# 7

# *MHD Turbulence at Low and High Magnetic Reynolds Number*

You asked, 'What is this transient pattern?'
If we tell the truth of it, it will be a long story;
It is a pattern that came up out of an ocean
And in a moment returned to that ocean's depth

*(Omar Khayyam)*

Turbulence is not an easy subject. Our understanding of it is limited, and those bits we do understand are arrived at through detailed and difficult calculation. G K Batchelor gave some hint of the difficulties when, in 1953, he wrote:

> It seems that the surge of progress which began immediately after the war has now largely spent itself, and there are signs of a temporary dearth of new ideas.... we have got down to the bedrock difficulty of solving non-linear partial differential equations.

Little has changed since 1953. Nevertheless, it is hard to avoid the subject of turbulence in MHD, since the Reynolds number, even in metallurgical MHD, is invariably very high. So at some point we simply have to bite the bullet and do what we can. This chapter is intended as an introduction to the subject, providing a springboard for those who wish to take it up seriously. In order not to demotivate the novice, we have tried to keep the mathematical difficulties to a minimum. Consequently, only schematic outlines are given of certain standard derivations and proofs. For example, deriving the standard form for second- and third-order velocity correlation tensors in isotropic turbulence can be hard work. Such derivations are well documented elsewhere and so there seems little point in giving a blow-by-blow description here. We have concentrated rather on trying to get the main physical ideas across.

Now the sceptic might say: 'if the theory of turbulence is so hard, why bother with it at all? After all we now have powerful computers available to us, which can compute both the mean flow and the motion of every turbulent eddy.' The experimentalist Corrsin had one answer to this.

Having estimated the computing resources required to simulate even the most modest of turbulent flows, and shown them to be well beyond the capacity of the time, he made the following whimsical comment:

> The foregoing estimate (of computing power) is enough to suggest the use of analog instead of digital computation; in particular, how about an analog consisting of a tank of water?

Corrsin said this in 1961, but actually it is still pertinent today. Despite the great advances which have occurred in computational fluid dynamics, forty years later our capacity to simulate accurately turbulent flows by computation is still rather poor, restricted to simple geometries and low Reynolds numbers (around 500). The problem, as you will see shortly, is that turbulent flows contain, at any instant, eddies (vortical structures) which have a wide range of sizes from the large to the minute, and it is difficult to capture this full spectrum of eddies in a numerical simulation.

## 7.1 A Survey of Conventional Turbulence

As a prelude to discussing MHD turbulence it seems prudent to summarise first the simpler features of conventional turbulence. Of course, turbulence is a vast subject, filling many erudite if forbidding texts. We have time to touch on only a few issues here. We start with a short historical introduction.

### *7.1.1 A historical interlude*

> At times water twists to the northern side, eating away the base of the bank; at times it overthrows the bank opposite on the south; at times it leaps up swirling and bubbling to the sky; at times revolving in a circle it confounds its course.... Thus without any rest it is ever removing and consuming whatever borders upon it. Going thus with fury it is **turbulent** and destructive.
>
> *Leonardo da Vinci*

So began man's study of turbulent fluid motion.

We start this section with a brief historical survey of turbulence, a survey which begins with Newton and the ideas of viscosity and eddy viscosity (Table 7.1). The relationship between shear stress and gradients in mean velocity has been a recurring theme in turbulence theory. In the laminar context this was established in 1687 by Newton who, in *Principia*, hypothesised that the resistance to relative movement in parts of a fluid

Table 7.1. *Comparison of the history of theories of turbulence with those of magnetism*

| Theory of turbulence | | | Electricity and magnetism |
|---|---|---|---|
| | | 11th century | Compass |
| 1500s | Leonardo's first observations | 1269 | Peregrinus: magnetic poles |
| | | 1600 | Gilbert: geomagnetism |
| | | 1750s | Coulomb: action at a distance |
| | | 1820s | Ampère: forces on currents |
| | | 1831 | Faraday: electromagnetic induction, concept of fields |
| 1850s | Boussinesq: eddy viscosity | | |
| | | 1860s | Maxwell's equations |
| 1880s | Reynolds: two types of flow, turbulent stresses | | |
| 1904 | Prandtl: boundary layers | 1889 | Hertz: emission of electromagnetic waves |
| 1920s | Prandtl: mixing-length theory | | |
| 1930s | Taylor, von Kármán: statistical theory of turbulence | | |
| 1940s | Kolmogorov: modern theory of turbulence | | |
| 1942 | beginning of MHD – Alfvén's waves discovered | | |

are *'proportional to the velocity with which the parts of fluid are separated from one another'*, i.e. the relative rate of sliding of layers in the fluid. The constant of proportionality is, of course, the coefficient of viscosity. Newton's idea of internal friction was somewhat overlooked by the 18th century mathematicians and it languished until 1823 when Navier, and a little later Stokes, introduced viscous forces into the equations of hydrodynamics.

Shortly after the introduction of Newton's law of viscosity, questions were raised as to the uniformity of $\nu$. For example, in 1851 Saint-Venant speculates that*:

> If Newton's assumption, . . . , which consists in taking interior friction proportional to the speed of the fluid elements sliding against one another, can be applied approximately to the set of points of a given fluid section, all the known facts lead us to

* Translation by U. Frisch, 1995.

> infer that the coefficient of this proportionality should increase with the size of transverse sections; this may be explained up to a point by noticing that the fluid elements are not progressing parallel to each other with regularly graded velocities, and that ruptures, eddies and other complex and oblique motions, which must strongly effect the magnitude of frictions, are formed.

There is clearly some embryonic notion of turbulence and of *eddy viscosity* here, albeit confused with molecular action. This was pursued by both Reynolds and Boussinesq, the latter being Saint-Venant's student. Boussinesq came first, noting that turbulence must greatly increase the (eddy) viscosity because: '*the (turbulent) friction experienced, being caused by finite sliding between adjacent layers, will be much larger than would be the case should velocities vary in a continuous way*' (1870). Shortly after, Reynolds' classic paper on pipe flow appeared (1883). This clearly differentiates between laminar and turbulent flow, and identifies the key role played by $ul/v$ in determining which state prevails. Later, Reynolds reaffirmed the idea of an eddy viscosity while introducing the notion that the fluid velocity might be decomposed into a mean and fluctuating component, the latter giving rise to the fictitious, time-averaged shear stresses which now bear Reynolds' name. Reynolds used the term *sinuous* to describe the appearance of turbulence.

By 1925 Prandtl clearly recognised the analogy between the turbulent transport of momentum (through turbulent eddies) and the laminar shear stress caused by molecular motion, as predicted by the kinetic theory of gases. He introduced the mixing length model of turbulence described in Chapter 3, which had some notable successes at the time (e.g. the log-law of the wall) but is now regarded as flawed. (The problem is that there is no real separation of length scales between the turbulent fluctuations and gradients in mean velocity as required by a mixing length theory. In fact, any result deduced by mixing length can also be deduced by purely dimensional arguments.)

The great breakthrough in turbulence theory came with the pioneering work of G I Taylor in the early 1930s, who for the first time fully embraced the need for a statistical approach to the subject. He introduced the idea of the velocity correlation function $Q_{ij}(\mathbf{r}) = \overline{u_i'(\mathbf{x})u_j'(\mathbf{x}+\mathbf{r})}$, a generalisation of the Reynolds stress, which is now the common currency of turbulence theory. The quantity $Q_{ij}$ tells us about the degree to which the fluctuating component of motion, $\mathbf{u}'$, is statistically correlated at two points separated by a distance $|\mathbf{r}|$. A strong correlation implies that there are eddies which span the gap $|\mathbf{r}|$. Conversely, if $Q_{ij}$ is very small, then $\mathbf{x}$ and $\mathbf{x}+\mathbf{r}$ are statistically independent. Thus $Q_{ij}$ contains information about the structure of the turbulence. Taylor also promoted the useful idealisation of statistically homogeneous and isotropic turbulence. This initiative was pursued by the engineer von Kármán, who showed that, with the help of the symmetry implied by isotropy, $Q_{ij}$ could be expressed in terms of a single scalar function, $f(|\mathbf{r}|)$, and that the Navier–Stokes equation could be manipulated into the form $\partial f/\partial t = (\ldots)$. At last there was the possibility of making rigorous, quantitative predictions about turbulence. Unfortunately the right-hand side of this equation includes new terms such as triple velocity correlations of the form $\overline{u_i(\mathbf{x})u_j(\mathbf{x})u_k(\mathbf{x}+\mathbf{r})}$. Consequently, it is not always possible to predict the evolution of $f$. Nevertheless, in certain circumstances the triple correlations can be finessed away, and so Karman's equation, now called the Karman–Howarth equation, can provide useful information.

The statistical theory of turbulence was greatly developed in the (then) USSR in the 1940s, particularly by Kolmogorov and Obukhov. These researchers realised that a vast range of scales (eddy sizes) exist in a turbulent flow, and that viscosity influences only the smallest eddies. They quantified the idea of the energy cascade, in which eddies continually break up into smaller and smaller vortices, until viscosity destroys the motion. This allowed them to predict how the energy of a turbulent flow is distributed between the various eddy sizes. Great strides were made, and by 1950 a physical and mathematical picture of homogeneous turbulence had emerged which is little different today. However, this picture is not completely deductive, but relies rather on certain (plausible) physical assumptions based on empirical evidence.

Turbulence plays a key part in MHD. Virtually all laboratory and industrial flows are turbulent. Moreover, turbulence is an essential ingredient of geo-dynamo theory, and it is needed in astrophysical MHD to explain the flux tube reconnections which are so hard to account for in terms of the vanishingly small molecular diffusivity. Comparing the development of turbulence and the laws of electromagnetism, we see that turbulence was rather a late developer, reflecting the formidable difficulties inherent in tackling a non-linear, random process. Even today there is no universal 'theory of turbulence'. We have a few theoretical results relating to various idealised configurations, and a great deal of experimental data. Sometimes, but not always, the two coincide. Of course, it is when theory and experiment differ, and we try to reconcile those differences, that we learn the most. As with all fluid mechanics, our understanding of turbulence has developed through a careful assessment of the experimental evidence; which brings us back to Leonardo da Vinci's observations.

One cannot help but be struck by the similarities between Reynolds' idea of two motions, a mean forward motion and a turbulent vortical motion, and his observation of the sinuous nature of turbulence in a pipe, and Leonardo da Vinci's note in 1513:

> Observe the motion of the surface of water, which resembles the behaviour of hair, which has two motions, of which one depends on the weight of the strands, the other on the line of its revolving; thus water makes revolving eddies, one part of which depends upon the impetus of the principle current, and the other depends on the incident and reflected motions.

[Note accompanying Leonardo's well-known sketches of water flow around obstacles.]

### *7.1.2 A note on tensor notation*

It is difficult to make much progress in turbulence without the use of tensor notation, something which we have managed to avoid so far. This sub-section is for those who have not met tensors before, or who have studiously avoided them.

Tensor notation is compact and efficient, but it can be off-putting to those who are unfamiliar with it. Luckily, to get through this chapter, there is only one thing you need to know about tensors, and that is the *implied summation convention*. A couple of examples will get the general idea across.

Consider the convective derivative $(\mathbf{u} \cdot \nabla)f$, where $f$ is some scalar function:

$$(\mathbf{u} \cdot \nabla)f = u_x \frac{\partial f}{\partial x} + u_y \frac{\partial f}{\partial y} + u_z \frac{\partial f}{\partial z}$$

In tensor notation we write this as simply $u_i(\partial f/\partial x_i)$. The rule is: if a suffix is repeated then there is an implied summation over that index. Thus, in the example above,

$$u_i \frac{\partial f}{\partial x_i} = u_x \frac{\partial f}{\partial x} + u_y \frac{\partial f}{\partial y} + u_z \frac{\partial f}{\partial z}$$

Sometimes there is more than one suffix. For example, $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is a symbolic representation of the vector having the three components $(\mathbf{u} \cdot \nabla)u_x$,

$(\mathbf{u}\cdot\nabla)u_y$ and $(\mathbf{u}\cdot\nabla)u_z$. In tensor notation, we would write $(\mathbf{u}\cdot\nabla)\mathbf{u}$ as $u_i(\partial u_j/\partial x_i)$, implying an automatic summation over the repeated suffix $i$ (but no summation over $j$). Put another way, $u_i(\partial u_j/\partial x_i)$ is the $j$th component of $(\mathbf{u}\cdot\nabla)\mathbf{u}$. Sometimes we have two repeated indices, in which case there are two implied summations. For example, $[(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{B}$ is written as $[u_i(\partial u_j/\partial x_i)]B_j$. That is, we take the $j$th component of $(\mathbf{u}\cdot\nabla)\mathbf{u}$, multiply it by the $j$th component of B, repeat the operation for $j = x$, $y$ and $z$, and sum the terms.

The Navier–Stokes equation

$$\rho\left[\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=-\nabla p+\rho\nu\nabla^2\mathbf{u}$$

is, in tensor notation,

$$\rho\left[\frac{\partial u_j}{\partial t}+u_i\frac{\partial u_j}{\partial x_i}\right]=-\frac{\partial p}{\partial x_j}+\rho\nu\frac{\partial^2 u_j}{\partial x_i^2} \tag{7.1}$$

This represents the $j$th component of the Navier–Stokes equation and the summation convention has appeared twice, once in the convective derivative, $u_i[\partial(\cdot)/\partial x_i]$, and once in the Laplacian, $\partial^2/\partial x_i^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. The continuity equation is, in tensor form,

$$\frac{\partial u_i}{\partial x_i}=0 \tag{7.2}$$

Some quantities, such as the stress tensor, $\tau_{ij}$, depend themselves on two indices. In the case of the stress tensor, $\tau_{ij}$ represents the component of stress pointing in the $j$th direction and evaluated on the surface whose normal points in the $i$th direction. This is illustrated in Figure 7.1.

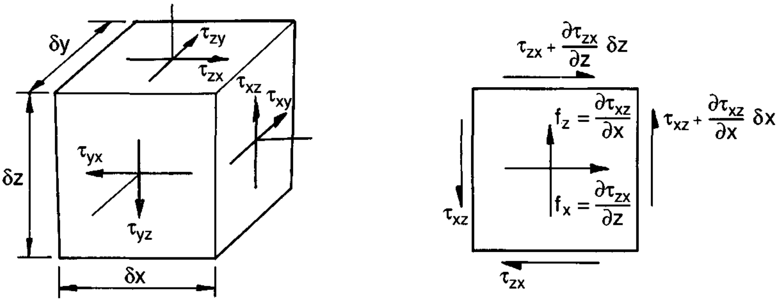
The Navier–Stokes equation written in terms of the viscous stress tensor is

$$\rho\left[\frac{\partial u_j}{\partial t}+u_i\frac{\partial u_j}{\partial x_i}\right]=-\frac{\partial p}{\partial x_j}+\frac{\partial\tau_{ij}}{\partial x_i} \tag{7.3}$$

where Newton's law of viscosity stipulates that

$$\tau_{ij}=\rho\nu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)=2\rho\nu S_{ij} \tag{7.4}$$

Here $S_{ij}$ is called the strain-rate tensor. (The reader might wish to check that substituting (7.4) into (7.3), and using (7.2), we arrive back at (7.1).) The term $\partial\tau_{ij}/\partial x_i$ in (7.3) arises from the fact that the net viscous force per unit volume acting on the cube shown in Figure 7.1 is




(a) Sign convention for stresses (note that $\tau_{ij} = \tau_{ji}$)

(b) Net effective body force, per unit volume, caused by an inbalance in stresses. Only components in th xz plane are shown

Figure 7.1 The stress tensor.

$$f_x = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

in the $x$-direction, with analogous expressions for $f_y$ and $f_z$.

Finally we introduce the symbol $\delta_{ij}$, which has the usual meaning of $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Armed with this brief introduction to tensor notation, we now start our survey of conventional (non-MHD) turbulence.

### *7.1.3 The structure of turbulent flows: the Kolmogorov picture of turbulence*

Let us start with a traditional question in turbulence theory. Suppose we have a (statistically) steady flow, say flow in a pipe. Then the turbulent eddies are continually subject to viscous dissipation yet the energy of the turbulence does not, on average, change. Where does the turbulence energy come from? Of course, in some sense it comes from the mean flow. The traditional way of quantifying this relies on the idea of dividing the flow into two distinct parts, a mean component and a turbulent motion, and then examining the exchange of energy between the two: which brings us back to the idea of a Reynolds stress.

You are already familiar with the concept of the Reynolds stress, $\tau_{ij}^R$. In Chapter 3 we showed that, when we time-average the Navier–Stokes equation in a turbulent flow, the presence of the turbulence gives rise to additional stresses, $\tau_{ij}^R = -\rho\overline{u_i' u_j'}$, which act on the mean flow. Here the prime on $\mathbf{u}'$ indicates that this is a fluctuating component of velocity, $\mathbf{u}' = \mathbf{u} - \bar{\mathbf{u}}$, and the overbar signifies a time average. Now these Reynolds stresses give rise to a net force acting on the mean flow, $f_i = \partial\tau_{ij}^R/\partial x_j$, and if the rate of working of this force, $f_i\bar{u}_i$, is negative, then the mean flow must lose mechanical energy to the agent which supplies the force, i.e. the turbulence. We say that energy, usually kinetic, is transferred from the mean flow to the turbulence. This is why the turbulence in a pipe, say, does not die away. The viscous dissipation of turbulent eddies is matched by the rate of working of $f_i$.

Of course, this is all a little artificial, in the sense that we have just one fluid and one flow. All we are saying is that when we decompose $\mathbf{u}$ into $\bar{\mathbf{u}}$ and $\mathbf{u}'$ then the total kinetic energy, which is conserved in the absence of viscosity, is like-wise divided between $\frac{1}{2}\bar{\mathbf{u}}^2$ and $\frac{1}{2}\overline{\mathbf{u}'^2}$. When $f_i\bar{u}_i$ is negative, energy is transferred from $\bar{\mathbf{u}}$ to $\mathbf{u}'$. Physically this corresponds to the creation of turbulent eddies through some form of instability in the mean flow. Now we can write $f_i\bar{u}_i$ as

$$f_i\bar{u}_i = \frac{\partial}{\partial x_j}\left[\bar{u}_i\tau_{ij}^R\right] - \tau_{ij}^R S_{ij}, \qquad S_{ij} = \frac{1}{2}\left[\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right] \tag{7.5}$$

Here $S_{ij}$ is the strain-rate tensor introduced in Chapter 7, Section 1.2. The first term on the right of (7.5) is just the divergence of $\bar{u}_i\tau_{ij}^R$. In a finite, closed domain, in which $\bar{u}_j$ is zero on the boundary, or else in a statistically homogeneous turbulent flow, this term integrates to zero. Thus the net rate of transfer of mechanical energy to the turbulence is just the volume integral of $\tau_{ij}^R S_{ij}$, which is sometimes called the *deformation work*. Usually $\tau_{ij}^R S_{ij}$ is a positive quantity, reflecting the tendency for parts of the mean flow to disintegrate into eddies due to inertially driven instabilities. Thus a finite strain-rate in the mean flow tends to keep the turbulence alive. Note that there are no viscous effects involved in this transfer of energy (if Re is large): it is a non-dissipative process. The next question, therefore, is where does this turbulent energy go to?

If we have a steady-on-average flow in a pipe, say, then there is a continual energy transfer from the mean flow, via $\tau_{ij}^R S_{ij}$, to the turbulence. However, the turbulence in such a situation will be statistically steady and so this energy must be dissipated somehow. Ultimately, of course, it is viscosity which destroys the mechanical energy of the eddies. However, when Re is large, the viscous stresses acting on the large eddies are negligible, so there must be some rather subtle process at work. This leads to the idea of the energy cascade, a concept first proposed by the British meteorologist L F Richardson in the 1920s.

It is an empirical observation that any turbulent flow comprises 'eddies' which have a wide range of sizes. That is to say, there is always a wide spectrum of length scales, velocity gradients etc. Richardson's idea is that the largest eddies, which are created by instabilities in the mean flow, are themselves subject to inertial instabilities and rapidly break up into yet smaller vortices. These smaller eddies then, in turn, become unstable and break up into even smaller vortices and so on. There is a continual *cascade* of energy from the large scale down to the small (Figure 7.2).

It should be emphasised, however, that viscosity plays no part in this cascade. That is, when Re is large (based on $\mathbf{u}'$ and a typical eddy size), then the viscous stresses acting on the larger eddies are negligible. The whole thing is essentially driven by inertia. The cascade is halted, however, when the eddies become so small that the Reynolds number based on the small-scale eddy size is of the order of unity. That is, the very smallest eddies are dissipated by viscous forces, and for the viscous forces to be significant we need a Reynolds number of order unity. We may think of viscosity as providing a dustbin for energy at the very end of the cascade. In this sense the viscous forces are passive in nature, mopping up whatever energy is fed down from above. This process of a progressive energy cascade from large to small eddies was nicely summed up by Richardson in his parody of Swift's 'Fleas Sonnet': '*Big whirls have little whirls, which feed on their velocity, and little whirls have lesser whirls and so to viscosity.*'

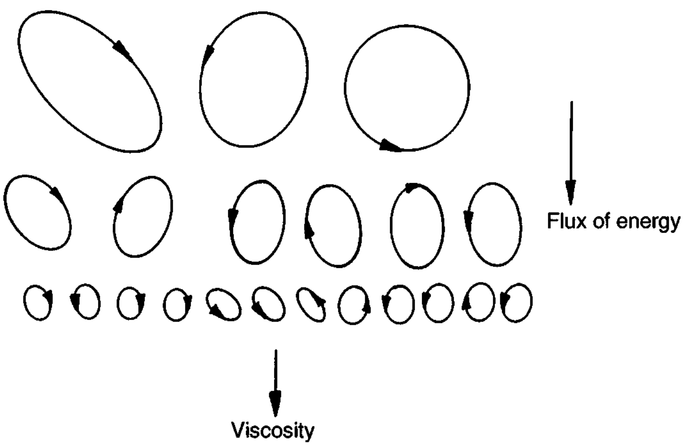



Figure 7.2 A schematic representation of the energy cascade.

Let us try to quantify this process. Let $l$ and $u'$ be typical length and velocity scales for the larger eddies. We might, for example, define $u'$ through $(u')^2 = \overline{(u_x')^2}$ or $\overline{(u_y')^2}$. Also, let $\varepsilon$ be the rate of dissipation of mechanical energy (per unit mass) due to viscosity acting on the small-scale eddies. In statistically steady turbulence $\varepsilon$ must also equal the rate at which energy is fed to the turbulence from the mean flow, $\tau_{ij}^R S_{ij}$. If it did not, the turbulence would either gain or lose energy. In fact, if we are to avoid a build-up of eddies of a particular size, $\varepsilon$ must equal the rate at which energy is past down the cascade at any point within that cascade. Let $G$ be the rate at which energy (per unit mass) is passed down the cascade. Symbolically, we have $G = \varepsilon$.

If we plot the energy contained in the eddies of a particular size against eddy size we might get something that looks like Figure 7.3. Remember, there is dissipation only at the smallest scales, and so $G$ has to be the same at all points between $A$ and $B$, i.e. $G_A = G_B$, where $G_A = \tau_{ij}^R S_{ij}/\rho$. Now it is an empirical observation that the rate of extraction of energy (per unit mass) from the large eddies to the energy cascade is of the order of

$$G_A \sim (u')^3/l$$

This is not a trivial result. As we shall see, it turns out to be very useful. Physically, it states that the largest eddies break up on a time scale of $l/u'$, their turn-over time.

We now try to determine the size of the smallest eddies. Let $v$ and $\eta$ be the velocity and length scales, respectively, of the smallest structures in

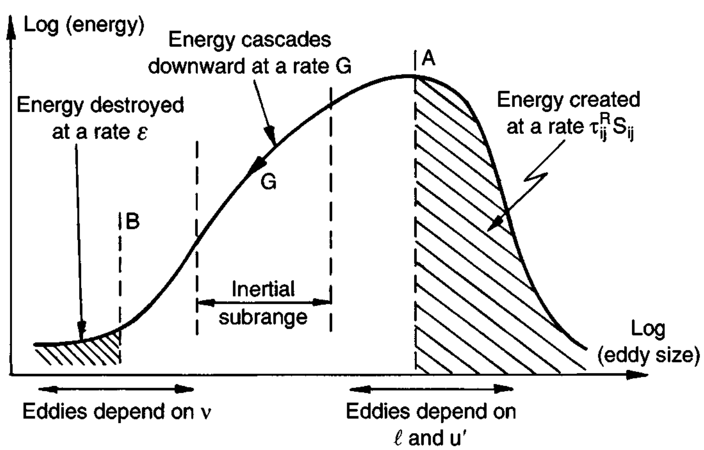



Figure 7.3 The energy cascade.

the flow. There are two things we can say about these eddies. First, $v\eta/\nu \sim 1$. That is, rather like a boundary layer, the size of the small eddies automatically adjusts to make the viscous forces an order-one quantity. Second, the energy dissipation rate per unit mass, which in a laminar flow is $-\nu(\nabla^2\mathbf{u}) \cdot \mathbf{u}$, must be of order $\varepsilon \sim \nu v^2/\eta^2$. Let us now summarise everything we know about the energy cascade.

1. The process is inviscid except at the smallest scales and so, in statistically steady turbulence,
$$\varepsilon = G, \qquad G_A = G_B \tag{7.6a, b}$$
2. Empirically it is observed that energy is extracted from the large scales, through eddy break-up, at a rate
$$G_A \sim (u')^3/l \tag{7.7}$$
3. The smallest scales must satisfy
$$v\eta/\nu \sim 1, \qquad \varepsilon \sim \nu v^2/\eta^2 \tag{7.8a, b}$$

We may eliminate either $\eta$ or $v$ from (7.8a) and (7.8b), and then use the fact of $\varepsilon \sim (u')^3/l$ to express $\eta$ or $v$ in terms of the large-scale parameters. Following this procedure we find that

$$\eta/l \sim (u'l/\nu)^{-3/4} = (\mathrm{Re})^{-3/4} \tag{7.9}$$

$$v/u' \sim (u'l/\nu)^{-1/4} = (\mathrm{Re})^{-1/4} \tag{7.10}$$

Here the Reynolds number is based on the large-scale velocity and length scales. Suppose, for example, that $\mathrm{Re} \sim 10^4$ and $l \sim 1$ cm, which is not untypical in a wind tunnel. Then $\eta \sim 0.01$ mm, which is very small! There is, therefore, a large spectrum of eddy sizes in a typical turbulent flow, and it is this which makes them so difficult to simulate numerically.

The quantities $v$ and $\eta$ are known as the *Kolmogorov microscales* of velocity and length, respectively, whereas $l$, the size of the large eddies, is known as the *integral scale* of the turbulence. (It is possible to give a rigorous definition of $l$, which we do later.)

There is something else of interest to be extracted from these simple estimates. Eliminating $v$ from (7.8a) and (7.8b), and then equating $\varepsilon$ to $G_B$, we find that the rate at which energy cascades downward at the tail end of the energy cascade is

$$G_B \sim \frac{v^3}{\eta} \tag{7.11}$$

Table 7.2. *Time and velocity scales for small eddies*

| Dimension | Ratio of Kolmogorov scale to large scale |
|---|---|
| Length | $\eta/l = \mathrm{Re}^{-3/4}$ |
| Velocity | $\mathrm{v}/u' = \mathrm{Re}^{-1/4}$ |
| Time | $u'\tau/l = \mathrm{Re}^{-1/2}$ |

Compare this with (7.7). The implication is that the small eddies, just like the largest ones, break up on a timescale of their turn-over time, $\tau = \eta/v$. Moreover, (7.9) and (7.10) give $\tau \sim (\mathrm{Re})^{-1/2} l/u' \ll l/u'$. So the characteristic timescale for the break up of the small eddies is very much faster than the turn-over time of the large eddies. Things happen very rapidly at the small scales. The relationship between the smallest and largest scales is given in Table 7.2.

Let us now try to predict the shape of the energy curve shown in Figure 7.3. We focus attention on the central region, well removed from the largest scales and the Kolmogorov microscale. For convenience we shall assume that this part of the eddy spectrum is statistically isotropic and homogeneous, an approximation which becomes increasingly sound as we move away from region $A$. We need to introduce some notation. Let $r$ be the size of some intermediate eddy in the cascade, $\eta < r < l$. Next, we introduce the so-called *velocity increment*, $\Delta v$, defined by $[\Delta v(r)]^2 = [u_x'(\mathbf{x}) - u_x'(\mathbf{x} + r\hat{\mathbf{e}}_x)]^2$, or else defined using the equivalent expression involving $y$ or $z$. Only eddies of size $r$ or smaller contribute to $\Delta v$, and so $(\Delta v)^2$ is an indication of the energy per unit mass contained in eddies of size $r$ and less.

We now try to predict $\Delta v$ at points well removed from regions $A$ and $B$, the so-called *inertial subrange*. If we are well away from $A$, then the eddies which concern us have a complicated heritage. They are the offspring of larger eddies which, in turn, come from yet bigger eddies, and so on. We would expect, therefore, that $\Delta v$ in the inertial subrange is independent of the structure of the very largest scales, and hence of $l$ and $u'$. Moreover, since we are well removed from region $B$, $\Delta v$ will not depend on $\nu$. Thus, provided $\eta \ll r \ll l$, $\Delta v$ will be a function of $G = \varepsilon$ and $r$ alone, there being no other relevant physical parameter. Symbolically, $\Delta v = \Delta v(\varepsilon, r)$. Now $\Delta v$ has dimensions of $ms^{-1}$, $\varepsilon$ has dimensions of $m^2 s^{-3}$ and $r$ has dimensions of $m$. The only dimensionless group which can be constructed from $\Delta v$, $\varepsilon$ and $r$ is $(\Delta v)/\varepsilon^{1/3} r^{1/3}$. It follows that $(\Delta v)/\varepsilon^{1/3} r^{1/3}$ is a pure number, presumably of the order of unity, and so

$$(\Delta v)^2 \sim \varepsilon^{2/3} r^{2/3} \tag{7.12}$$

This is known as *Kolmogorov and Obukhov's two-thirds law* and it is an excellent fit to the experimental data!

Sometimes (7.12) is expressed in a slightly different way. Many researchers work in Fourier space and introduce an *energy spectrum*, $E(k)$, where $k$ is a wave number, $k \sim 1/r$. $E(k)$ is defined by the requirement that $E(k)dk$ gives all the energy contained in eddies whose size lies in the range $k \rightarrow k + dk$. Since

$$(\Delta v)^2 \sim \int_k^\infty E(k)dk \sim (\text{energy in eddies of size smaller than } k^{-1})$$

we find

$$E(k) \sim \varepsilon^{2/3} k^{-5/3} \tag{7.13}$$

In this form it is known as Kolmogorov's five-thirds law.

Now the arguments above are all rather qualitative and more than a little heuristic. What, for example, do we mean by an eddy? It would be a mistake, however, to dismiss them lightly. The Kolmogorov–Richardson picture of turbulence gives an excellent qualitative description of conventional turbulence. Still, there is a need to introduce a more formal descriptive framework, and we start with the idea of velocity correlation functions.

### *7.1.4 Velocity correlation functions and the Karman–Howarth equation*

In order to simplify matters we now restrict ourselves to a form of idealised turbulence. We consider flows which are statistically homogeneous and isotropic, i.e. their statistical properties do not depend on position or direction. Also, we shall take the mean velocity to be zero. Since, in the absence of a mean shear, there is no mechanism for injecting energy into the turbulence, such a flow will always decay in the course of time. We might picture this as a fluid which is subjected to vigorous stirring and then left to itself. The properties of the turbulence are now time-dependent and so we need to introduce a different means of performing averages. We rely on ensemble averages, i.e. an average over many realisations of the flow. This is represented by $\langle\cdot\rangle$. In homogeneous turbulence, such an average can be shown to be equivalent to a spatial average, while in statistically steady turbulence, ensemble averages are equivalent to time averages, $\langle\cdot\rangle = \overline{(\cdot)}$.

From a practical point of view, the easiest way of generating homogeneous turbulence is to pass air uniformly through a wire mesh in a wind tunnel and adopt a frame of reference moving with the mean flow. (Actually, such a turbulence is not strictly homogeneous because of the boundaries and because it decays as we move downstream, but it is not a bad approximation.) The workhorse of turbulence theory is the *velocity correlation tensor*, sometimes called the velocity correlation function (Figure 7.4). This plays the same rôle in turbulence theory as velocity or momentum does in laminar flow. The *second-order velocity correlation tensor* is defined as

$$Q_{ij}(\mathbf{r}) = \langle u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})\rangle \tag{7.14}$$

(Actually $Q_{ij}$ also depends on $t$, so strictly this should be written as $Q_{ij}(\mathbf{r}, t)$.) Note that, because the turbulence is homogeneous, $Q_{ij}$ does not depend on $\mathbf{x}$. Also, since the mean velocity is zero, there is no need to use a prime to indicate a fluctuating velocity component. This correlation function has the geometric property $Q_{ij}(\mathbf{r}) = Q_{ji}(-\mathbf{r})$ and is related to the kinetic energy per unit mass by $\frac{1}{2}\langle\mathbf{u}^2\rangle = \frac{1}{2}Q_{ii}(0)$.

So what does $Q_{ij}(\mathbf{r})$ represent? Conceptually it is easier to think in terms of time averages rather than ensemble averages, and so we temporarily move back to thinking about steady-on-average flows and write

$$Q_{ij}(\mathbf{r}) = \overline{u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})}$$

The first thing to note about $Q_{ij}$ is that, when $\mathbf{r} = 0$, it is proportional to the Reynolds stress, $\tau_{ij}^R = -\rho Q_{ij}(0)$. Yet what if $\mathbf{r} \neq 0$? In this case $Q_{ij}$ simply tells us if two scalar quantities, $f = u_i(\mathbf{x})$ and $h = u_j(\mathbf{x}+\mathbf{r})$, are statistically correlated. We say that $f$ and $h$ are correlated if $\overline{fh} \neq 0$ and uncorrelated if $\overline{fh} = 0$. Often a *correlation coefficient*, $c$, is introduced, defined by

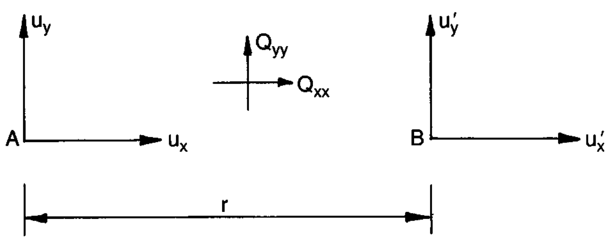



Figure 7.4 (a) Definition of velocity correlation functions.

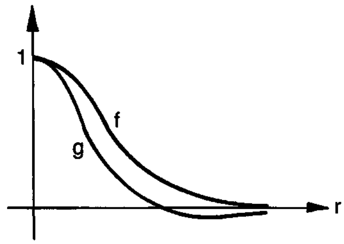



Figure 7.4 (b) Shape of the velocity correlation functions.

$$c^2 = \frac{(\overline{fh})^2}{\overline{f^2}\,\overline{h^2}}$$

If $c = \pm 1$, the correlation is said to be perfect. (Any variable is, of course, perfectly correlated to itself.) If $\overline{fh} = 0$ then it means that both $f$ and $h$ fluctuate in time in a manner quite independent of each other.

We now go back to ensemble averages. Consider two points, $A$ and $B$, separated by $\mathbf{r} = r\hat{\mathbf{e}}_x$ (Figure 7.4(a)). The correlation function $Q_{xx}(r\hat{\mathbf{e}}_x)$ represents the degree to which the horizontal velocities at $A$ and $B$ (at some particular instant) are correlated when averaged over many realisations. If the velocity fluctuations at $A$ and $B$ were statistically independent then $Q_{xx}(r)$ would be zero. On the other hand, if precisely the same thing is happening at $A$ and $B$ (the two points are perfectly correlated) then $Q_{xx} = \langle u_x^2 \rangle$. We expect that $Q_{xx} \to \langle u_x^2 \rangle$ as $r \to 0$ and $Q_{xx} \to 0$ as $r \to \infty$, remote points in a turbulent flow being uncorrelated. We now introduce some additional notation. Let $u$ be a characteristic turbulence velocity, defined by

$$u^2 = \langle u_x^2 \rangle = \langle u_y^2 \rangle = \langle u_z^2 \rangle \tag{7.15}$$

and write $Q_{xx}(r)$ and $Q_{yy}(r)$ in the form

$$Q_{xx}(r\hat{\mathbf{e}}_x) = u^2 f(r) \tag{7.16}$$

$$Q_{yy}(r\hat{\mathbf{e}}_x) = u^2 g(r) \tag{7.17}$$

The functions $f$ and $g$ are known as the longitudinal and lateral velocity correlation functions (or coefficients), respectively. They are dimensionless, satisfy $f(0) = g(0) = 1$, and have the shape shown in Figure 7.4(b). The integral scale, $l$, of the turbulence is often defined as

$$l = \int_0^\infty f(r)dr$$

This provides a convenient measure of the extent of the region within which velocities are appreciably correlated, i.e. the size of the large eddies. In fact, $f$ and $g$ are not independent functions. It may be shown that the continuity equation demands

$$g = f + \frac{1}{2} r f'$$

Moreover, symmetry and continuity arguments allow us to express $Q_{ij}(\mathbf{r})$ purely in terms of $f(r)$ and $\mathbf{r}$. The details are long and tedious and we merely state the end result. For isotropic turbulence it may be shown that

$$Q_{ij}(\mathbf{r}) = \frac{u^2}{2r}\left[\frac{d}{dr}\left(r^2 f\right)\delta_{ij} - f' r_i r_j\right] \tag{7.18}$$

(see suggested reading at end of the chapter).

The *third-order (or triple) velocity correlation function (or tensor)* is defined as

$$S_{ijl}(\mathbf{r}) = \langle u_i(\mathbf{x}) u_j(\mathbf{x}) u_l(\mathbf{x} + \mathbf{r})\rangle$$

It too can be written in terms of a single scalar function, $k(r)$, defined by

$$u^3 k(r) = \langle u_x^2(\mathbf{x}) u_x(\mathbf{x} + r\hat{\mathbf{e}}_x)\rangle$$

Again, symmetry and continuity arguments may be used to show that, in homogeneous, isotropic, turbulence

$$S_{ijl} = u^3\left[\left(\frac{k - rk'}{2r^3}\right) r_i r_j r_l + \left(\frac{2k + rk'}{4r}\right)(r_i\delta_{jl} + r_j\delta_{il}) - \frac{k}{2r} r_l \delta_{ij}\right] \tag{7.19}$$

The function $k$ is known as the longitudinal triple-velocity correlation function.

So far we have made lots of definitions and exploited certain kinematic relationships. However, this has not really got us very far. To make progress we need to introduce some dynamics in the form of the Navier–Stokes equation. Let $\mathbf{x}' = \mathbf{x} + \mathbf{r}$ and $\mathbf{u}' = \mathbf{u}(\mathbf{x}')$. (From now on, a prime will indicate a quantity at position $\mathbf{x}'$ and has nothing to do with a fluctuating, as distinct from mean, variable.) Then we have

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_k}(u_i u_k) - \frac{\partial}{\partial x_i}(p/\rho) + \nu \nabla_x^2 u_i \tag{7.20}$$

$$\frac{\partial u_j'}{\partial t} = -\frac{\partial}{\partial x_k'}(u_j' u_k') - \frac{\partial}{\partial x_j'}(p'/\rho) + \nu \nabla_{x'}^2 u_j' \tag{7.21}$$

On multiplying the first of these by $u'_j$, and the second by $u_i$, adding the two and averaging, we obtain

$$\frac{\partial}{\partial t}\langle u_i u'_j\rangle = -\left\langle u_i \frac{\partial u'_j u'_k}{\partial x'_k} + u'_j \frac{\partial u_i u_k}{\partial x_k}\right\rangle - \frac{1}{\rho}\left\langle u_i \frac{\partial p'}{\partial x'_j} + u'_j \frac{\partial p}{\partial x_i}\right\rangle \\ + \nu\left\langle u_i \nabla^2_{x'} u'_j + u'_j \nabla^2_x u_i\right\rangle \tag{7.22}$$

This looks a bit of a mess! However, it may be tidied up considerably. We note that, (i) operations of taking averages and differentiation permute, (ii) $\partial/\partial x_i$ and $\partial/\partial x'_j$ operating on averages may be replaced by $-\partial/\partial r_i$ and $\partial/\partial r_j$, respectively, and (iii) $u_i$ is independent of $\mathbf{x}'$ while $u'_j$ is independent of $\mathbf{x}$. (7.22) then simplifies to something a lot neater:

$$\frac{\partial Q_{ij}}{\partial t} = \frac{\partial}{\partial r_k}\left[S_{ikj} + S_{jki}\right] + 2\nu\nabla^2 Q_{ij} \tag{7.23}$$

(Consult one of the suggested texts at the end of the chapter for the details.) We have dropped the terms involving pressure since it may be shown that $\langle p u'_j\rangle = 0$ in isotropic turbulence.

We have managed to relate the rate of change of the second-order velocity correlation tensor to the third-order one. We might now go on to obtain an equation for the rate of change of $S_{ijk}$. Unfortunately, however, this contains terms involving fourth-order correlations, which in turn depend on fifth-order correlations and so on. We have come up against the *closure problem* of turbulence. So let us stick with (7.23) and see where it leads. Substituting for $Q_{ij}$ and $S_{ijk}$ in terms of the scalar functions $f(r)$ and $k(r)$ yields, after a considerable amount of algebra,

$$\boxed{\frac{\partial}{\partial t}\left[u^2 r^4 f(r)\right] = u^3 \frac{\partial}{\partial r}\left[r^4 k(r)\right] + 2\nu u^2 \frac{\partial}{\partial r}\left[r^4 f'(r)\right]} \tag{7.24}$$

This is known as the Karman–Howarth equation and it constitutes one of the central results in the theory of isotropic turbulence. We shall see in the next section that it can be used to estimate the rate of decay of freely evolving turbulence.

We close this section with a brief discussion of a different type of velocity correlation function, sometimes called a second-order structure function. It is defined by

$$B_{ik} = \left\langle (u'_i - u_i)(u'_k - u_k)\right\rangle \tag{7.25}$$

For example, $B_{ii} = \left\langle (\mathbf{u} - \mathbf{u}')^2 \right\rangle$. This is closely related to the velocity increment, $\Delta\mathbf{v}$, introduced in the last section. It is easy to show that the second-order correlation tensors $Q_{ij}$ and $B_{ij}$ are related by

$$B_{ij} = \frac{2}{3}\langle \mathbf{u}^2 \rangle \delta_{ij} - 2Q_{ij} \tag{7.26}$$

and so $B_{ij}$ may be expressed in terms of $f$, just like $Q_{ij}$. However, the tensor $B_{ij}$ has an advantage over $Q_{ij}$. Only eddies of size less than or equal to $|\mathbf{r}|$ can contribute to $B_{ij}(\mathbf{r})$. Consequently, by making $|\mathbf{r}|$ progressively smaller, we can move down the energy cascade, focusing on smaller and smaller eddies.

### *7.1.5 Decaying turbulence: Kolmogorov's law, Loitsyansky's integral, Landau's angular momentum and Batchelor's pressure forces*

It is natural to suppose that well-separated points in a turbulent flow are statistically uncorrelated, and so we expect $f(r)$ and $k(r)$ to decrease rapidly with distance. In fact, prior to 1956 it was assumed that $f$ and $k$ decay exponentially at large $r$. *If* this is the case, then the Karman–Howarth equation may be integrated to yield

$$I = u^2 \int_0^\infty r^4 f(r) dr = \text{constant} \tag{7.27}$$

This is known as Loitsyansky's integral. A N Kolmogorov took advantage of the (supposed) invariance of $I$ to predict the rate of decay of energy in freely evolving, isotropic turbulence. The argument goes as follows. The integral scale, $l$, is defined as $\int_0^\infty f dr$. We would expect, therefore, that in freely evolving (decaying) turbulence

$$I \sim u^2 l^5 = \text{constant} \tag{7.28}$$

We also know that the large eddies tend to break up on a timescale of their turn-over time, so that the large scales lose energy at a rate

$$\frac{du^2}{dt} \sim -\frac{u^3}{l} \tag{7.29}$$

and this energy is not replenished. Combining (7.28) and (7.29) yields *Kolmogorov's decay laws* for isotropic turbulence:

$$u^2 = u_0^2 [1 + (7/10)(u_0 t / l_0)]^{-10/7} \tag{7.30}$$

$$l \sim l_0 [1 + (7/10)(u_0 t / l_0)]^{2/7} \tag{7.31}$$

Here $u_0$ and $l_0$ are initial values of $u$ and $l$. In fact, these predictions are reasonably in line with the experimental data, which typically give $l \sim t^{0.35}$ and $u \sim t^{-1.26} \to t^{-1.34}$, depending on the Reynolds number.

The supposed invariance of $I$ has another consequence. It can be shown that for many, if not most, types of turbulence the energy spectrum at small $k$ has the form $E \sim (I/3\pi)k^4$ (plus higher order terms in $k$). We would expect, therefore, that the conservation of $I$ should lead to the energy spectrum at small $k$ being invariant during the decay, and this is indeed observed. This phenomenon is termed the *permanence of the large eddies* (since $E$ at small $k$ represents the energy of the largest eddies). All-in-all, it would seem that the experiments support (7.27).

There remain two problems, however. First, if we are to trust (7.27), then we would really like some *physical* explanation for the invariance of $I$. Second, we need some evidence that $f$ and $k$ decay exponentially, rather than algebraically, at large $r$. The physicist L D Landau resolved the first of these issues. He showed that, provided $f$ and $k$ decay exponentially, as assumed by Loitsyansky and Kolmogorov, then the invariance of $I$ is a direct consequence of the conservation of angular momentum. He argued as follows.

In general, a patch of turbulence will contain a finite amount of angular momentum. Consider, for example, turbulence which has been created in a wind tunnel by passing air through a wire grid. The turbulence is created because vortices are randomly shed from the wires, just like Karman vortices are shed from a cylinder. This ensemble of coherent vortices interact as the fluid is swept downstream until eventually a full field of turbulence emerges. Now each time a vortex is shed from a wire a finite amount of angular momentum is injected into the flow. (An eddy contains angular momentum.) This is evident from the shuddering of a loosely suspended grid, which is a manifestation of the back reaction (torque) exerted by the fluid on the grid. Thus, in grid-produced turbulence, we inject angular momentum into the fluid in the form of a sequence of randomly orientated vortices.

Now this angular momentum is important since, as the fluid moves downstream, its energy decays according to

$$\frac{du^2}{dt} \sim -\frac{u^3}{l}$$

yet this decay is subject to the constraint that the angular momentum of a given mass of turbulent fluid is conserved. (We ignore the viscous torque exerted by the boundaries.) Landau's great achievement was to show that the conservation of $I$ is simply a manifestation of the conservation of angular momentum.

There are two hurdles to overcome, however, in establishing this fact. First, $I =$ constant is a statistical statement about the turbulence, in the sense that it says something about the local, quadratic quantity $\langle \mathbf{u} \cdot \mathbf{u}' \rangle$. However, the angular momentum

$$\mathbf{H} = \int \mathbf{x} \times \mathbf{u} dV$$

is a global, linear measure of the velocity field, which is clearly not a statistical quantity. We must find someway of relating $\mathbf{H}$ to $\langle \mathbf{u} \cdot \mathbf{u}' \rangle$. The second problem is that as our field of turbulence becomes infinite ($V \to \infty$) we would expect the angular momentum per unit volume, $\mathbf{H}/V$, associated with a field of randomly orientated vortices to tend to zero. How can an infinitesimally small quantity influence the dynamical behaviour of the turbulence?

The trick is to consider a large but finite volume of turbulent fluid which has been stirred up and then left to itself. In this finite volume there will, in general, be an imperfect cancellation of the angular momentum associated with the vortices. In fact, as we shall see shortly, $\mathbf{H}/V$ does tend to zero as $V$ tends to infinity, but at finite rate: $V^{-1/2}$. Thus a finite volume contains a finite global angular momentum of order $V^{1/2}$. (The relationship $\langle \mathbf{H}^2 \rangle \sim V$ follows from the central limit theorem. This states that, if $\mathbf{x} \times \mathbf{u}$ at each location, $\mathbf{x}$, can be considered as an independent random variable, which might be the case at $t = 0$, then the variance of the volumetric average of $\mathbf{x} \times \mathbf{u}$ over some large volume $V$ will tend to zero at the rate $V^{-1}$ as $V \to \infty$. It follows that, if we take $\langle \cdot \rangle$ to be a volume average, $\langle \mathbf{H}^2 \rangle \sim V$.)

We are still left with the problem of how to convert the conservation of $\mathbf{H}$ into a statistical statement. To this end it is useful to take $\langle \cdot \rangle$ as an ensemble average and to consider a large number of realisations of the turbulence in our large but finite volume. That is, we stir the fluid up $N$ times and examine the subsequent decay for each realisation. Now if the size of $V$ is very much larger than the eddy size, $l$, then the turbulence should behave in a way which does not depend on the boundaries. We may then ignore the torque associated with the shear stresses exerted on the fluid by the boundaries. In each realisation, then, $\mathbf{H}^2/V$ will be independent of $t$. It follows that, when we average over all of the realisations, $\langle \mathbf{H}^2 \rangle / \mathrm{V}$ will be independent of time and of the size of the domain. It is the invariance of$\langle \mathbf{H}^2 \rangle / V$, rather than $\mathbf{H}$, which leads to (7.27). The exact relationship between $\langle \mathbf{H}^2 \rangle / V$ and $I$ may be established as follows.

Suppose the turbulent flow evolves in a large, closed sphere, whose radius $R$ greatly exceeds $l$ (Figure 7.5). The global angular momentum of the turbulence is then

$$\mathbf{H} = \int_V \mathbf{x} \times \mathbf{u} dV$$

(We will not bother carrying the density $\rho$ through the calculation.) The square of $\mathbf{H}$

$$\mathbf{H}^2 = \int_V \mathbf{x} \times \mathbf{u} dV \cdot \int_V \mathbf{x}' \times \mathbf{u}' dV'$$

can, with a little effort, be rearranged into the form

$$\mathbf{H}^2 = -\int_V \int_{V'} (\mathbf{x}' - \mathbf{x})^2 \mathbf{u} \cdot \mathbf{u}' dV dV'$$

We now ensemble average over each pair of points separated by a fixed distance $\mathbf{r} = \mathbf{x}' - \mathbf{x}$ to give

$$\langle \mathbf{H}^2 \rangle = -\int \int \mathbf{r}^2 \langle \mathbf{u} \cdot \mathbf{u}' \rangle d^3\mathbf{r} dV$$

Next, Landau assumed that $\langle \mathbf{u} \cdot \mathbf{u}' \rangle$ decays rapidly with $\mathbf{r}$ so that far-field contributions to the integral

$$\int r^2 \langle \mathbf{u} \cdot \mathbf{u}' \rangle d^3\mathbf{r}$$

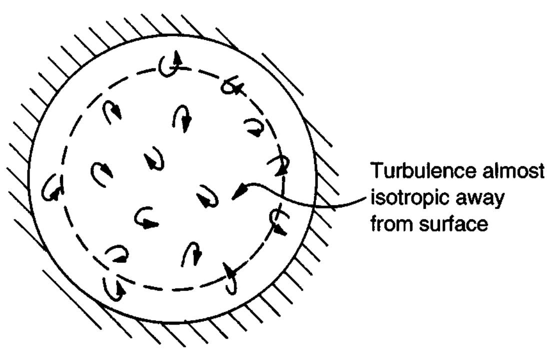



Figure 7.5 Landau's thought experiment.

are small. In such a situation only those velocity correlations taken close to the surface $R$ are aware of the presence of the boundary, and in this sense the turbulence is approximately homogeneous and isotropic. To leading order in $l/R$ we then have

$$\boxed{\langle H^2 \rangle / V = -\int \mathbf{r}^2 \langle \mathbf{u} \cdot \mathbf{u}' \rangle d^2\mathbf{r}} \tag{7.32}$$

However, (7.18) allows us to evaluate the integral on the left, which turns out to be $8\pi I$. The invariance of $I$ then follows from conservation of angular momentum, the viscous stresses on the boundary having negligible effect as $R/l \to \infty$. So, according to L D Landau, Kolmogorov's decay law is a direct consequence of the conservation of angular momentum. Given that the predictions of (7.30) and (7.31) are reasonably in line with the experimental data, and that there is a firm physical basis for the conservation of $I$, there was, for some time, a general feeling of satisfaction with the $t^{-10/7}$ decay law.

However, in 1956, G K Batchelor opened a can of worms when he showed that, at least in *anisotropic* turbulence, $k \sim r^{-4}$ as $r \to \infty$. *If* this is also true of isotropic turbulence, then the Karman–Howarth equation gives

$$\frac{dI}{dt} = \left[u^3 r^4 k\right]_\infty \neq 0 \tag{7.33}$$

and Loitsyansky's integral becomes time-dependent. The reason for the relatively slow decline in $k$ (algebraic rather than exponential) is interesting and subtle. It arises from the action of long-range pressure forces. A fluctuation in $\mathbf{u}$ at one point in a flow sends out pressure waves, which travel infinitely fast in an incompressible fluid, and these produce pressure forces, and hence accelerations, which fall off algebraically with distance from the source. Thus, because of pressure, a fluctuation in $\mathbf{u}$ at one point is felt everywhere within the fluid. Now Batchelor argued that turbulence in, say, a wind-tunnel would behave as if it had emerged from initial conditions in which remote points were statistically independent. However, because of the long-range pressure forces such a situation cannot persist, and long-range (algebraic) velocity correlations, $\langle \mathbf{u} \cdot \mathbf{u}' \rangle$, inevitably develop. At least this is the case in anisotropic turbulence (Figure 7.6).

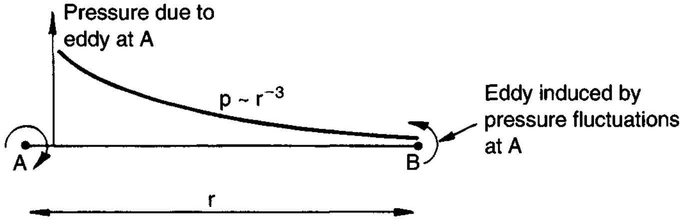



Figure 7.6 A schematic representation of Batchelor's long-range effects.

For *isotropic* turbulence, however, there is a high degree of symmetry in the statistics, and this symmetry is sufficiently strong to cause the direct effect of the long-range pressure forces on $\langle \mathbf{u} \cdot \mathbf{u}' \rangle$ to cancel. (This is why the pressure terms disappeared as we moved from (7.22) to (7.23).) Nevertheless, the pressure forces can still influence the triple correlations in isotropic turbulence and these, in turn, can influence the double correlations. In particular, it may be shown that,[1]

$$\frac{d}{dt}\left[u^3 r^4 k\right]_\infty = 3\int r^2 \langle ss' \rangle dr$$

where $s = u_x^2 - u_y^2$. Combining this with (7.33) yields

$$\frac{d^2 I}{dt^2} = \frac{d}{dt}\left[u^3 r^4 k\right]_\infty = 3\int r^2 \langle ss' \rangle dr = J$$

Thus, in general, we would expect $I$ to be time dependent. This is a direct result of the pressure forces which induce a $k_\infty \sim r^{-4}$ algebraic tail in the triple correlations and thus an $r^{-6}$ tail in $f_\infty$. (Remember that such algebraic tails invalidate both Landau's and Loitsyansky's arguments.) Crucially, however, we have failed to determine the magnitude of $J$. Now, over the years, $J$ has been estimated using a variety of *closure hypotheses*, such as the so-called quasi-normal approximation. However, these closure hypotheses are unreliable. In fact, the safest thing to do is to examine the experimental evidence. Interestingly, this suggests that $J$ is rather small. There are no direct measurements of $J$, but there is some indirect evidence. This comes in three parts.

[1] This equation only holds if the so-called cummulants of the fourth-order velocity correlations are negligible for well-separated points. Such a situation occurs when fourth-order velocity correlations are statistically independent for well-separated points. There is some experimental evidence to support this assertion.

- First, there is the predicted invariance of $E(k)$ at small $k$ (permanence of the large eddies) which comes from $E \sim (I/3\pi)k^4$. Is this in accordance with observation?
- Second, there have been measurements of $u^2(t)$. How do these compare with Kolmogorov's decay law?
- Third, there exist measurements of $Q_{ij}$ in the so-called final period of decay (when the turbulence is weak and viscosity is important). Do these show exponential or algebraic behaviour at large $r$?

It seems that, by and large, the experiments support Landau, Loitsyansky and Kolmogorov to the extent that they suggest that, once the turbulence is fully developed, $J$ is negligible. The permanence of the large eddies is indeed observed, and the form of $Q_{ij}$ at large $r$ is exponential in the final period of decay. Moreover, the measured decay rate of isotropic turbulence is not too far out of line with Kolmogorov's law. In 1960, Corrsin found $u^2 \sim t^{-n}$ where $n$ lies in the range $1.2 \rightarrow 1.4$ with an average value of 1.26. Later, Lumley, in 1978, found $u^2 \sim t^{-1.34}$ and $l \sim t^{0.35}$. ( Kolmogorov's law predicts $u^2 \sim t^{-1.43}$ and $l \sim t^{0.29}$) The observed exponential decay of $Q_{ij}$ in particular seemed to have surprised Batchelor who, having just established the existence of these long-range pressure forces and the associated long-range correlations in anisotropic turbulence, noted that: '*it is disconcerting that the present more extensive analysis cannot do as well as the old*'.

Interestingly, some authors suggest that Loitsyansky's integral is strongly time-dependant, or else does not exist (i.e. diverges). There are two reasons for this. First, a turbulence closure model which was popular in the 1960s, the quasi-normal (QN) model, predicts that $I$ varies as

$$\frac{d^2 I}{dt^2} \propto \int \frac{E^2}{k^2} dk \neq 0$$

where $E(k)$ is the energy spectrum. (This was shown by Proudman and Reid in 1954.) However, the quasi-normal model has no real physical basis and is known to produce anomalous effects, such as a negative energy spectrum. A variant of this, called the EDQNM model, avoids some of the worst excesses of the QN model, while still predicting a (slight) time-dependance for $I$. However, the EDQNM model automatically assumes $k_\infty \sim r^{-4}$ and so builds in long-range effects from the outset. In short, it prejudges the issue.

The second reason often given for doubting the approximate conservation of $I$ is the discovery by Saffman in 1967 that, *for suitable initial* *conditions*, long-range correlations can exist in a turbulent motion which are even stronger than those of Batchelor. This leads to an energy spectrum at small $k$ of the form $E \sim k^2$, unlike the usual assumption of $E \sim k^4$. In such a situation Loitsyansky's integral diverges. However, these particularly potent long-range correlations are too strong to have emerged from Batchelor's pressure forces, and so if they are to exist they must be imbedded in the initial conditions. Saffman himself argued that such initial conditions are unlikely to be met in, say, wind-tunnel turbulence, and so we would expect 'conventional' turbulence to have a Batchelor spectrum, $E \sim k^4$. Certainly, this is in accord with measurements of the decline of $u^2$ in the final period of decay, which clearly shows results compatible with $E \sim k^4$ and incompatible with Saffman's spectrum.

All-in-all, it would seem likely that the Landau–Loitsyansky equation

$$\langle \mathbf{H}^2 \rangle / V = 8\pi I = \text{constant} \tag{7.34}$$

is approximately valid in isotropic turbulence provided the initial conditions are of the form assumed by Batchelor (i.e. those where remote points are statistically independent). Moreover, such initial conditions are probably typical of, say, wind-tunnel turbulence.

### *7.1.6 On the difficulties of direct numerical simulations (DNS)*

For some time now people have been computing the evolution of turbulent flows in a cubic domain in which the boundaries have very special properties; they are periodic. That is to say, whatever is happening at one face of the cube happens on the opposite face. Such domains are called periodic cubes and they lend themselves to particularly efficient numerical algorithms for solving the Navier–Stokes equations. So far these simulations have been restricted to Reynolds numbers of around $ul/\nu \sim 100 \rightarrow 500$. Higher values of Re are difficult to achieve because of the computational cost of computing all of the turbulent scales down to the Kolmogorov microscale. (As Re increases so the range of scales increases.) Still, many people believe that turbulence at, say, $\text{Re} = 500$ might capture some of the features of high-Re turbulence, and so considerable attention has been given to these simulations.

It might be thought, therefore, that issues such as the rate of dissipation of energy, or the invariance (or otherwise) of Loitsyansky's integral could be settled by computer simulations in a periodic cube. After all, such simulations are now routinely performed and it is usually assumed, either implicitly or else explicitly, that turbulence in a periodic cube is representative of homogeneous, isotropic turbulence in an infinite domain. Unfortunately, such an assumption is somewhat misleading. In fact, turbulence in a periodic cube represents a rather special dynamical system, the large scales of which are somewhat different from real-life turbulence. It is this which makes it difficult to investigate the behaviour of $u^2(t)$ or of $I$.

There are two important points to note. First, turbulence in a periodic cube is anisotropic at the large scales. To see that this is so, simply consider $Q_{ij}(\mathbf{r})$. Suppose that $r = L_{\text{box}}$ and choose $\boldsymbol{x}$ and $\boldsymbol{x}'$ to lie at the bottom corners of one of the vertical faces of the cube. Then $Q_{ii}(\mathbf{r}) = 3u^2$ since the two points are perfectly correlated. Now rotate $\mathbf{r}$ by some angle, say, 45°. One point lies at the corner of the box and the other in the interior. $Q_{ii}(\mathbf{r})$ is now less than $3u^2$ since there is no longer a perfect correlation. It follows that the turbulence is anisotopic, at least at the large scales. Worse still, strong, long-range correlations, which are quite unphysical, are built into the periodic cube at the scale of the box.

Still, it seems plausible that if $L_{\text{box}}$ is, say, two orders of magnitude greater than the integral scale, $l$, then there may be some sub-domain within the box in which the influence of the boundaries are not felt. The bulk of the turbulence might then be homogeneous and isotropic. It seems likely, therefore, that the requirements for a simulation to be representative of real-life turbulence are

(i) $\text{Re} \gg 1$
(ii) $l \ll L_{\text{box}}$

Unfortunately, because of limitations in computer power, it is difficult to satisfy both of these criteria. In order to obtain $\text{Re} \sim 500$, it is normally required to have $l \sim L_{\text{box}}/3$. Conversely, if we require a value of $l \sim L_{\text{box}}/100$, then it is difficult to get Re much larger than $\sim 20$. In short, turbulence in a periodic cube usually knows it is in a periodic cube and the large scales behave accordingly. At least, that is the story to date.

This concludes our introduction to turbulence. We have omitted a great deal in our brief survey, including many of the details of the derivations of (7.18), (7.19) and the Karman–Howarth equation, as well as the proof of

$$\langle \mathbf{H}^2 \rangle / V = 8\pi I$$

However, the interested reader can readily fill the gaps using one or more of the many excellent texts which exist on turbulence.

We now turn to MHD turbulence, which is our main interest. We shall see that Landau's ideas prove particularly fruitful, but that G K Batchelor's warnings of long-range statistical correlations continue to haunt the subject.

## 7.2 MHD Turbulence

We now examine the influence of a uniform, imposed magnetic field on the decay of (initially isotropic) freely evolving turbulence. We start by returning to the model problem discussed in §5.3, extending it, with the help of Landau's ideas, to a formal statistical theory (Figure 7.7).

### *7.2.1 The growth of anisotropy at low and high $R_m$*

Suppose that a conducting fluid is held in an insulated sphere of radius $R$. The sphere sits in a uniform, imposed field $\mathbf{B}_0$, so that the total magnetic field is $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, $\mathbf{b}$ being associated with the currents induced by $\mathbf{u}$ within the sphere. For simplicity, we take the fluid to be inviscid (we shall put viscosity back in later). However, we place no restriction on $R_m$, nor on the interaction parameter which we define here to be $N = \sigma B_0^2 l / \rho u$, $l$ being the integral scale of the turbulence. When $R_m$ is small we have $|\mathbf{b}| \ll |\mathbf{B}_0|$, but in general $|\mathbf{b}|$ may be as large as, or possibly even larger than, $|\mathbf{B}_0|$. At $t = 0$ the fluid is vigorously stirred and then left to itself. The questions of interest are: (i) can we characterise the anisotropy introduced into the turbulence by $\mathbf{B}_0$; (ii) how does the energy decay?

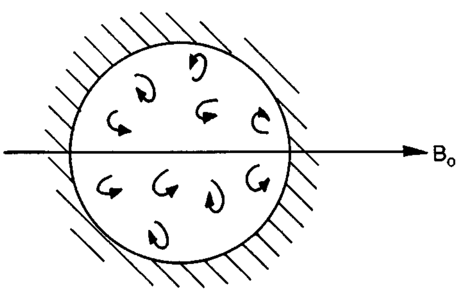



Figure 7.7 MHD turbulence evolving in a sphere.

We shall attack the problem in precisely the same way as in § 5.3. We start by noting that the global torque exerted on the fluid by the Lorentz force is

$$\mathbf{T} = \int \mathbf{x} \times (\mathbf{J} \times \mathbf{B}_0) dV + \int \mathbf{x} \times (\mathbf{J} \times \mathbf{b}) dV \tag{7.35}$$

However, a closed system of currents produces zero net torque when they interact with their self-field, $\mathbf{b}$. (This follows from conservation of angular momentum.) It follows that the second integral on the right is zero. We now transform the first integral using the identity

$$2\mathbf{x} \times [\mathbf{G} \times \mathbf{B}_0] = [\mathbf{x} \times \mathbf{G}] \times \mathbf{B}_0 + \nabla \cdot [(\mathbf{x} \times (\mathbf{x} \times \mathbf{B}_0))\mathbf{G}] \tag{7.36}$$

(Here $\mathbf{G}$ is any solenoidal vector field.) Setting $\mathbf{G} = \mathbf{J}$ we find

$$\mathbf{T} = \left\{ \frac{1}{2} \int (\mathbf{x} \times \mathbf{J}) dV \right\} \times \mathbf{B}_0 = \mathbf{m} \times \mathbf{B}_0 \tag{7.37}$$

and consequently the global angular momentum evolves according to

$$\rho \frac{d\mathbf{H}}{dt} = \mathbf{T} = \mathbf{m} \times \mathbf{B}_0, \qquad \mathbf{H} = \int (\mathbf{x} \times \mathbf{u}) dV \tag{7.38}$$

By implication, $\mathbf{H}_{//}$ is conserved. This, in turn, gives a lower bound on the total energy of the system,

$$E = E_{\mathbf{b}} + E_{\mathbf{u}} > \mathbf{H}_{//}^2 \left( 2 \int \mathbf{x}_\perp^2 dV \right)^{-1} \tag{7.39}$$

(Expression (7.39) follows from the Schwarz inequality in the form $\mathbf{H}_{//}^2 \geq \int \mathbf{u}_\perp^2 dV \int \mathbf{x}_\perp^2 dV$. See Chapter 5, Section 3.) However, the total energy declines due to Joule dissipation and so we also have

$$\frac{dE}{dt} = \frac{d}{dt} \int_{V_R} \frac{\rho \mathbf{u}^2}{2} dV + \frac{d}{dt} \int_{V_\infty} \frac{b^2}{2\mu} dV = -\frac{1}{\sigma} \int_{V_R} \mathbf{J}^2 dV \tag{7.40}$$

We have the makings of a paradox. One component of angular momentum is conserved, requiring that $E$ is non-zero, yet energy is dissipated as long as $\mathbf{J}$ is finite. The only way out of this paradox is for the turbulence to evolve to a state in which $\mathbf{J} = 0$, yet $E_{\mathbf{u}}$ is non-zero (to satisfy (7.39)). However, if $\mathbf{J} = 0$ then Ohm's law requires $\mathbf{E} = -\mathbf{u} \times \mathbf{B}_0$, while Faraday's law requires that $\nabla \times \mathbf{E} = 0$. It follows that, at large times, $\nabla \times (\mathbf{u} \times \mathbf{B}_0) = (\mathbf{B}_0 \cdot \nabla)\mathbf{u} = 0$, and so $\mathbf{u}$ becomes independent of $\mathbf{x}_{//}$ as $t \to \infty$. The final state is therefore strictly two-dimensional, of the form $\mathbf{u}_\perp = \mathbf{u}_\perp(\mathbf{x}_\perp)$, $\mathbf{u}_{//} = 0$. In short, the turbulence ultimately reaches a state which consists of one or more columnar eddies, each aligned with $\mathbf{B}_0$. Note that all of the components of $\mathbf{H}$, other than $\mathbf{H}_{//}$ are destroyed during this evolution.

At low $R_m$ this transition will occur on the timescale of $\tau = \left(\sigma B_0^2/\rho\right)^{-1}$, the magnetic damping time. This was demonstrated in § 5.3 and the argument is straightforward. A low $R_m$, the current density is governed by

$$\mathbf{J} = \sigma(-\nabla V + \mathbf{u} \times \mathbf{B}_0) \tag{7.41}$$

and so the global dipole moment, $\mathbf{m}$, is

$$\mathbf{m} = \frac{1}{2}\int \mathbf{x} \times \mathbf{J} dV = (\sigma/2)\int \mathbf{x} \times (\mathbf{u} \times \mathbf{B}_0) dV - (\sigma/2)\oint (V\mathbf{x}) \times d\mathbf{S}$$

The surface integral vanishes while the volume integral transforms, with the aid of (7.36), to give

$$\mathbf{m} = (\sigma/4)\mathbf{H} \times \mathbf{B}_0$$

Substituting into (7.38) we obtain

$$\frac{d\mathbf{H}}{dt} = -\frac{\mathbf{H}_\perp}{4\tau}, \qquad \tau^{-1} = \sigma B_0^2/\rho \tag{7.42}$$

and so $\mathbf{H}_{//}$ is conserved (as expected) while $\mathbf{H}_\perp$ declines exponentially on a timescale of $\tau$.

In summary then, whatever the initial condition, and for any $R$m or $N$, the flow evolves towards the anisotropic state (Figure 7.8)

$$\mathbf{u}_\perp = \mathbf{u}_\perp(\mathbf{x}_\perp), \qquad \mathbf{H}_{//} = \mathbf{H}_{//}(0), \qquad \mathbf{H}_\perp = \mathbf{u}_{//} = \mathbf{b} = \mathbf{J} = \mathbf{0} \tag{7.43}$$

From the point of view of turbulence theory, the two most important points are: (i) $\mathbf{B}_0$ introduces severe anisotropy into the turbulence, and (ii)

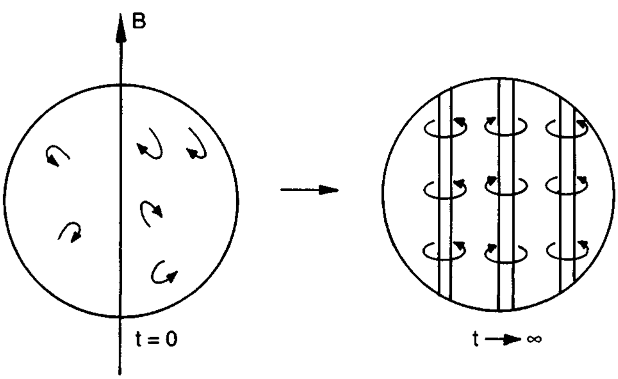



Figure 7.8 Growth of anistropy in MHD turbulence.

$\mathbf{H}_{//}$ is conserved during the decay. Following Landau's arguments, the latter point implies that

$$\langle \mathbf{H}_{//}^2 \rangle = -\int\int \mathbf{r}_\perp^2 \langle \mathbf{u}_\perp \cdot \mathbf{u}'_\perp \rangle d^3\mathbf{r}\, d^3\mathbf{x} = \text{constant} \tag{7.44}$$

where $\mathbf{r} = \mathbf{x}' - \mathbf{x}$. If (and it is a significant if) we can ignore Batchelor's long-range statistical correlations, then, for as long as $R \gg l$, we have the invariant

$$\langle \mathbf{H}_{//}^2 \rangle / V = -\int \mathbf{r}_\perp^2 \langle \mathbf{u}_\perp \cdot \mathbf{u}'_\perp \rangle d^3\mathbf{r} = \text{constant} \tag{7.45}$$

This is Loitsyansky's integral for MHD turbulence. (When $\mathbf{B}_0 = 0$ and the turbulence is isotropic, we can replace (7.45) by (7.34).)

Of course, in these arguments we have ignored $\nu$ and hence the process of energy removal via the energy cascade. In reality, for a finite $\nu$, the predicted growth of anisotropy will occur only if the turbulence lives for long enough and this, in turn, requires $\mathbf{J} \times \mathbf{B} \geq\sim \rho(\mathbf{u} \cdot \nabla)\mathbf{u}$, i.e. $N \geq\sim 1$. Note, however, that (7.45) is valid for any $N$ provided that the long-range statistical correlations are weak.

### 7.2.2 Decay laws at low $R_m$

We now restrict ourselves to low values of $R_m$, and reintroduce viscosity. We would like to develop the MHD equivalent of Kolmogorov's decay laws (7.30) and (7.31)

$$u^2 \sim u_0^2[1 + (7/10)(u_0 t/l_0)]^{-10/7}$$

$$l \sim l_0[1 + (7/10)(u_0 t/l_0)]^{2/7}$$

Recall that these were based on the estimates

$$\frac{du^2}{dt} \sim -\frac{u^3}{l}, \qquad \int \mathbf{r}^2 \langle \mathbf{u} \cdot \mathbf{u}' \rangle d^3\mathbf{r} \sim u^2 l^5 = \text{constant} \tag{7.46a, b}$$

We require MHD analogues of these equations.

In MHD turbulence the total energy, which at low $R_m$ is dominated by kinetic energy ($\mathbf{b}$ being vanishingly small), declines due to both Joule dissipation and viscosity:

$$\frac{dE}{dt} = -\frac{1}{\sigma}\int \mathbf{J}^2 dV - \rho\nu \int \omega^2 dV \tag{7.47}$$

(Here we have used the fact that the viscous dissipation is minus the rate of working of the viscous force, $-\nu(\nabla^2\mathbf{u}) \cdot \mathbf{u}$, and this is related to the vorticity by $-(\nabla^2\mathbf{u})\cdot\mathbf{u} = \boldsymbol{\omega}^2 + \nabla\cdot(\boldsymbol{\omega}\times\mathbf{u})$, the latter term integrating to zero.) Now let us suppose that the energy cascade proceeds as usual,[2] on a timescale of $l/u$. Then the MHD analogue of (7.46a) is

$$\frac{du^2}{dt} \sim -\frac{u^3}{l_\perp} - \left(\frac{l_\perp}{l_{//}}\right)^2 \frac{u^2}{\tau} \tag{7.48}$$

Here $l_\perp$ and $l_{//}$ represent suitably defined transverse and longitudinal integral scales for the turbulence. The new term in (7.47) represents the Joule dissipation $\langle \mathbf{J}^2 \rangle / \rho\sigma$, $\langle \mathbf{J}^2 \rangle$ having been estimated from the curl of the low-$R_m$ form of Ohm's law, $\nabla\times\mathbf{J} = \sigma(\mathbf{B}_0\cdot\nabla)\mathbf{u}$. We now need the analogue of (7.46b). This is provided by our conservation law (7.45), which, in the absence of long-range statistical correlations, yields the estimate

$$u^2 l_{//} l_\perp^4 = \text{constant} \tag{7.49}$$

Expressions (7.48) and (7.49) are the analogues of Kolmogorov's equations (7.46a, b). However, because of the anisotropy of MHD turbulence, we have three, rather than two, unknowns: $u$, $l_{//}$, $l_\perp$. We need a third relationship if we are to predict the rate of decay of energy. This comes from the fact that $l_{//}/l_\perp = 1$ if $N$ is small, and obeys (5.16) if $N$ is large. For example, in the high-$N$ examples given in Chapter 5, Section 2, where isolated vortices evolve in a uniform magnetic field, $l_{//}$ increases due to $\mathbf{B}_0$ but $l_\perp$ is unaffected by the field. The end result is $l_{//}/l_\perp \sim (t/\tau)^{1/2}$. Both limits (low and high $N$) are captured by the heuristic expression

$$\frac{d}{dt}\left(\frac{l_{//}}{l_\perp}\right)^2 \sim \frac{2}{\tau} \tag{7.50}$$

Expressions (7.48) $\to$ (7.50) represent a closed system for $u$, $l_{//}$ and $l_\perp$. They contain two timescales, $\tau$ and $l_\perp/u$, the ratio of which is $N$, and they predict very different kinds of behaviour depending on the initial value of $N$. For example, whenever inertia is negligible by comparison with $\mathbf{J}\times\mathbf{B}_0$, (7.48) $\to$ (7.50) reduce to

$$\frac{du^2}{dt} \sim -\left(\frac{l_\perp}{l_{//}}\right)^2 \frac{u^2}{\tau}$$

$$u^2 l_{//} l_\perp^4 = \text{constant}$$

$$\frac{d}{dt}\left(\frac{l_{//}}{l_\perp}\right)^2 = \frac{2}{\tau}$$

[2] This is plausible since the Lorentz force is felt only by the largest eddies, the turnover time of the bulk of the eddies being much shorter than $\tau$. See Appendix 4.

These are readily integrated to yield $u^2 \sim (t/\tau)^{-1/2}$, $l_{//} \sim (t/\tau)^{1/2}$, results which coincide with our study of isolated vortices at high $N$ (see Section 5.2). In fact, these turbulence scalings may be verified by exact analysis through integration of the linearised (inertia-less) Navier–Stokes equation. However, the procedure is complicated, involving three-dimensional Fourier transforms, and so we shall not reproduce the results here.

When $N$ is small, on the other hand, (7.48) → (7.50) yield Kolmogorov's law, $u^2 \sim t^{-10/7}$, with a small correction due to Joule dissipation. For intermediate values of $N$, however, the situation is rather different. In general there is no power law decay behaviour, although for the particular case $N(t = 0) = 7/15$ we find $u^2 \sim (t/\tau)^{-11/7}$, $l_{//} \sim (t/\tau)^{5/7}$and $l_\perp \sim (t/\tau)^{3/14}$. This compares favourably with laboratory experiments performed at $N \sim 1$.[3]

So the general theme here is that the eddies tend to elongate in the direction of $B_0$, causing $l_{//}$ to grow faster than $l_\perp$, as anticipated in Chapter 7, Section 2.1. There are three distinct but related explanations for the growth of $l_{//}$ given in the literature. One is the argument presented in the preceding section, the essence of which is that the conservation of $\mathbf{H}_{//}$, in the face of continual Joule dissipation, is possible only if $l_{//}$ grows. That is to say, at high $N$

$$\frac{du^2}{dt} \sim -\left(\frac{l_\perp}{l_{//}}\right)^2 \frac{u^2}{\tau}$$

which implies that $u^2$ declines according to

$$u^2 \sim u_0^2 \exp\left[-\frac{1}{\tau}\int_0^t (l_\perp/l_{//})^2 dt\right] \tag{7.51}$$

If $l_\perp/l_{//}$ were to remain of order unity, then $u^2$ would decline exponentially, in direct contradiction to

$$u^2 l_{//} l_\perp^4 = \text{constant}$$

It is inevitable, therefore, that $l_{//}/l_\perp$ grows, thus introducing anisotropy into the turbulence.

An alternative argument relies on the fact that the curl of the Lorentz force (per unit mass) may be written in the form

$$\nabla \times \mathbf{F} = \nabla \times [\mathbf{J} \times \mathbf{B}_0/\rho] = -\tau^{-1}\nabla^{-2}\left[\partial^2 \boldsymbol{\omega}/x_{//}^2\right] \tag{7.52}$$

which looks a bit like

[3] See Appendix 4.

$$\nabla \times [\mathbf{F}] \sim \left(l_\perp^2/\tau\right)\partial^2\omega/\partial x_{//}^2 \tag{7.53}$$

(When $l_{//} \gg l_\perp$, this may be made rigorous by Fourier transforming the vorticity equation in the transverse plane, so that (7.52) transforms to $\left(k_\perp^2\tau\right)^{-1}\partial^2\omega/\partial x_{//}^2$, $k_\perp$ being a wavenumber in the transverse plane). The implication is that, provided inertia is small, so that $\nabla \times (\mathbf{u} \times \boldsymbol{\omega})$ is much weaker than (7.52), the vorticity will diffuse along the $\mathbf{B}_0$-lines with a diffusion coefficient of $l_\perp^2/\tau$. This pseudo-diffusion is the last vestige of Alfvén wave propagation, as discussed in Chapter 6, Section 1.

A third, more mechanistic, argument is the following. Suppose we have a vortex, as shown in Figure 7.9, in which $\boldsymbol{\omega}$ is aligned with $\mathbf{B}_0$. (We use local cylindrical polar coordinates as shown.) Then the vortex will spread along the $\mathbf{B}$-line. The mechanism for this elongation is as follows. The term $\mathbf{u}_\theta \times \mathbf{B}$ tends to drive a current, $J_r$, in accordance with Ohm's law. Near the centre of the vortex, where axial gradients in $u_\theta$ are small, this is counter-balanced by an electrostatic potential, $\nabla V$, and so no current flows. However, near the top and bottom of the vortex, the current can return through regions of small or zero swirl, as shown. The resulting inward flow of current above and below the vortex gives rise to an azimuthal torque which, in turn, creates swirl in previously stagnant regions. In this way vorticity diffuses out along the $\mathbf{B}$-lines. (We will return to this issue in Chapter 9, where we look at vortices of arbitrary orientation.)

We close this section on a note of caution. Because of anisotropy, great care must be taken in the definition of $N$. A nominal definition might be

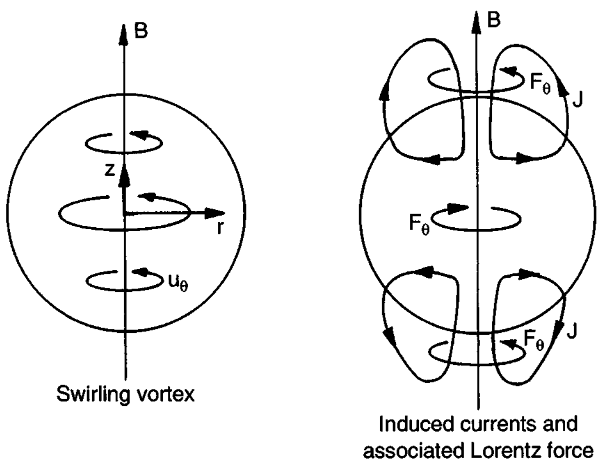



Figure 7.9 Mechanism for the elongation of vortices in a magnetic field.

$$N_{\perp} = \sigma B_0^2 l_{\perp} / \rho u \tag{7.54a}$$

or perhaps

$$N_{//} = \sigma B_0^2 l_{//} / \rho u \tag{7.54b}$$

However, it is readily demonstrated that the true ratio of $\mathbf{J} \times \mathbf{B}_0$ to inertia is

$$N_{\text{true}} = N_{//}(l_{\perp}/l_{//})^3 \tag{7.54c}$$

In practice, the difference between $N_{//}$ and $N_{\text{true}}$ can be large. Suppose, for example, that $l_{//} \sim 3l_{\perp}$ and $N_{//} \sim 10$ (which is not untypical in the laboratory). It might be thought, naively, that $\mathbf{J} \times \mathbf{B}_0$ is the dominant force. In fact $N_{\text{true}}$ in this case is less than unity, so that inertia is dominant! Such misconceptions occur commonly in the literature.

Interestingly, whatever the initial value of $N$, $N_{\text{true}}$ always evolves towards unity, representing a balance between $\mathbf{J} \times \mathbf{B}_0$ and inertia. For example, if $N$ is initially very large, then $u^2 \sim t^{-1/2}$, $l_{//} \sim t^{1/2}$ and $l_{\perp} = \text{constant}$. As a result $N_{\text{true}} = N_{\perp}(l_{\perp}/l_{//})^2 \sim N_0(t/\tau)^{-3/4}$, $N_0$ being the initial value of $N$ (the initial conditions are assumed to be isotropic). Thus, $N_{\text{true}}$ will fall as the eddies elongate, essentially because $\mathbf{J} \times \mathbf{B}_0$ declines due to a fall in $\mathbf{J}$. Conversely, if $N$ is initially very small, so the turbulence remains (almost) isotropic, then $u^2 \sim t^{-10/7}$, $l \sim t^{2/7}$ and $N_{\text{true}} \sim N_0(u_0 t/l_0)$. In this case $N_{\text{true}}$ rises as the inertia of the eddies becomes weaker. In either case, for large or small $N_0$, $N_{\text{true}} \to \sim 1$ as $t \to \infty$.

### 7.2.3 *The spontaneous growth of a magnetic field at high $R_m$*

We now turn to high-$R_m$ turbulence and consider the case where the imposed field, $\mathbf{B}_0$, is zero. We are interested in whether or not a small 'seed' field, present in the fluid at $t = 0$, will grow or decay in statistically steady turbulence. An intriguing argument, proposed by G K Batchelor, suggests a seed field will grow if $\lambda < \nu$ and decay if $\lambda > \nu$.

Batchelor noted that the fate of the seed field is determined by the balance between the random stretching of the flux tubes by $\mathbf{u}$, which will tend to increase $\langle B^2 \rangle$, and Ohmic dissipation, which operates mainly on the small-scale flux tubes (which have large spatial gradients in $\mathbf{B}$). He also noted the analogy between $\boldsymbol{\omega}$ and $\mathbf{B}$ in the sense that they are governed by similar equations:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \lambda \nabla^2 \mathbf{B}$$

If $\lambda = \nu$, there exists a solution for the seed field of the form $\mathbf{B} = \text{constant} \times \boldsymbol{\omega}$. Thus, since $\langle \omega^2 \rangle$ is steady, so is $\langle \mathbf{B}^2 \rangle$. It follows that, if $\lambda = \nu$, flux-tube stretching and Ohmic dissipation have equal but opposite influences on $\langle \mathbf{B}^2 \rangle$. If $\lambda$ exceeds $\nu$, however, we would expect enhanced Ohmic dissipation and a decline in $\langle \mathbf{B}^2 \rangle$, while $\lambda < \nu$ should lead to spontaneous growth in the seed field, a growth which is curtailed only when $\mathbf{J} \times \mathbf{B}$ is large enough to suppress the turbulence significantly. (Note that the threshold $\lambda = \nu$ is a very stringent condition. In most liquid metals, for example, $\nu/\lambda \sim 10^{-6}$. Since $\sigma$ and $\nu$ increase with the mean free path lengths of the charge and mass carriers, respectively, the condition $\lambda < \nu$ is likely to be met only in the astrophysical context, perhaps in the solar corona or the interstellar gas.)

These arguments are intriguing but imperfect. The problems are two-fold. First, the analogy between $\mathbf{B}$ and $\boldsymbol{\omega}$ is not exact: $\boldsymbol{\omega}$ is functionally related to $\mathbf{u}$ in a way in which $\mathbf{B}$ is not. Second, if the turbulence is to be statistically steady, then a forcing term must appear in the vorticity equation representing some kind of mechanical stirring (which is required to keep the turbulence alive). Since the corresponding term is absent in the induction equation, the analogy between $\mathbf{B}$ and $\boldsymbol{\omega}$ is again broken. One might try to circumvent this objection by considering freely decaying turbulence. Unfortunately, this also leads to problems, since the turbulence will die on a timescale of $l/u$, and if $R_m = u_0 l_0/\lambda$ is large, this implies we can get a growth in $\langle B^2 \rangle$ only for times much less than the Ohmic timescale, $l^2/\lambda$. However, in the dynamo context, such transient growths are of little interest. Thus the conditions under which $\langle \mathbf{B}^2 \rangle$ will spontaneously grow are still unclear.

If we accept the argument that a seed field is amplified for sufficiently small $\lambda$, it is natural to ask what the spatial structure of this field might be. Will it have a very fine-scale structure due to flux tube stretching, or a large-scale structure due to flux-tube mergers? In this context it is interesting to note that arguments have been put forward to suggest that there is an *inverse cascade* of the magnetic field in freely evolving, high-$R_m$ turbulence. That is to say, the integral scale for $\mathbf{B}$ grows as the flow evolves because small-scale flux tubes merge to produce a large-scale field. The arguments are rather tentative, and rest on the approximate conservation of magnetic helicity which, in turn, relies on the three equations:

$$\frac{D}{Dt}\left(\frac{\rho\mathbf{u}^2}{2}\right) = -\nabla\cdot[p\mathbf{u}] - \rho\nu\left[\boldsymbol{\omega}^2 + \nabla\cdot(\boldsymbol{\omega}\times\mathbf{u})\right] + \left[\mathbf{J}\cdot\mathbf{E} - \mathbf{J}^2/\sigma\right]$$

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu}\right) = -\mathbf{J}\cdot\mathbf{E} - \nabla\cdot[(\mathbf{E}\times\mathbf{B})/\mu]$$

$$\frac{D}{Dt}(\mathbf{A}\cdot\mathbf{B}) = \nabla\cdot[(\mathbf{u}\cdot\mathbf{A})\mathbf{B}] - \sigma^{-1}[2\mathbf{J}\cdot\mathbf{B} + \nabla\cdot(\mathbf{J}\times\mathbf{A})]$$

The first of these equations comes from taking the product of $\mathbf{u}$ with the Navier–Stokes equation, noting that the rate of working of the Lorentz force is $(\mathbf{J}\times\mathbf{B})\cdot\mathbf{u} = -(\mathbf{u}\times\mathbf{B})\cdot\mathbf{J}$, and then using Ohm's law to write $\mathbf{u}\times\mathbf{B}$ in terms of $\mathbf{E}$ and $\mathbf{J}$. The second arises from the product of $\mathbf{B}$ with Faraday's law, and noting that

$$\mathbf{B}\cdot\nabla\times\mathbf{E} = \mathbf{E}\cdot\nabla\times\mathbf{B} + \nabla\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{E}\cdot(\mu\mathbf{J}) + \nabla\cdot(\mathbf{E}\times\mathbf{B})$$

The third relates to magnetic helicity which, as we saw in Chapter 4, Section 4, is globally conserved when $\lambda = 0$. We now take averages, and assume that the turbulence is statistically homogeneous so that the divergences of averaged quantities disappear. Adding the first two equations to eliminate $\langle\mathbf{J}\cdot\mathbf{E}\rangle$ yields

$$\frac{d}{dt}\left[\frac{1}{2}\rho\langle\mathbf{u}^2\rangle + \langle B^2\rangle/2\mu\right] = -\rho\nu\langle\boldsymbol{\omega}^2\rangle - \langle J^2\rangle/\sigma$$

$$\frac{d}{dt}[\langle\mathbf{A}\cdot\mathbf{B}\rangle] = -2\langle\mathbf{J}\cdot\mathbf{B}\rangle/\sigma$$

We recognise the first of these as representing the decline of energy through viscous and Ohmic dissipation. Let us write these symbolically as

$$\frac{dE}{dt} = -\rho\nu\langle\boldsymbol{\omega}^2\rangle - \langle J^2\rangle/\sigma$$

$$\frac{dH_B}{dt} = -2\langle\mathbf{J}\cdot\mathbf{B}\rangle/\sigma$$

The next step is to show that, as $\sigma\to\infty$, $dE/dt$ remains finite while $dH_B/dt$ tends to zero. We proceed as follows. The Schwarz inequality (see Chapter 5, Section 3) tells us

$$\left\{\int \mathbf{J}\cdot\mathbf{B}dV\right\}^2 \le \int J^2 dV \int B^2 dV$$

This may be rewritten as

$$|\langle\mathbf{J}\cdot\mathbf{B}\rangle|/\sigma \le (2\mu/\sigma)^{1/2}\left[|\dot{E}|E\right]^{1/2}$$

and so we can place an upper bound on the rate of destruction of magnetic helicity:

$$|\dot{H}_B|/\mu \le (8\lambda)^{1/2}|\dot{E}|^{1/2}E^{1/2}$$

We now let $\sigma \to \infty$. In the process, however, we assume that $\dot{E}$ remains finite. We might try to justify this as follows. We expect that, as $\sigma \to \infty$ more and more of the Joule dissipation is concentrated into thin current sheets. However, by analogy with viscous dissipation at small $\nu$, we might expect that $\langle J^2\rangle/\sigma$ remains finite in the limiting process. (This is, however, an assumption.) If this is true, it follows that, in the limit $\lambda \to 0$, $H_B$ is conserved. Thus, for small $\lambda$, we have the destruction of energy subject to the conservation of magnetic helicity. In finite domains this presents us with a well-defined variational problem. Minimising $E$ subject to the conservation of $H_B$ in a bounded domain gives us (see Chapter 4, Section 4)

$$\nabla\times\mathbf{B} = \alpha\mathbf{B}, \qquad \mathbf{u} = 0$$

where $\alpha$ is an eigenvalue of the variational problem. The implication is that $\mathbf{B}$ ends up with a large length scale, comparable with the domain size.

In summary then, the assumption that $E$ remains finite as $\sigma \to \infty$ leads to the conservation of helicity, and minimising energy subject to the invariance of $H_B$ gives, for a finite domain, a large-scale static magnetic field with $\mathbf{J}$ and $\mathbf{B}$ aligned. However, this picture of high-$R_m$ turbulence raises as many questions as it answers. What, for example, is the physical mechanism for the inverse cascade of $\mathbf{B}$?

This completes our survey of MHD turbulence. We have left a great deal out. For example, we have not discussed the growth of anisotropy in high-$R_m$ turbulence, where two-dimensionality is thought to be related to the propagation of Alfvén waves. However, the reader will find many useful references at the end of the chapter. We now turn to one of the extreme consequences of an intense magnetic field – two-dimensional turbulence.

## 7.3 Two-Dimensional Turbulence

Everything should be made as simple as possible, but not simpler.

*A Einstein*

Probably the most common statement made about two-dimensional turbulence is that it does not exist. While factually correct, it rather misses the point. There are many flows whose large-scale behaviour is, in some sense, two-dimensional. Large-scale atmospheric and oceanic flows fall into this category, if only because of the thinness of the atmosphere and oceans in comparison with their lateral dimensions. Moreover, both rapid rotation and strong stratification tend to promote two-dimensional flows through the propagation of internal waves, and, of course, strong magnetic fields promote two-dimensionality. While no flow will ever be truly two-dimensional, it seems worthwhile to examine the dynamics of strictly two-dimensional motion in the hope that it sheds light on certain aspects of real, 'almost' two-dimensional phenomena.

In moving from three- to two-dimensions we greatly simplify the equations. Most importantly, we throw out vortex stretching. One might expect, therefore, that two-dimensional turbulence should be much simpler than isotropic turbulence. Mathematically, this is correct, as it must be. Curiously though, the physical characteristics of two-dimensional turbulence are, in many ways, more counter-intuitive than conventional turbulence. At least, this is the case for one brought up in the tradition of Richardson and Kolmogorov. For example, in two dimensions, there is an *inverse cascade* of energy, from the small to the large, as small vortices merge to form larger ones!

### *7.3.1 Batchelor's self-similar spectrum and the inverse energy cascade*

When a number of vortices having the same sense of rotation exist in proximity to one another, they tend to approach one another, and to amalgamate into one intense vortex.

*(Aryton, 1919)*

We shall restrict ourselves to strictly two-dimensional turbulence, $\mathbf{u}(x, y) = (u_x, u_y, 0)$, $\boldsymbol{\omega} = (0, 0, \omega)$, and to turbulence which is homogeneous and isotropic (in a two-dimensional sense). We shall ignore all body forces, such as Lorentz forces or the Coriolis force, and address the problem of freely evolving turbulence. As before, we define the characteristic velocity $u$ through $u^2 = \langle u_x^2 \rangle = \langle u_y^2 \rangle$.

All existing phenomenological theories are based on the two equations

$$\frac{d}{dt}\left[\frac{1}{2}\langle \mathbf{u}^2\rangle\right] = -\nu\langle\omega^2\rangle \tag{7.55}$$

$$\frac{d}{dt}\left[\frac{1}{2}\langle \omega^2\rangle\right] = -\nu\langle(\nabla\omega)^2\rangle \tag{7.56}$$

These state that the kinetic energy density, $\frac{1}{2}\langle \mathbf{u}^2\rangle$, and the so-called enstrophy, $\langle\omega^2\rangle$, both decline monotonically in freely evolving two-dimensional turbulence. The first of these relationships comes from taking the product of $\mathbf{u}$ with

$$\frac{D\mathbf{u}}{Dt} = -\nabla\left(\frac{p}{\rho}\right) - \nu\nabla\times\boldsymbol{\omega}$$

which yields

$$\frac{D}{Dt}\left[\frac{\mathbf{u}^2}{2}\right] = -\nabla\cdot\left[\frac{p\mathbf{u}}{\rho}\right] - \nu\left\{\omega^2 - \nabla\cdot(\mathbf{u}\times\boldsymbol{\omega})\right\}$$

We now average this equation, noting that an ensemble average is equivalent to a spatial average, and that statistical homogeneity of the turbulence ensures that all divergences integrate to zero. The end result is (7.55). Similarly, starting with

$$\frac{D\omega}{Dt} = \nu\nabla^2\omega$$

from which

$$\frac{D}{Dt}\left(\frac{\omega^2}{2}\right) = -\nu\left\{(\nabla\omega)^2 - \nabla\cdot(\omega\nabla\omega)\right\}$$

we obtain, on forming a spatial average, (7.56).

Now the key point about (7.55) and (7.56) is that, as $\mathrm{Re}\to\infty$, $u^2$ is conserved, since the enstrophy remains finite and bounded by its initial value. This is in stark contrast to three-dimensional turbulence, where a decline in $\nu$ is accompanied by a rise in $\langle\omega^2\rangle$ in such a way that the dissipation of kinetic energy remains finite (of order $u^3/l$) as $\mathrm{Re}\to\infty$. This conservation of energy in two-dimensional turbulence implies a long-lived evolution for these flows.

In the limit $\mathrm{Re}\to\infty$ diffusion becomes small (except at the smaller scales) and so the isovortical lines become material lines, and are continually teased out as the flow evolves so that the vorticity field rapidly adopts the structure of thin, sinuous sheets, like cream stirred into coffee.

This filamentation of vorticity feeds an enstrophy cascade (lumps of vorticity are teased out to smaller and smaller scales) which is halted at the small scales only when the transverse dimensions of the sheets are small enough for viscosity to act, destroying the enstrophy and diffusing the vorticity. As in three dimensions, viscosity plays a passive rôle, mopping up the enstrophy (energy in three dimensions) which has cascaded down from above. The dynamics are controlled by the large scales, and even as $\nu \to 0$ the destruction of enstrophy remains finite.

This passive rôle of viscosity led G K Batchelor to propose a self-similar distribution of energy for the large and intermediate scales. In terms of the velocity increment, $\Delta v$, which represents the r.m.s. difference in velocity between two points separated by a distance $r$ (see Chapter 7, Section 1.3), this self-similar energy spectrum takes the form

$$[\Delta v(r)]^2 = u^2 g(r/ut) \tag{7.57}$$

The argument behind (7.57) is essentially a dimensional one. If the turbulence has evolved long enough for the influence of the initial conditions to be erased, and viscosity controls only the smallest scales, then all that the large scales remember is $u$. It follows that $u$, $r$ and $t$ are the only parameters determining $\Delta v(r)$, and (7.57) is then inevitable. In this model then, the integral scale grows as $l \sim ut$. That is, if we divide $\Delta v$ by $u$ and $r$ by $l = ut$, we obtain a self-similar energy spectrum valid throughout the evolution of the flow (Figure 7.10) and so the size of the most energetic eddies must grow as $ut$.

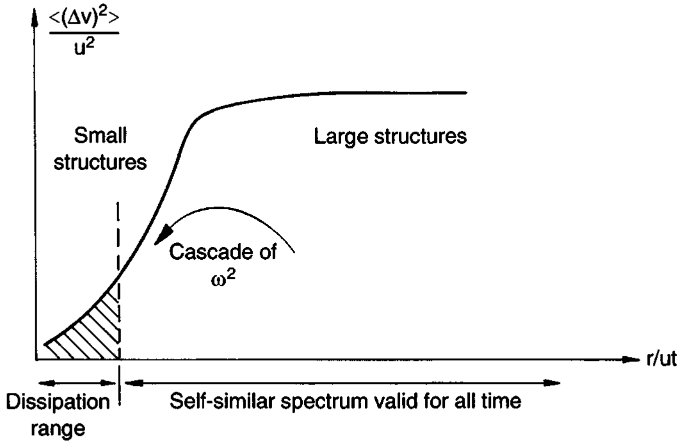


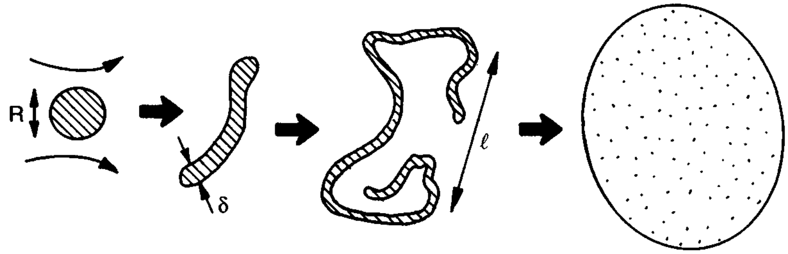

Figure 7.10 Batchelor's universal energy spectrum for two-dimensional turbulence.

For almost thirty years, dating from its introduction in 1969, Batchelor's self-similar energy spectrum, and associated theories by Kraichnan, dominated the literature on two-dimensional turbulence. Note, however, that this dimensional argument hinges on the flow remembering nothing other than $u$. It might, for example, also remember $\langle H^2 \rangle$, but this is a whole new story to which we shall return later.

In the Batchelor–Kraichnan picture we have two cascades: a direct cascade of enstrophy from the large scale to the small, going hand-in-hand with an *inverse cascade* of energy (as anticipated by Ayrton in 1919) as more and more energy moves to larger scales, the total energy being conserved. Physically, we can picture this in terms of the filamentation of vorticity, as shown in Figure 7.11. A (red) blob of vorticity, such as that shown in Figure 7.11(a), will be teased out into a strip of thickness $\delta$ by eddies whose dimensions are comparable with the blob size, $R$. Area is conserved by the vortex patch and so $\delta$ falls as the characteristic integral dimension, $l$, increases. The strip is then further teased and twisted by the flow ((b) $\rightarrow$ (c)), and in the process $l$ continues to grow at a rate $l \sim ut$ while $\delta$ declines. The process ceases, for this particular vortex patch, when $\delta$ becomes so small that diffusion sets in, and the red spaghetti of Figure 7.11(c) becomes the pink cloud of (d). The direct cascade of enstropy is associated with the reduction in $\delta$, while the inverse cascade of energy is associated with the growth of $l$, which characterises the eddy size associated with the vortex patch.

### 7.3.2 *Coherent vortices*

In Batchelor's theory the vorticity is treated essentially like a passive tracer in the flow. However, following the rapid development of compu-




(a) Red blob (b) Red strip (c) Red spaghetti (d) Pink cloud

Figure 7.11 Destruction of a lump of vorticity in two-dimensional turbulence.

tational fluid dynamics in the 1980s, and its application to two-dimensional turbulence, it soon became clear that this was not the whole story. While filamentation of vorticity does indeed occur, numerical experiments suggested that intense patches of vorticity, embedded in the initial conditions, survive the filamentation process (process (a) → (b) in Figure 7.11), forming long-lived *coherent vortices*. These coherent vortices obey a different set of dynamical rules, interacting with each other, sometimes merging and sometimes being destroyed by a stronger vortex. The picture is now one of two sets of dynamical processes coexisting in the same vorticity field. Weak vorticity is continually filamented, feeding the enstrophy cascade in the manner suggested by Batchelor. However, within this sea of quasi-passive vorticity filaments, bullets of coherent vorticity fly around, rather like point vortices, increasing in size and decreasing in number through a sequence of mergers.

The emergence of coherent vortices is generally attributed to the equation

$$\frac{D^2}{Dt^2}[\nabla\omega] + \frac{1}{4}\left[\omega^2 - S_1^2 - S_2^2\right]\nabla\omega = \text{terms of order}\left(\frac{\mathrm{D}S_{1,2}}{\mathrm{D}t}\nabla\omega\right) \quad (7.58)$$

where $S_1$ and $S_2$ represent the strain fields $2\partial u_x/\partial x$ and $(\partial u_x/\partial y + \partial u_y/\partial x)$, respectively. ((7.58) follows from $D\omega/Dt = 0$). If the rate of change of $S_1$ and $S_2$ (following a material particle) is much less than the corresponding rate of change of $\nabla\omega$, then the right-hand side of (7.58) may be neglected. It then follows that vorticity gradients will grow exponentially in regions where the strain field dominates, or else oscillate in regions where the vorticity dominates. The latter regime leads to coherent vortices, or at least that is the idea. It should be stressed, however, that there is no real justification for neglecting the terms on the right of (7.58) and so this is an imperfect explanation. Nevertheless, we have the empirical observation that the peaks in vorticity, say $\hat{\omega}$, survive the filamentation and so are remembered by the flow. This leads to the idea that Batchelor's energy spectrum should be generalised to

$$[\Delta v(r)]^2 = u^2 g(r/ut, \hat{\omega}t) \quad (7.59)$$

### 7.3.3 *The governing equations of two-dimensional turbulence*

The arguments above are essentially heuristic, although the evidence of the numerical experiments suggest that they are reasonably sound. However, it seems natural to go further and establish the governing equations for two-dimensional turbulence to see if they tell us anything more.

The two-dimensional analogues of (7.18) and (7.19) are

$$Q_{ij}(\mathbf{r}) = u^2\left\{\frac{d}{dr}(rf)\delta_{ij} - \frac{r_i r_j}{r}f'(r)\right\} \tag{7.60a}$$

$$S_{ijl} = u^3\left\{\frac{1}{2r}\frac{d}{dr}(rk)\left(r_i\delta_{jl} + r_j\delta_{il}\right) - \frac{r_i r_j r_l}{r}\frac{d}{dr}\left(\frac{k}{r}\right) - \frac{r_l\delta_{ij}k}{r}\right\} \tag{7.60b}$$

where $f$ and $k$ are the usual longitudinal velocity correlation functions. Substituting these into the dynamic equation

$$\frac{\partial Q_{ij}}{\partial t} = \frac{\partial}{\partial r_k}\left[S_{ikj} + S_{jki}\right] + 2\nu\nabla^2 Q_{ij} \tag{7.61}$$

yields the two-dimensional analogue of the Karman–Howarth equation:

$$\boxed{\frac{\partial}{\partial t}\left[u^2 r^3 f\right] = u^3\frac{\partial}{\partial r}\left[r^3 k\right] + 2\nu u^2\frac{\partial}{\partial r}\left[r^3\frac{\partial f}{\partial r}\right]} \tag{7.62}$$

(Compare this with (7.24).) Next we integrate over all space. This furnishes a result reminiscent of Loitsyansky's integral equation

$$\frac{d}{dt}\left\{u^2\int_0^\infty r^3 f dr\right\} = u^3\left[r^3 k\right]_{r\to\infty} + 2\nu u^2\left[r^3 f'(r)\right]_{r\to\infty} \tag{7.63}$$

Now, if we follow Batchelor's argument and look at the long-range pressure forces in order to determine the form of $k_\infty$, then it can be shown that $k_\infty \sim r^{-3}$, or less. This is the analogue of the three-dimensional result, $k_\infty \sim r^{-4}$ (or less) (see Chapter 7, Section 1.4). It follows from (7.62) that, at most, $f_\infty \sim r^{-5}$, and so our integral equation simplifies to

$$\frac{d}{dt}\left\{u^2\int_0^\infty r^3 f dr\right\} = u^3\left[r^3 k\right]_{r\to\infty} \tag{7.64}$$

Owing to the similarity between (7.64) and Loitsyansky's integral (7.27) it seems natural to investigate the angular momentum of two-dimensional turbulence. (Remember, Loitsyansky's integral is a measure of angular momentum.) In two dimensions, the global angular momentum of a flow is $H = \int_V (\mathbf{x}\times\mathbf{u})_z dV = 2\int_V \psi dV$, where $\psi$ is the streamfunction. This, in turn, suggests that we introduce the correlation function $\langle\psi\psi'\rangle$, which is related to $Q_{ii}$ by

$$Q_{ii} = -\nabla^2\left[\langle\psi\psi'\rangle\right] \tag{7.65}$$

(see references at the end of this chapter). Substituting for $Q_{ii}$ using (7.60a) we find

$$\langle \psi\psi' \rangle = \langle \psi^2 \rangle - u^2 \int_0^r rf dr \tag{7.66}$$

We now introduce the two-dimensional analogue of Loitsyansky's integral

$$I = u^2 \int_0^\infty r^3 f dr = (4\pi)^{-1} \left\{ 4 \int \langle \psi\psi' \rangle d^2\mathbf{r} \right\} \tag{7.67}$$

where the second equality comes from (7.66). In terms of $I$, (7.64) becomes

$$\frac{dI}{dt} = u^3 \left[ r^3 k \right]_{r \to \infty} \tag{7.68}$$

So far we have made no assumption about $k_\infty$, other than noting that it decreases no more slowly than $k_\infty \sim r^{-3}$. Now it turns out that, just like in three-dimensional turbulence, Batchelor's long-range pressure forces cannot directly influence $\langle \mathbf{u} \cdot \mathbf{u}' \rangle$, although they can create an algebraic tail in the triple correlations. This, in turn, can produce an algebraic tail in $\langle \mathbf{u} \cdot \mathbf{u}' \rangle$. In fact it may be shown that

$$\frac{d}{dt}\left[u^3 r^3 k\right]_\infty = \int r \langle ss' \rangle dr$$

where $s = u_x^2 - u_y^2$, from which,

$$\frac{d^2 I}{dt^2} = \frac{d}{dt}\left[u^3 r^3 k\right]_\infty = \int r \langle ss' \rangle dr$$

Thus a $k_\infty \sim r^{-3} (f_\infty \sim r^{-5})$ tail is kinematically feasible. Of course, this would invalidate a Loitsyansky-type argument for the invariance of $I$. However, there is some slight evidence that, for *certain initial conditions* (i.e. those where the long-range correlations are absent), the long-range pressure forces remain weak as the flow evolves. This leads to precisely the conditions assumed by Loitsyansky and Kolmogorov prior to Batchelor's discovery of long-range, pressure-induced forces. Under these conditions Landau's angular momentum argument of Chapter 7, Section 1.4, adapted to two-dimensions, yields

$$\langle H^2 \rangle / V = 4 \int \langle \psi\psi' \rangle d^2\mathbf{r} = \text{constant} \tag{7.69}$$

This is consistent with (7.68) which, for $k_\infty < 0(r^{-3})$, becomes

$$4\pi I = 4 \int \langle \psi\psi' \rangle d^2\mathbf{r} = \text{constant} \tag{7.70}$$

Combining these we obtain the Landau–Loitsyansky equation for two-dimensional turbulence

$$\boxed{I = u^2 \int_0^\infty r^3 f dr = \langle H^2 \rangle / (4\pi V) = \text{constant}} \tag{7.71}$$

[conservation of angular momentum]

Of course, we also have conservation of energy (at high Re) and so

$$\boxed{u^2 = \text{constant}} \tag{7.72}$$

[conservation of energy]

These conservation laws provide powerful constraints on the evolution of freely decaying turbulence. If valid, they invalidate Batchelor's self-similar energy spectrum which relies on the existence of only one invariant, $u^2$. However, it is believed by many that the long-range effects can be significant in two-dimensional turbulence, in which case (7.71) is incorrect and the most that we can say is

$$\boxed{u^2 = \text{constant}, \qquad \frac{d}{dt}\int_0^\infty r^3 f dr = u[r^3 k]_{r\to\infty}} \tag{7.73}$$

(long-range effects significant)

The whole issue of freely evolving two-dimensional turbulence is still a matter of considerable debate and, as of now, it does not seem possible to progress much beyond this point.

### *7.3.4 Variational principles for predicting the final state in confined domains*

We now turn to freely decaying turbulences in confined domains (at high Re). Unlike three-dimensional turbulence, the conservation of $u^2$, and the continual growth of $l$, means that two-dimensional turbulence in a finite domain will evolve to a quasi-steady state, containing (almost) the same energy as the initial conditions, but with an integral scale comparable with the domain size. In short, a two-dimensional turbulent flow will eventually evolve into just one or two eddies which fill the domain. Although contrary to intuition, this is precisely what is observed in the numerical simulations. Once this quasi-steady state has been reached, which takes a time $t \sim R/u$, $R$ being the domain size, the flow then settles down to a laminar motion which decays slowly due to friction on the boundary (Figure 7.12).

Heuristic theories have been developed which, given the initial conditions, purport to identify the quasi-steady state reached at the end of the cascade-enhanced destruction of enstrophy. These theories are essentially all variational principles and we shall discuss them in the context of circular domains, where $H$ is (almost) conserved.

The simplest model for predicting the quasi-steady state (state (c) in Figure 7.12) is the so-called *minimum enstrophy theory*. The idea is that the enstrophy falls monotonically during the cascade-enhanced evolution and this occurs on the fast (inertial) timescale of the eddy turn-over time. Once a quasi-steady state is reached, the enstrophy, as well as the energy and angular momentum, evolve on the much slower diffusive timescale, $R^2/\nu$. It is plausible, therefore, that the quasi-steady state corresponds to a minimum in $\langle \omega^2 \rangle$ subject to the conservation of $u^2$ and of $H$. In practice, though, this (and all other similar variational principles) suffer from three major drawbacks. First, while seeming plausible, they are all ultimately heuristic. Second, the transition from a cascade-enhanced evolution to a slow diffusive evolution is not always clear-cut. Third, at finite Re, $H$ and $u^2$ will not be exactly conserved. Nevertheless, let us see where the minimum enstrophy theory leads.

Minimising enstrophy subject to the conservation of $u^2$ and $H = 2\int \psi dV$is equivalent to minimising the functional

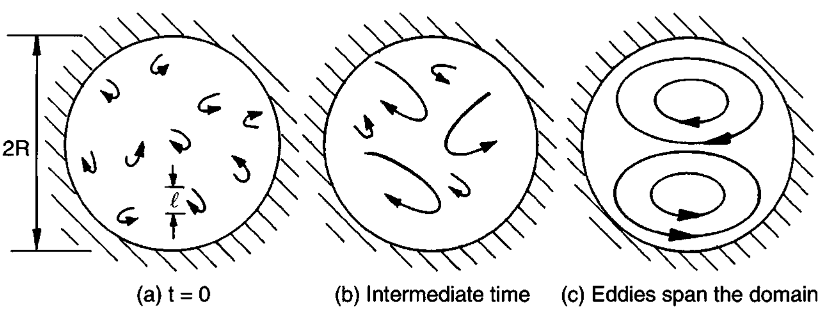



Figure 7.12 Two-dimensional turbulence in a confined domain.

$$F = \int \left[R^2\omega^2 - \lambda^2(\nabla\psi)^2 + 2\lambda^2\Omega\psi\right]dV$$

where $\lambda$ and $\Omega$ are constants (Lagrange multipliers) which we shall determine from the initial condition. The use of the calculus of variations shows that the minimum value of $F$, compatible with no-slip boundary conditions on $r = R$, is obtained when $\omega$ is given by

$$\frac{\omega}{\Omega} = 1 - \frac{\lambda J_0(\lambda r/R)}{2J_1(\lambda)} \tag{7.74}$$

Here $J_0$, $J_1$, etc are the usual Bessel functions denoted by these symbols. The Lagrange multipliers are now fixed by the initial values of $H$ and $u^2$. On integration of (7.74) we find

$$H = -(\pi/4)\Omega R^4 J_3(\lambda)/J_1(\lambda) \tag{7.75}$$

$$\left[\pi R^3 u/H\right]^2 = \left[2J_2^2(\lambda) - 3J_1(\lambda)J_3(\lambda)\right]/J_3^2(\lambda) \tag{7.76}$$

The second of these equations fixes $\lambda$ in terms of $u^2$ and $H$, so that the first determines $\Omega$. The vorticity distribution (and by implication the velocity distribution) of the quasi-steady state is now uniquely determined by the initial conditions through (7.74)–(7.76). Somewhat surprisingly, (7.74) compares well with numerical experiments of two-dimensional turbulence (provided $H$ is not too small), so that, at least for this simple geometry, the minimum enstrophy-theory works well.

There are other variational principles designed to do the same as minimum enstrophy. One has the impressive name: *maximum entropy*. In effect, this defines some measure of mixing and then assumes that the turbulence maximises this mixing (rather than minimising enstrophy) subject to the conservation of $u^2$ and $H$. The maximum entropy theory seems, at first sight, appealing because there are analogues in statistical physics. In practice, however, it is a heuristic model and has all the same advantages and disadvantages of the minimum entrophy theory. In fact, as often as not, maximum entropy and minimum enstrophy give virtually identical predictions.

## Suggested Reading

L D Landau & E M Lifshitz, *Course of theoretical physics, vol. 6, Fluid mechanics*. 1st Edition, 1959. Butterworth–Heinemann Ltd. (Chapter 3, § 38 for a discussion of angular momentum in turbulence.)

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## Examples

7.1 Derive Kolmogorov's five-thirds law by dimensional analysis.

7.2 Show that $\langle pu_i' \rangle = 0$ is homogeneous, isotropic turbulence.

7.3 Sketch the shape of the second-order structure function $B_{xx}(r\hat{\mathbf{e}}_x)$.

7.4 Show that the idea of a self-similar energy spectrum, $E(k/l)$, in freely decaying isotropic turbulence is incompatible with conservation of Loitsyansky's integral.

7.5 Show that low-$R_m$ MHD turbulence in a large spherical domain (of radius much greater than the integral scale) which is subject to a uniform magnetic field, $\mathbf{B}_0$, and has angular momentum $\mathbf{H}$, satisfies

$$\langle (\mathbf{H} \cdot \mathbf{B}_0)^2 \rangle = -\int\int \mathbf{r}_\perp^2 \langle \mathbf{u}_\perp \cdot \mathbf{u'}_\perp \rangle d^3\mathbf{r} d^3\mathbf{x} B_0^2$$

7.6 Show that low-$R_m$ turbulence always tends to a state where $N_{\text{true}} \sim 1$.

7.7 Give a physical explanation for the growth of the integral scale, $l \sim ut$, in Batchelor's self-similar spectrum for 2D turbulence.