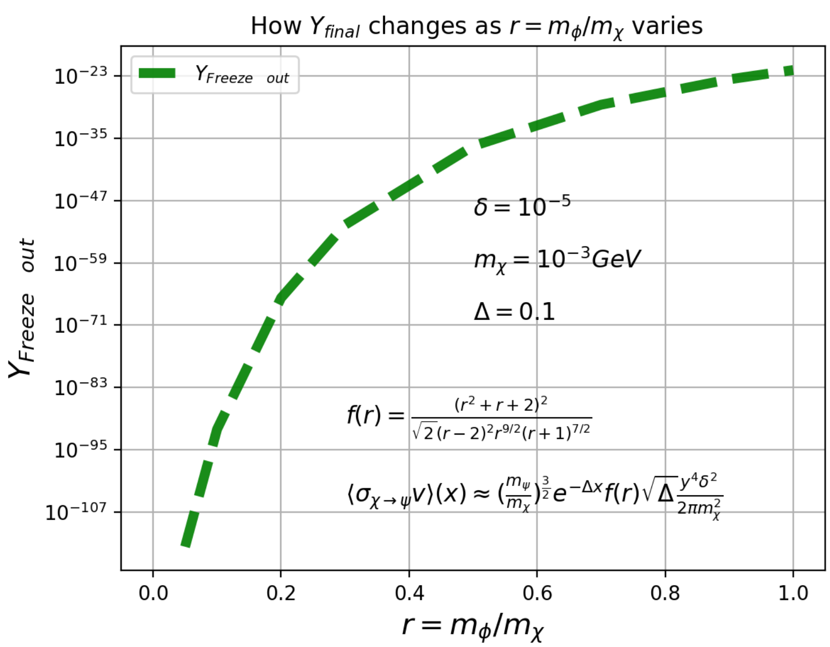

Figure 6: sa

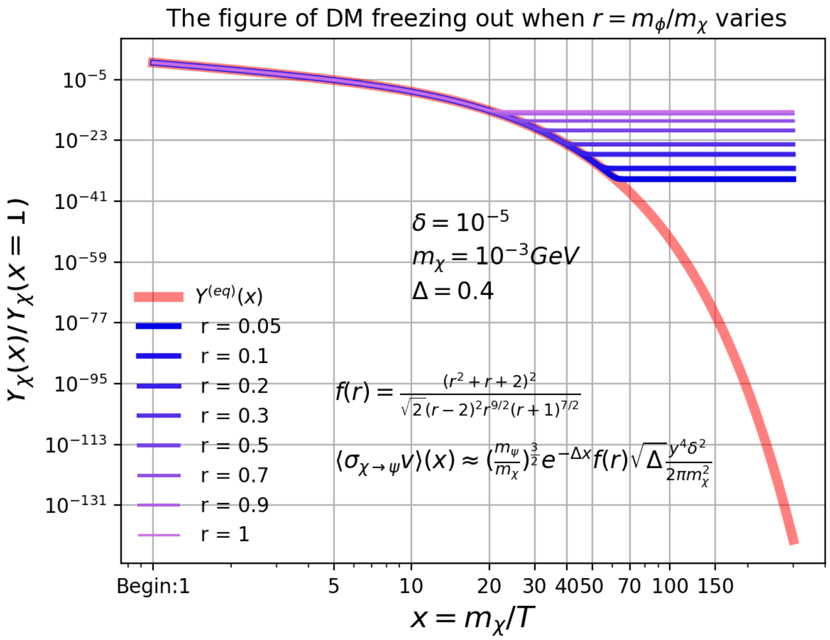

Figure 7: sa

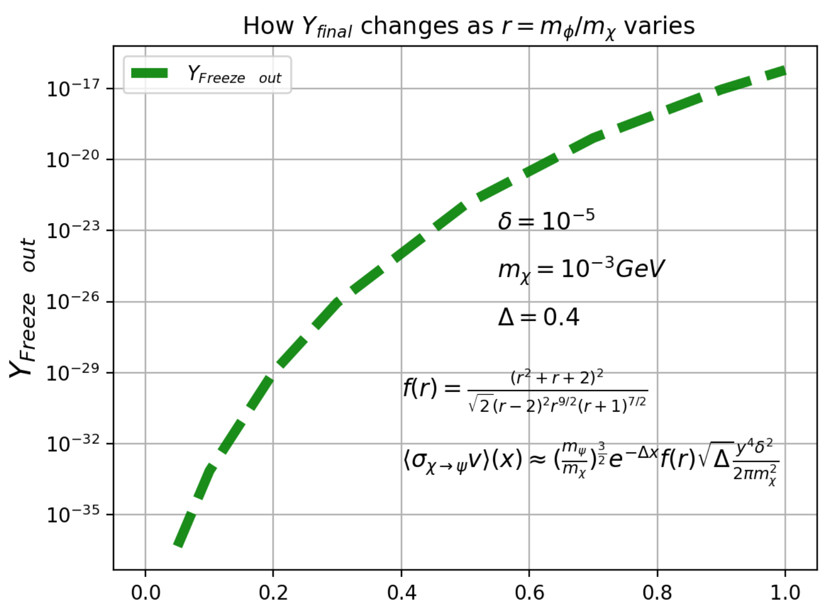

Figure 8: sa

3.4 Change parameter $\Delta = (m_\psi - m_\chi)/m_\chi$

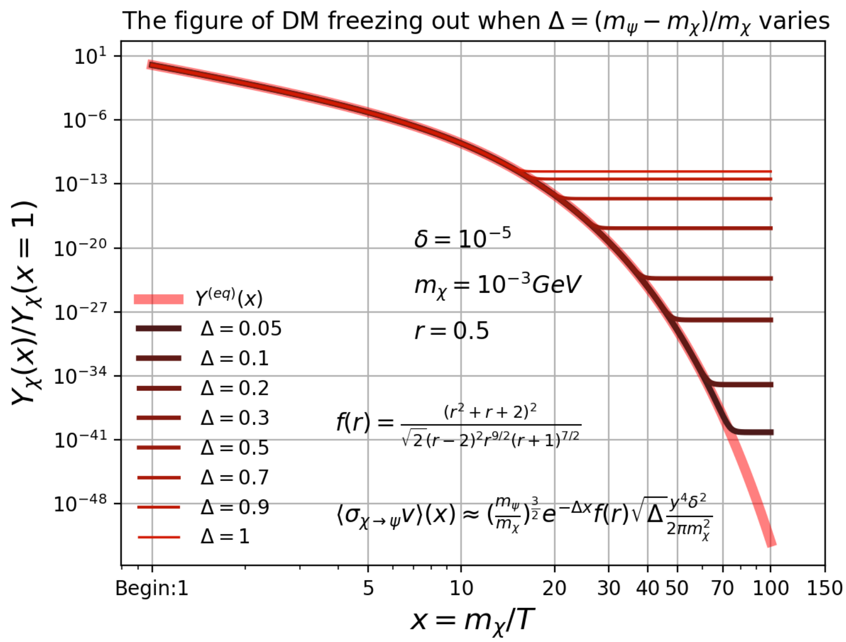

Figure 9: sa

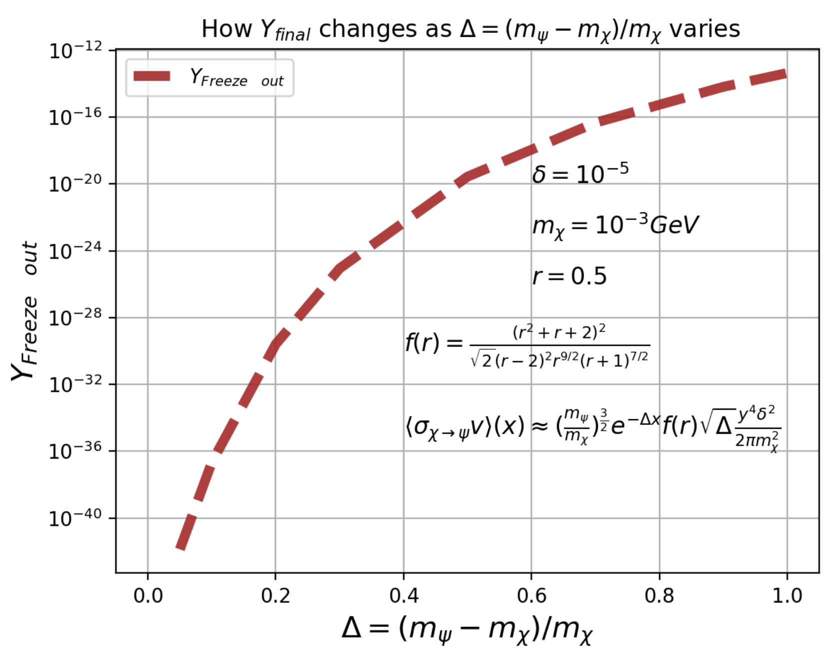

Figure 10: sa

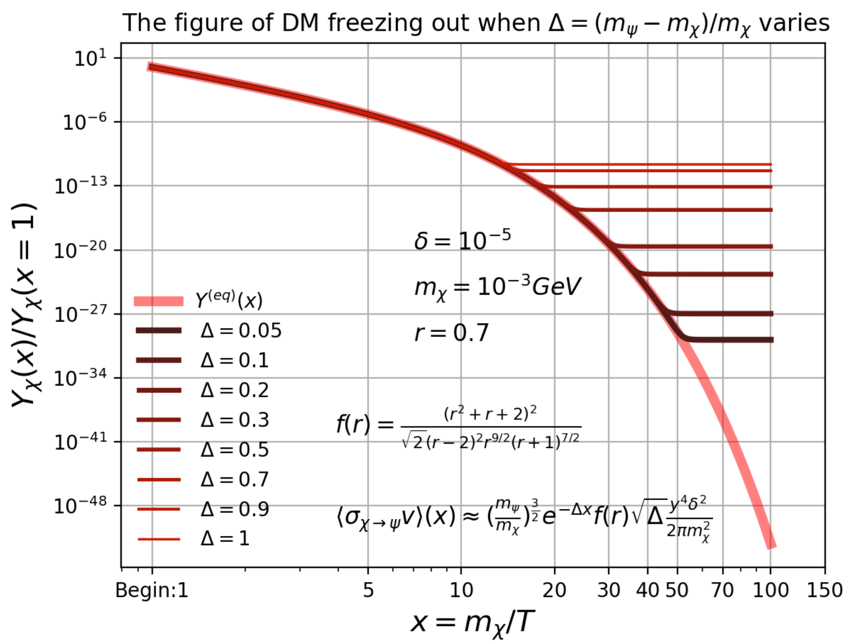

Figure 11: sa

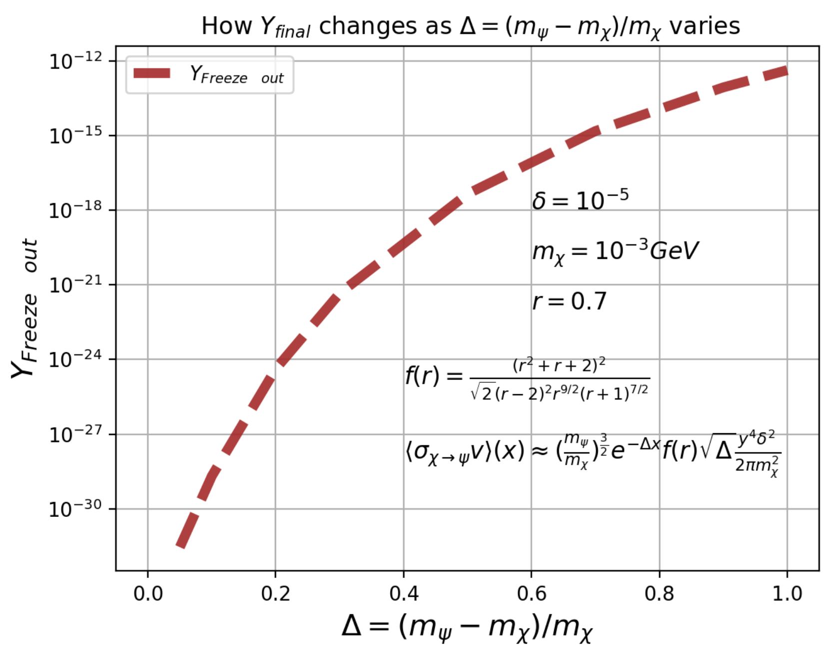

Figure 12: sa

4 Self-Consistency check

4.1 Change time step number N_t

I am trying to add a title above a table, without success. Here the code:

Table 1: dsd

N_t	50	70	100	200
$Y_{Freeze-out}$	10.003×10^{-37}	5.915×10^{-37}	4.079×10^{-37}	2.712×10^{-37}
N_t	1000	5000	10000	50000
$Y_{Freeze-out}$	1.996×10^{-37}	1.882×10^{-37}	1.868×10^{-37}	1.857×10^{-37}

$r = 0.5$, $\Delta = 0.1$, $m_\chi = 10^{-3} GeV$, $x_0 = 1$

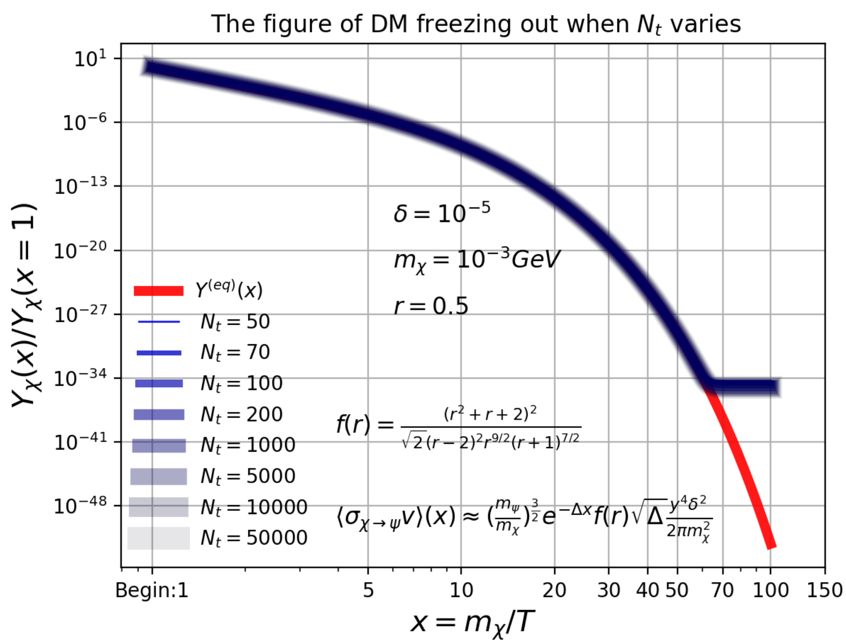

Figure 13: sa

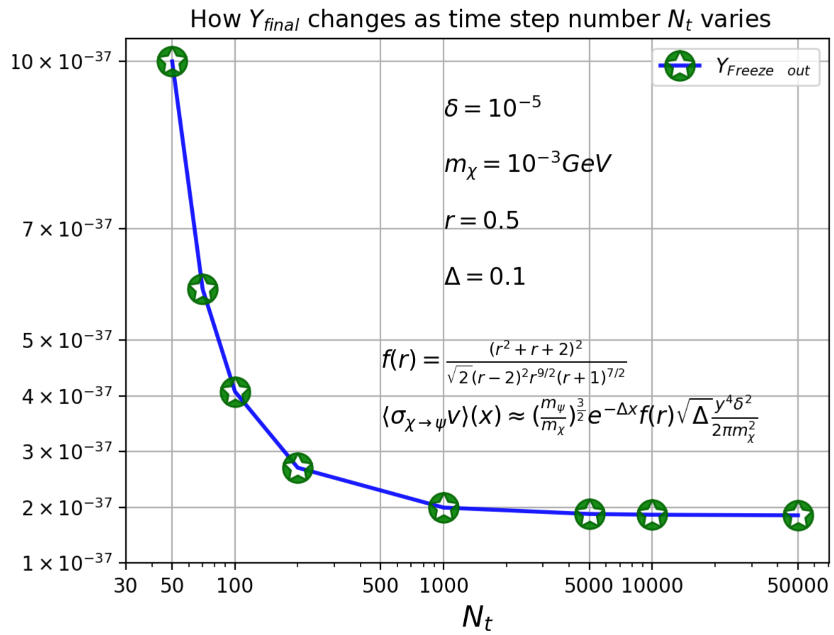

Figure 14: sa

4.2 Change initial x

5 Appendix A: WIMP Miracle

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