GEODESICS AND THE PENROSE PROCESS

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Contents

GR FUNDAMENTALS
Lesson 1: Gravity is not a force

- Special relativity: “Everything is relavant (even space and time ≡ spacetime) except $c \equiv 299\,792\,458\; m/s$”
- General relativity: “Everything is falling except me” a.k.a. “(Strong) Equivalence principle”
  - In other words, “Locally, spacetime looks flat”
- Implications: Gravity is not a real force. Everything moves freely!
  - Flat spacetime: Free motion = Straight line
  - Curved spacetime: Free motion = Straightest possible line ≡ Geodesic

    →Non-”linearities” in trajectory interpreted as gravitational force measured from a local observer

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GR FUNDAMENTALS
Lesson 2: Sources of gravity

- Sources of gravity/curvature: Energy and momentum
- Einstein's field equations (EFE):

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- $G_{\mu\nu}$: "Einstein tensor" → Contains information about curvature
- $T_{\mu\nu}$: "Energy-momentum tensor" → Contains information about energy and momentum of matter, EM radiation, etc.
- John Archibald Wheeler: "Spacetime tells matter how to move (geodesics); matter tells spacetime how to curve (EFE)"

GR FUNDAMENTALS
Lesson 3: Metric tensor

- Invariant length ds: An invariant measure of how close two events are:

$$ds^2 = -c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$$

  - $\{x^\mu | \mu = 0,1,\ldots,D-1\}$: Coordinate system ($D$: # of spacetime dimensions)
  - $\mu = 0$: Temporal index
  - $\mu = i, i = 1,\ldots,D-1$: Spatial indices
  - $x^0 \equiv t$: Coordinate time (time used by static observer)
  - τ: Proper time (time used by local/"falling" observer)
  - $g_{\mu\nu}$: Metric tensor (inner product of coordinate basis vectors)
    → Contains information about lengths and relative angles

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GEODESICS

- Trajectory of point particles in background spacetime = Path of extreme invariant length:

$$S \sim \int ds = \int d\lambda \sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}$$

- λ: "Affine parameter" → Makes acceleration perpendicular to velocity
  - Essentially a rescaling of proper time to also work for massless particles for which $ds = d\tau = 0$
- Principle of extreme action $\delta S = 0 \Rightarrow$ Geodesics:

$$\boxed{\ddot{x}^{\rho} + \Gamma^{\rho}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0}$$

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_{\mu}g_{\sigma\nu} + \partial_{\nu}g_{\mu\sigma} - \partial_{\sigma}g_{\mu\nu}\right)$$

- $\Gamma^{\rho}_{\mu\nu}$: "Affine Connection" or "Christoffel Symbols"

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- For a static observer the geodesics read:

$$\boxed{\boldsymbol{\frac{dx^i}{dt} = v^i}}$$

$$\boxed{\boldsymbol{\frac{dv^i}{dt} = -\Gamma^i_{00} + \left(\Gamma^0_{00}\delta^i_j - 2\Gamma^i_{0j}\right)v^j - \left(\Gamma^i_{jk} - 2\Gamma^0_{0j}\delta^i_k\right)v^j v^k + \Gamma^0_{jk} v^i v^j v^k}}$$

- These are the equations we want to solve!
- Important observables: (Conjugate) momenta p_μ

$$p_\mu = m u_\mu = m \frac{g_{\mu\nu} v^\nu}{\sqrt{-g_{\rho\sigma} v^\rho v^\sigma}}$$

$$v^\mu = (1, \vec{v})$$

→**PART (A) OF PROJECT:**

SOLVE THESE EQUATIONS FOR ANY BACKGROUND IN ANY SPACETIME DIMENSIONALITY D

PART (A) PROGRAM

- To solve the geodesics, we first have to generate them
- Inputs:
  - Coordinate system $\{x^\mu\}$
  - Background geometry, i.e. $g_{\mu\nu}$
- Python library sympy:
  - Python version of MATHEMATICA
  - Allows to define symbolic functions and, thus, take exact derivatives
  - Allows to convert symbolic functions to actual python functions: “lambdification”
- After generating geodesics, input initial conditions and solve using an integrator
- Integrators implemented:
  - Forward Euler, RK2, RK4, Leapfrog, Verlet, Adaptive RK2, Adaptive RK4
- Units used: $G = c = 1$ (Geometrical units)

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Schwarzschild solution

- Vacuum solution of EFE for a spherically symmetric, stationary, electrically neutral star of mass M in asymptotically flat spacetime
  - Schwarzschild solution in "spherical" coordinates:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2$$

$$f(r) = 1 - \left(\frac{R_S}{r}\right)^{D-3}$$

  - $d\Omega_{D-2}^2$ = initesimal length on a unit $(D-2)$-dimensional sphere
  - R_S: "Schwarzschild radius"

$$R_S^{D-3} = \frac{16\pi M}{(D-2)\Omega_{D-2}}$$

  - In $D = 4$, $R_S = 2M$
- If the star has a radius $R < R_S$, it forms a **black hole**
  - R_S marks the **event horizon**: Limit of "no-return"
  - The star's mass is concentrated at $r \to 0$: Singularity (Physics break)
- **Bound trajectories for $E < m$**

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RUN 1: BOUND TRAJECTORY

- Geodesics generator: Schwarzschild solution in $D = 4$
- Initial conditions:
  - $t_0 = 0,\ r_0 = 10.0,\ \theta_0 = \frac{\pi}{2},\ \phi_0 = 0$
  - $v_{r0} = -0.03,\ \ v_{\theta 0} = 0,\ v_{\phi 0} = 0.03$
- Solver: RK4 with $N = 10000$ steps
- Periodic trajectory with period $T \simeq 240$
- Clear perihelion/aspidal precession
- Beautiful picture after many rotations!

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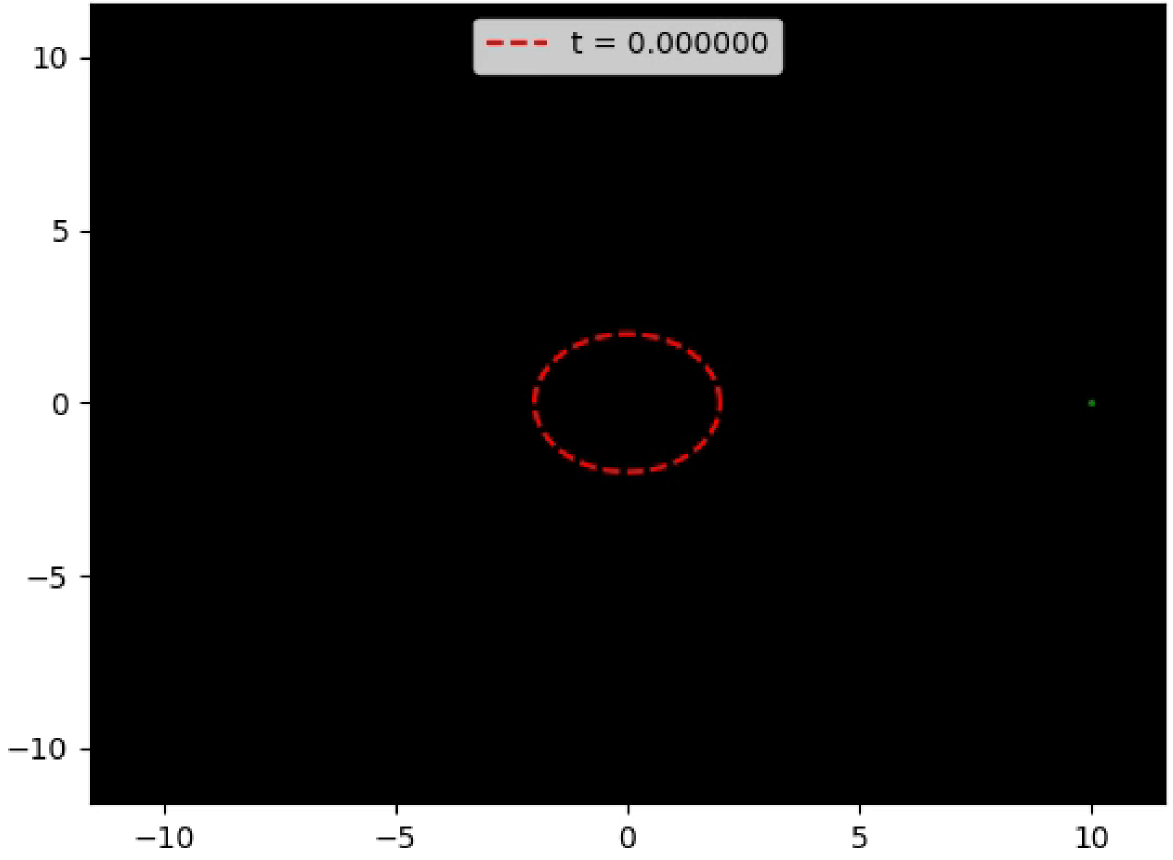

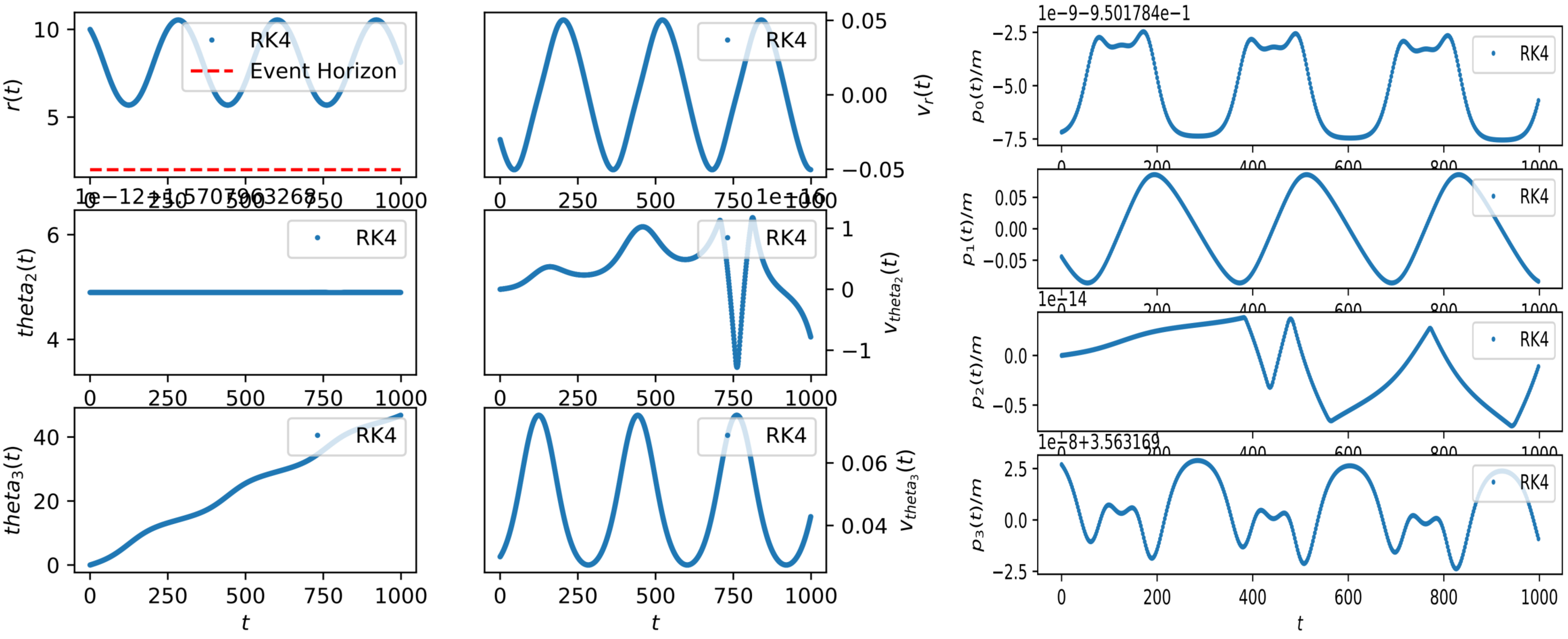
r(t)
RK4
Event Horizon
v_r(t)
1e−12+1.570796326
theta_2(t)
1e−16
v_theta_2(t)
theta_3(t)
v_theta_3(t)
t
1e−9−9.501784e−1
p_0(t)/m
p_1(t)/m
1e−14
p_2(t)/m
1e−8+3.563169
p_3(t)/m

RUN 2: INFALLING TRAJECTORY

- Geodesics generator: Schwarzschild solution in $D = 4$
- Initial conditions:
  - $t_0 = 0,\ r_0 = 3.0,\ \theta_0 = \frac{\pi}{2},\ \phi_0 = 0$
  - $v_{r0} = -0.3,\ \ v_{\theta 0} = 0,\ v_{\phi 0} = 0.03$
- Solver: RK4 with $N = 300$ steps
- Particle falls towards the event horizon but NEVER crosses it!
- Colored trajectory indicates toy-redshift
  - As the particle approaches the event horizon, it slows down and becomes more and more redshifted

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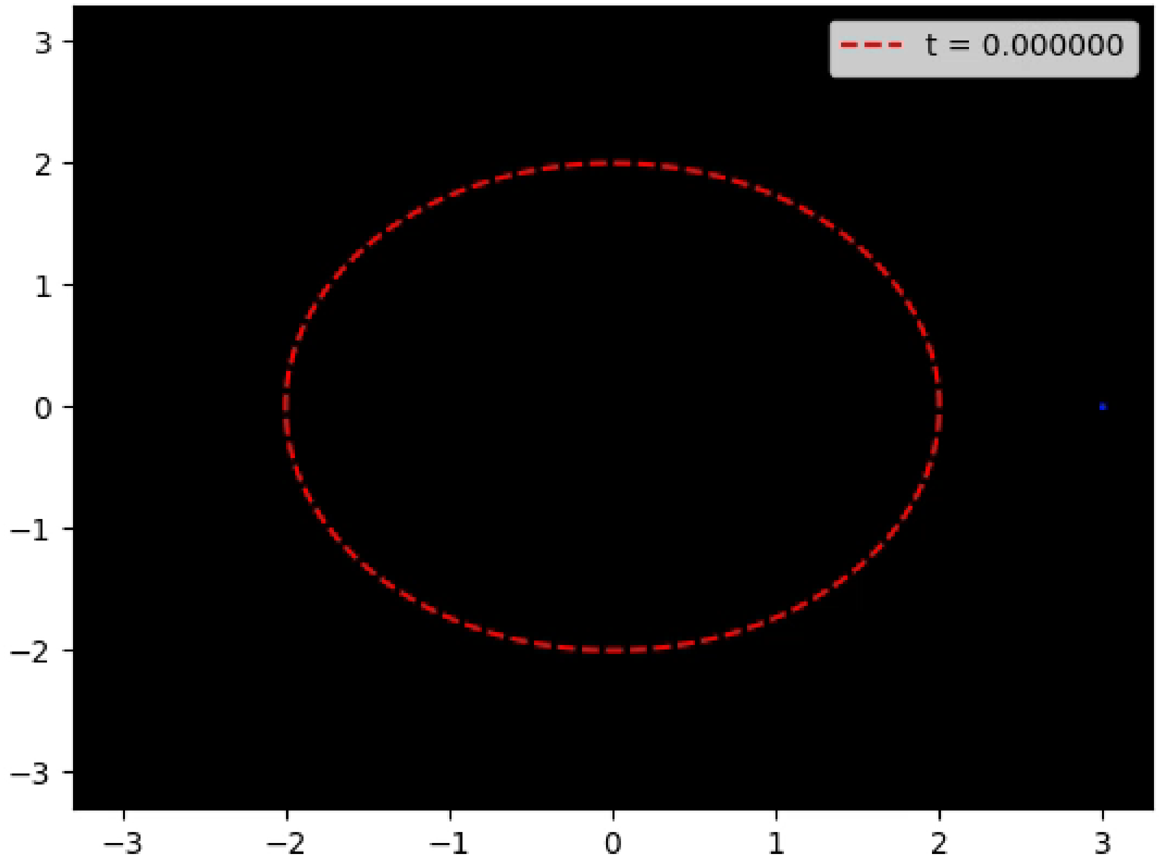

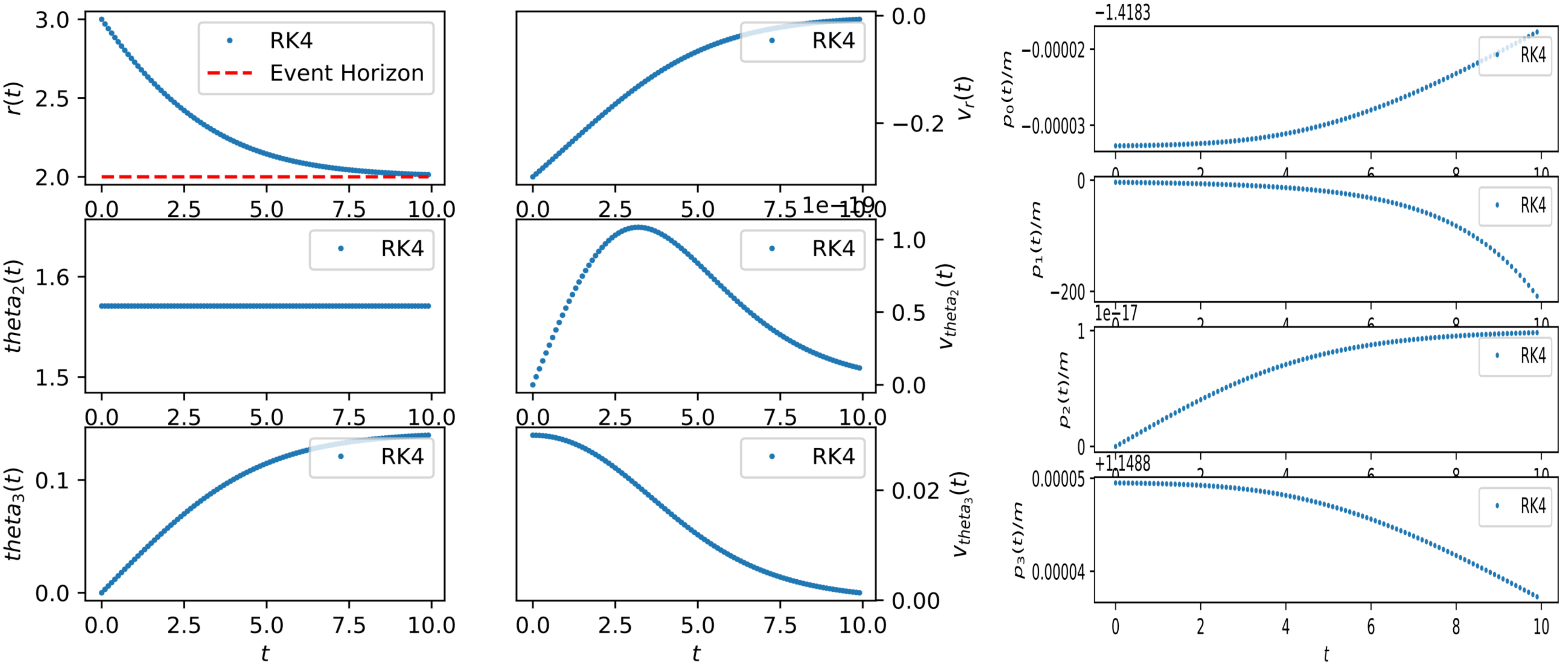
RK4
Event Horizon
r(t)
theta_2(t)
theta_3(t)
v_r(t)
v_theta_2(t)
v_theta_3(t)
1e−19
t
p_0(t)/m
p_1(t)/m
p_2(t)/m
p_3(t)/m
−1.4183
1e−17
+1.1488

Rotating Black holes in $D = 4$

- Vacuum solution of EFE for a stationary, electrically neutral star of mass M rotating around z-axis with angular momentum J in asymptotically flat spacetime
- Kerr's solution in "spherical" coordinates ($G = c = 1$):

$$ds^2 = -\left(1 - \frac{R_S r}{\Sigma}\right)dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{R_S r a^2}{\Sigma}\sin^2\theta\right)\sin^2\theta\, d\phi^2 - \frac{2R_S r a \sin^2\theta}{\Sigma}dt d\phi$$

  - $R_S = 2M$
  - $a = \frac{J}{M}$
  - $\Sigma = r^2 + a^2\cos^2\theta$
  - $\Delta = r^2 - R_S r + a^2$
- Event horizon(s):
  - $g^{rr} = 0 \Rightarrow r_H^{\pm} = M \pm \sqrt{M^2 - a^2}$
  - Physical event horizon: $r_H = r_H^{+} = M + \sqrt{M^2 - a^2}$ → For $a = 0,\ r_H = 2M = R_S$

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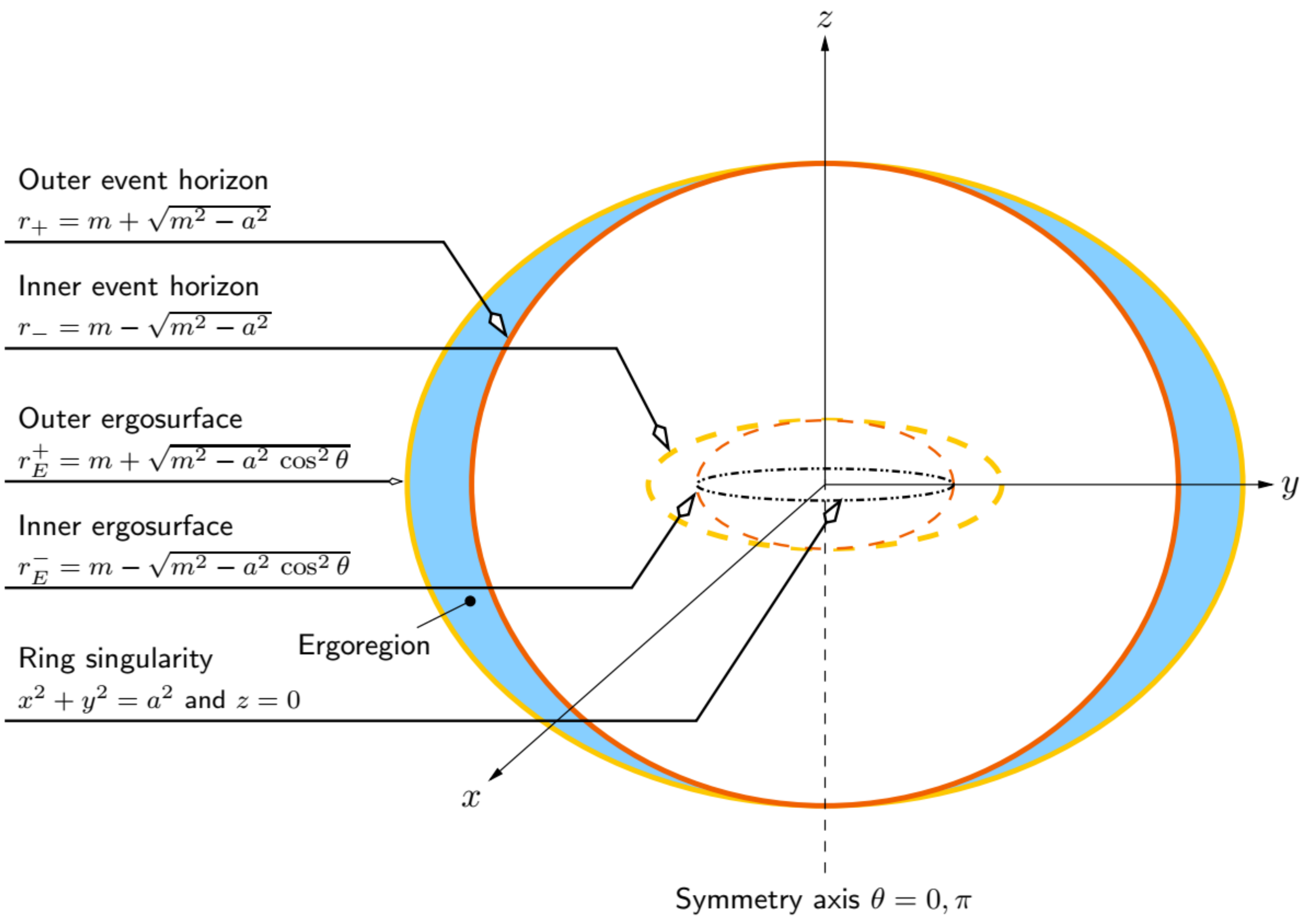


Figure 1 from Matt Visser. "*The Kerr spacetime: A Brief introduction*". 2007

- Ergosphere: Region within which the particle cannot stay still
  - $g_{tt} < 0 \Rightarrow r_E^- < r_{ERGO} < r_E^+$

$$r_E^\pm = M \pm \sqrt{M^2 - a^2 \cos^2\theta}$$

- Physical ergosphere:
  - $(r_H <)\, r_{ERGO} < r_E^+ \equiv r_E$

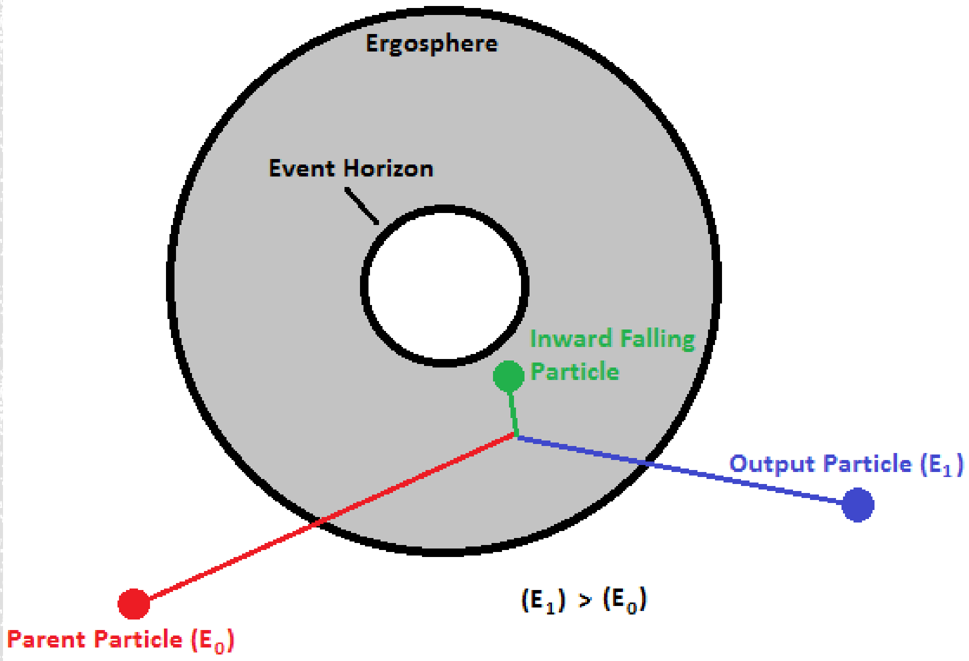
Penrose Process

- An infalling particle with energy E_1 enters the ergosphere and decays into an outgoing particle with energy E_2 and a new infalling particle with energy $E_* = E_1 - E_2$.

⚠ Energies E_1, E_2 are those measured by a distant observer

- Actual energy of the particle:
  - $E \equiv -p_t = m \frac{-g_{t\nu}v^{\nu}}{\sqrt{-g_{\rho\sigma}v^{\rho}v^{\sigma}}}$
  - For $r \to \infty$, $g_{\mu\nu} \to \eta_{\mu\nu} = diag(-1,1,1,1)$

$$\Rightarrow E \to \frac{m}{\sqrt{1 - \vec{v}_{\infty}^{\,2}}} = \gamma m$$



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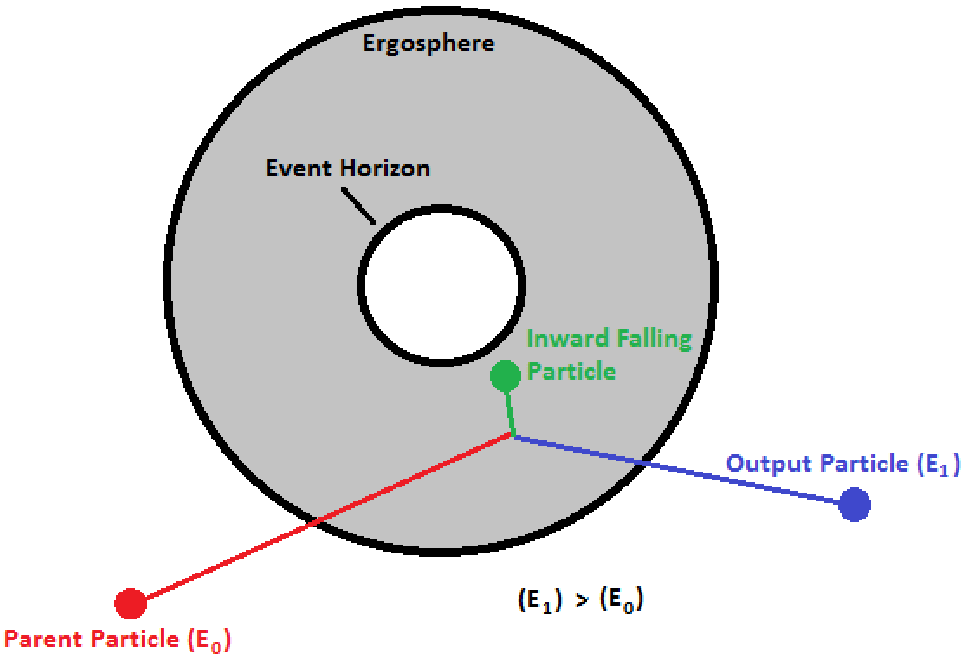

- $E_* = E_1 - E_2$ can become negative for $r < r_E$

  ⚠ The actual energy of the infalling particle is positive!

- $\Rightarrow E_2 > E_1$: The observed outgoing particle has gained energy! : Mechanical Penrose process

- Where did the energy come from?
  - $E_* < 0$ iff $L_* < 0$ and $r^2 - 2rR_S < \Delta \frac{m_*^2}{L_*^2}$ ($L \equiv p_\phi$ = Angular momentum of particle)
  - The infalling particle reduces the angular momentum of the rotating black hole: $J' = J + L_* < J$
  - Extra energy for outgoing particle by slowing down the rotation of the black hole

→**PART (B) OF PROJECT:**

SIMULATE THE MECHANICAL PENROSE PROCESS

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PART (B) PROGRAM

- Step 1: Generate geodesics using PART (A) for the Kerr solution
- Step 2: Solve geodesics 2 times
  - 1 for infalling particle
  - 1 for outgoing particle after decay in the ergosphere
- Initial conditions for initial infalling particle must be such that the particle reaches the ergosphere
- Initial conditions for outgoing particle:
  - $\vec{x}_{20} = \vec{x}_{*0} = \vec{x}_{1f}$
  - Initial velocity of outgoing particle must satisfy conservation of 4-momentum:

$$p_{1\mu} = p_{2\mu} + p_{*\mu}$$

$\Rightarrow$ Rest masses of new particles must satisfy:

$$\left(\frac{m_2}{m_1}\right)^2 = 1 + \left(\frac{m_*}{m_1}\right)^2 + 2\frac{m_*}{m_1}\frac{g_{\mu\nu}v_{1f}^{\mu}v_{*0}^{\nu}}{\sqrt{g_{\rho\sigma}g_{\kappa\lambda}v_{1f}^{\rho}v_{1f}^{\sigma}v_{*0}^{\kappa}v_{*0}^{\lambda}}}$$

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$$p_{1\mu} = p_{2\mu} + p_{*\mu}$$

$\Rightarrow$ Rest masses of new particles must satisfy:

$$\left(\frac{m_2}{m_1}\right)^2 = 1 + \left(\frac{m_*}{m_1}\right)^2 + 2\frac{m_*}{m_1}\frac{g_{\mu\nu}v_{1f}^{\mu}v_{*0}^{\nu}}{\sqrt{g_{\rho\sigma}g_{\kappa\lambda}v_{1f}^{\rho}v_{1f}^{\sigma}v_{*0}^{\kappa}v_{*0}^{\lambda}}}$$

- To gain energy from the outgoing particle, the following condition must hold:

$$E_2 > E_1$$

$$\Rightarrow \frac{m_2}{m_1}\sqrt{\frac{1-v_{1\infty}^2}{1-v_{2\infty}^2}} > 1$$

- Still not consciously implemented…
- PLAN:
  - Let initial particle fall from rest from infinity
  - **Analyze better what the initial conditions for the outgoing particle must be**
  - RUN!
- Numerically, we cannot let a particle fall from infinity or sent an outgoing particle there
  - Effectively, set numerical infinity very far away
  - How far?
    - Such that $\Delta g_{\mu\nu} \equiv \left|g_{\mu\nu} - \eta_{\mu\nu}\right| = \sigma \ll 1$
    - For equatorial ($\theta = \frac{\pi}{2}$) motion $\Rightarrow r_\infty \simeq \frac{R_S}{\sigma}$

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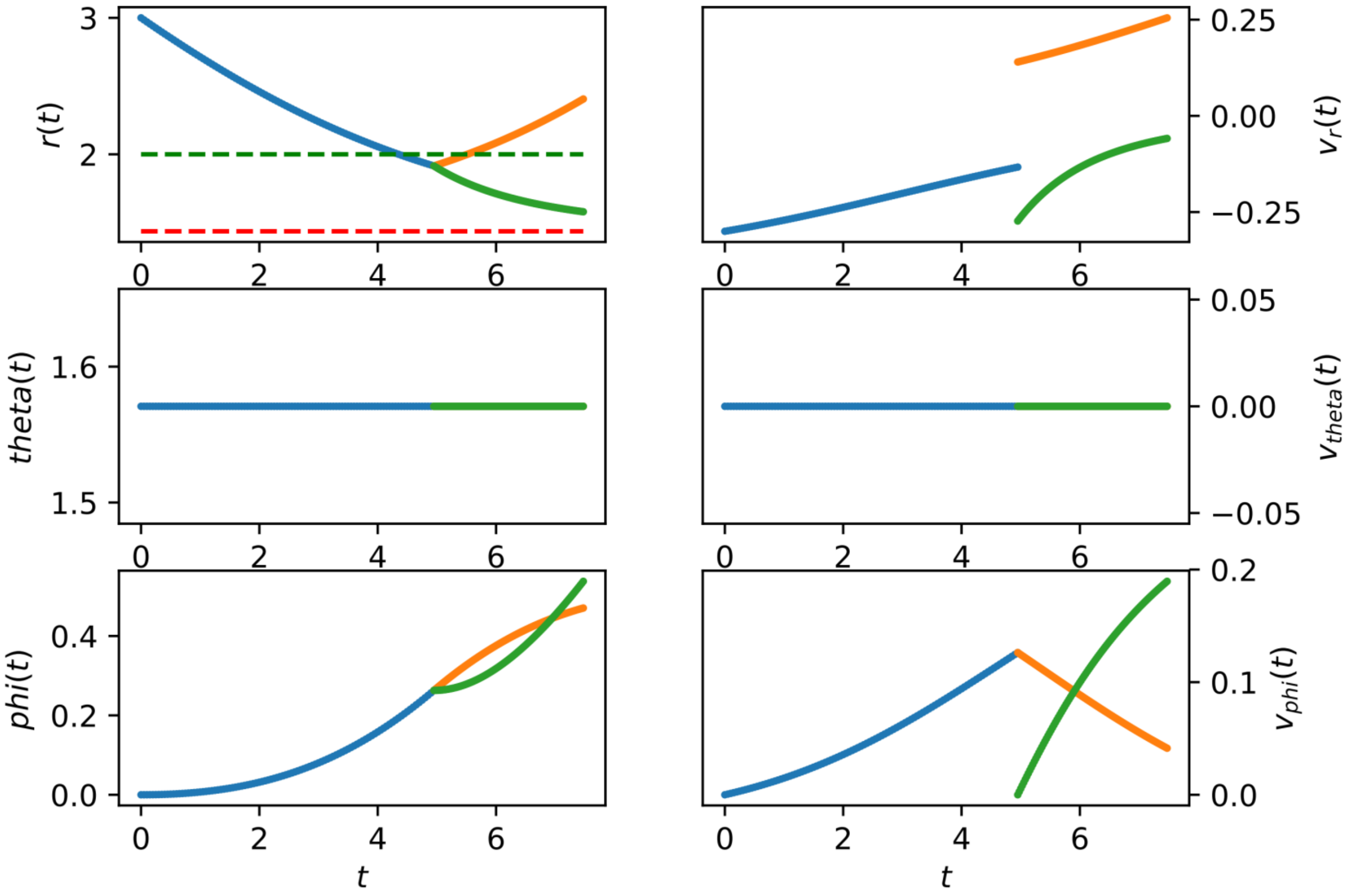

- Just a taste…
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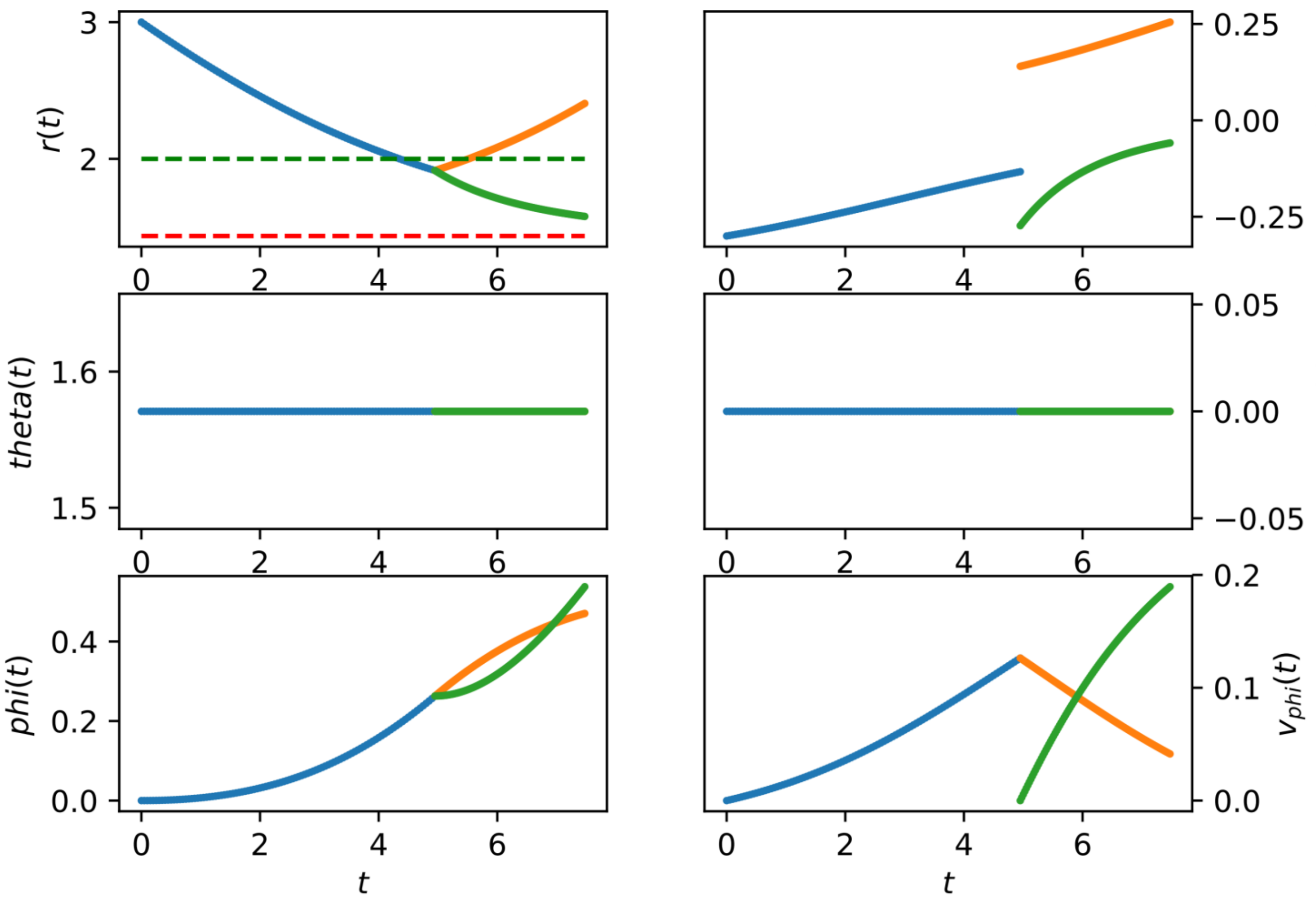

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QUESTIONS?

Schwarzschild radius in $D = 4$

- Newtonian mechanics: Energy for motion in the presence of a spherically symmetric mass distribution:

$$E = \frac{1}{2}mv^2 - \frac{GmM}{r}$$

- Escape velocity: $E = 0$
- When $v = c$, $r = R_S$:

$$\frac{1}{2}mc^2 - \frac{GmM}{R_S} = 0 \Rightarrow R_S = \frac{2GM}{c^2}$$

- In $D = 4$, $d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2 = dx^2 + dy^2 + dz^2$ (for $r = 1$)