Fundamentals of Magnetohydrodynamics (MHD)

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Aim

Derivation of MHD equations from conservation laws

Quasi-neutrality

Validity of MHD

MHD equations in different forms

MHD waves

Alfven's Frozen Flux Theorem

Line Conservation Theorem

Characteristics

Shocks

Applications of MHD, i.e. all the interesting stuff!, will be in later lectures covering Waves, Reconnection and Dynamos etc.

Derivation of MHD

Possible to derive MHD from

- N-body problem to Klimotovich equation, then take moments and simplify to MHD
- Louiville theorem to BBGKY hierarchy, then take moments and simplify to MHD
- Simple fluid dynamics and control volumes

First two are useful if you want to study kinetic theory along the way but all kinetics removed by the end

Final method followed here so all physics is clear

Ideal MHD

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \wedge \mathbf{E}$$

Maxwell equations

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \mathbf{v}) = 0$$

Mass conservation

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B}$$

F = ma for fluids

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j}$$

Low frequency Maxwell

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$

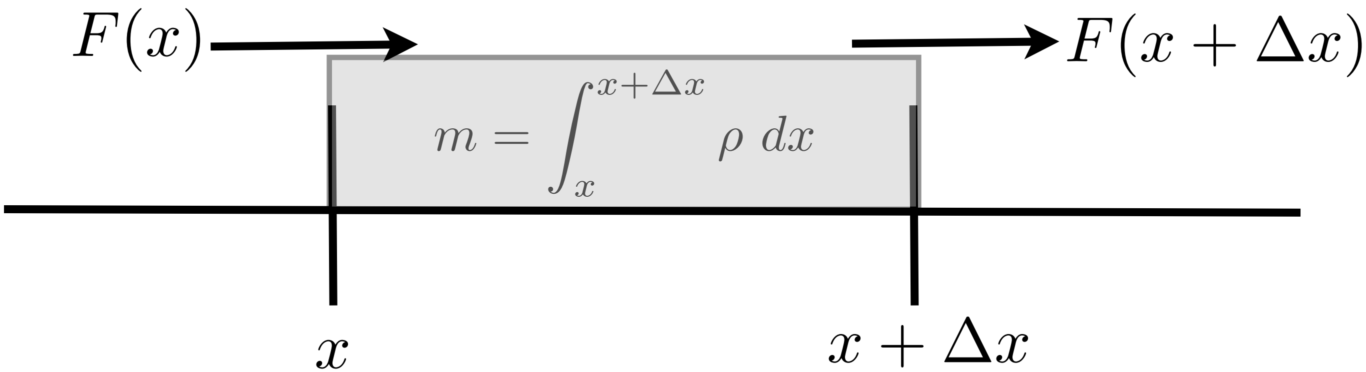
Adiabatic equation for fluids

$$\mathbf{E} + \mathbf{v} \wedge \mathbf{B} = 0$$

Ideal Ohm's Law for fluids

8 equations with 8 unknowns

Mass Conservation - Continuity Equation



Mass *m* in cell of width Δx changes due to rate of mass leaving/entering the cell $F(x)$

$$\frac{\partial}{\partial t}\left(\int_x^{x+\Delta x} \rho \, dx\right) = F(x) - F(x+\Delta x)$$

$$\frac{\partial \rho}{\partial t} = \lim_{\Delta x \to 0} \frac{F(x) - F(x+\Delta x)}{\Delta x}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial F(x)}{\partial x} = 0$$

Mass flux - conservation laws

$$\frac{\partial \rho}{\partial t} + \frac{\partial F(x)}{\partial x} = 0$$

Mass flux per second through cell boundary

$$F(x,t) = \rho(x,t)\ v_x(x,t)$$

In 3D this generalizes to

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0}$$

This is true for any conserved quantity so if $\int \mathbf{U}\ dx$ conserved

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla.\mathbf{F} = 0$$

Hence applies to mass density, momentum density and energy density for example.

Convective Derivative

In fluid dynamics the relation between total and partial derivatives is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}.\nabla$$

Convective derivative:

Rate of change of quantity at a point moving with the fluid.

Rate of change of quantity at a fixed point in space

Often, and frankly for no good reason at all, write

$$\frac{D}{Dt} \text{ instead of } \frac{d}{dt}$$

Adiabatic energy equation

If there is no heating/conduction/transport then changes in fluid element's pressure and volume (moving with the fluid) is adiabatic

$$PV^{\gamma} = constant$$

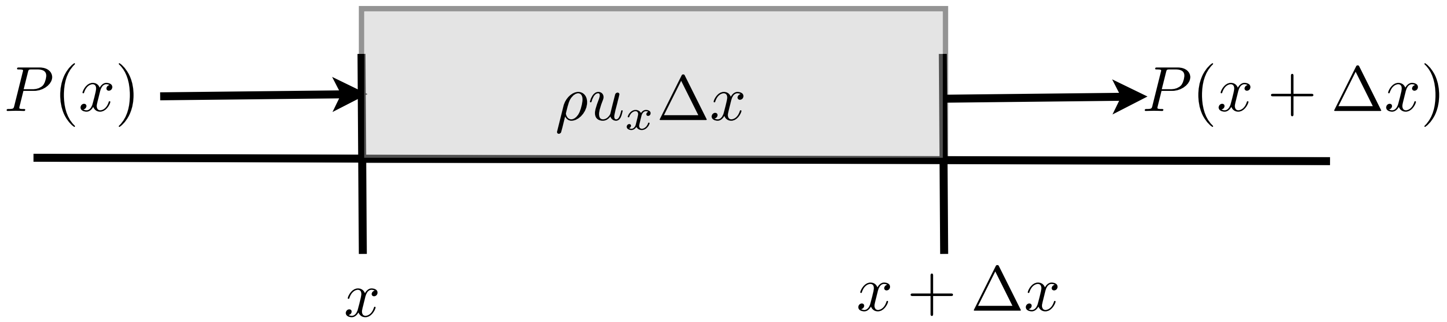
Where γ is ration of specific heats

$$\frac{d}{dt}(PV^{\gamma}) = 0$$

Moving with a packet of fluid the mass is conserved so $V \propto \rho^{-1}$

$$\boxed{\frac{d}{dt}\left(\frac{P}{\rho^{\gamma}}\right) = 0}$$

Momentum equation - Euler fluid



Total momentum in cell changes due to pressure gradient

$$\frac{\partial}{\partial t}(\rho u_x \Delta x) = F(x) - F(x+\Delta x) + P(x) - P(x+\Delta x)$$

Now *F* is momentum flux per second $F = \rho u_x u_x$

$$\frac{\partial}{\partial t}(\rho u_x) + \frac{\partial F}{\partial x} = -\nabla P$$

Momentum equation - Euler fluid

Use mass conservation equation to rearrange as

$$\rho \frac{\partial u_x}{\partial t} + \rho u_x \frac{\partial u_x}{\partial x} = -\nabla P$$

$$\rho \left(\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} \right) = -\nabla P$$

$$\rho \frac{du_x}{dt} = -\nabla P$$

Since by chain rule

$$\frac{du_x(x,t)}{dt} = \frac{\partial u_x}{\partial t} + \frac{\partial x}{\partial t} \frac{\partial u_x}{\partial x}$$

Momentum equation - MHD

For Euler fluid $\rho \frac{d\mathbf{u}}{dt} = -\nabla P$ how does this change for MHD?

Force on charged particle in an EM field is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Hence total EM force per unit volume on electrons is

$$-n_e e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

and for ions (single ionized) is

$$n_i e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Where n_e and n_i are the electron and ion number densities

Momentum equation - MHD

Hence total EM force per unit volume

$$e(n_i - n_e)\mathbf{E} + (en_i\mathbf{u_i} - en_e\mathbf{u_e}) \times \mathbf{B}$$

If the plasma is quasi-neutral (see later) then this is just

$$en(\mathbf{u_i} - \mathbf{u_e}) \times \mathbf{B} = \mathbf{j} \times \mathbf{B}$$

Where ***j*** is the current density. Hence

$$\boxed{\rho\frac{d\mathbf{u}}{dt} = -\nabla P + \mathbf{j} \times \mathbf{B}}$$

Note ***jxB*** is the only change to fluid equations in MHD. Now need an equation for the magnetic field and current density to close the system

Maxwell equations

$$\nabla . \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Not allowed in MHD!

$$\nabla . \mathbf{B} = 0$$

Initial condition only

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Used to update **B**

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

'Low' frequency version used to find current density **j**

Low-frequency Maxwell equations

$$\boxed{\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}}$$

$$\varepsilon_0 \mu_0 \frac{E_0}{T} = \frac{1}{c^2} \frac{E_0}{T}$$

$$\text{Magnitude} \sim \frac{\text{typical B}}{\text{length-scale}} \sim \frac{B_0}{L}$$

$$\frac{\left| \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \right|}{\left| \nabla \wedge \mathbf{B} \right|} \sim \frac{L}{T} \frac{1}{c^2} \frac{E_0}{B_0}$$

Displacement current

$$\frac{\left|\varepsilon_0\mu_0\dfrac{\partial \mathbf{E}}{\partial t}\right|}{\left|\nabla\wedge\mathbf{B}\right|}\sim\frac{L}{T}\frac{1}{c^2}\frac{E_0}{B_0}$$

Later we will show that in MHD $\dfrac{E_0}{B_0}\sim V$

$$\frac{\left|\varepsilon_0\mu_0\dfrac{\partial \mathbf{E}}{\partial t}\right|}{\left|\nabla\wedge\mathbf{B}\right|}\sim\frac{L}{T}\frac{V}{c^2}\sim\frac{V^2}{c^2}$$

So for low velocities/frequencies we can ignore the displacement current

Quasi-neutrality

For a pure hydrogen plasma we have

$$\frac{\partial n_i}{\partial t} + \nabla.(n_i \mathbf{u}_i) = 0 \qquad \frac{\partial n_e}{\partial t} + \nabla.(n_e \mathbf{u}_e) = 0$$

Multiply each by their charge and add to get $$\frac{\partial \sigma}{\partial t} + \nabla.\mathbf{j} = 0$$

where σ is the charge density and **j** is the current density $\mathbf{j} = en_i\mathbf{u_i} - en_e\mathbf{u_e}$

From Ampere's law $\nabla \wedge \mathbf{B} = \mu_\mathbf{0}\mathbf{j} + \varepsilon_0\mu_\mathbf{0}\dfrac{\partial \mathbf{E}}{\partial t}$ if we look only at low frequencies

$$\nabla \wedge \mathbf{B} = \mu_\mathbf{0}\mathbf{j} \longrightarrow \nabla.(\nabla \wedge \mathbf{B}) = \mu_\mathbf{0}\nabla.\mathbf{j} \longrightarrow \nabla.\mathbf{j} = 0$$

$$\frac{\partial \sigma}{\partial t} = 0$$

Hence for low frequency processes $n_i \approx n_e$ this is quasi-nuetrality

MHD

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \wedge \mathbf{E}$$ ← Maxwell equations

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\, \mathbf{v}) = 0$$ ← Mass conservation

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B}$$ ← Momentum conservation

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j}$$ ← Low frequency Maxwell

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$ ← Energy conservation

8 equations with 11 unknowns! Need an equation for $\mathbf{E}$

Ohm's Law

Equations of motion for ion fluid is

$$m_i n_i \frac{d\mathbf{u_i}}{dt} = en_i\left(\mathbf{E} + \mathbf{u_i} \wedge \mathbf{B}\right) - \nabla P_i + \mathbf{F}_{ie}$$

Assume quasi-neutrality, subtract electron equation

$$\frac{m_e}{ne^2}\frac{\partial \mathbf{j}}{\partial t} = \mathbf{E} + \mathbf{v} \wedge \mathbf{B} - \frac{1}{ne}(\mathbf{j} \wedge \mathbf{B}) - \frac{1}{ne}\nabla P_e - \eta\mathbf{j}$$

This is called the generalized Ohm's law

Note that Ohm's law for a current in a wire (V=IR) when written in terms of current density becomes $\mathbf{E} = \eta\,\mathbf{j}$

When fluid is moving this becomes $\boxed{\mathbf{E} + \mathbf{v} \wedge \mathbf{B} = \eta\,\mathbf{j}}$

Magnetohydrodynamics (MHD)

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \wedge \mathbf{E}$$

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\,\mathbf{v}) = 0$$

$$\rho\,\frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j}\times\mathbf{B}$$

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j}$$

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$

$$\mathbf{E} + \mathbf{v} \wedge \mathbf{B} = \eta\,\mathbf{j}$$

Valid for:

- Low frequency
- Large scales

If $\eta=0$ called ideal MHD

Missing viscosity, heating, conduction, radiation, gravity, rotation, ionisation etc.

Validity of MHD

Assumed quasi-neutrality therefore must be low frequency and speeds << speed of light

Assumed scalar pressure therefore collisions must be sufficient to ensure the pressure is isotropic. In practice this means:

- mean-free-path << scale-lengths of interest
- collision time << time-scales of interest
- Larmor radii << scale-lengths of interest

However as MHD is just conservation laws plus low-frequency MHD it tends to be a good first approximation to much of the physics even when all these conditions are not met.

Eulerian form of MHD equations

$$\frac{\partial \rho}{\partial t} = -\nabla.(\rho \mathbf{v})$$

$$\frac{\partial P}{\partial t} = -\gamma P \nabla.\mathbf{v}$$

$$\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{v}.\nabla.(\mathbf{v}) - \frac{1}{\rho}\nabla.P + \frac{1}{\rho}\mathbf{j} \times \mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

$$\mathbf{j} = \frac{1}{\mu_0}\nabla \times \mathbf{B}$$

Final equation can be used to eliminate current density so 8 equations in 8 unknowns

Lagrangian form of MHD equations

$$\frac{D\rho}{Dt} = -\rho\nabla.\mathbf{v}$$

$$\frac{DP}{Dt} = -\gamma P\nabla.\mathbf{v}$$

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla.P + \frac{1}{\rho}\mathbf{j}\times\mathbf{B}$$

$$\frac{D\mathbf{B}}{Dt} = (\mathbf{B}.\nabla)\mathbf{v} - \mathbf{B}(\nabla.\mathbf{v})$$

$$\mathbf{j} = \frac{1}{\mu_0}\nabla\times\mathbf{B}$$

Alternatives

$$\frac{D\epsilon}{Dt} = -\frac{P}{\rho}\nabla.\mathbf{v}$$

Specific internal energy density

$$\epsilon = \frac{P}{\rho(\gamma-1)}$$

$$\frac{D}{Dt}\left(\frac{\mathbf{B}}{\rho}\right) = \frac{\mathbf{B}}{\rho}.\nabla\mathbf{v}$$

Conservative form

$$\frac{\partial \rho}{\partial t} = -\nabla.(\rho \mathbf{v})$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} = -\nabla.\left(\rho \mathbf{v}\mathbf{v} + \mathbf{I}(P + \frac{B^2}{2}) - \mathbf{B}\mathbf{B}\right)$$

$$\frac{\partial E}{\partial t} = -\nabla.\left(\left(E + P + \frac{B^2}{2\mu_0}\right)\mathbf{v} - \mathbf{B}(\mathbf{v}.\mathbf{B})\right)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla(\mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v})$$

$$E = \frac{P}{\gamma - 1} + \frac{\rho v^2}{2} + \frac{B^2}{2\mu_0}$$

The total energy density

Plasma beta

A key dimensionless parameter for ideal MHD is the plasma-beta

It is the ratio of thermal to magnetic pressure

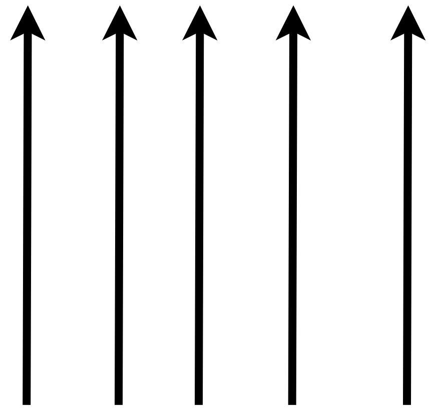
$$\beta = \frac{2\mu_0 P}{B^2}$$

Low beta means dynamics dominated by magnetic field, high beta means standard Euler dynamics more important

$$\beta \propto \frac{c_s^2}{v_A^2}$$

MHD Waves

Univorm **B** field



Constant density, pressure

Zero initial velocity

Apply small perturbation to system

Assume initially in stationary equilibrium

$$\nabla . P_0 = \mathbf{j_o} \times \mathbf{B_o}$$

Simplify to easiest case with $\rho_0, P_0, \mathbf{B}_0 = B_0 \hat{\mathbf{z}}$ constant and no equilibrium current or velocity

Apply perturbation, e.g. $P = P_0 + P_1$

MHD Waves

Ignore quadratic terms, e.g. $P_1 \nabla . \mathbf{v}_1$

Linear equations so Fourier decompose, e.g.

$$P_1(\mathbf{r}, t) = P_1 \exp i(\mathbf{k}.\mathbf{r} - \omega t)$$

Gives linear set of equations of the form $\bar{\bar{A}}.\bar{u} = \lambda \bar{u}$

Where $\bar{u} = (P_1, \rho_1, v_1, B_1)$

Solution requires $det|\bar{\bar{A}} - \lambda \bar{\bar{I}}| = 0$

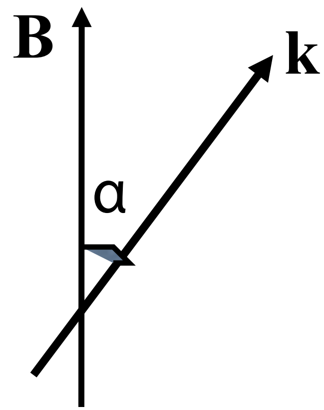
Dispersion relation

$$(\omega^2 - C_A^2 k^2 \cos^2\alpha)(\omega^2 - C_s^2 k^2) - C_A^2 \omega^2 k^2 \sin^2\alpha = 0$$

(Fast and slow magnetoacoustic waves)

$$\omega^2 - C_A^2 k^2 \cos^2\alpha = 0$$

(Alfvén waves)



$C_A = B_0 / \sqrt{\mu_0 \rho_0}$ Alfvén speed

$C_s = \sqrt{\gamma p_0 / \rho_0}$ Sound speed

Alfven Waves

$$\omega = C_A k \cos\alpha$$

$$v_A = \omega / k = C_A \cos\alpha$$

$$\mathbf{v_g} = \nabla_{\mathbf{k}}\omega = v_A \hat{\mathbf{B}}$$

- Incompressible – no change to density or pressure
- Group speed is along **B** – does not transfer energy (information) across B fields

Fast magneto-acoustic waves

For zero plasma-beta – no pressure

$$\omega = C_A k$$

$$v_A = \omega / k = C_A$$

$$\mathbf{v_g} = \nabla_{\mathbf{k}} \omega = v_A \hat{\mathbf{k}}$$

- Compresses the plasma – c.f. a sound wave
- Propagates energy in all directions

Magnetic pressure and tension

$$\rho\frac{\partial \mathbf{v}}{\partial t}+\rho(\mathbf{v}.\nabla)\mathbf{v}=-\nabla p+\mathbf{j}\times\mathbf{B}$$

But…

$$\mathbf{j}\times\mathbf{B}=-\frac{1}{2\mu_0}\nabla B^2+\frac{1}{\mu_0}(\mathbf{B}.\nabla)\mathbf{B}$$

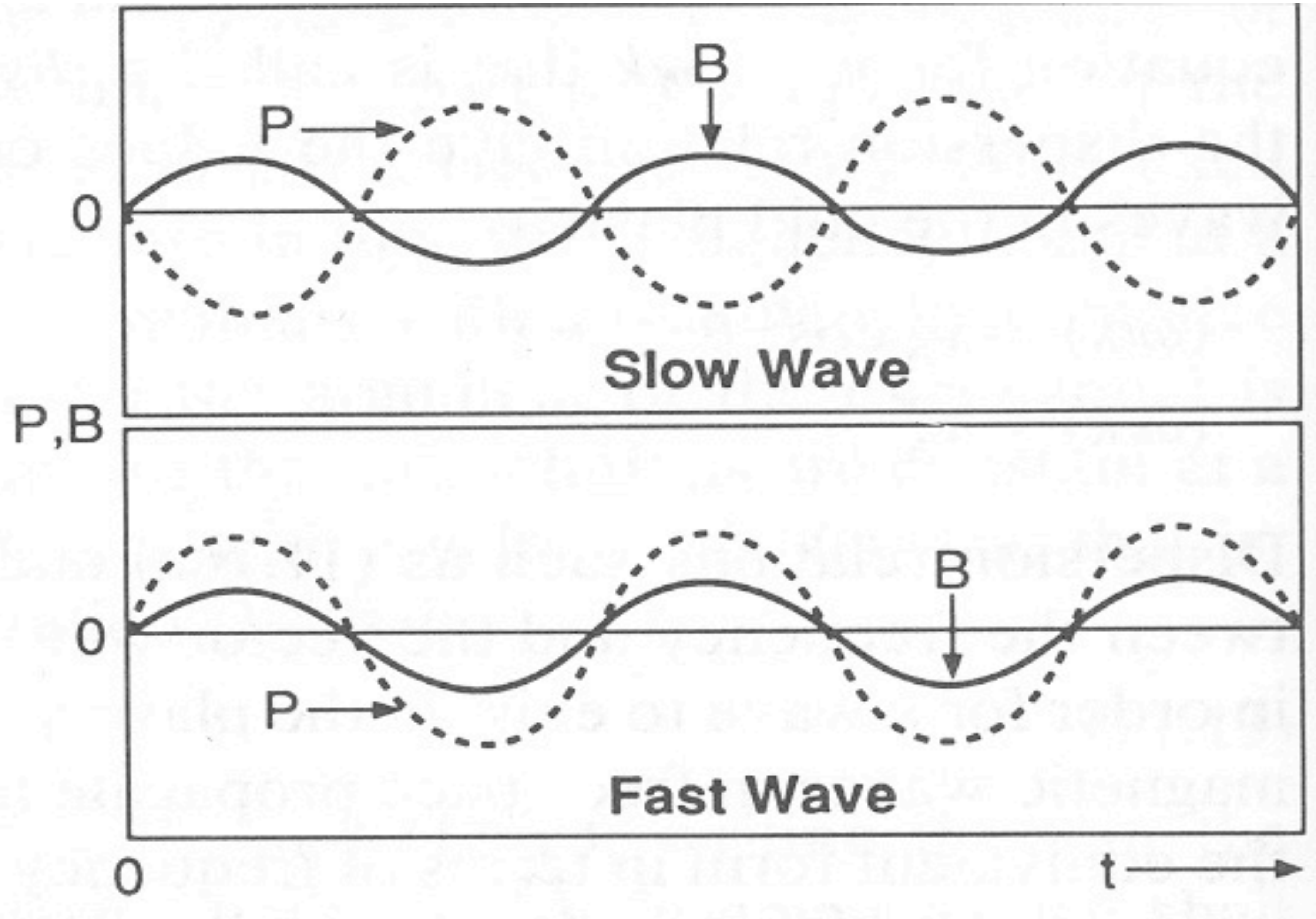
So…

$$\rho\frac{\partial \mathbf{v}}{\partial t}+\rho(\mathbf{v}.\nabla)\mathbf{v}=-\nabla\left(p+\frac{1}{2\mu_0}B^2\right)+\frac{1}{\mu_0}(\mathbf{B}.\nabla)\mathbf{B}$$

Magnetic pressure

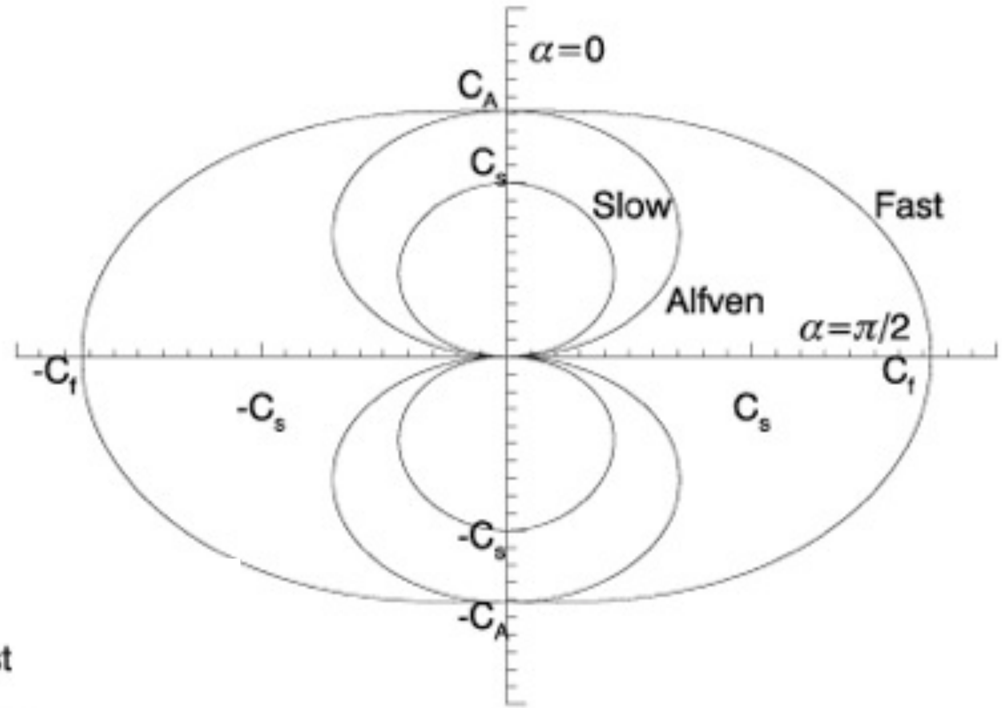
Magnetic tension

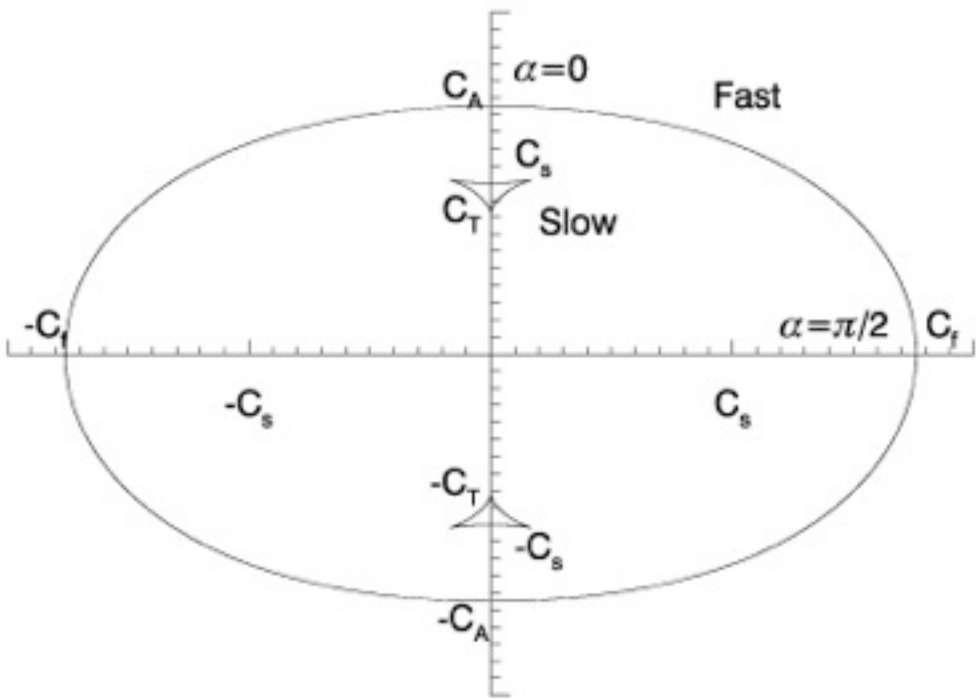
Pressure perturbations



Phase and group speeds

Phase speeds:

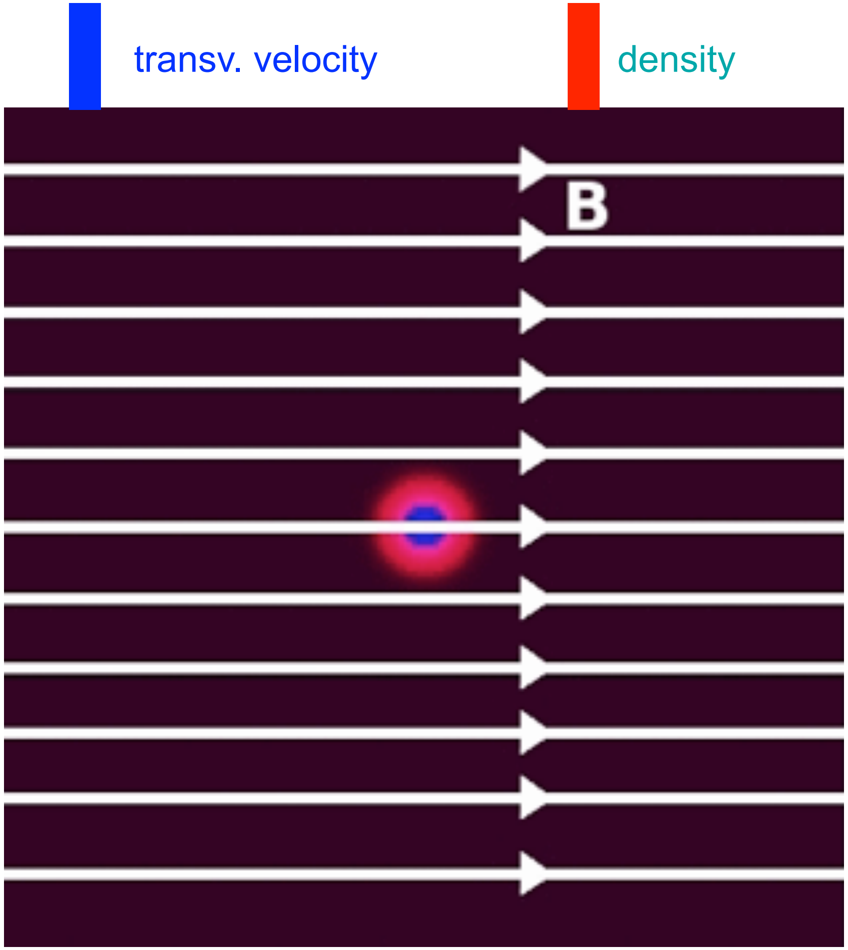




: Group speeds

MHD Waves Movie

Throwing a 'pebble' into a 'plasma lake'...
For low plasma beta $v_A >> c_s$



Three types of MHD waves

- **Alfvén waves**

  magnetic tension ($\omega = V_A k_{||}$)

- **Fast magnetoacoustic waves**

  magnetic with plasma pressure ($\omega \approx V_A k$)

- **Slow magnetoacoustic waves**

  magnetic against plasma pressure ($\omega \approx C_S k_{||}$)

Linear MHD for uniform media

1. The perturbations are waves
2. Waves are dispersionless
3. ω and k are always real
4. Waves are highly anisotropic
5. There are incompressible - Alfvén waves - and compressible - magnetoacoustic – modes

However, natural plasma systems are usually highly structured and often unstable

Non-ideal terms in MHD

Ideal MHD is a set of conservation laws

Non-ideal terms are dissipative and entropy producing

- Resistivity
- Viscosity
- Radiation transport
- Thermal conduction

Resistivity

Electron-ion collisions dissipate current

$$\mathbf{E} + \mathbf{v} \wedge \mathbf{B} = \eta \mathbf{j}$$

If we assume the resistivity is constant then

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{\eta}{\mu_0} \nabla^2 \mathbf{B}$$

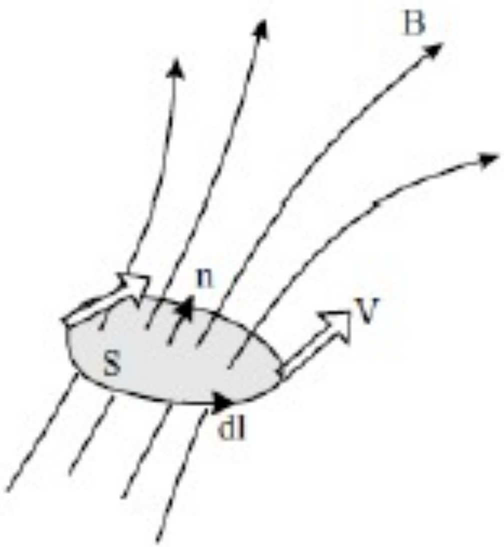
Ratio of advective to diffusive terms is the magnetic Reynolds number

$$R_m = \frac{\mu_0 L_0 v_0}{\eta}$$

Usually in space physics $R_m >> 1$ (10^6-10^{12}). This is based on global scale lengths L_0. If L_0 is over a small scale with rapidly changing magnetic field, i.e. a current sheet, then $R_m \simeq 1$

Alfven's theorem

Rate of change of flux through a surface moving with fluid



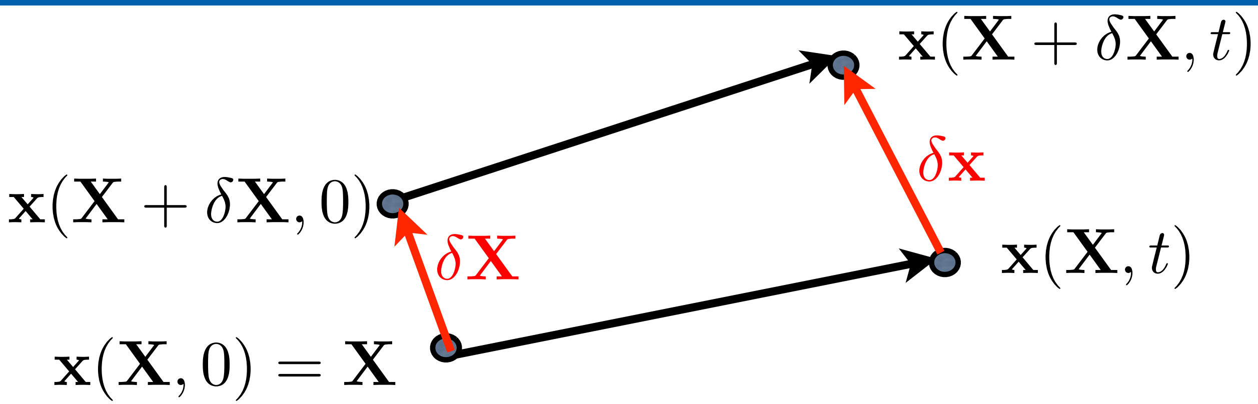
$$\frac{d}{dt}\int_S \mathbf{n}.\mathbf{B}\; dS = \int_S \mathbf{n}\frac{\partial \mathbf{B}}{\partial t}\; dS - \oint_l \mathbf{v}\times\mathbf{B}.\mathbf{dl}$$

$$= -\int_S \nabla\times(\mathbf{E}+\mathbf{v}\times\mathbf{B}).\mathbf{n}\; dS$$

Magnetic flux through a surface moving with the fluid is conserved if ideal MHD Ohm's law, i.e. no resistivity

Often stated as- the flux is frozen in to the fluid

Line Conservation



Consider two points which move with the fluid

$$\delta x_i = \frac{\partial x_i}{\partial X_j}\delta X_j$$

$$\frac{D}{Dt}\delta\mathbf{x} = \frac{\partial u_i}{\partial X_j}\delta X_j$$

$$= \frac{\partial u_i}{\partial x_k}\frac{\partial x_k}{\partial x_j}\delta X_j$$

$$= (\delta\mathbf{x}.\nabla)\mathbf{u}$$

Line Conservation -2

Equation for evolution of the vector between two points moving with the fluid is

$$\frac{D}{Dt}\delta\mathbf{x} = (\delta\mathbf{x}.\nabla)\mathbf{u}$$

Also for ideal MHD

$$\frac{D}{Dt}\left(\frac{\mathbf{B}}{\rho}\right) = \frac{\mathbf{B}}{\rho}.\nabla\mathbf{v}$$

Hence if we choose $\delta\mathbf{x}$ to be along the magnetic field at t = 0 then it will remain aligned with the magnetic field.

Two points moving with the fluid which are initially on the same field-line remain on the same field line in ideal MHD

Reconnection not possible in ideal MHD

Cauchy Solution

Shown that $\frac{\mathbf{B}}{\rho}$ and $\delta\mathbf{x}$ satisfy the same equation hence

$$\delta x_i = \frac{\partial x_i}{\partial X_j}\delta X_j$$

Implies

$$\frac{B_i}{\rho} = \frac{\partial x_i}{\partial X_j}\frac{B_j^0}{\rho^0}$$

Where superscript zero refers to initial values

$$B_i = \frac{\partial x_i}{\partial X_j}B_j^0\frac{\rho}{\rho^0}$$

$$\frac{\rho^0}{\rho} = \Delta = \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)}$$

Cauchy solution

$$\boxed{B_i = \frac{\partial x_i}{\partial X_j}\frac{B_j^0}{\Delta}}$$

MHD based on Cauchy

$$B_i = \frac{\partial x_i}{\partial X_j} \frac{B_j^0}{\Delta}$$

$$\frac{\rho^0}{\rho} = \Delta = \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)}$$

$$P = const \ \rho^\gamma$$

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla.P + \frac{1}{\mu_0 \rho}(\nabla \times \mathbf{B}) \times \mathbf{B}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}$$

Only need to know position of fluid elements and initial conditions for full MHD solution

Non-ideal MHD

Resistivity

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}$$

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\,\mathbf{v}) = 0$$

Coriolis

Gravity

"Other"

$$\rho\,\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}.\nabla)\mathbf{v} + 2\rho\,\boldsymbol{\Omega}\times\mathbf{v} = -\nabla p + \mathbf{j}\times\mathbf{B} + \rho\,\mathbf{g} + \mathbf{F},$$

$$p = \frac{1}{\tilde{\mu}}\rho\Re T$$

Thermal Conduction

Radiation

Ohmic heating

"Other"

$$\frac{\rho^{\gamma}}{\gamma-1}\frac{D}{Dt}\left(\frac{p}{\rho^{\gamma}}\right) = \nabla.(\kappa\nabla T) - \rho^2 Q(T) + \frac{j^2}{\sigma} + H$$

MHD Characteristics

Sets of ideal MHD equations can be written as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial}{\partial x}\left[\mathbf{F}(\mathbf{U})\right] = 0$$

All equations sets of this types share the same properties

- they express conservation laws
- can be decomposed into waves
- non-linear solutions can form shocks
- satisfy $L1$ contraction, TVD constraints

Characteristics

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial}{\partial x}\left[\mathbf{F}(\mathbf{U})\right] = 0$$

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial \mathbf{U}} \cdot \frac{\partial \mathbf{U}}{\partial x} = 0$$

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}(\mathbf{U}) . \frac{\partial \mathbf{U}}{\partial x} = 0$$

$\mathbf{A}(\mathbf{U})$ is called the Jabobian matrix

For linear systems can show that Jacobian matrix is a function of equilibria only, e.g. function of p_0 but not p_1

Properties of the Jacobian

Left and right eigenvectors/eigenvalues are real

$$\mathbf{A}.\mathbf{r}_\mathbf{i} = \lambda_i \mathbf{r}_\mathbf{i} \quad \text{and} \quad \mathbf{l}_\mathbf{i}\mathbf{A} = \lambda_i \mathbf{l}_\mathbf{i}$$

Diagonalisable: $\mathbf{A} = \mathbf{R}\,\Lambda\,\mathbf{R}^{-1}$

$$\Lambda = \mathbf{diag}(\lambda_1, \lambda_2 \ldots \lambda_m)$$

$$\mathbf{R} = \left[\mathbf{r}_1 \,|\, \mathbf{r}_2 \,|\, \ldots \,|\, \mathbf{r}_m\right]$$

$$\mathbf{R}^{-1} = \mathbf{L} = \begin{bmatrix} \mathbf{l}_1 \\ \mathbf{l}_2 \\ \vdots \\ \mathbf{l}_m \end{bmatrix}$$

Characteristic waves

This example is for linear equations with constant **A**

$$\mathbf{R}^{-1}.\left(\frac{\partial \mathbf{U}}{\partial t}+\mathbf{A}.\frac{\partial \mathbf{U}}{\partial x}\right)=0$$

$$\frac{\partial}{\partial t}\left(\mathbf{R}^{-1}\mathbf{U}\right)+\left(\mathbf{R}^{-1}\mathbf{A}\right).\frac{\partial \mathbf{U}}{\partial x}=0$$

But $\mathbf{A}=\mathbf{R}\boldsymbol{\Lambda}\mathbf{R}^{-1}$ so $\mathbf{R}^{-1}\mathbf{A}=\boldsymbol{\Lambda}\mathbf{R}^{-1}$

$$\frac{\partial}{\partial t}\left(\mathbf{R}^{-1}\mathbf{U}\right)+\boldsymbol{\Lambda}.\frac{\partial}{\partial x}\left(\mathbf{R}^{-1}\mathbf{U}\right)=0$$

$$\frac{\partial \mathbf{w}}{\partial t}+\boldsymbol{\Lambda}.\frac{\partial \mathbf{w}}{\partial x}=0 \quad \text{with} \quad \mathbf{w}=\mathbf{R}^{-1}\mathbf{U}$$

$\mathbf{w}$ is called the characteristic field

Riemann problems

Λ is diagonal so all equations decouple

$$\frac{\partial w_i}{\partial t} + \lambda_i . \frac{\partial w_i}{\partial x} = 0$$

i.e. characteristics w_i propagate with speed λ_i

In MHD the characteristic speeds are $v_x, v_x \pm c_f, v_x \pm v_A, v_x \pm c_s$
i.e. the fast, Alfven and slow speeds

Solution in terms of original variables $\mathbf{U}$

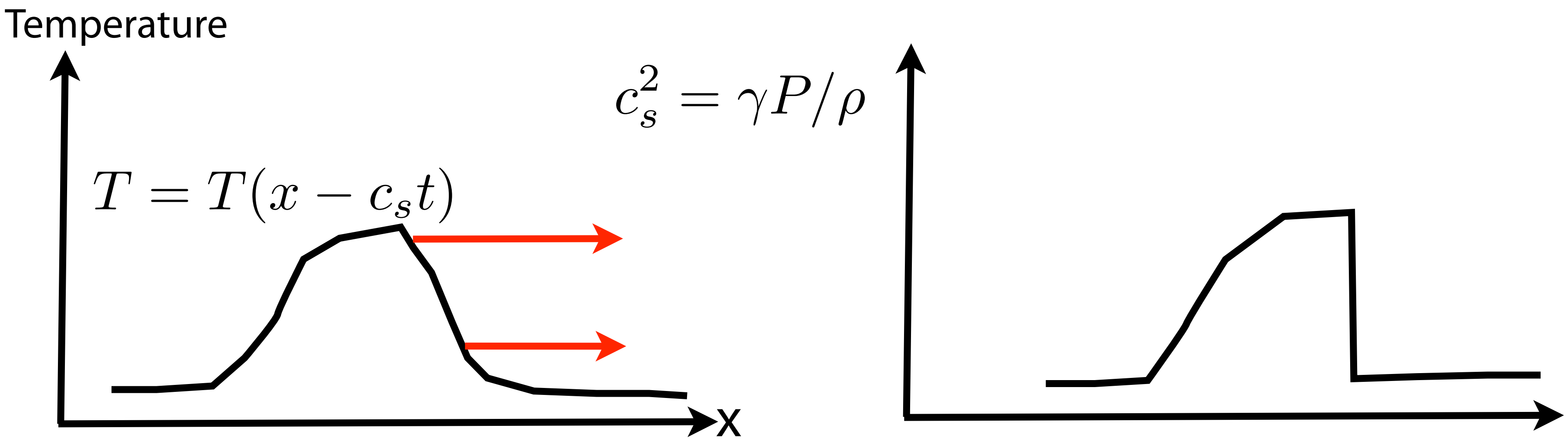
$$\mathbf{U} = \mathbf{R}.\mathbf{w}$$

$$\mathbf{U} = \sum_{i=1}^{m} w_i(x,t)\,\mathbf{r_i}$$

$$\boxed{\mathbf{U} = \sum_{i=1}^{m} w_i^{t=0}(x - \lambda_i t)\,\mathbf{r_i}}$$

This analysis forms the basis of Riemann decomposition used for treating shocks, e.g. Riemann codes in numerical analysis

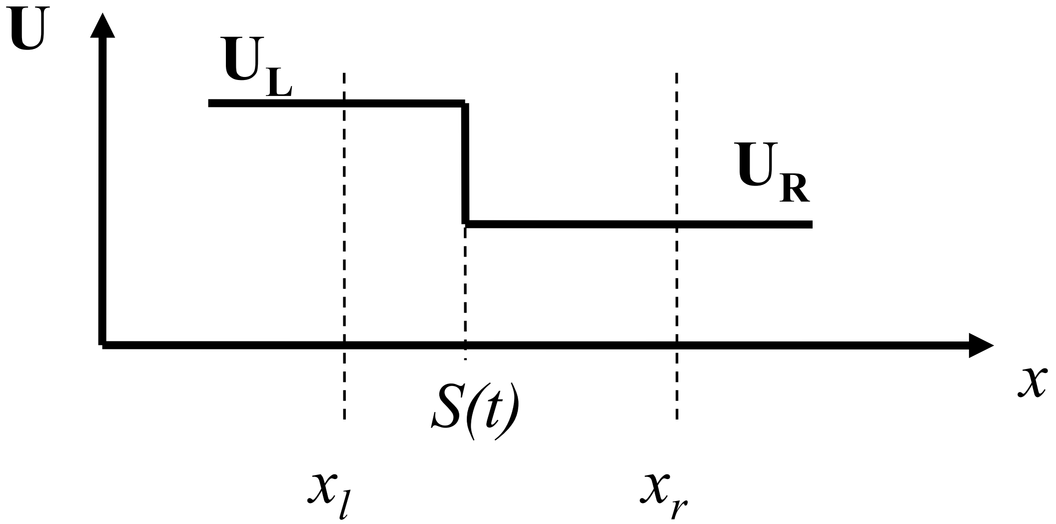
Basic Shocks



Without dissipation any 1D traveling pulse will eventually, i.e. in finite time, form a singular gradient. These are shocks and the differentially form of MHD is not valid.

Also formed by sudden release of energy, e.g. flare, or supersonic flows.

Rankine-Hugoniot relations



$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial}{\partial x}\left[\mathbf{F}(\mathbf{U})\right] = 0$$

Integrate equations from x_l to x_r across moving discontinuity $S(t)$

Jump Conditions

Use

$$\frac{d}{dt}\int_{x_l}^{x_r}\mathbf{U}(x,t)\,dx=\frac{d}{dt}\int_{x_l}^{S(t)}\mathbf{U}\,dx+\frac{d}{dt}\int_{S(t)}^{x_r}\mathbf{U}\,dx$$

$$=\left[\mathbf{U}(s_-,t)-\mathbf{U}(s_+,t)\right]\frac{ds}{dt}+\int_{x_l}^{S(t)}\frac{\partial\mathbf{U}}{\partial t}dx+\int_{S(t)}^{x_r}\frac{\partial\mathbf{U}}{\partial t}dx$$

Let x_l and x_r tend to $S(t)$ and use conservative form to get

$$\boxed{(\mathbf{U}_\mathbf{L}-\mathbf{U}_\mathbf{R})v_s=\mathbf{F}(\mathbf{U}_\mathbf{L})-\mathbf{F}(\mathbf{U}_\mathbf{R})}$$

- Rankine-Hugoniot conditions for a discontinuity moving at speed v_s
- All equations must satisfy these relations with the same v_s

The End