Appendix 2

Stability Criteria for Ideal MHD Based on the Hamiltonian

This appendix is a generalisation of the discussion of stability given in §6.4. There are several systematic approaches to developing sufficient conditions for the stability of ideal (conservative) flows. The Kelvin–Arnold[1] variational principle, and the energy–Casimir[2] method are, perhaps, the best known (see Morrison, 1998, or Davidson, 1998, for a review of these). Both methods are, in effect, elaborate procedures for constructing an (energy-like) functional which is (i) quadratic in the disturbance, and (ii) conserved by the linearised dynamics. Provided the resulting integral invariant is non-zero for all possible disturbance shapes, it can be used like a Lyaponov functional to bound the growth of disturbances. That is to say, if $\|\delta\mathbf{u}\|$ is some norm for the disturbance, and $\delta^2 F$ a conserved quadratic function of $\delta\mathbf{u}$ then the flow will be unstable if $\|\delta\mathbf{u}\|$ grows despite the conservation of $\delta^2 F$, and so for instability we require $\|\delta\mathbf{u}\|^2/\delta^2 F \to \infty$. Consequently, if there exist bounds of the form $|\delta^2 F| \geq \lambda\|\delta\mathbf{u}\|^2$for all $\delta\mathbf{u}$, then the flow cannot be unstable. In short, stability is ensured if $\delta^2 F$ is positive or negative definite.

However, as we shall see, there exists a third procedure for creating a conserved, quadratic functional. Like the Kelvin–Arnold and energy–Casimir methods it relies (in some sense) on the conservation of energy. However, unlike these other methods, it is the Lagrangian, L, rather than the energy, E, which plays the central rôle. We shall describe this procedure in a moment, but we might note in passing that it relies on expanding the Lagrangian up to quadratic terms in particle displacement, using Lagrange's equations to discard the first variation in L, and then con-

[1] This principle is often attributed to Arnold, but actually is was first stated (without proof) by Kelvin in 1887 (see Phil. Mag. **23**, 529–539.) Indeed, Kelvin used it to prove what is now known as Rayleigh's inflection point theorem. It was later rediscovered by Arnold in 1966.

[2] A Casimir is any integral invariant other than energy.

structing a conserved Hamiltonian for the truncated system. In order to differentiate this procedure from the Kelvin–Arnold and energy–Casimir methods we briefly summarise these other approaches.

In the Kelvin–Arnold method the appropriate functional is the disturbance energy $\Delta E = E - E_0$, where E_0 is the energy of the base flow. Evidently, ΔE is conserved by the perturbed flow. However, in order to ensure that ΔE is quadratic in the disturbance it is necessary to insist that $\delta^1 E = 0$. It turns out that this can be achieved by restricting the choice of disturbances to those which conserve the topological (frozen-in) invariants of the flow. (Such perturbations are termed isovortical perturbations in the case of Euler flows, or generalised isovortical perturbations for other systems.) In such cases, $\delta^2 E$ provides a conserved, quadratic measure of the disturbance (as far as the linearised dynamics are concerned), and stability to infinitesimal disturbances is then ensured if $\delta^2 E$ is positive or negative definite. The art of applying the Kelvin–Arnold variational principle lies in spotting how to conserve all of the topological (frozen-in) invariants when calculating ΔE, i.e. knowing how to construct the generalised isovortical perturbations. This is readily achieved for Euler flows where it is necessary only to ensure that Ω is frozen-in during the disturbance. However, it becomes quite intricate when it comes to MHD, where it becomes necessary to ensure that $\mathbf{B}$ is frozen-in as well as to conserve the cross-helicity of $\mathbf{B}$ and $\mathbf{u}$.

In the energy–Casimir method, on the other hand, the appropriate functional is $A = E + C$ where C (the Casimir) is an integral invariant for the flow which reflects, as generally as possible, the frozen-in (topological) invariants such as helicity, cross-helicity etc. If C is constructed in a sufficiently general way, then it is usually possible to choose the precise form of C such that $\delta^1 A = 0$ at equilibrium (i.e. we choose C so that $\delta^1 C = -\delta^1 E$). Linear stability is then ensured if $\delta^2 A$ is positive or negative definite. The Kelvin–Arnold and energy–Casimir methods are, in fact, closely related, with C playing the rôle of a Lagrange multiplier, effectively building in the topological constraints required by the Kelvin–Arnold method (see, e.g. Davidson, 1998)

The use of conserved, quadratic functionals (which are non-zero for all possible disturbance shapes) to bound the growth of perturbations is often referred to as establishing *formal stability*.

We now differentiate our procedure from the Kelvin–Arnold and energy–Casimir methods. A trivial example taken from mechanics will suffice to show the difference. Consider a particle of mass m moving in a circular orbit of radius r_0 under the influence of the radial force

$\mathbf{F} = f(r)\hat{\mathbf{e}}_r$. Suppose that f has potential $V, f = -V'(r)$ and let $\Gamma = r^2\dot{\theta}$ be the angular momentum of the particle. (We restrict ourselves to two-dimensional motion and use polar coordinates r and θ.) We now perturb the trajectory, $r = r_o + \eta$, $\theta = \theta_0 + \zeta/r_0$, and examine the linear stability of the perturbed trajectory. For this simple system a conventional perturbation analysis provides the necessary and sufficient conditions for stability. The trajectory is stable if and only if $[r_0^3 V_0'' + 3r_0^2 V_0'] > 0$.

Let us now see if we can obtain the same information using the energy principles described above. The energy of the particle on the perturbed path is

$$E = T + V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V = E_0 + \delta^1 E + \delta^2 E + \ldots$$

where δ^1 and δ^2 represent terms which are linear and quadratic in the disturbance, respectively. For arbitrary values of η and ζ, $\delta^1 E$ is non-zero. Thus, despite the conservation of E, $\delta^2 E$ does not, in general, provide a conserved, quadratic measure of the disturbance. (Remember, formal stability requires that we can find a conserved, quadratic measure of the disturbance which is positive or negative definite.) In the Kelvin–Arnold procedure we remedy this as follows. We note that the particle conserves not only E but also Γ. We now restrict ourselves to initial perturbations in which $\delta\Gamma = 0$. Since $\delta\Gamma = 0$ at $t = 0$, it must remain zero for all t. Thus we write E as

$$E = \frac{1}{2}m\left(\dot{r}^2 + \Gamma^2/r^2\right) + V(r)$$

and treat Γ as a constant, $\Gamma = \Gamma_0$. For this restricted set of disturbances we find $\delta^1 E = 0$ and $\delta^2 E = \frac{1}{2}m\dot{\eta}^2 + \frac{1}{2}\eta^2\left[V'' + 3V'/r\right]_0$. In this case conservation of E does indeed ensure that $\delta^2 E$ is conserved by the disturbance (to quadratic order), and so we have formal stability if $\delta^2 E > 0$ for all possible η and $\dot{\eta}$. Thus stability is ensured if $\left[r^3 V'' + 3r^2 V'\right]_0 > 0$, which coincides with our conventional perturbation analysis. Note that the Kelvin–Arnold method only provides a stability criterion for a restricted set of perturbations (in this case ones where $\delta\Gamma = 0$), although it is readily verified that the value of $\delta\Gamma$ at $t = 0$ does not influence the stability of the perturbed trajectory.

The energy–Casimir method also requires that we spot that Γ is conserved by the particle, although this time there is no need to restrict the form of the initial disturbance. It proceeds as follows. We introduce the generalised invariant, $A = E + C(\Gamma)$ where C is an arbitrary function of Γ (a Casimir). We now choose C such that $\delta^1 A = 0$ for all possible choices of η and ζ (this requires $C = -m\theta_0\Gamma$). It follows that $\delta^2 A$ is conserved by the motion. It is readily confirmed that

$$\delta^2 A = \frac{1}{2}m(\dot{\eta}^2 + \dot{\zeta}^2) + \frac{1}{2}\eta^2[V_0'' - V_0'/r_0]$$

We have formal stability if $\delta^2 A > 0$ for all (η, ζ) and this requires that $V_0'' - V_0'/r_0 > 0$. This coincides with our perturbation analysis since $V_0'' - V_0'/r_0 > 0$ ensures that $V_0'' + 3V_0'/r_0 > 0$. Thus the energy–Casimir method has provided a sufficient (though not necessary) condition for stability.

The third approach does not require that the Casimir invariants of the system (in this case Γ) be identified, although it still relies on the conservation of energy. We proceed as follows. Let $L = T - V$ and η and ζ be generalised coordinates, q_i. We now evaluate

$$L = L_0 + \delta^1 L + \delta^2 L + \dots$$

and calculate the generalised momenta, $p_i = \partial L/\partial \dot{q}_i$. The final step is to evaluate the Hamiltonian, H:

$$H = \sum (p_i \dot{q}_i) - L$$

Since L is not an explicit function of time, H is an invariant. It turns out that

$$e = H + L_0 = \frac{1}{2}m(\dot{\eta}^2 + \dot{\zeta}^2) + \frac{1}{2}\eta^2[V_0'' - V_0'/r_0]$$

Once again we have a conserved quadratic measure of the disturbance and the motion is stable provided that e is positive definite.

Now in this simple example the third procedure offers no obvious advantage over the others. However, when it comes to more complex systems, where it is by no means obvious what the Casimir invariants are, it does provide an advantage, as we shall see.

We shall now show that for any conservative system

$$\boxed{e = \frac{1}{2}\int \dot{\boldsymbol{\eta}}^2 dV - d^2 L(\eta) = \text{constant}} \tag{A2.1}$$

which is a generalisation of (6.51). (Here $\boldsymbol{\eta}$ is the virtual displacement field introduced in Section 6.4.2.) This furnishes a variety of stability criteria. The proof of (A2.1) relies on expanding the Lagrangian up to second order in the particle displacements, invoking Lagrange's equation to dispense with the first variation in L, and then performing a transformation to create a conserved Hamiltonian, which is quadratic in the disturbance. The first and most important step is to introduce the Lagrangian displacement,

$$\boldsymbol{\zeta}(\mathbf{x}, t) = \mathbf{x}_p(t) - \mathbf{x}_{p0}(t)$$

where $\mathbf{x}_{p0}$ is the position vector of particle p in the base flow and $\mathbf{x}_p$ is the position of the same particle in the perturbed flow. The generalisation of (6.15) is then

$$\frac{\partial \boldsymbol{\zeta}}{\partial t} + \mathbf{u}_0(\mathbf{x}) \cdot \nabla \boldsymbol{\zeta} = \frac{D\boldsymbol{\zeta}}{Dt} = \mathbf{u}(\mathbf{x} + \boldsymbol{\zeta}, t) - \mathbf{u}_0(\mathbf{x}) \tag{A2.2}$$

In the linear (small amplitude) approximation, this becomes

$$\frac{\partial \boldsymbol{\zeta}}{\partial t} + \mathbf{u}_0 \cdot \nabla \boldsymbol{\zeta} = \delta \mathbf{u}(\mathbf{x}, t) + \mathbf{u}_0(\mathbf{x} + \boldsymbol{\zeta}) - \mathbf{u}_0(\mathbf{x}) \tag{A2.3}$$

which, using the approximation $\mathbf{u}_0(\mathbf{x} + \boldsymbol{\zeta}) - \mathbf{u}_0(\mathbf{x}) = \boldsymbol{\zeta} \cdot \nabla \mathbf{u}_0$, simplifies to

$$\delta^1 \mathbf{u} = \dot{\boldsymbol{\zeta}} + \nabla \times [\boldsymbol{\zeta} \times \mathbf{u}_0] \tag{A2.4}$$

The key step is now to switch from $\boldsymbol{\zeta}$ to $\boldsymbol{\eta}(\mathbf{x}, t)$, the virtual displacement field. (The two are related by (6.19).) This greatly simplifies the subsequent analysis. Since $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ are equal to leading order, (A2.4) yields

$$\delta^1 \mathbf{u} = \dot{\boldsymbol{\eta}}(\mathbf{x}, t) + \nabla \times [\boldsymbol{\eta} \times \mathbf{u}_0] \tag{A2.5}$$

Returning to (A2.2), but retaining terms up to second order, we find

$$\delta^2 \mathbf{u} = \frac{1}{2} \nabla \times [\boldsymbol{\eta} \times \dot{\boldsymbol{\eta}}] + \frac{1}{2} \nabla \times [\boldsymbol{\eta} \times (\nabla \times (\boldsymbol{\eta} \times \mathbf{u}_0))] \tag{A2.6}$$

We now introduce some notation. We take δ to represent an arbitrary (physically realisable) variation of some field, say $\delta\mathbf{u}$. We take d, on the other hand, to represent a frozen-in variation of any field. In the case of the **B**-field, the two coincide ($\delta\mathbf{B} = d\mathbf{B}$) since (6.42) demands that, if **B** is frozen into the fluid during the initial perturbation, then it is frozen in for all subsequent time. In the case of **u**, however, $d\mathbf{u}$ does not represent a dynamically meaningful perturbation. Nevertheless, we are still free to ask what happens to **u** and T (the kinetic energy) in the event of a variation in which the **u**-lines are frozen in. What we choose to do with that information is another matter. From (6.17) and (6.20) (with **u** replacing **B**) we have, in terms of the virtual displacement field,

$$d^1\mathbf{u} = \nabla\times(\boldsymbol{\eta}\times\mathbf{u}_0),\qquad d^2\mathbf{u} = \frac{1}{2}\nabla\times[\boldsymbol{\eta}\times d^1\mathbf{u}] \tag{A2.7}$$

$$d^1T = \int(\mathbf{u}_0\cdot d^1\mathbf{u})dV,\qquad d^2T = \frac{1}{2}\int[(d^1\mathbf{u})^2 + 2\mathbf{u}_0\cdot d^2\mathbf{u}]dV \tag{A2.8}$$

Evidently,

$$\delta^1\mathbf{u} = \dot{\boldsymbol{\eta}} + d^1\mathbf{u},\qquad \delta^2\mathbf{u} = \frac{1}{2}\nabla\times[\boldsymbol{\eta}\times\dot{\boldsymbol{\eta}}] + d^2\mathbf{u} \tag{A2.9}$$

(The equivalent expressions in terms of ζ are far more complicated.) We shall return to these expressions shortly. In the meantime, let us try to understand the significance of d-perturbation as applied to $\mathbf{u}$. We shall use the term 'd-variation' to mean a perturbation of the equilibrium configuration in which: (i) $\mathbf{u}$ is perturbed according to (A2.7), i.e. the $\mathbf{u}$-lines are frozen in during the perturbations; (ii) any auxiliary field, such as $\mathbf{B}$, is perturbed in a manner compatible with the governing equations, e.g. $\mathbf{B}$ is frozen in. (This requires that the perturbations in $\mathbf{B}$ are given by (6.17).) Also, let us introduce a generalised version of the Euler equation in the form

$$\frac{\partial\mathbf{u}}{\partial t} = \mathbf{u}\times\boldsymbol{\Omega} - \nabla C + \mathbf{f} \tag{A2.10}$$

where $\mathbf{f}$ is any conservative body force, such as $[\mathbf{J}\times\mathbf{B}]/\rho$. Let V be the potential energy associated with $\mathbf{f}$. This could, for example, be magnetic energy, gravitational energy, or some combination of these. From (A2.7) and (A2.8),

$$d^1T = \int[\boldsymbol{\Omega}_0\cdot(\boldsymbol{\eta}\times\mathbf{u}_0)]d\mathbf{x} = -\int\boldsymbol{\eta}\cdot\mathbf{f}_0 d\mathbf{x} = d^1V \tag{A2.11}$$

It follows that $d^1L = 0$ under this type of variation, which is the first hint that there is, in fact, some significance to our d-variation. Actually, in two dimensions, the physical significance of $d\mathbf{u}$ is that, by advecting the $\mathbf{u}$-lines, we create a new set of particle trajectories with the special property that the time of flight between two fixed points is preserved. This is precisely the sort of perturbation demanded by Hamilton's principle and $d^1L = 0$ is, in fact, a direct consequence of Hamilton's principle (see Davidson, 1998.) In three dimensions we must do a little more work to explain the significance of $d^1L = 0$. Once again it rests on the fact that the time of flight of a fluid particle is preserved by the d-variation. To see that this is so, consider the time of flight equation

$$t_B - t_A = \int_A^B \frac{d\mathbf{l}}{|\mathbf{u}|} = \frac{1}{\Phi}\int_A^B dV \tag{A2.12}$$

Here Φ is the volume flux down a stream-tube which surrounds a path-line linking A and B, and $\int dV$ is the volume of the stream-tube (of rectangular cross section) which may be constructed from pairs of intersecting stream surfaces which, in turn, might be locally represented by Clebsch variables. Such stream surfaces are frozen into the fluid during a d-perturbation and so, as in two dimensions, the time of flight of fluid particles is preserved. This ensures that the first variation of the action integral is zero, and it is this which lies behind (A2.11).

So the idea of a *d-variation* has some physical basis. We now examine second variations, and this will lead to our stability criterion (A2.1). The first step is to calculate $\Delta T = T - T_0$ and $\Delta L = L - L_0$ for an arbitrary (physically realisable) δ-variation of the equilibrium state. We have

$$\delta^1 T = \int \mathbf{u}_0 \cdot \delta^1 \mathbf{u} dV, \qquad \delta^2 T = \frac{1}{2}\int \left[\left(\delta^1 \mathbf{u}\right)^2 + 2\mathbf{u}_0 \cdot \delta^2 \mathbf{u} \right] dV$$

Next, using (A2.9) to substitute for $\delta^1\mathbf{u}$ and $\delta^2\mathbf{u}$, we find

$$\delta^1 T = d^1 T(\boldsymbol{\eta}) + \int \mathbf{u}_0 \cdot \dot{\boldsymbol{\eta}} dV \tag{A2.13}$$

$$\delta^2 T = d^2 T(\boldsymbol{\eta}) + \frac{1}{2}\int \dot{\boldsymbol{\eta}}^2 dV + \hat{I}(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}) \tag{A2.14}$$

where $\hat{I}$ is bi-linear in $\boldsymbol{\eta}$ and $\dot{\boldsymbol{\eta}}$ and is given by

$$\hat{I} = \frac{1}{2}\int \dot{\boldsymbol{\eta}} \cdot [2d^1\mathbf{u} + \boldsymbol{\Omega}_0 \times \boldsymbol{\eta}] dV \tag{A2.15}$$

Now if $\mathbf{f}$ is conservative, then the potential energy, V, will depend only on the instantaneous configuration of the flow and not on its history. Thus,

$$\Delta V = V - V_0 = \delta^1 V(\boldsymbol{\eta}) + \delta^2 V(\boldsymbol{\eta}) + \text{H.O.T.}$$

This gives us an expression for ΔL in terms of $\boldsymbol{\eta}$ and $\dot{\boldsymbol{\eta}}$:

$$\Delta L = \frac{1}{2}\int \dot{\boldsymbol{\eta}}^2 dV + \left[d^1 T(\boldsymbol{\eta}) - \delta^1 V(\boldsymbol{\eta})\right] + \left[d^2 T(\boldsymbol{\eta}) - \delta^2 V(\boldsymbol{\eta})\right] + I(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}) + \text{H.O.T}$$

where $I(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}) = \hat{I} + \int \mathbf{u}_0 \cdot \dot{\boldsymbol{\eta}} dV$. Now recall that we defined our *d-variation* such that $\mathbf{u}$ is perturbed according to (A2.7), but the auxiliary fields, such as $\mathbf{B}$ and ρ, are varied in a physically realisable manner. (This requires that $\mathbf{B}$ is frozen in.) It follows that, as a matter of notation, we can write $\delta^1 V = d^1 V$ and $\delta^2 V = d^2 V$. Our expression for the Lagrangian becomes

$$\Delta L = \frac{1}{2}\int \dot{\boldsymbol{\eta}}^2 dV + \left[d^1 T(\boldsymbol{\eta}) - d^1 V(\boldsymbol{\eta})\right] + \left[d^2 T(\boldsymbol{\eta}) - d^2 V(\boldsymbol{\eta})\right] + I(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}})$$

We now use $\boldsymbol{\eta}$ as a set of generalised coordinates describing the instantaneous state of the system. Note that ΔL is a function only of $\boldsymbol{\eta}$ and $\dot{\boldsymbol{\eta}}$. It is not an explicit function of time. Now for a system with a finite number of degrees of freedom, $\boldsymbol{\eta}_i$, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right) - \frac{\partial L}{\partial \eta_i} = 0 \tag{A2.16}$$

so that steady solutions are represented by $\partial L/\partial \eta_i = 0$. Also if $L = L(\eta_i, \dot{\eta}_i)$ is not an explicit function of time, the system possesses a conserved Hamiltonian:

$$e = \dot{\eta}_i \frac{\partial L}{\partial \dot{\eta}_i} - L = \text{constant}$$

The equivalent results for our continuous system are that $d^1 L = 0$ for an equilibrium solution and

$$e = \left[\int \dot{\boldsymbol{\eta}}^2 dV + I(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}})\right] - \Delta L = \text{constant}$$

The fact that $d^1 L = 0$ follows directly from Lagrange's equations is reassuring since (for two-dimensional flows) we have already noted that this may be deduced from Hamilton's principle. Next, substituting for ΔL yields, at last, our conserved, quadratic functional:

$$\boxed{e = \frac{1}{2}\int \dot{\boldsymbol{\eta}}^2 dV - d^2 L(\boldsymbol{\eta}) = \text{constant}, \qquad d^1 L(\boldsymbol{\eta}) = 0} \tag{A2.17}$$

$$\boxed{d^2 L(\boldsymbol{\eta}) = \frac{1}{2}\int \left[\left(d^1 \mathbf{u}\right)^2 + \mathbf{u}_0 \cdot \nabla \times \left(\boldsymbol{\eta} \times d^1 \mathbf{u}\right)\right] dV - \delta^2 V(\boldsymbol{\eta})} \tag{A2.18}$$

This is the key result. Since e is a conserved quadratic measure of the disturbance, many stability criteria may be established on the back of (A2.17). We might refer to (A2.17) as a principle of maximum action.

The following two theorems follow directly from (A2.17) and (A2.18).

Theorem 1

The equilibrium of any conservative, incompressible flow possesses (formal) stability provided that

$$d^2L(\boldsymbol{\eta}) = \frac{1}{2}\int\left[\left(d^1\mathbf{u}\right)^2 + \mathbf{u}_0 \cdot \nabla \times \left(\boldsymbol{\eta} \times d^1\mathbf{u}\right)\right]dV - \delta^2 V(\boldsymbol{\eta}),$$

$$d^1\mathbf{u} = \nabla \times (\boldsymbol{\eta} \times \mathbf{u}_0)$$

is negative definite for all possible $\boldsymbol{\eta}$.

Theorem 2

The equilibrium of any conservative, incompressible flow posses (formal) stability provided that

$$e = \frac{1}{2}\int \dot{\boldsymbol{\eta}}^2 dV - d^2L(\boldsymbol{\eta}), \qquad \dot{\boldsymbol{\eta}} = \delta^1\mathbf{u} - \nabla \times (\boldsymbol{\eta} \times \mathbf{u}_0)$$

is positive or negative definite for all possible perturbations of the equilibrium.

It is easy to show that special cases of Theorem 1 are Rayleigh's circulation criterion, the Rayleigh–Taylor criterion for stratified fluids, Bernstein's (1958) principle for magnetostatics (c.f.(6.32)), and Friedlander & Vishik's (1990) and Frieman & Rottenberg's (1960) stability tests for ideal MHD equilibria (c.f. (6.51)). A special case of Theorem 2 is Arnold's (1966) variational principle for Euler flows.

Note that (A2.16) also furnishes the governing equation for $\boldsymbol{\eta}(\mathbf{x}, t)$. Substituting for ΔL in (A2.16) yields,

$$\ddot{\boldsymbol{\eta}} + 2\mathbf{u}_0 \cdot \nabla\dot{\boldsymbol{\eta}} = \nabla(\dot{\boldsymbol{\eta}} \cdot \mathbf{u}_0) + \mathbf{F}(\boldsymbol{\eta}) \qquad \text{(A2.19)}$$

where

$$F_i(\boldsymbol{\eta}) = \frac{\delta\left[d^2L(\boldsymbol{\eta})\right]}{\delta\eta_i} \qquad \text{(A2.20)}$$

The form of $\mathbf{F}(\eta)$ depends on the nature of the body force. When $\mathbf{f} = \mathbf{J} \times \mathbf{B}$, as in ideal MHD, the $\mathbf{B}$-field is frozen in during the perturbation and we have

$$\delta^2 V(\boldsymbol{\eta}) = \frac{1}{2}\int\left[\left(d^1\mathbf{B}\right)^2 + \mathbf{B}_0 \cdot \nabla \times \left(\boldsymbol{\eta} \times d^1\mathbf{B}\right)\right]dV, \qquad d^1\mathbf{B} = \nabla \times (\boldsymbol{\eta} \times \mathbf{B}_0)$$

(This is just $\delta^2 E_B$ given by (6.20).) In this case (A2.20) yields, after a little algebra,

$$\begin{aligned}\mathbf{F} = \mathbf{u}_0 \times \lfloor \nabla \times (d^1\mathbf{u}) \rfloor + d^1\mathbf{u} \times [\nabla \times \mathbf{u}_0] \\ - \mathbf{B}_0 \times \left[\nabla \times (d^1\mathbf{B})\right] - d^1\mathbf{B}) \times [\nabla \times \mathbf{B}_0] + \nabla(\sim)\end{aligned} \tag{A2.21}$$

which is identical to (6.45).

We conclude by showing that the Kelvin–Arnold variational principle, as applied to Euler flows, is a special case of Theorem 2. We start by noting that, when $\mathbf{f} = 0$, (A2.19) becomes

$$\ddot{\boldsymbol{\eta}} + 2\mathbf{u}_0 \cdot \nabla\dot{\boldsymbol{\eta}} = \mathbf{u}_0 \times \left(\nabla \times d^1\mathbf{u}\right) + d^1\mathbf{u} \times (\nabla \times \mathbf{u}_0) + \nabla(\cdot)$$

This may be integrated once to give

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\eta} \times \boldsymbol{\Omega}_0 - \nabla \times (\boldsymbol{\eta} \times \mathbf{u}_0) + \mathbf{m} + \nabla(\cdot)$$

where $\mathbf{m}$ is independent of $\boldsymbol{\eta}$ and is governed by

$$\partial\mathbf{m}/\partial t = \nabla \times (\mathbf{u}_0 \times \mathbf{m})$$

It follows from (A2.9) that

$$\delta^1\mathbf{u} = \boldsymbol{\eta} \times \boldsymbol{\Omega}_0 + \mathbf{m} + \nabla(\cdot)$$

If, at $t = 0$, we specify that $\mathbf{m} = 0$, then $\mathbf{m}$ will be zero for all time. In such a case

$$\delta^1\boldsymbol{\Omega} = \nabla \times (\boldsymbol{\eta} \times \boldsymbol{\Omega}_0)$$

Evidently, this is a perturbation in which the $\boldsymbol{\Omega}$-lines are frozen into the fluid – an *isovortical perturbation*. The Kelvin–Arnold principle states that a steady Euler flow is stable provided that $\delta^2 T$ is positive definite or negative definite under an isovortical perturbation. Let us denote such a perturbation by $\hat{d}$, to distinguish it from a general perturbation, δ. However, using (A2.9) it is readily confirmed that

$$\hat{d}^2 T = \frac{1}{2}\int \dot{\boldsymbol{\eta}}^2 dV - d^2 T = e$$

Thus the Kelvin–Arnold principle is simply a special case of Theorem 2.