1D MHD

An HLL approximate Riemann Solver

Computational Physics Project

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1D Magnetohydrodynamics

- Studys: electrically conducting fluids (plasmas, salt water ...
- 1D: consider plane wave solution, variables only depend on x,t
- —> $\partial y = \partial z = 0$
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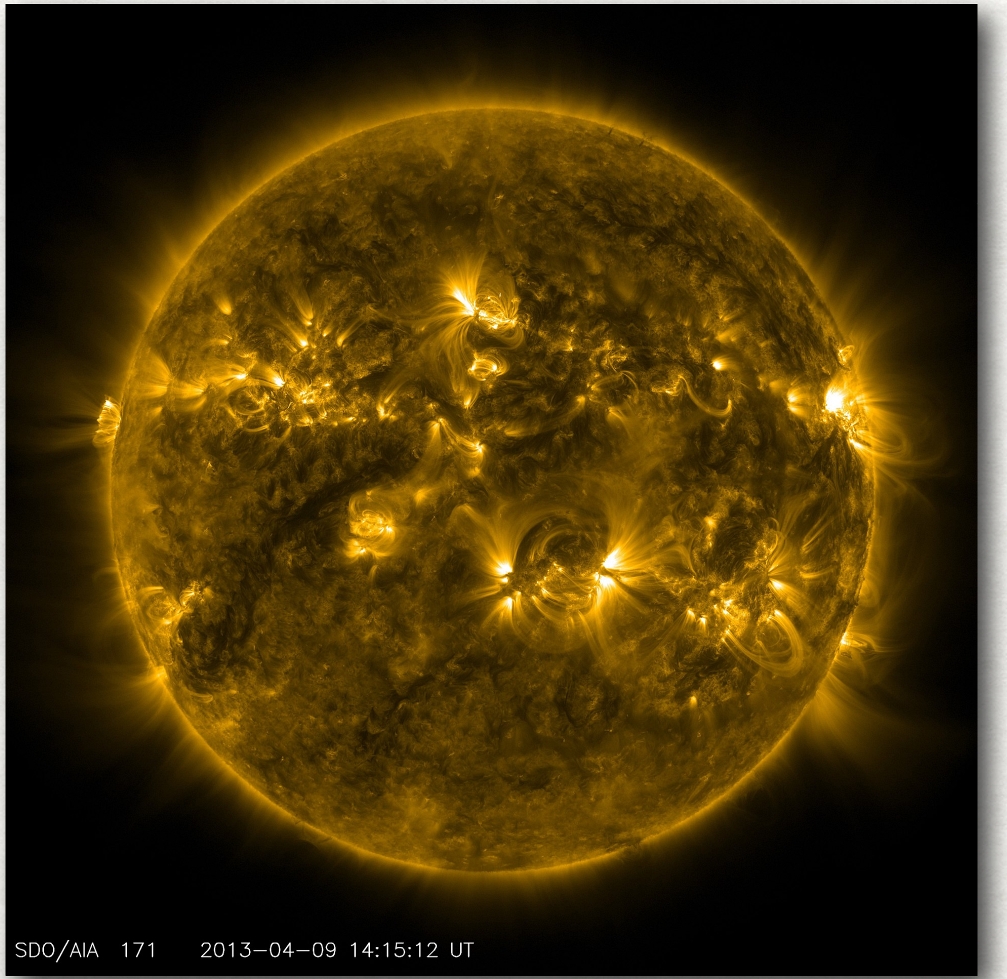


The sun is a MHD system

The MHD equations

- hydrodynamics + Maxwell's equations —> ideal MHD eqns:

$$\frac{\partial}{\partial t}\begin{bmatrix} \varrho \\ \varrho u \\ B \\ E \end{bmatrix} + \nabla\cdot\begin{bmatrix} \varrho u \\ \varrho uu + I\left(\left(p+\frac{1}{2}B^2\right) - BB\right) \\ uB - Bu \\ \left(E+p+\frac{1}{2}B^2\right)u - B(u\cdot B) \end{bmatrix} = 0$$

Magneto-fluid <—> magnetic field

- All components:

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ B_y \\ B_z \\ E \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho v_x \\ \rho v_x^2 + p + \frac{1}{2}B^2 - B_x^2 \\ \rho v_x v_y - B_x B_y \\ \rho v_x v_z - B_x B_z \\ B_z v_x - B_x v_z \\ B_z v_x - B_x v_z \\ v_x\left(E+p+\frac{1}{2}B^2\right) - B_x\left(v_x B_x + v_y B_y + v_z B_z\right) \end{bmatrix} = 0$$

Where $p = (\gamma - 1)\left(E - \frac{1}{2}\rho v^2\right) - \frac{1}{2}B^2 \qquad B_x = \text{ constant}$

Hyperbolicity Of Linear System

- Simple case: linear advection Eqn

$$\partial_t u + A \partial_x u = 0$$

- Discontinuity propagates along the characteristic curve

- PDE of the form:

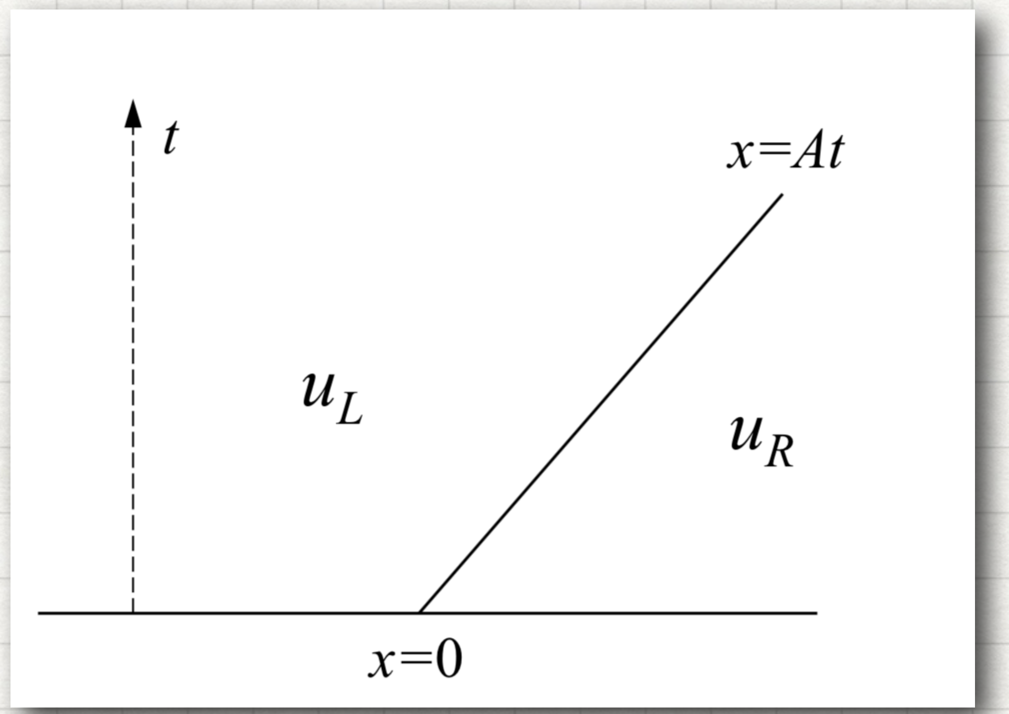
$$\partial_t \boldsymbol{u} + \mathbf{A} \cdot \partial_x \boldsymbol{u} = 0$$

- Diagonalize A

$$R^{-1} A R = \Lambda = \operatorname{diag}\left(\lambda^1, \lambda^2, \ldots, \lambda^m\right)$$

$$\boldsymbol{w} = R^{-1} \boldsymbol{u}$$

$$\partial_t w^p + \lambda^p \partial_x w^p = 0$$



1D MHD Equation is hyperbolic

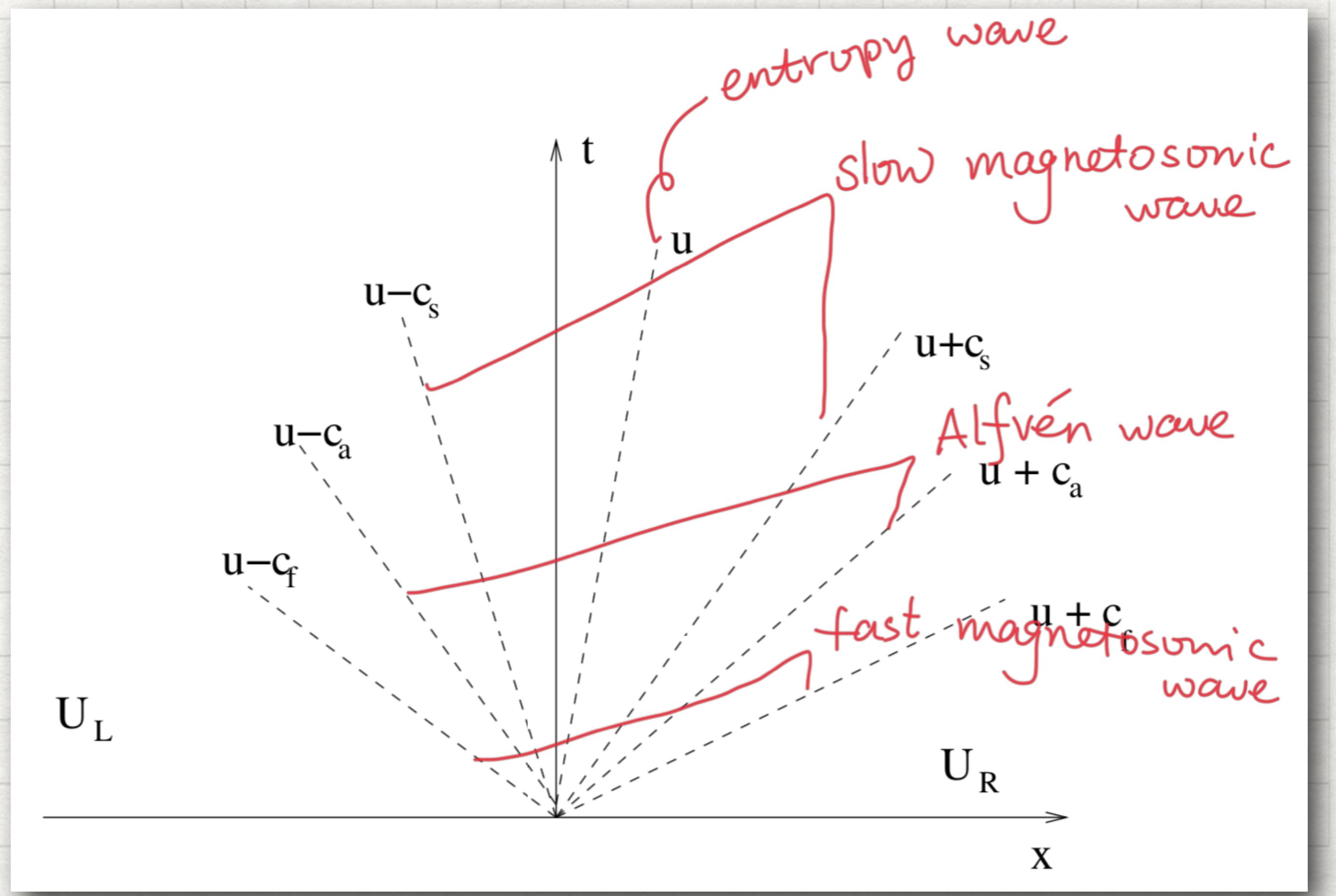
- Solving for eigenvalues/wave speeds of 1D MHD Eqns:

$$\lambda_{1,7} = u \mp c_f$$
$$\lambda_{2,6} = u \mp c_a$$
$$\lambda_{3,5} = u \mp c_s$$
$$\lambda_4 = u$$

$$c_{f,s}^2 = \frac{1}{2}\left[\left(a^2 + v_A^2\right) \pm \sqrt{\left(a^2 + v_A^2\right)^2 - 4a^2 c_a^2}\right]$$

$$c_a = \frac{B_x}{\sqrt{4\pi\rho}} \quad v_A = \frac{B}{\sqrt{4\pi\rho}} \quad a^2 = \gamma P/\rho$$

- 1,7 : fast Magnetosonic waves
- 2,6: Alfvén waves
- 3,5: slow Magnetosonic waves
- 4: contact discontinuity

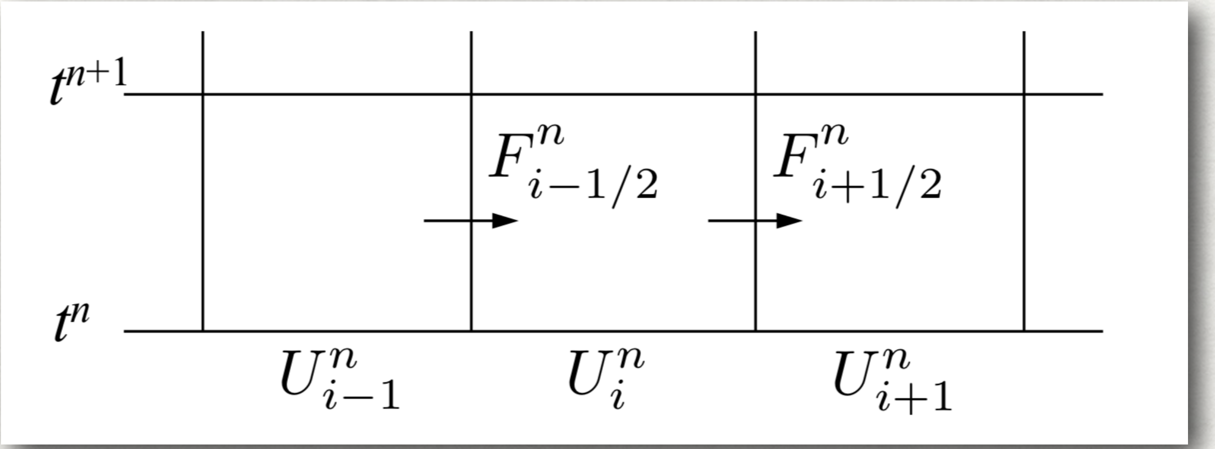


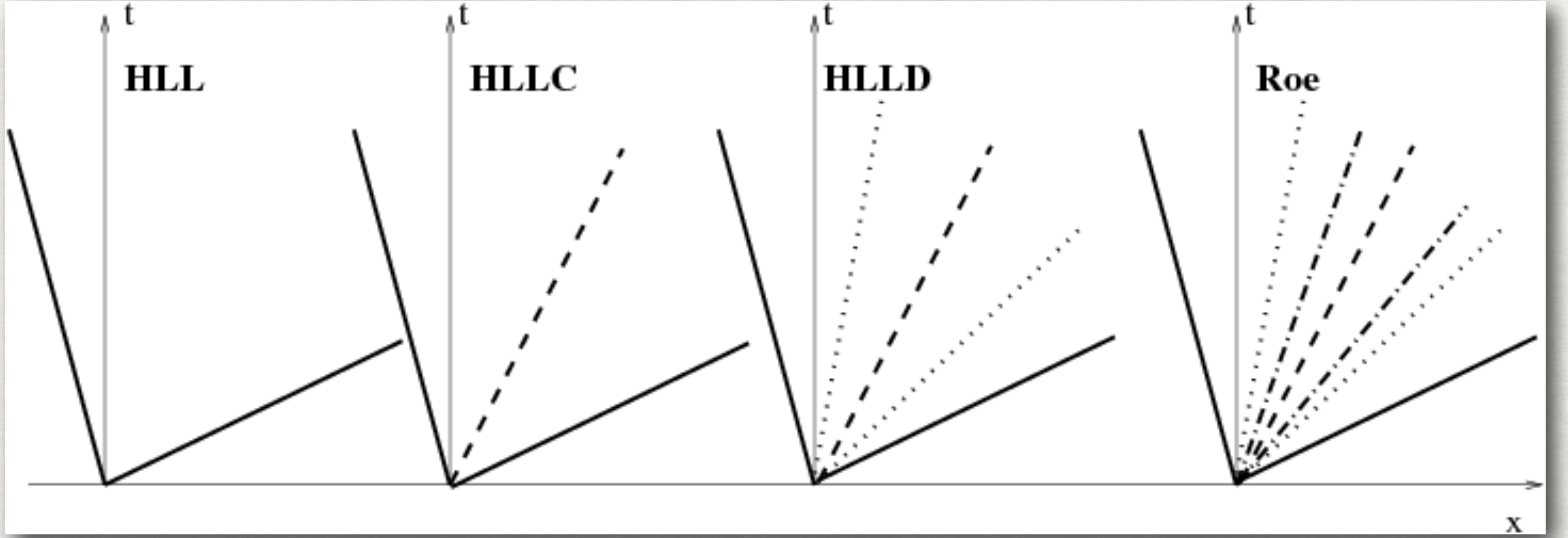
Numerically solving the Eqns

- Similar to hydro code

$$\mathbf{U}^{n+1} = \mathbf{U}^n - \frac{\Delta t}{\Delta x}(\mathbf{F}^n_{i+1/2} - \mathbf{F}^n_{i-1/2})$$

- Estimate F : HLL flux
- Make it High order
- Try other flux, eg, HLLC flux





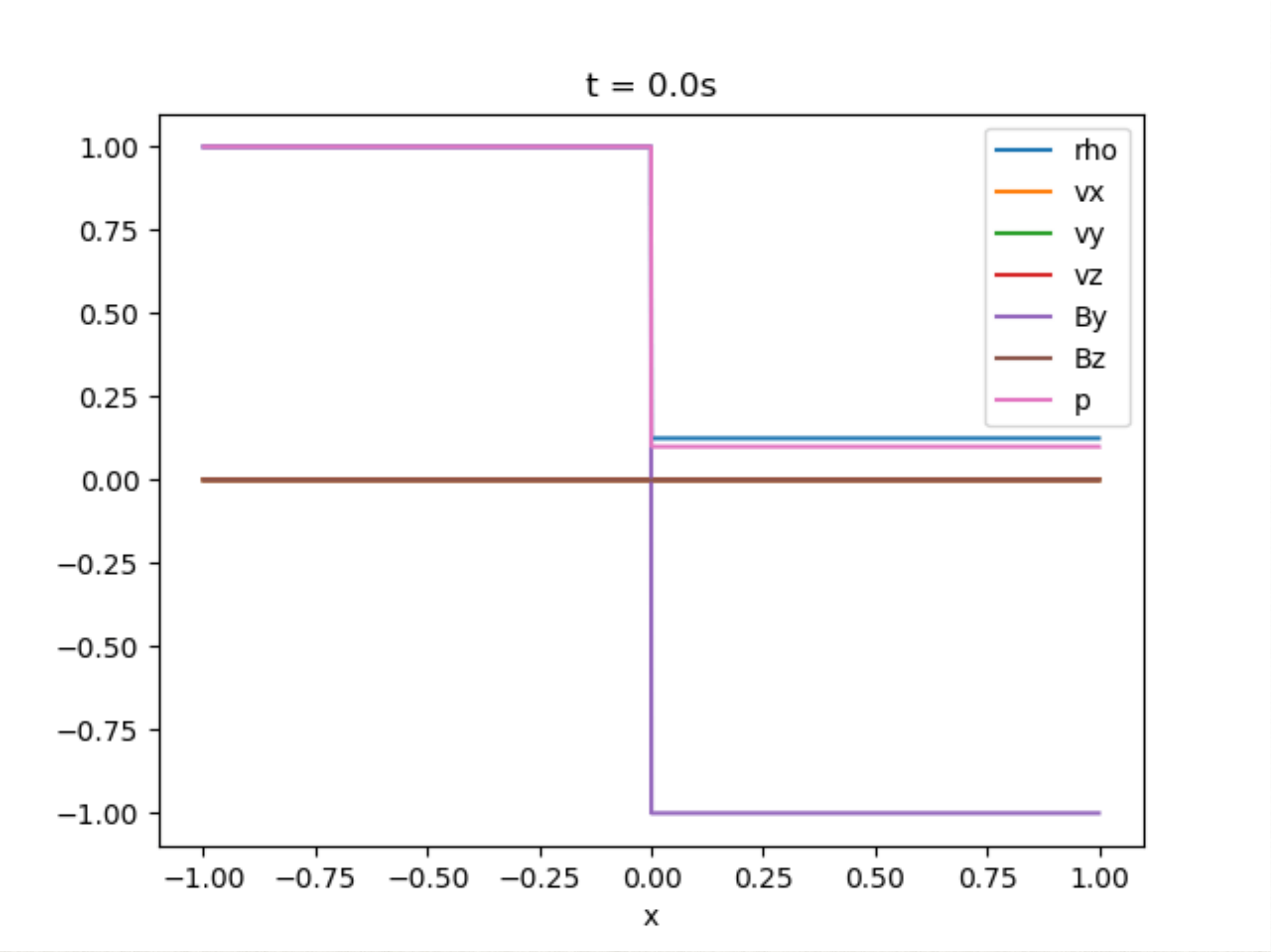
Numerical Tests

Brio-Wu shock tube

Set up:domain: [-1,1], 2000 cells, gamma = 2.0, Bx = 0.75, final = 0.2s, Dirichlet boundary conditions, cfl number = 0.475

Left state: ($\boldsymbol{\rho}$, v_x, v_y, v_z, B_y, B_z, E)=[1.0,0.0,0.0,0.0,1.0,0.0,1.0],

right state:($\boldsymbol{\rho}$, v_x, v_y, v_z, B_y, B_z, E)=[0.125,0.0,0.0,0.0,-1.0,0.1]



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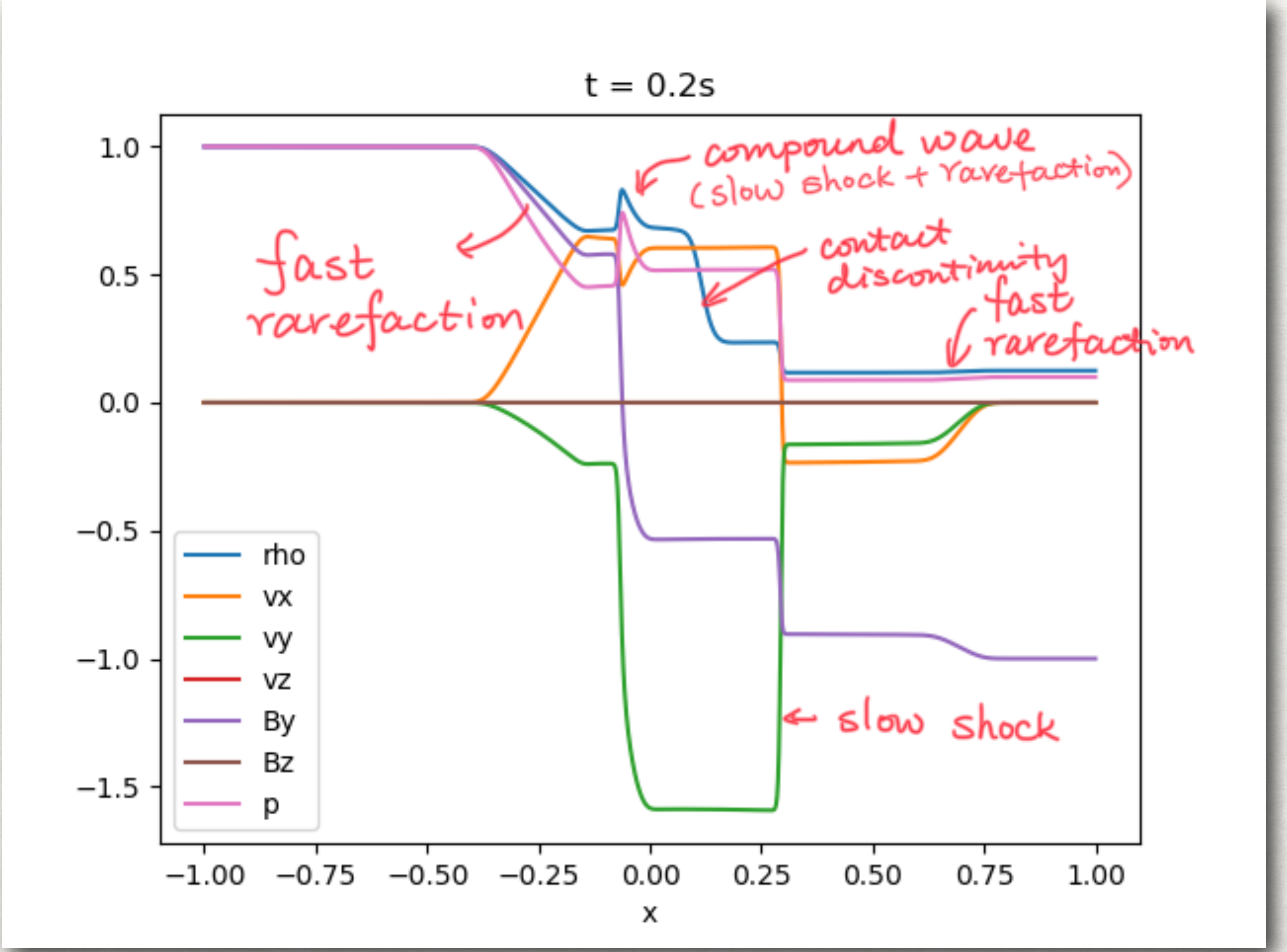
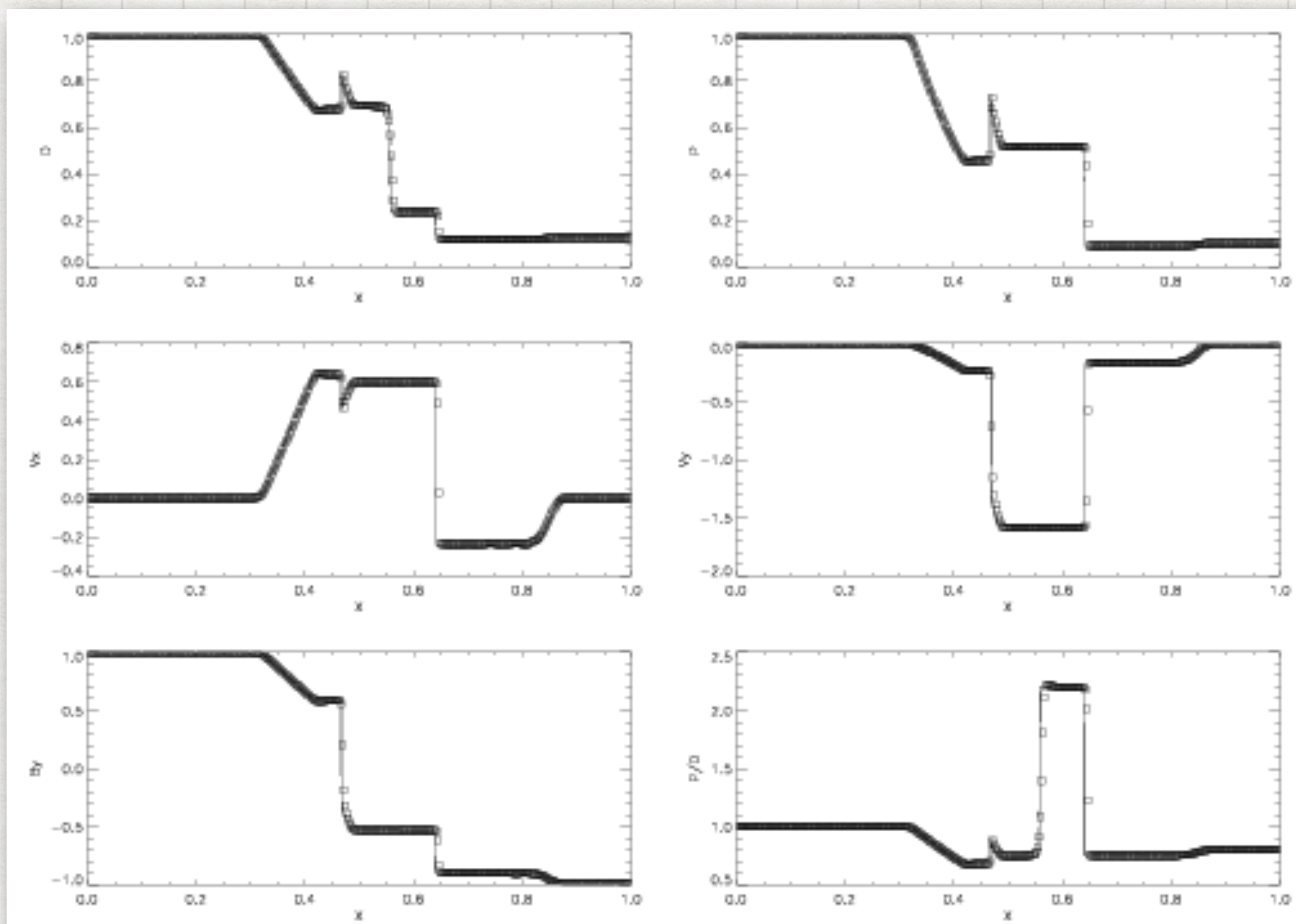
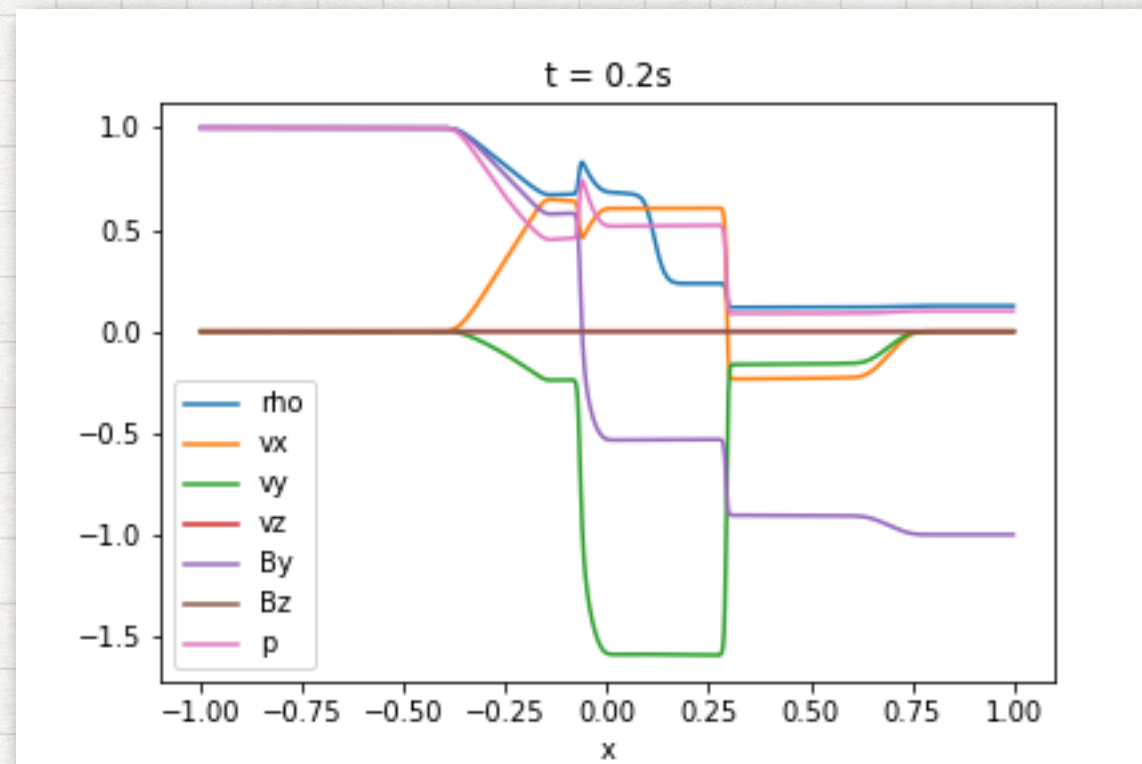


Brio-Wu Shock Tube

Compare with results from other method



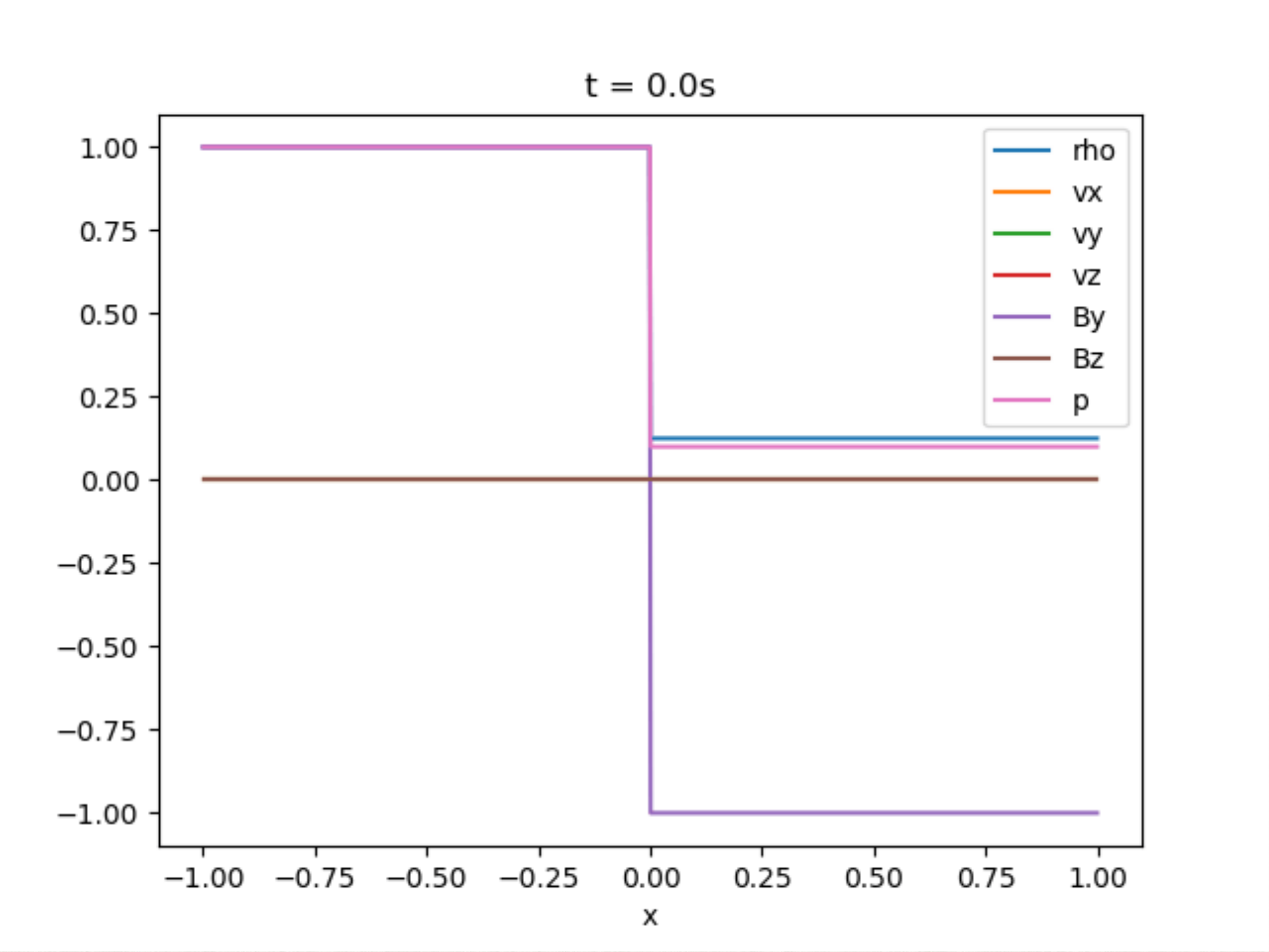
Results computed with Athena using the second order Roe solver



Results computed using HLL Riemann solver

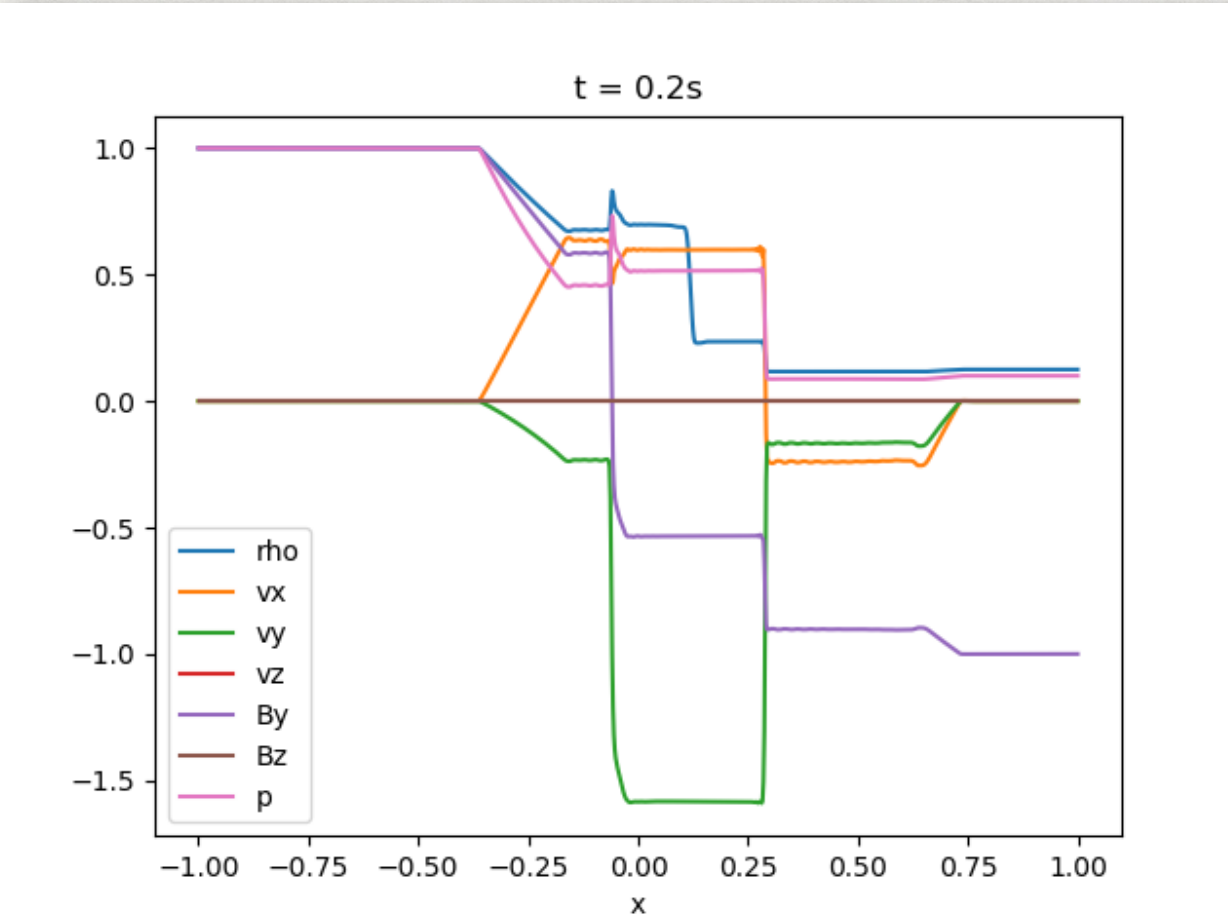
Brio-Wu shock tube

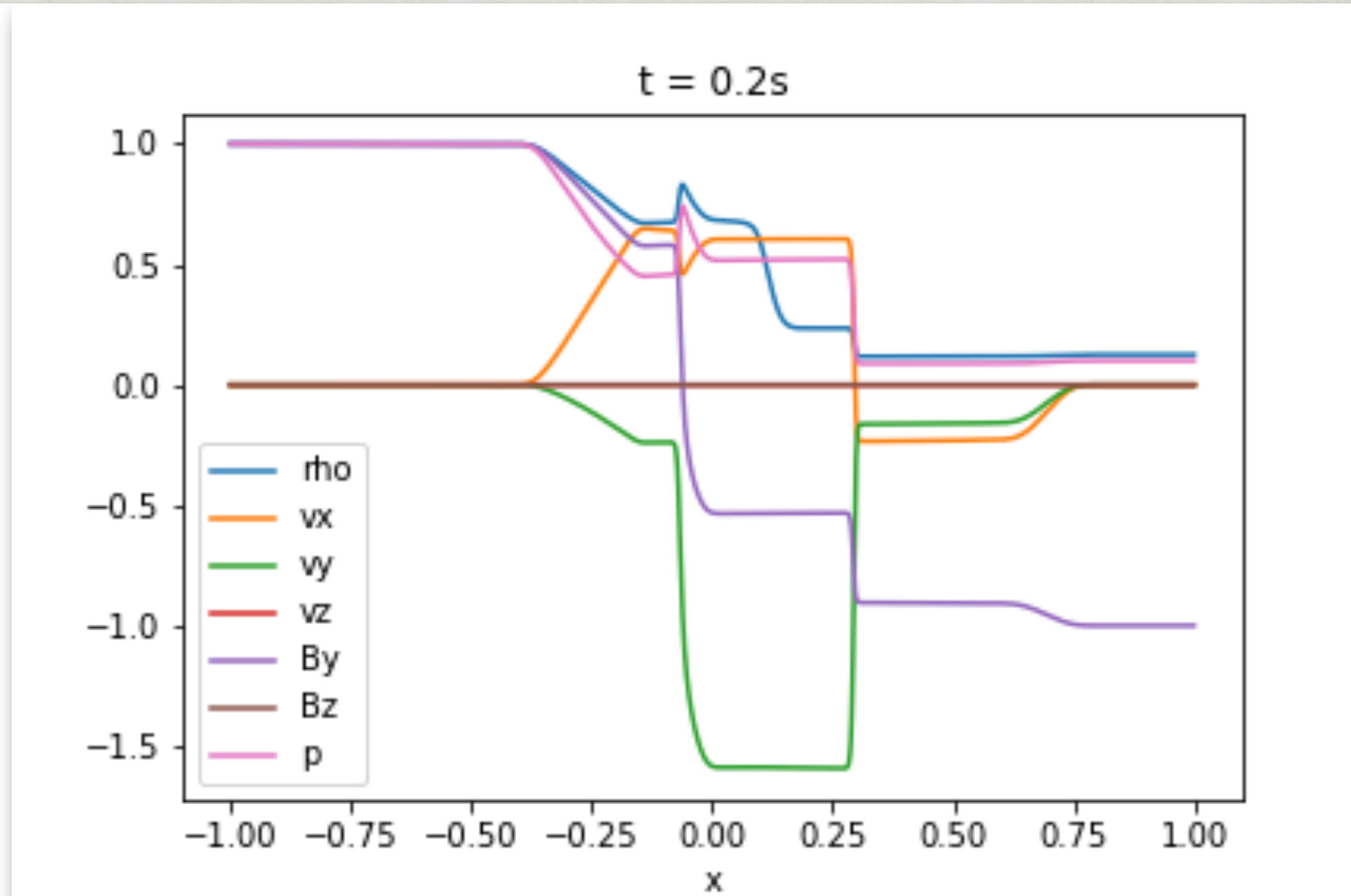
High order scheme(500 cells)



Brio-Wu shock tube

High order scheme



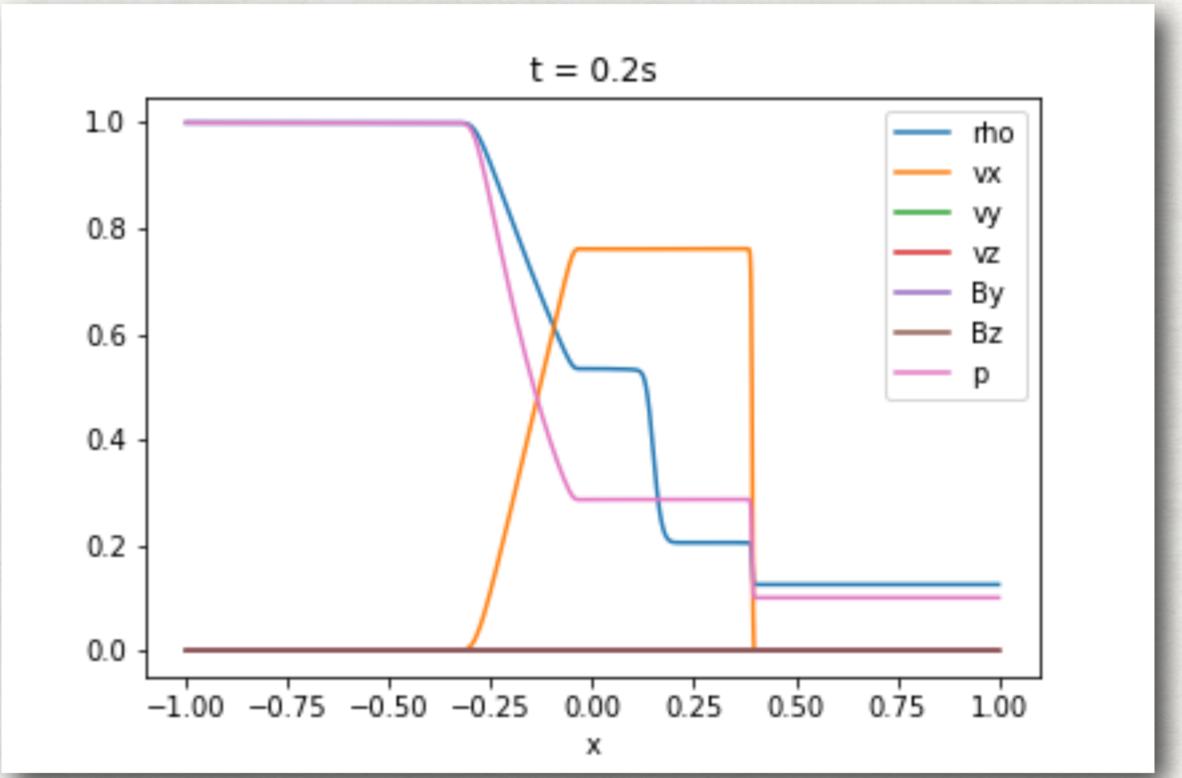


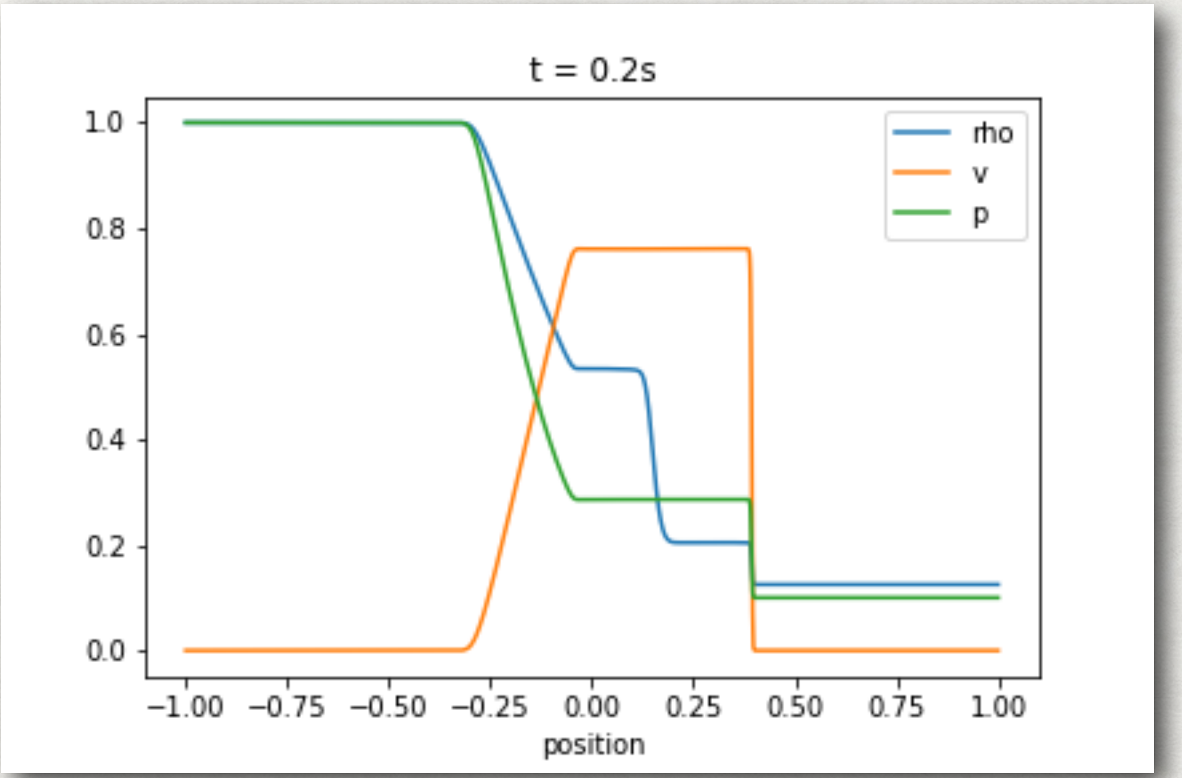
High order

First order

Brio-Wu shock tube

If we let B = 0, should return to Euler fluid





Magneto-fluid

Euler fluid

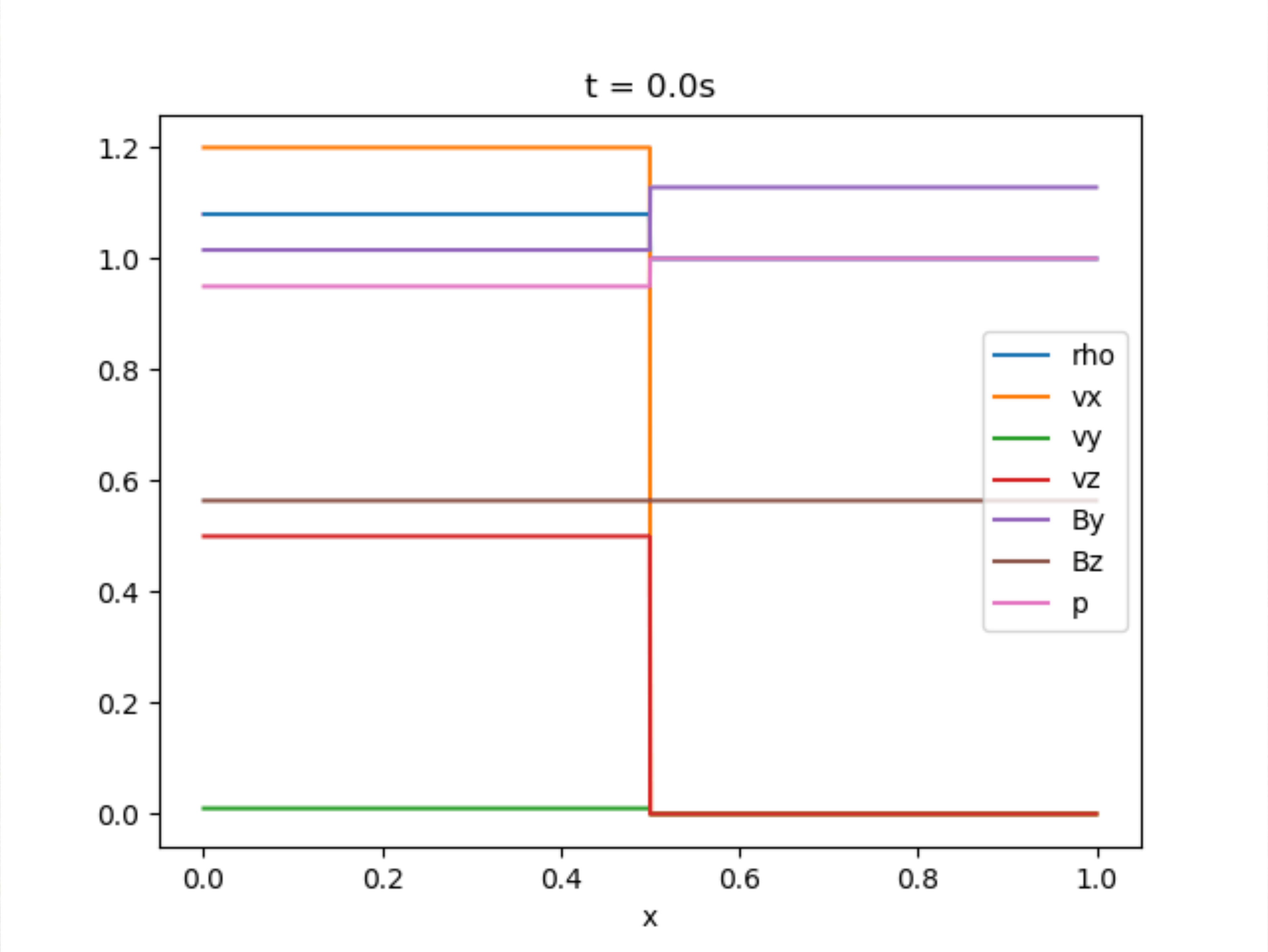
Numerical Tests

Ryu and Jones Test 2A

Set up:domain: [0,1], 2000 cells, gamma = 5/3, Bx =$2/(4\pi)^{1/2}$, final = 0.2s, Dirichlet boundary conditions

Left state: (ρ, v_x, v_y, v_z, B_y, B_z, E) = [1.08, 1.2, 0.01, 0.5, $3.6/(4\pi)^{1/2}$,$2/(4\pi)^{1/2}$,0.95]

Right state: (ρ, v_x, v_y, v_z, B_y, B_z, E) = [1,0,0,0,$4/(4\pi)^{1/2}$,$2/(4\pi)^{1/2}$,1]



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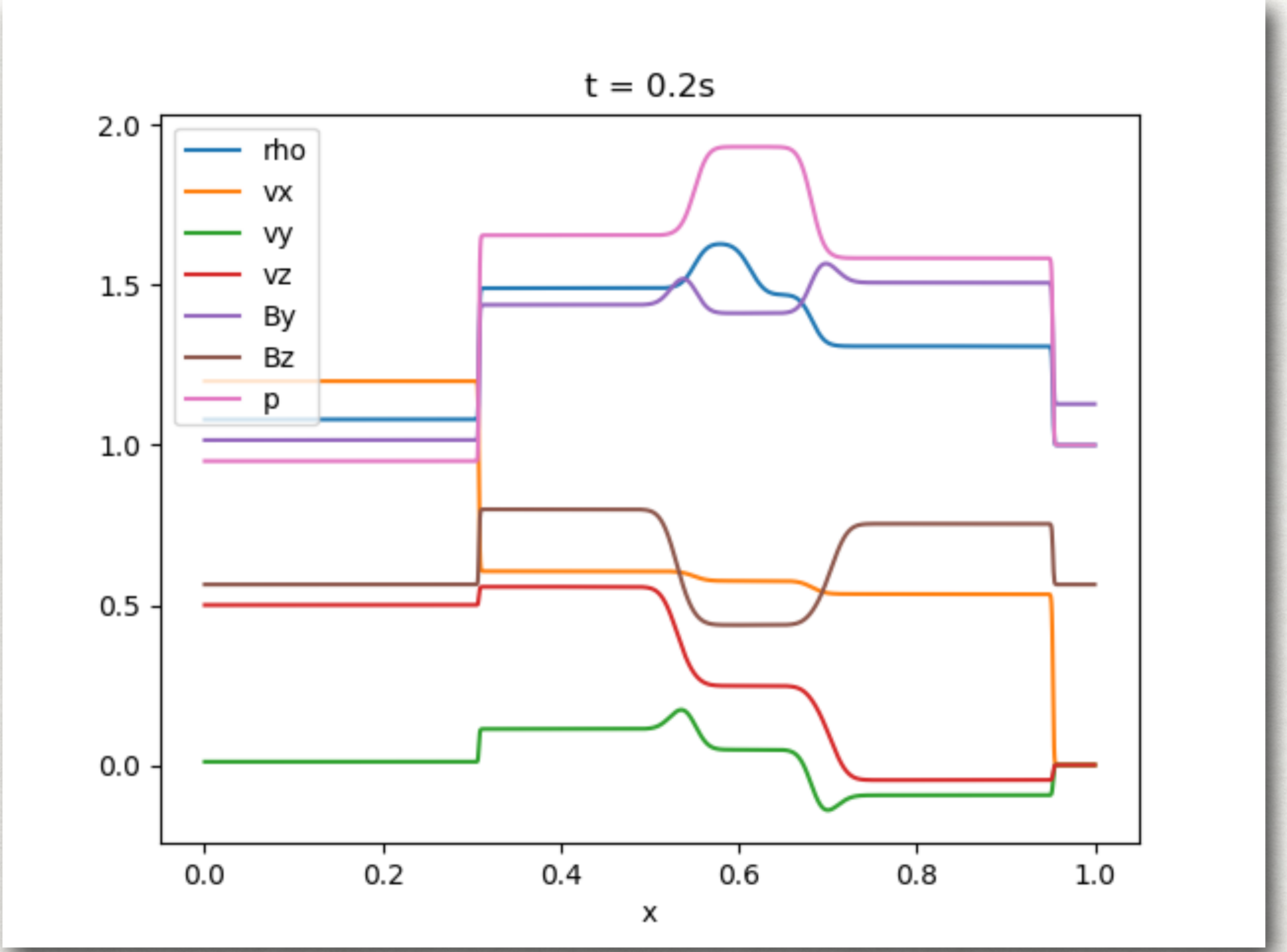

Numerical Tests

Ryu and Jones Test 2A+high order

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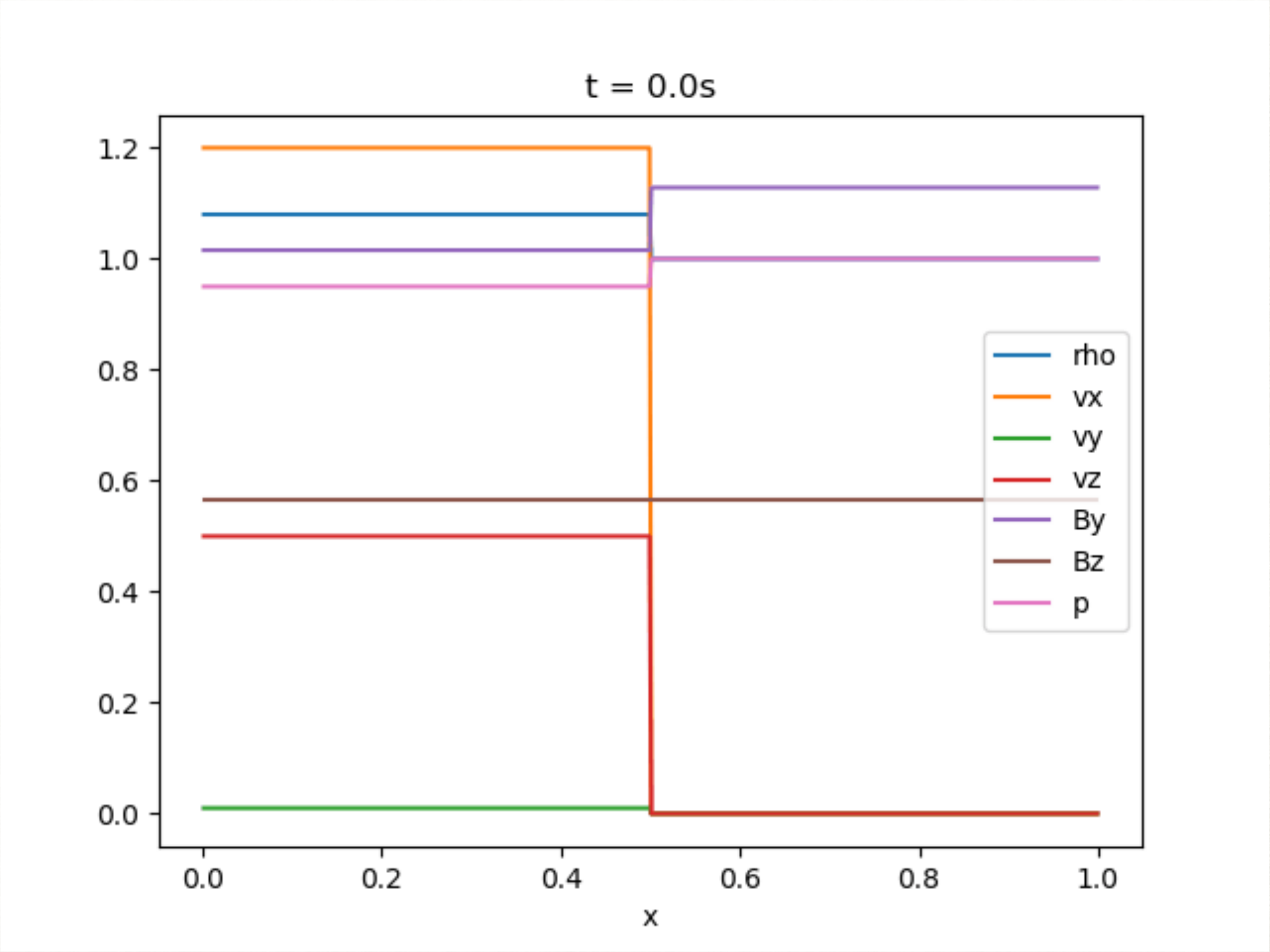

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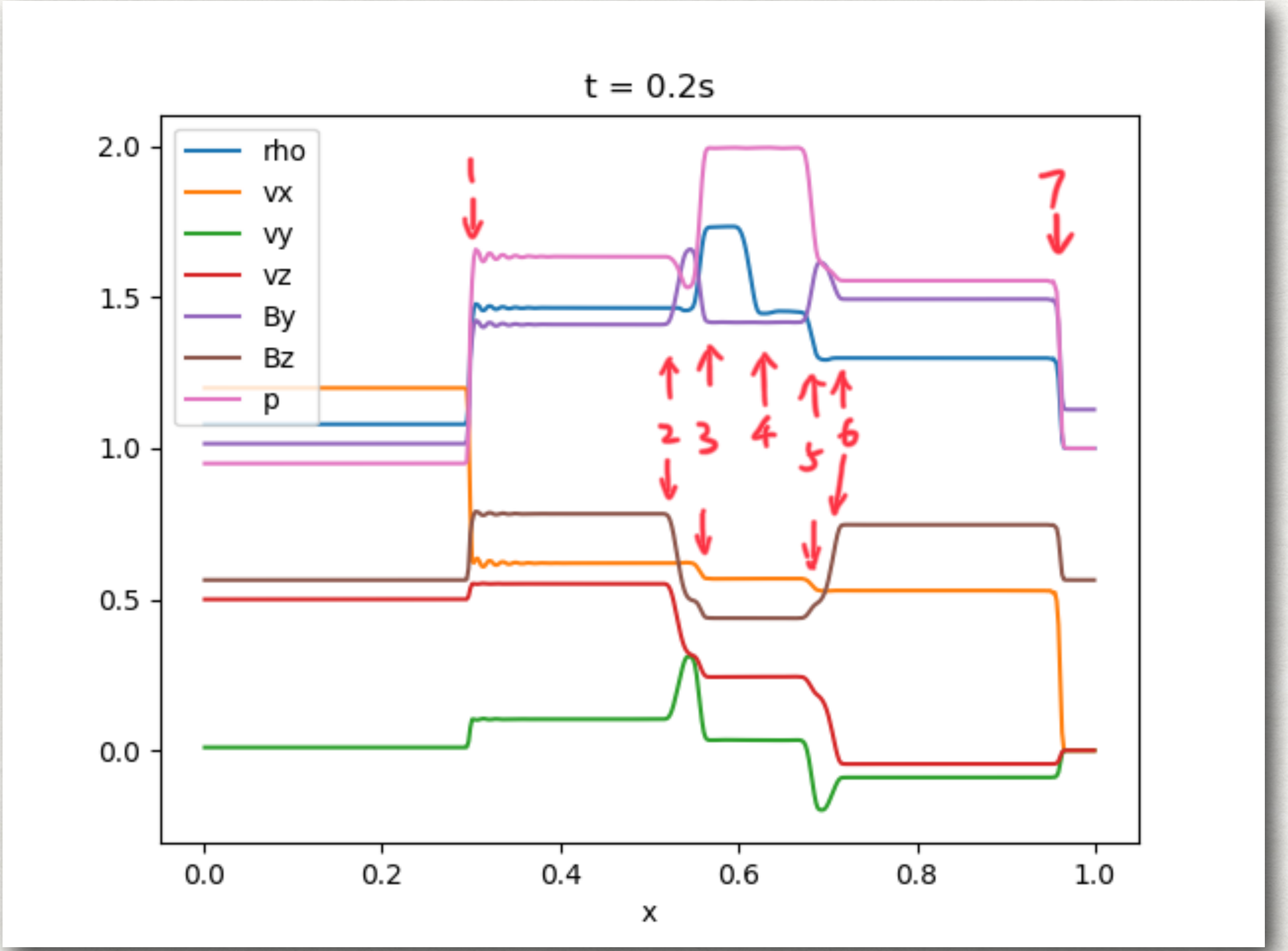

All 7 waves can be observed From this single test

FUTURE WORK

- Use HLLC/HLLD/Roe approximate Riemann Solver (As HLL solver can not resolve isolated contact discontinuities very well)
- High order
- 2D/3D MHD
- ...