PHYS-GA-2000 Computational Physics
FINAL PROJECT
Simulating the Penrose Process

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December 20, 2018

Abstract

We have written a `python` program solving geodesics with only inputs the coordinate system used along with the metric tensor components in that particular coordinate basis and non-gravitational forces. After demonstrating the code's correctness, we use it to simulate the Penrose process, observable in rotating black holes.

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1   Introduction

We aim to investigate the geodesic equations for the motion of a massive particle in the presence of a background gravitational field. The dimensionality of the spacetime itself is left as an input parameter along with the metric tensor. For a formal introduction to General Relativity, we recommend the reader to be advised from [5] or [3].

   The geodesic equations are in general,

$$\ddot{x}^\rho + \Gamma^\rho_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \tag{1}$$

where, "dots" represent derivatives with respect to the affine parameter λ, e.g. $\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}$, and $\Gamma^\rho_{\mu\nu}$ are the Christoffel symbols whose components in a coordinate basis are obtained by first derivatives of the metric tensor,

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \tag{2}$$

The geodesic equations in $(D = d + 1)$-dimensions are a set of D coupled second-order ordinary differential equations and can equivalently be written as $2D$ coupled first-order ordinary differential equations by introducing the D-velocities, $u^\mu \equiv \dot{x}^\mu$, allowing to write our problem as,

$$
\begin{aligned}
\dot{x}^\rho &= u^\rho & : \quad & D \text{ equations} \\
\dot{u}^\rho &= -\Gamma^\rho_{\mu\nu}(x)u^\mu u^\nu & : \quad & D \text{ equations}
\end{aligned}
\tag{3}
$$

We are more interested to see how the massive particle propagates according to a stationary observer, that is, we set the affine parameter to the proper time τ and measure time using the coordinate time $x^0 \equiv t$ by also introducing the coordinate velocity, $v^\mu \equiv \frac{dx^\mu}{dt} = (1, \vec{v})$. In terms of these, the equations of motion read (Appendix A),

$$
\begin{aligned}
\frac{dx^i}{dt} &= v^i \\
\frac{dv^i}{dt} &= -\Gamma^i_{00} + \left(\Gamma^0_{00}\delta^i_j - 2\Gamma^i_{0j}\right)v^j + \left(2\Gamma^0_{0j}\delta^i_k - \Gamma^i_{jk}\right)v^j v^k + \Gamma^0_{jk}v^i v^j v^k
\end{aligned}
\tag{4}
$$

If there are additional, non-gravitational, forces, e.g. electromagnetic interaction of a charged particle with a charged black hole, then the D-acceleration a^μ associated with non-gravitational interactions adds a new term in the equations of motion. The geodesics become,

$$
\ddot{x}^\rho + \Gamma^\rho_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = a^\rho
\tag{5}
$$

and the subsequent coordinate equations of motion turn out to be,

$$
\begin{aligned}
\frac{dx^i}{dt} &= v^i \\
\frac{dv^i}{dt} &= -\left(\Gamma^i_{00} + a^i g_{00}\right) + \left(\left(\Gamma^0_{00} + a^0\right)\delta^i_j - 2\left(\Gamma^i_{0j} + a^i g_{0j}\right)\right)v^j + \\
&\quad + \left(2\left(\Gamma^0_{0j} + a^0 g_{0j}\right)\delta^i_k - \left(\Gamma^i_{jk} + a^i g_{jk}\right)\right)v^j v^k + \left(\Gamma^0_{jk} + a^0 g_{jk}\right)v^i v^j v^k
\end{aligned}
\tag{6}
$$

We treat the extra D-acceleration a^μ as an additional input. We leave as a future development to take as an input the interaction Lagrangian rather than the D-acceleration (see Appendix C). This would allow to directly construct both the D-acceleration a^μ *and* the conjugate momenta P^μ associated with useful observables such as the energy and the angular momentum of the particle (see Appendix C).

1.1   Notation

We work in geometric units with $G = c = 1$. We also take the mass of the simulated black holes to be $M = 1$ for simplicity. To restore units, just multiple lengths, masses and times with the following quantities L_{GU}, M_{GU} and T_{GU} respectively,

$$
\begin{aligned}
L_{GU} &= \frac{GM}{c^2} \\
M_{GU} &= M \\
T_{GU} &= \frac{L_{GU}}{c} = \frac{GM}{c^3}
\end{aligned}
\tag{7}
$$

2   Part A: Solving Geodesics

Using the python library `sympy`[1], we evaluate the Christoffel symbols (2) and then use them to construct the force functions on the right-hand side of (4) for the coordinate velocities by also inputting the additional D-acceleration a^μ.

We have implemented different integration schemes seen in class. Namely, we have implemented Forward Euler, RK2, RK4, Leapfrog and Verlet as well as the adaptive step size versions for the explicit integrators.

The inputs and outputs of the program are as the following tables 1 and 2 indicate respectively,

Input Parameters	Description
x^μ	Coordinate system to be used
$g_{\mu\nu}(x)$	Metric tensor components in the $\{x^\mu\}$-coordinate system
a^μ	D-acceleration associated with non-gravitational interaction
t_0	Initial coordinate time instant
t_N	Final coordinate time instant
N	Number of time steps to be taken
$x_0^i, \, i = 1, \ldots, D-1$	Initial spatial position at coordinate time t_0
$v_0^i, \, i = 1, \ldots, D-1$	Initial coordinate spatial velocity at coordinate time t_0
INTEGRATOR	Integration scheme to be used

Table 1: Input parameters

Output Parameters	Description
$x^i(t), \, i = 1, \ldots, D-1$	Spatial position at coordinates times $t \in [t_0, t_N]$
$v^i(t), \, i = 1, \ldots, D-1$	Spatial coordinate velocity at coordinate times $t \in [t_0, t_N]$
$u_\mu(t), \, i = 0, \ldots, D-1$	Covariant D-velocity at coordinate times $t \in [t_0, t_N]$

Table 2: Output parameters

From the covariant D-velocity, one can also read the energy and angular momentum of the particle around the x_{D-1}-axes whose rotations are associated with the azimuthal angle $\phi \equiv \theta_{D-1}$ when using spherical coordinates,

$$E \equiv -mu_0$$
$$L \equiv mu_{D-1} \tag{8}$$

but these expressions have additional terms when considering non-gravitational forces (see Appendix C).

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[1]This does not come directly by installing python using the `pip` command. To install it, simply type, `pip install --user sympy`.

2.1 Schwarzschild Metric

As a first test, we solve the geodesics for motion around a Schwarzschild black hole whose geometry is,

$$ds_S^2 = -f_S(r)dt^2 + \frac{dr^2}{f_S(r)} + r^2 d\Omega_{D-2}^2$$

$$f_S(r) = 1 - \left(\frac{R_S}{r}\right)^{D-3} \tag{9}$$

where R_S is the Schwarzschild radius related to the mass M of the black hole (ADM mass) according to,

$$R_S^{D-3} = \frac{16\pi GM}{(D-2)\Omega_{D-2}} \tag{10}$$

with Ω_{D-2} the surface of a unit $(D-1)$-dimensional sphere,

$$\Omega_{D-2} = Area(S^{D-2}) = (D-1)\frac{\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D+1}{2}\right)} \tag{11}$$

and $d\Omega_{D-2}^2$ is the infinitesimal length on S^{D-2},

$$d\Omega_{D-2}^2 = \sum_{n=2}^{D-1}\left(\prod_{m=2}^{n-1}\sin^2\theta_m\right)d\theta_n^2 \tag{12}$$

Note that we have indexed the angle variables from 2 to $D-1$ to remind that the x^1 coordinate is r.

The Schwarzschild black hole is equipped with an event horizon[2] at precisely the Schwarzschild radius,

$$r_H = R_S \tag{13}$$

and signifies the limit of no return. Quite remarkably, a stationary observe will never see the massive particle falling inside the black hole; the particle will appear to slow down near the event horizon until it finally freezes at $r = r_H$ and disappears due to infinite redshift. This is a property of all black hole solutions.

2.1.1 Run 1: Bound trajectory

For the first run, we take $D = 4$ and the following initial conditions,

---

[2]The event horizons are null surfaces. For the case of the event horizons located at the surfaces of constant radial coordinate $r_H = const.$, the condition they must obey is $g^{rr}\mid_{r=r_H} = 0$.
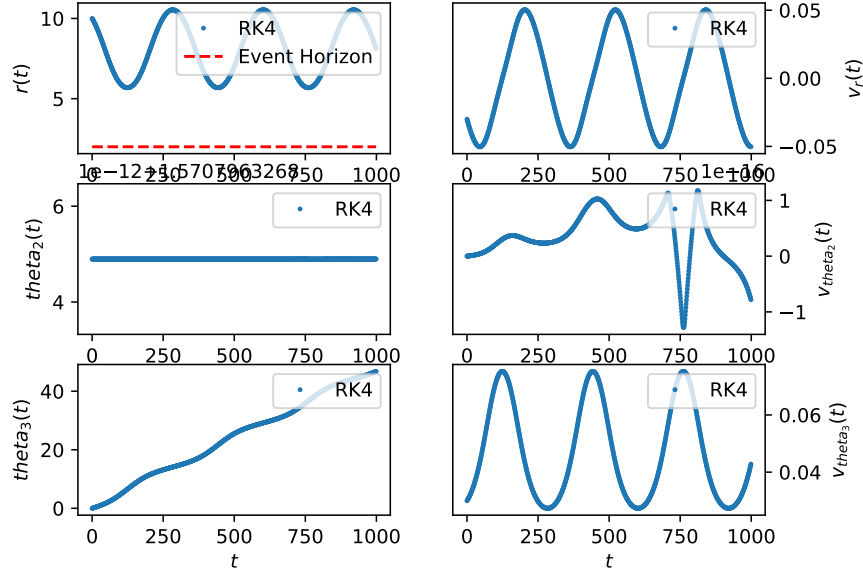
Figure 1: Run 1: Coordinate positions and velocities for a bound trajectory in 4-dimensional Schwarzschild solution.

t_0	0.0
r_0	10.0
θ_0	$\frac{\pi}{2}$
ϕ_0	0.0
v_{r0}	-0.03
$v_{\theta 0}$	0.0
$v_{\phi 0}$	0.03

Table 3: Run 1: Initial Conditions.

These initial conditions ensure a specific energy $\frac{E}{m} = 0.9501784 < 1$ allowing a bound trajectory. Running using the RK4 integrator with $N = 10000$ iterations and final time instant $t_N = 1000.0$, we output figures 1 and 2.

2.1.2 Run 2: Doomed trajectory

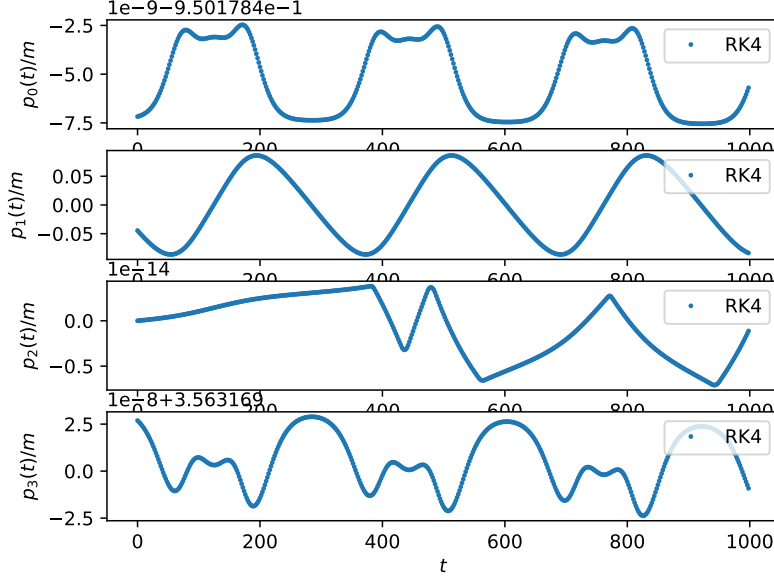For the second run, we take $D = 4$ and the following initial conditions,

Figure 2: Run 1: 4-velocity components for a bound trajectory in 4-dimensional Schwarzschild solution.

t_0	0.0
r_0	3.0
θ_0	$\frac{\pi}{2}$
ϕ_0	0.0
v_{r0}	-0.3
$v_{\theta 0}$	0.0
$v_{\phi 0}$	0.0

Table 4: Run 2: Initial Conditions.

These initial conditions account for an unbound trajectory with a specific energy $\frac{E}{m} = 1.41833 > 1$ but the angular velocity $v_{\phi 0}$ is small enough to not escape the black hole. Running using the RK4 integrator again with $N = 300$ steps and final time instant $t_N = 10.0$, we output figures 3 and 4.

2.2 Reissner-Nordström Metric

A second test, would be to run our program for motion around a Reissner-Nordstroöm black hole,

$$ds_{RN}^2 = -f_{RN}(r)dt^2 + \frac{dr^2}{f_{RN}(r)} + r^2 d\Omega_{D-2}^2$$

$$f_{RN}(r) = 1 - \left(\frac{R_S}{r}\right)^{D-3} + \left(\frac{A}{r}\right)^{2(D-3)}$$
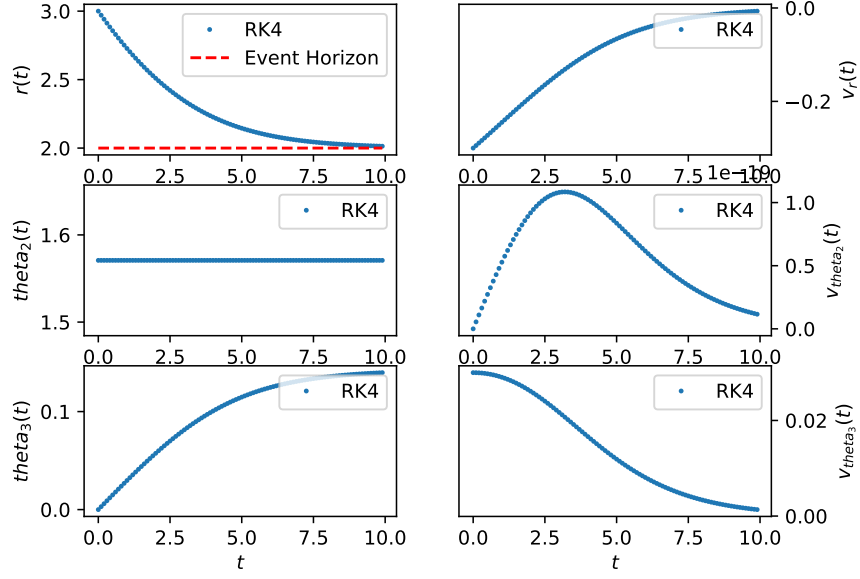
(14)

7

Figure 3: Run 2: Coordinate positions and velocities for an unbound, doomed trajectory in 4-dimensional Schwarzschild solution.
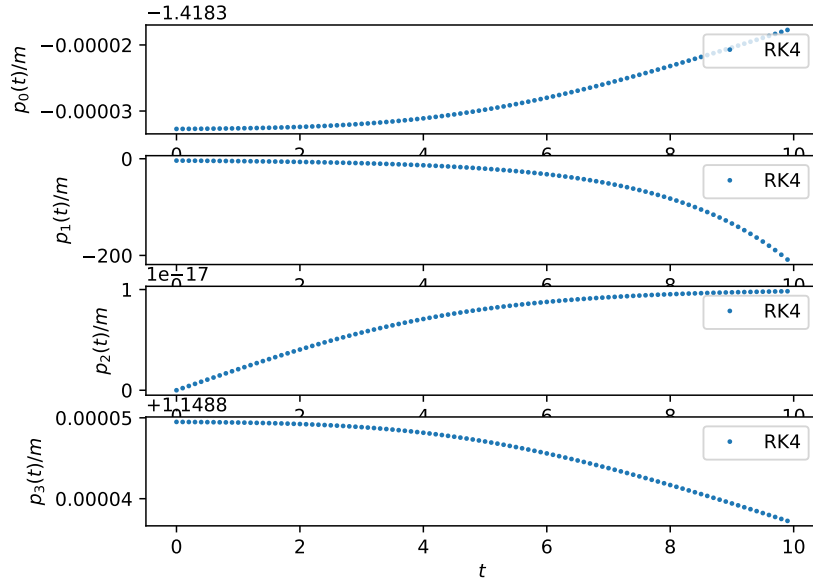

Figure 4: Run 2: 4-velocity components for a unbound, doomed trajectory in 4-dimensional Schwarzschild solution.

where A is related to the total charge Q of the black hole according to,

$$A^{2(D-3)} = \frac{8\pi G Q^2}{(D-2)(D-3)} \tag{15}$$

In contrast to the Schwarzschild solution, the Reissner-Nordström black hole has two event horizons,

$$r_H^{\pm} = \frac{R_S}{2}\left(1 \pm \sqrt{1 - 4\left(\frac{A}{R_S}\right)^{2(D-3)}}\right)^{\frac{1}{D-3}} \tag{16}$$

from which, however, only the outer horizon can be formed from physical processes, $r_H = r_H^+$; the inner event horizon is a mathematical artifact.

This run would also test the implementation of additional non-gravitational forces. We leave this analysis for future development.

3   Part B: Simulating the Penrose Process

3.1   Kerr-Newman Metric

For the final attempt, we are interested in the 4-dimensional rotating black hole and the investigation of the Penrose process. The metric of a 4-dimensional axisymmetric, stationary black hole in asymptotically flat spacetime is given by the Kerr-Newman solution,

$$\begin{aligned}
ds^2 = &- \frac{\Delta - a^2 \sin^2\theta}{\Sigma}dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \\
&+ \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 - 2\frac{R_S r a \sin^2\theta}{\Sigma}dt d\phi
\end{aligned} \tag{17}$$

with,

$$\begin{aligned}
\Delta &= r^2 + a^2 - R_S r + Q^2 \\
\Sigma &= r^2 + a^2\cos^2\theta
\end{aligned} \tag{18}$$

The parameters R_S, a and Q are the Schwarzschild radius, $R_S = 2GM$, the spin parameter, $a = \frac{J}{M}$, with J the angular momentum of the black hole around the z-axes, and the electric charge of the black hole respectively. This solution also has two event horizons,

$$r_H^{\pm} = \frac{R_S}{2}\left(1 \pm \sqrt{1 - 4\frac{a^2 + Q^2}{R_S^2}}\right) \tag{19}$$

with only the outer horizon being physical, $r_H = r_H^+$.

Apart from the event horizons, the rotating black hole possesses a unique region, called the ergoregion. This is the region within which a particle cannot stay still and is given by

the set of those points satisfying $g_{tt} > 0$. As a result, the ergoregion is bounded by two ergospheres $r_E^{\pm}$ ($r_E^- < r < r_E^+$),

$$r_E^{\pm} = \frac{R_S}{2} \left(1 \pm \sqrt{1 - 4\frac{a^2 \cos^2 \theta + Q^2}{R_S^2}} \right) \tag{20}$$

but only $r_E^+ \equiv r_E$ is physical, while $r_E^- < r_H^-$ so, the real ergoregion is $r_H < r < r_E$.

3.2 Penrose Process

In 1971, Roger Penrose and R. M. Floyd ([4]) theorized the extraction of energy from a rotating black hole through a simple mechanism. The procedure is as in figure 5. A parent particle (particle 1) with some energy E_1 decays into an outgoing particle (particle 2) with energy E_2 and an ingoing particle (particle 3) with energy E_3. Under special conditions, E_3 can be negative allowing particle 2 to be detected by a distant observer with an increased energy.

The condition for a negative energy is a ϕ-component of the coordinate velocity smaller than a critical value,

$$v^{\phi} < -\frac{g_{tt}}{g_{t\phi}} \tag{21}$$

This also sets an allowed range of the specific angular momentum $l \equiv \frac{L}{m}$,

$$l < -\sqrt{\frac{-\psi g_{\phi\phi}}{g_{t\phi}^2 g_{tt}}} \tag{22}$$

where $\psi = g_{tt}g_{t\phi} - g_{t\phi}^2 = -\Delta \sin^2 \theta < 0$ is the determinant of the sub-metric 2×2 matrix constructed only from t and ϕ components,

$$\tilde{g} = \begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{\phi t} & g_{\phi\phi} \end{pmatrix} \Rightarrow \tilde{g}^{-1} = \frac{1}{\psi} \begin{pmatrix} g_{\phi\phi} & -g_{t\phi} \\ -g_{\phi t} & g_{tt} \end{pmatrix} \tag{23}$$

From the specific angular momentum condition (22), it is obvious that the Penrose process can be produced only when $g_{tt} > 0$, i.e. only within the ergoregion. The negative sign in (22) also shows that the particle must have an opposite angular momentum ($L < 0$) from the angular momentum of the black hole ($J > 0$), while the ϕ-component of the coordinate velocity can still be positive. Evidentially, a careful analysis of the process reveals that the extra energy particle 2 has gained comes from the rotational energy of the black hole; particle 3 is absorbed by the black hole lowering its total angular momentum to $J' = J + L < J$.

At the point of split, conservation of 4-momentum must be satisfied,

$$p_{1\mu} = p_{2\mu} + p_{3\mu} \tag{24}$$

To simplify the problem, we consider equatorial motion, that is, we set $\theta = \frac{\pi}{2}$ and $v_1^{\theta} = v_2^{\theta} = v_3^{\theta} = 0$[3]. In addition, to further simplify the simulation, we take $v_3^r = 0$. As a result,

---

[3]It is simple to prove that, if the initial θ-component of the coordinate velocity is zero at the equatorial plane, then it stays zero along the entire propagation. This follows from the conservation of the angular momentum L which ensures that the motion is restricted to the plane perpendicular to $\vec{L} = L\hat{z}$
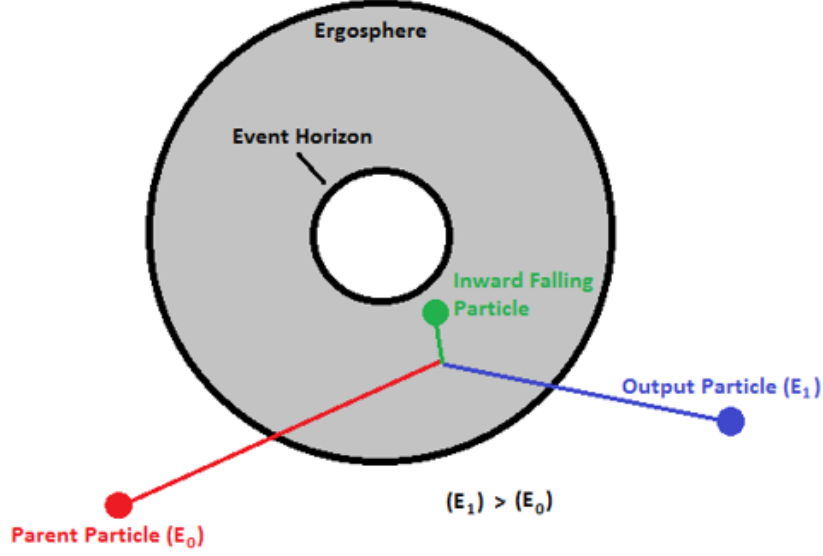
Figure 5: Mechanical Penrose Process. A parent particle falls from infinity towards the black hole and decays into a new ingoing particle and an outgoing particle. The second in-falling particle can have negative energy within the ergoregion allowing the outgoing particle to be detected with increased energy from a distant observer.

conservation of 4-momentum is expanded to the following 3 constraints,

$$\epsilon_1 = \mu_2 \epsilon_2 + \mu_3 \epsilon_3 \Rightarrow \boxed{\epsilon_2 = \frac{\epsilon_1 - \mu_3 \epsilon_3}{\mu_2}} \tag{25}$$

$$\dot{t}_1 v_1^r = \mu_2 \dot{t}_2 v_2^r \Rightarrow \boxed{v_2^r = -\frac{\dot{t}_1 v_1^r}{\mu_2}} \tag{26}$$

$$l_1 = \mu_2 l_2 + \mu_3 l_3 \Rightarrow \boxed{l_2 = \frac{l_1 - \mu_3 l_3}{\mu_2}} \tag{27}$$

where $\epsilon \equiv \frac{E}{m}$ and $l \equiv \frac{L}{m}$ are the specific energy and specific angular momentum respectively and μ_2 and μ_3 are the ratios of the masses of particles 2 and 3 over the mass of the parent particle, $\mu_2 \equiv \frac{m_2}{m_1}$, $\mu_3 \equiv \frac{m_3}{m_1}$. In addition, we have a constraint following from $p^2 = -m^2$,

$$p_2^2 = (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 \Rightarrow \boxed{\mu_2 = \sqrt{1 + \mu_3^2 + 2\mu_3 u_1 \cdot u_3}} \tag{28}$$

From these 4 equations, we can solve for v_2^r, v_2^ϕ, μ_2 and μ_3 given the ϕ-component of the coordinate velocity v_3^ϕ of particle 3. This is chosen to ensure that it has negative energy, that is, we take the splitting to take place within the ergoregion $r_H < r < r_E$ and $v_3^\phi < -\frac{g_{tt}}{g_{t\phi}}$.

To sum up so far, the known quantities are $\vec{v}_1 = (v_1^r, 0, v_1^\phi)$ and $\vec{v}_3 = (0, 0, v_3^\phi)$, while the unknown quantities are $\vec{v}_2 = (v_2^r, 0, v_2^\phi)$, μ_2 and μ_3.

Solving (26), we find that,

$$v_2^r = \sqrt{-\frac{\lambda_1^2}{\lambda_1^2 g_{rr} + \mu_2^2}\left(g_{tt} + 2g_{t\phi}v_2^\phi + g_{\phi\phi}(v_2^\phi)^2\right)} \tag{29}$$

where $\lambda_1 \equiv \dot{t}_1 v_1^r$. We have chosen the "+" sign to ensure that particle 2 heads out and is not in danger of getting too close to the event horizon before escaping to infinity. Plugging this expression into the expression of $l_2 \equiv u_{2\phi}$ and using (27), we find that,

$$v_2^\phi = g_{tt}\frac{-g_{t\phi} + \sqrt{-\psi}\sqrt{k(k+1)}}{g_{t\phi}^2 + kg_{tt}g_{\phi\phi}} \tag{30}$$

where the parameter k is given by,

$$k = \left(\frac{\mu_3 l_3}{\mu_2}\right)^2\left(1 - \frac{\lambda_1^2}{\lambda_1^2 g_{rr} + \mu_2^2}\right) \tag{31}$$

Finally, use this result of v_2^ϕ to also write v_2^r in terms of μ_2 and μ_3 and use (25) and (28) to solve for μ_2 and μ_3. This could in principle be done analytically but the calculations are too much so we instead be lazy and solve the system computationally using the Newton-Raphson method.

3.3 Constraints on μ_2 and μ_3

We can see that $v_2^r > 0$ always exists[4]. The existence of v_2^ϕ, however, requires that,

$$k(k+1) \geq 0 \Rightarrow k \geq 0 \ \ or \ \ k \leq -1 \tag{33}$$

The first case of $k \geq 0$ is always satisfied unless $g_{rr} \leq 1$ but that would require the particle to be inside the black hole[5]. On the other hand, $k \leq -1$ can never be achieved because it yields the condition,

$$\mu_3^2 \leq \frac{\mu_2^2}{g_{tt}l_3^2}\frac{\lambda_1^2 g_{rr} + \mu_2^2}{\lambda_1^2(1 - g_{rr}) - \mu_2^2} < 0 \tag{34}$$

The only true constraint on μ_2 and μ_3 is the requirement of particle 2 reaching infinity. This requires $\epsilon_2 \geq 1$ so,

$$\boxed{\mu_2 \leq \epsilon_1 - \mu_3\epsilon_3} \tag{35}$$

---

[4]One could argue that $g_{tt} + 2g_{t\phi}v_1^\phi + g_{\phi\phi}(v_1^\phi)^2$ must be negative, but this follows directly from the fact that the particles are massive. This means that,

$$-\dot{t}^{-2} = g_{tt} + 2g_{t\phi} + g_{rr}(v^r)^2 + g_{\phi\phi}(v^\phi)^2 < 0$$
$$\Rightarrow g_{tt} + 2g_{t\phi} + g_{\phi\phi}(v^\phi)^2 < -g_{rr}(v^r)^2 < 0 \tag{32}$$

given that the particle is outside the black hole ($r > r_H \Rightarrow g_{rr} > 0$).

[5]Expanding $g_{rr} \leq 1$ from (17), we find that $r \leq \frac{a^2+Q^2}{R_S}$. Physical black holes are not naked singularities, meaning that the event horizon exists, i.e. $\frac{a^2+Q^2}{R_S} \leq \frac{R_S}{4}$. Consequently, if $g_{rr} \leq 1$, then $r \leq \frac{R_S}{4}$ but $\frac{R_S}{4} < r_H$.

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A natural process will also have $\mu_2^2 + \mu_3^2 < 1$ because,

$$p_1^2 = p_2^2 + p_3^2 + 2p_2 \cdot p_3 \Rightarrow \mu_2^2 + \mu_3^2 = 1 + 2p_2 \cdot p_3 < 1 \tag{36}$$

since particles 2 and 3 are timelike-separated, i.e. $p_2 \cdot p_3 < 0$.

3.4   Run 3: Results

The rotating black hole we simulated has $J = 0.9$ and $Q = 0.0$. We firstly solved the geodesics using part A to simulate the propagation of the parent particle from very far $r = 100R_S = 200.0$ until somewhere near the black hole $r \sim r_H$. Then we used those values of $\{t, \vec{x}, \vec{v}\}$ for the parent particle at $r \sim 6.0$ as initial conditions to simulate the process. We propagated the parent particle until $r_H < r < r_E$ and then perform the splitting into particles 2 and 3 according to the above discussion.

The initial conditions for the parent particle were,

t_{10}	3176.535
r_{10}	6.103719
θ_{10}	$\frac{\pi}{2}$
ϕ_{10}	0.072048
v_{1r0}	-0.386224
$v_{1\theta 0}$	0.0
$v_{1\phi 0}$	0.00769361

Table 5: Run 3: Initial Conditions for parent particle.

We set the final time instant to $t_N = 3191.4575$ for which the final position for particle 1 and initial positions for particles 2 and 3 are,

$t_{02,3}$	3191.4425775
$r_{02,3}$	1.72223189
$\theta_{02,3}$	$\frac{\pi}{2}$
$\phi_{02,3}$	1.15216234

Table 6: Run 3: Initial positions for particles 2 and 3.

After solving the equations for μ_3, μ_2 and the initial velocity for particle 2, we arrive at the following results,

v_{2r0}	0.08684806
$v_{2\theta 0}$	0.0
$v_{2\phi 0}$	0.22415412
v_{3r0}	0.0
$v_{3\theta 0}$	0.0
$v_{3\phi 0}$	0.09949532

Table 7: Run 3: Initial velocities for particles 2 and 3.

μ_2	0.738561712
μ_3	$1.47921968 \cdot 10^{-5}$

Table 8: Run 3: Masses of particles 2 and 3.

The specific energies of particles 1,2 and 3 are then,

ϵ_1	0.994987723
ϵ_2	1.37619187
ϵ_3	$-1.447716 \cdot 10^3$

Table 9: Run 3: Specific energies of particles 1, 2 and 3.

Finally, we solved the geodesics for particles 2 and 3 separately with final time instants $t_{2N} = 5000.0$, $t_{3N} = 3206.442575$ and number of time steps $N_2 = N_3 = 15000$. The relevant graphs are figures

4 Comments and Discussion

4.1 Part A: Solving Geodesics

As is obvious, the trajectories simulated are in agreement with the expected observations. For the first run, the trajectory is bounded and the energy ($E = -mu_0$) and angular momentum ($L = mu_3$) are conserved. The conservation laws also hold in the second in-falling trajectory but some expected instabilities seem to add up as the particle approaches more and more the event horizon at $r = 2.0$. This just a computational artifact coming from the singular behavior of the metric at the event horizon. What is remarkable is that the particle slows down as it approaches the event horizon until it eventually stop at $r = 2.0$; this is the expected behavior as seen by a stationary observer.

4.2 Part B: Simulating the Penrose Process

The results speak for themselves. Particle 3 does indeed have a negative energy, while particle 2 has sufficient energy to escape to infinity ($\epsilon_2 > 1$). The final terminal velocity will have
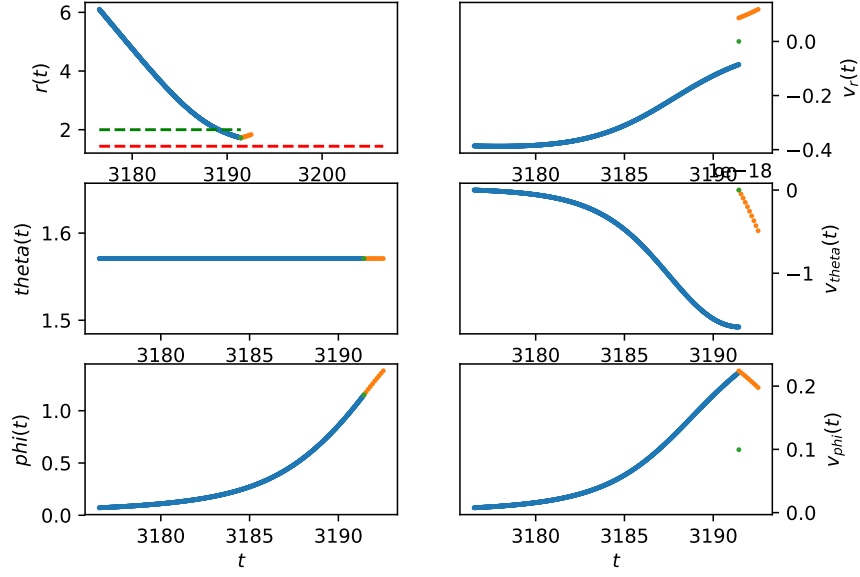
Figure 6: Run 3: Coordinate positions and velocities for the simulated Penrose process.
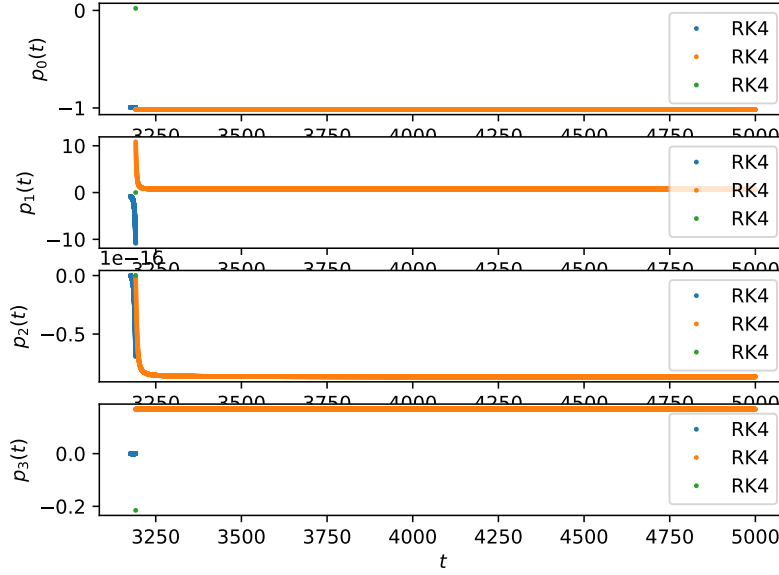

Figure 7: Run 3: 4-momentum for the simulated Penrose process.

only a radial non-zero component with value[6],

$$v_{2\infty} = \sqrt{1 - \frac{1}{\epsilon_2^2}} = 0.687015393 \tag{38}$$

The efficiency of the particular process is,

$$\eta = \frac{E_2}{E_1} = \mu_2 \frac{\epsilon_2}{\epsilon_1} = 102.15\% \tag{39}$$

In other words, particle 2 has gained 2.15% more energy than the initial parent particle!

5   Perspectives

The analysis demonstrated has been successful to simulate the Penrose process. Further applications would be to investigate more into the limits of the process in terms of its efficiency. An even more interesting extension would be to add a non-zero electrical charge on the particles. Then, there will be an additional Lorentz force that actually favors the Penrose process ([1]). Simulation of this generalized Penrose process could shine more light on the natural mechanisms allowing a physical black hole to appear energetic to a distant observer.

Eventually, we would like to also simulate the Blandford-Znajek process ([2]) which is believed to be the main mechanism explaining the incredible power emitted by quasars but for now, we submit the process so far.

A   Derivation of equations of motion

In this appendix, we derive the equations of motion (6) for a massive particle. To derive them, we start from the geodesics equations,

$$\ddot{x} + \Gamma^\rho_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = a^\rho \tag{40}$$

and write all the derivatives with respect to the proper time τ in term of derivatives with respect to the coordinate time t. For the D-velocity, we have,

$$\dot{x}^\mu = \dot{t}v^\mu \tag{41}$$

---

[6]Since energy is conserved, we simply set the asymptotic behavior of the metric $g_{\mu\nu} \to \eta_{\mu\nu}$,

$$\epsilon = u_0 \to \frac{1}{\sqrt{1 - v_{r\infty}}} \tag{37}$$

while, for the D-acceleration, we have,

$$\ddot{x}^\mu = (\dot{x}^\mu)^{\cdot}$$

$$= \ddot{t}v^\mu + \dot{t}^2 \frac{dv^\mu}{dt} \tag{42}$$

$$= \dot{t}^2 \left(\frac{dv^\mu}{dt} - \Gamma^0_{\mu\nu} v^\mu v^\nu \right) + a^0 v^\mu$$

where we used the 0-component of the geodesics in the third line. Lastly, we use the normalization of the D-velocity for massive particles to express $\dot{t}$ in terms of the coordinate velocity,

$$u_\mu u^\mu = g_{\mu\nu} u^\mu u^\nu = \dot{t}^2 g_{\mu\nu} v^\mu v^\nu = -1$$

$$\Rightarrow \dot{t} = \frac{1}{\sqrt{-g_{\mu\nu} v^\mu v^\nu}} \tag{43}$$

and plug everything back to the geodesics to extract the equations of motion for the coordinate velocity,

$$\frac{dv^\rho}{dt} = \left(\Gamma^0_{\mu\nu} v^\rho - \Gamma^\rho_{\mu\nu} \right) v^\mu v^\nu + \frac{a^\rho - a^0}{\dot{t}^2} \tag{44}$$

$$= \left(\Gamma^0_{\mu\nu} v^\rho - \Gamma^\rho_{\mu\nu} - g_{\mu\nu}(a^\rho - a^0) \right) v^\mu v^\nu$$

B   Coordinate velocitiy Vs D-velocity

In this small appendix, we analyze the equivalence of giving the coordinate velocity components $v^\mu = (1, \vec{v})$ and giving the covariant[7] D-velocity u_μ. Although there are only $D - 1$ independent components for the coordinate velocity, there is the additional parameter of the rest mass m of the massive particle resulting in an equal number of spatial components of the coordinate velocity $\vec{v}$ and independent components of the D-velocity u_μ. The rest mass can be immediately obtained from,

$$p^2 = -m^2 \tag{45}$$

from which the condition $u^2 = -1$ for massive particles follows. The notation of the norm of a D-vector A^μ here is,

$$A^2 \equiv A_\mu A^\mu = g_{\mu\nu} A^\mu A^\nu \tag{46}$$

As a result we can determine the coordinate velocity $\vec{v}$ from $D - 1$ components of the of the D-velocity u_μ. To simplify things to situations that are very common, we assume that we know all the vectorial components of the coordinate velocity apart from the i'th component, that is, we know v^l, $l = 1, \ldots, i - 1, i + 1, \ldots, D - 1$ but not v^i. Then, given, one of the

---

[7]We consider the covariant D-velocity because it includes the usual conserved charges. For example, the energy is defined by $E = -mu_0$ and the angular momentum associated with rotations around the x_d-axes is given by $L = mu_d$ when using spherical coordinates.

components of the D-velocity, say the μ'th component, it can be shown that v^i is given by,

$$
\begin{aligned}
v^i &= \frac{-\beta_{i\mu} \pm \sqrt{\beta_{i\mu}^2 - \alpha_{i\mu}\gamma_{i\mu}}}{\alpha_{i\mu}} \\
\alpha_{i\mu} &= u_\mu^2 g_{ii} + g_{\mu i}^2 \\
\beta_{i\mu} &= u_\mu^2 \left(g_{0i} + g_{il}v^l \right) + g_{\mu i} \left(g_{\mu 0} + g_{\mu l}v^l \right) \\
\gamma_{i\mu} &= u_\mu^2 \left(g_{00} + 2g_{0l}v^l + g_{lm}v^m v^l \right) + \left(g_{\mu 0} + g_{\mu l}v^l \right)^2
\end{aligned}
\tag{47}
$$

where there is no sum over the mu and i indices, while the summation over m and l runs from 1 to $D-1$ excluding the value i. To decide the "+" or "−" sign, one just plugs v^i in the expression for u_μ and checks which sign gives the right answer.

For more practical applications of this procedure, we take the $\mu = i$. In this case, the results simplify and become,

$$
\begin{aligned}
v^i &= -\frac{g_{0i} + g_{il}v^l}{g_{ii}} + u_i\sqrt{\Delta_i} \\
\Delta_i &= \frac{\left(g_{0i} + g_{il}v^l \right)^2 - g_{ii} \left(g_{00} + 2g_{0l}v^l + g_{lm}v^m v^l \right)}{g_{ii}^2 \left(u_i^2 + g_{ii} \right)}
\end{aligned}
\tag{48}
$$

This prescription will prove useful for determining the initial conditions for the Penrose process to be observed.

C D-momentum Vs Conjugate momenta

Here, we emphasize on the difference between the D-momentum p_μ appearing in General Relativity versus the conjugate momenta of a particle system described by a Lagrangian L. The D-momentum components for a massive particle are defined by,

$$
p_\mu = mu_\mu
\tag{49}
$$

with u_μ the covariant D-velocity components. In terms of the coordinate velocity components, p_μ are written as,

$$
p_\mu = m\frac{g_{\mu\nu}v^\nu}{\sqrt{-g_{\rho\sigma}v^\rho v^\sigma}} = m\frac{g_{\mu 0} + g_{\mu i}v^i}{\sqrt{-\left(g_{00} + 2g_{0j}v^i + g_{jk}v^j v^k \right)}}
\tag{50}
$$

In the absence of any non-gravitational interactions, the D-momentum components are also the conjugate momenta,

$$
P_\mu = \frac{\partial \mathcal{L}}{\partial u^\mu}
\tag{51}
$$

with $\mathcal{L}$ the Lagrangian describing the system. The gravitational part reads,

$$
\mathcal{L}_{gr} = \frac{1}{2}mu^2 = \frac{1}{2}mg_{\mu\nu}u^\mu u^\nu
\tag{52}
$$

where we have adopted the notation for the norm of a D-vector A^μ,

$$A^2 \equiv A_\mu A^\mu = g_{\mu\nu} A^\mu A^\nu = g^{\mu\nu} A_\mu A_\nu \tag{53}$$

If the Lagrangian contains additive new terms $\mathcal{L}_{int}$ associated with non-gravitational interactions, e.g. electromagnetic interactions,

$$\mathcal{L} = \mathcal{L}_{gr} + \mathcal{L}_{int} \tag{54}$$

then the conjugate momenta are,

$$P_\mu = p_\mu + \frac{\partial \mathcal{L}_{int}}{\partial u^\mu} \tag{55}$$

The conjugate momenta are useful observables from which ordinary constants of motion can be read from. For example, the energy is defined by $E \equiv -P_0$.

The Euler-Lagrange equations then become the geodesics in the presence of non-gravitational forces,

$$\dot{u}^\rho + \Gamma^\rho_{\mu\nu} u^\mu u^\nu = a^\rho \tag{56}$$

with the non-gravitational D-acceleration a^ρ constructed entirely from the interaction Lagrangian,

$$
\begin{aligned}
a^\rho &= \frac{g^{\rho\mu}}{m} \left(\partial_\mu \mathcal{L}_{int} - \frac{d}{d\tau} \left(\frac{\partial \mathcal{L}_{int}}{\partial u^\mu} \right) \right) \\
&= \frac{g^{\rho\mu}}{m} \left(\partial_\mu \mathcal{L}_{int} - u^\nu \partial_\nu \left(\frac{\partial \mathcal{L}_{int}}{\partial u^\mu} \right) - \dot{u}^\nu \frac{\partial \mathcal{L}_{int}}{\partial u^\mu \partial u^\nu} \right)
\end{aligned}
\tag{57}
$$

In the above expansion of the D-acceleration we made the very natural assumption that the interaction Lagrangian only depends on the spacetime position x^μ and the D-velocity u^μ.

C.1   Example: Lorentz force

If the background metric describes an electrically charged mass distribution, e.g. a charged black hole, then a massive particle that also carries an electric charge q will also experience a Lorentz force. This is inputted in the Lagrangian by the term,

$$\mathcal{L}_{int} = q A_\mu u^\mu \tag{58}$$

with A_μ the x^μ-dependent covariant D-potential created by the charged mass configuration. We assume here that the background electromagnetic field created is stationary. Then the D-acceleration is,

$$a^\rho = \frac{q}{m} F^\rho{}_\sigma u^\sigma \tag{59}$$

with $F_{\mu\nu}$ the electromagnetic field strength tensor[8],

$$F_{\mu\nu} = 2\nabla_{[\mu} A_{\nu]} = 2\partial_{[\mu} A_{\nu]} \tag{61}$$

---

[8]The operation of anti-symmetrization of a rank-2 tensor $T_{\mu\nu}$ is,

$$T_{[\mu\nu]} \equiv \frac{1}{2} \left(T_{\mu\nu} - T_{\nu\mu} \right) \tag{60}$$

while the conjugate momenta read,

$$P_\mu = p_\mu + qA_\mu \tag{62}$$

References

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