Appendix 1

Vector Identities and Theorems

(1) Grad, div and curl in Cartesian coordinates:

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

$$\nabla\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\nabla\times\mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}$$

In cylindrical polar coordinates

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{\partial\phi}{\partial z}\hat{\mathbf{e}}_z$$

$$\nabla\cdot\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}$$

$$\nabla\times\mathbf{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{\mathbf{e}}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(rF_\theta) - \frac{1}{r}\frac{\partial F_r}{\partial\theta}\right)\hat{\mathbf{e}}_z$$

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

(2) Vector identities:

$$
\begin{aligned}
\nabla \cdot (\phi \mathbf{u}) &= \phi \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \phi \\
\nabla \times (\phi \mathbf{u}) &= \phi \nabla \times \mathbf{u} + \nabla \phi \times \mathbf{u} \\
\nabla (\mathbf{u} \cdot \mathbf{v}) &= \mathbf{u} \times \nabla \times \mathbf{v} + \mathbf{v} \times \nabla \times \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} \\
\nabla \cdot (\mathbf{u} \times \mathbf{v}) &= \mathbf{v} \cdot \nabla \times \mathbf{u} - \mathbf{u} \cdot \nabla \times \mathbf{v} \\
\nabla \times (\mathbf{u} \times \mathbf{v}) &= \mathbf{u}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{u}) + (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} \\
\nabla \times (\nabla \times \mathbf{u}) &= \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u} \\
\nabla \times \nabla \phi &= 0 \\
\nabla \cdot \nabla \times \mathbf{u} &= 0
\end{aligned}
$$

(3) Integral theorems:

$$
\begin{aligned}
\int_V \nabla \cdot \mathbf{F} dV &= \oint_S \mathbf{F} \cdot \mathbf{n} da \\
\int_V \nabla \phi dV &= \oint_S \phi \mathbf{n} da \\
\int_V (\nabla \times \mathbf{F}) dV &= \oint_S (\mathbf{n} \times \mathbf{F}) da \\
\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} da &= \oint_C \mathbf{F} \cdot d\mathbf{l} \\
\int_S (\mathbf{n} \times \nabla \phi) da &= \oint_C \phi d\mathbf{l}
\end{aligned}
$$

(4) Navier–Stokes equations in cylindrical polar coordinates:

$$
\begin{aligned}
\frac{\partial}{\partial t} u_r + (\mathbf{u} \cdot \nabla) u_r - \frac{u_\theta^2}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) \\
\frac{\partial u_\theta}{\partial t} + (\mathbf{u} \cdot \nabla) u_\theta + \frac{u_r u_\theta}{r} &= -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \nu \left(\nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right) \\
\frac{\partial u_z}{\partial t} + (\mathbf{u} \cdot \nabla) u_z &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 u_z
\end{aligned}
$$