3

The Governing Equations of Fluid Mechanics

In his 1964 lectures on physics, R P Feynman noted that:

> The efforts of a child trying to dam a small stream flowing in the street, and his surprise at the strange way the water works its way out, has its analog in our attempts over the years to understand the flow of fluids. We have tried to dam the water by getting the laws and equations... but the water has broken through the dam and escaped our attempt to understand it.

In this chapter we build the dam and write down the equations. Later, particularly in Chapter 7 where we discuss turbulence, we shall see how the dam bursts open.

Part 1: Fluid Flow in the Absence of Lorentz Forces

In the first seven sections of this chapter we leave aside MHD and focus on fluid mechanics in the absence of the Lorentz force. We return to MHD in Section 3.8. Readers who have studied fluid mechanics before may be familiar with much of the material in Sections 3.1 to 3.7, and may wish to proceed directly to Section 3.8. The first seven sections provide a self-contained introduction to the subject, with particular emphasis on vortex dynamics, which is so important in the study of MHD.

3.1 Elementary Concepts

3.1.1 Different categories of fluid flow

The beginner in fluid mechanics is often bewildered by the many diverse categories of fluid flow which appear in the text books. There are entire books dedicated to such subjects as potential flow, boundary layers, turbulence, vortex dynamics and so on. Yet the relationship between these different types of flow, and their relationship to 'real' flows, is often unclear. You might ask, if I want to understand natural convection in a room do I want a text on boundary layers, turbulence or vortex dynamics? The answer, probably, is all three. These subjects rarely exist in isolation, but rather interact in some complex way. For example, a turbulent wake is usually created when the turbulent fluid within one or more thin boundary layers is ejected from the boundaries into the main flow. The purpose of this section is to give some indication as to what expressions such as boundary layers, turbulence and vorticity mean, how these subjects interact, and when they are likely to be important in practice. The discussion is essentially qualitative, and anticipates some of the results proved in the subsequent sections. So the reader will have to take certain facts at face value. Nevertheless, the intention is to provide a broad framework into which the many detailed calculations of the subsequent sections fit.

We shall describe why, for good physical reasons, fluid mechanics and fluid flows are often divided into different regimes. In particular, there are three very broad sub-divisions in the subject. The first relates to the issue of when a fluid may be treated as inviscid, and when the finite viscosity possessed by all fluids (water, air, liquid metals) must be taken into account. Here we shall see that, typically, viscosity and shear stresses are of great importance close to solid surfaces (within so-called boundary layers) but often less important at a large distance from a surface. Next there is the sub-division between laminar (organised) flow and turbulent (chaotic) flow. In general, low speed or very viscous flows are stable to small perturbations and so remain laminar, while high speed or almost inviscid flows are unstable to the slightest perturbation and rapidly develop a chaotic component of motion. The final, rather broad, subdivision which occurs in fluid mechanics is between irrotational (sometimes called potential) flow and rotational flow. (By irrotational flow we mean flows in which $\nabla \times \mathbf{u} = 0$.) Turbulent flows and boundary layers are always rotational. Sometimes, however, under very particular conditions, an external flow may be approximately irrotational, and indeed this kind of flow dominated the early literature in aerodynamics. In reality, though, such flows are extremely rare in nature, and the large space given over to potential flow theory in traditional texts probably owes more to the ease with which such flows are amenable to mathematical description than to their usefulness in interpreting real events.

Let us now explore in a little more detail these three sub-divisions. We need two elementary ideas as a starting point. We need to be able to quantify shear stress and inertia in a fluid.

Let us start with inertia. Suppose, for the sake of argument, that we have a steady flow. That is to say, the velocity field $\mathbf{u}$, which we normally write as $\mathbf{u}(\mathbf{x},t)$, is a function of $\mathbf{x}$ but not of t. It follows that the speed of the fluid at any one point in space is steady, the flow pattern does not change with time, and the streamlines (the analogue of **B**-lines) represent particle trajectories for individual fluid 'lumps'. Now consider a particular streamline, C, as shown in Figure 3.1, and focus attention on a particular fluid blob as it moves along the streamline. Let s be a curvilinear coordinate measured along C, and $V(s)$ be the speed $|\mathbf{u}|$. Since the streamline represents a particle trajectory, we can apply the usual rules of mechanics and write

$$(\text{acceleration of lump}) = V\frac{dV}{ds}\hat{\mathbf{e}}_t - \frac{V^2}{R}\hat{\mathbf{e}}_n$$

where R is the radius of curvature of the streamline, and $\hat{\mathbf{e}}_t$, $\hat{\mathbf{e}}_n$ represent unit vectors tangential and normal to the streamline. In general, then, the acceleration of a typical fluid element is of order $|\mathbf{u}|^2/l$, where l is a characteristic length scale of the flow pattern.

Next we turn to shear stress in a fluid. In most fluids this is quantified using an empirical law known as Newton's law of viscosity. This is most simply understood in a one-dimensional flow, $u_x(y)$, as shown in Figure 3.2. Here fluid layers slide over each other due to the fact that u_x is a function of y. One measure of this rate of sliding is the angular distortion rate, $d\gamma/dt$, of an initially rectangular element. (See Figure 3.2 for the definition of γ.) Newton's law of viscosity says that a shear stress, τ, is required to cause the relative sliding of the fluid layers. Moreover it states that τ is directly proportional to $d\gamma/dt$: $\tau = \mu(d\gamma/dt)$. The coefficient of proportionality is termed the absolute viscosity. However, it is clear from the diagram that $d\gamma/dt = \partial u/\partial y$, and so this expression is usually rewritten as

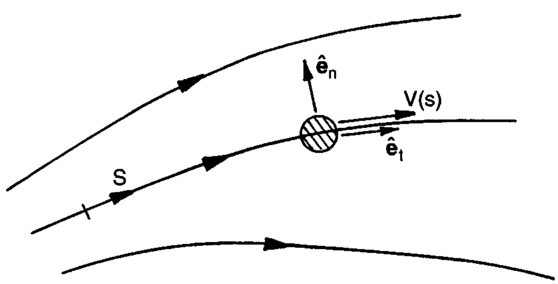


Figure 3.1 Acceleration of a fluid element in a steady flow.

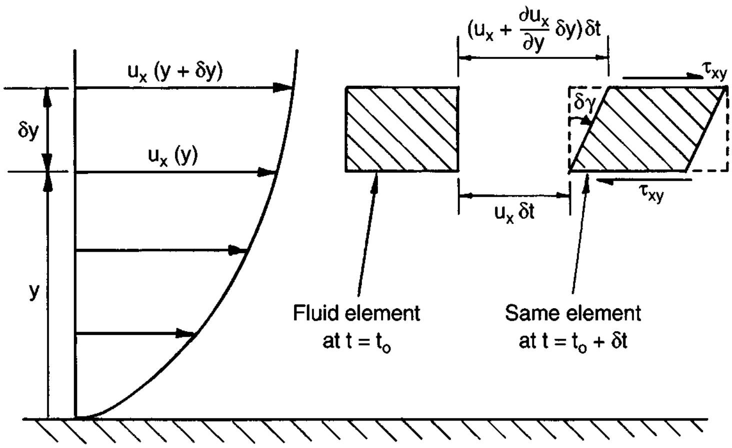


Figure 3.2 Distortion of a fluid element in a parallel flow.

$$\tau = \rho\nu\frac{\partial u_x}{\partial y}, \qquad \nu = \mu/\rho$$

where ν is called the kinematic viscosity. (The choice of kinematic viscosity rather than absolute viscosity is arbitrary, but has the benefit of avoiding confusion between permeability and viscosity.)

In a more general two-dimensional flow, $\mathbf{u}(x, y) = (u_x, u_y, 0)$, it turns out that γ, and hence $d\gamma/dt$, has two components, one arising from the rotation of vertical material lines, as shown above, and one arising from the rotation of horizontal material lines. A glance at Figure 3.9 will confirm that the additional contribution to $d\gamma/dt$ is $\partial u_y/\partial x$. Thus, in two dimensions, Newton's law of viscosity becomes

$$\tau_{xy} = \rho\nu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

This generalises in an obvious way to three dimensions (see Chapter 3, Section 1.2). Now shear stresses are important not just because they cause fluid elements to distort, but because an imbalance in shear stress can give rise to a net force on individual fluid elements. For example, in Figure 3.2 a net horizontal force will be exerted on the element if τ_{xy} at the top of the element is different to τ_{xy} at the bottom of the element. In fact, in this simple example it is readily confirmed that the net horizontal shear force per unit volume is $f_x = \partial\tau_{xy}/\partial y = \rho\nu\partial^2 u_x/\partial y^2$.

We are now in a position to estimate the relative sizes of inertial and viscous forces in a three-dimensional flow. The viscous forces per unit volume are of the form of gradients in shear stress, such as $\partial\tau_{xy}/\partial y$, and have a size: $f_\nu \sim \rho\nu|\mathbf{u}|/l_\perp^2$, where $l_\perp$is a characteristic length scale normal to the streamlines. The inertial forces per unit volume, on the other hand, are of the order of $f_{in} \sim \rho \times$ (acceleration) $\sim \rho u^2/l$ where l is a typical geometric length scale. The ratio of the two is of order

$$\mathrm{Re} = \frac{ul}{\nu}$$

This is the Reynolds number. When Re is small, viscous forces outweigh inertial forces, and when Re is large viscous forces are relatively small. Now we come to the key point. When Re is calculated using some characteristic geometric length scale, it is almost always very large. This reflects the fact that the viscosity of nearly all common fluids, including liquid metal, is minute, of the order of 10^{-6} m^2/s. Because of the large size of Re, it is tempting to dispense with viscosity altogether. However, this is extremely dangerous. For example, inviscid theory predicts that a sphere sitting in a uniform cross-flow experiences no drag (d'Alembert's paradox) and this is clearly not the case, even for 'thin' fluids like air.

Something seems to have gone wrong. The problem is that, no matter how small ν might be, there are always regions near surfaces where the shear stresses are significant, i.e. of the order of the inertial forces. These *boundary layers* give rise to the drag on, say, an aerofoil. Consider the flow over an aerofoil shown in Figure 3.3. Here we use a frame of reference moving with the foil. The value of Re for such a flow, based on the width of the aerofoil, will be very large, perhaps around 10^8. Consequently, away from the surface of the aerofoil, the fluid behaves as if it is inviscid. Close to the aerofoil, however, something else happens, and this is a direct result of a boundary condition called the *no-slip condition*. The no-slip condition says that all fluids are 'sticky', in the sense that there can be no relative motion between a fluid and a surface

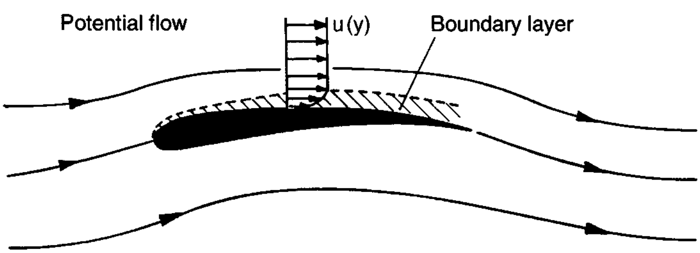


Figure 3.3 Boundary layer on an aerofoil.

with which it comes into contact. The fluid 'sticks' to the surface. In the case of the aerofoil, this means that there must be some transition region near the surface where the fluid velocity drops down from its *free-stream* value (the value it would have if the fluid were inviscid) to zero on the surface. This is the boundary layer. Boundary layers are usually very thin. We can estimate their thickness as follows. Within the boundary layer there must be some force acting on the fluid which pulls the velocity down from the free-stream value to zero at the surface. The force which does this is the viscous force, and so within the boundary layer the inertial and viscous forces must be of similar order. Let δ be the boundary layer thickness and l be the span of the aerofoil. The inertial forces are of order $\rho u^2/l$ and the viscous forces are of order $\rho \nu u/\delta^2$. Equating the two gives

$$\delta/l \sim (ul/\nu)^{-1/2} \ll 1$$

Thus we see that, no matter how small we make ν, there is always some (thin) boundary layer in which shear stresses are important. This is why an aerofoil experiences drag even when ν is very small.

We have reached the first of our three general sub-divisions in fluid mechanics. That is to say, often (but not always) a high-Re flow may be divided into an external, inviscid flow plus one or more boundary layers. Viscous effects are then confined to the boundary layer. This idea was introduced by Prandtl in 1904 and works well for external flow over bodies, particularly streamlined bodies, but can lead to problems in confined flows. (It is true that boundary layers form at the boundaries in confined flows, and that shear stresses are usually large within the boundary layers and weak outside the boundary layers. However, the small but finite shear in the bulk of a confined fluid can, over long periods of time, have a profound influence on the overall flow pattern (see Chapter 3, Section 5.))

Boundary layers have another important characteristic, called *separation*. Suppose that, instead of an aerofoil, we consider flow over a sphere. If the fluid were inviscid (which no real fluid is!) we would get a symmetric flow pattern as shown in Figure 3.4(a). The pressure at the stagnation points A and C in front of and behind the sphere would be equal (by symmetry), and from Bernoulli's equation the pressure at these points would be high, $P_A = P_\infty + \frac{1}{2}\rho V_\infty^2$, with P_∞ and V_∞ being the upstream pressure and velocity, respectively. The real flow looks something like that shown in Figure 3.4(b). A boundary layer forms at the leading stagnation point and this remains thin as the fluid moves to the edges of the sphere. However, towards the rear of the sphere something unexpected happens. The boundary layer separates. That is to say, the fluid in the boundary layer is ejected into the external flow and a turbulent wake forms. This separation is caused by pressure forces. Outside the boundary layer the fluid, which tries to follow the inviscid flow pattern, starts to slow down as it passes over the outer edges of the sphere (points B and D) and heads towards the rear stagnation point. This deceleration is caused by pressure forces which oppose the external flow. These same pressure forces are experienced by the fluid within the boundary layer and so this fluid also begins to decelerate (Figure 3.4(c)). However, the fluid in the boundary layer has less momentum than the corresponding external flow and very quickly it comes to a halt, reverses direction, and moves off into the external flow, thus forming a wake. Thus we see that the flow over a body at high Re can generally be divided into three regions: an inviscid external flow, boundary layers, and a turbulent wake.

Now the fact that Re is invariably large has a second important consequence: most flows in nature are *turbulent*. This leads to a second classification in fluid mechanics. It is an empirical observation that at low values of Re flows are laminar, while at high values of Re they are turbulent (chaotic). This was first demonstrated in 1883 by Reynolds, who studied flow in a pipe. In the case of a pipe the transition from laminar to turbulent flow is rather sudden, and occurs at around Re $\sim$2000, which usually constitutes a rather slow flow rate.

A turbulent flow is characterised by the fact that, superimposed on the mean (time-averaged) flow pattern, there is a random, chaotic motion. The velocity field is often decomposed into its time-averaged component and random fluctuations about that mean: $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'$. The transition from laminar to turbulent flow occurs because, at a certain value of Re, instabilities develop in the laminar flow, usually driven by the inertial

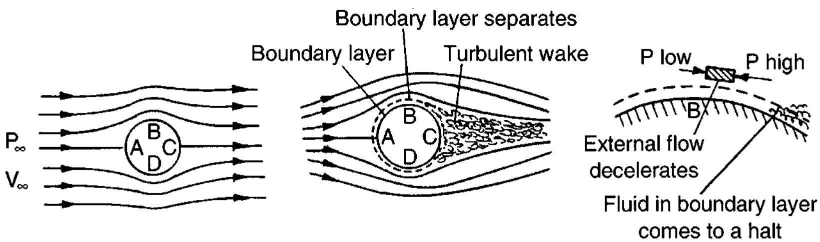


Figure 3.4 Flow over a sphere: (a) inviscid flow; (b) real flow at high Re; (c) pressure forces which cause separation.

forces. At low values of Re these potential instabilities are damped out by viscosity, while at high values of Re the damping is inadequate.

There is a superficial analogy between turbulence and the kinetic theory of gases. The steady laminar flow of a gas has, at the macroscopic level, only a steady component of motion. However, at the molecular level, individual atoms not only possess the mean velocity of the flow, but also some random component of velocity which is related to their thermal energy. It is this random fluctuation in velocity which gives rise to the exchange of momentum between molecules and thus to the macroscopic property of viscosity. There is an analogy between individual atoms in a laminar flow and macroscopic blobs of fluid in a turbulent flow. Indeed, this (rather imperfect) analogy formed the basis of most early attempts to characterise turbulent flow. In particular, it was proposed that one should replace ν in Newton's law of viscosity, which for a gas arises from thermal agitation of the molecules, by an 'eddy viscosity' ν_t, which arises from macroscopic fluctuations.

The transition from laminar to turbulent flow is rarely clear cut. For example, often some parts of a flow field are laminar while, at the same time, other parts are turbulent. The simplest example of this is the boundary layer on a flat plate (Figure 3.5). If the front of the plate is streamlined, and the turbulence level in the external flow is low, the boundary layer usually starts off as laminar. Of course, eventually it becomes unstable and turns turbulent.

Often periodic (non-turbulent) fluctuations in the laminar flow precede the onset of turbulence. This is illustrated in Figure 3.6, which shows flow over a cylinder at different values of Re. At low values of Re we get a

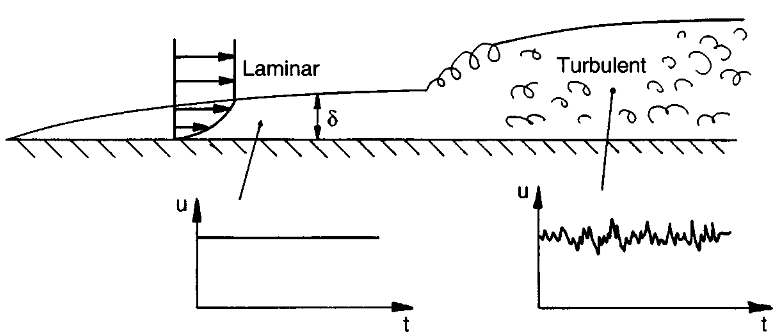


Figure 3.5 Development of a boundary layer on a flat plate.

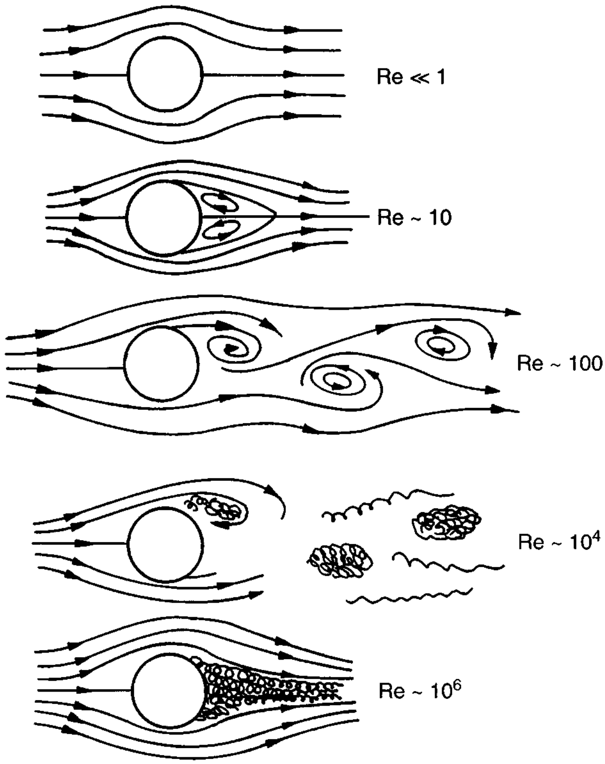


Figure 3.6 Flow behind a cylinder at various values of Re.

symmetric flow pattern. This is called creeping flow. As Re rises above unity, steady vortices appear at the rear of the cylinder. By the time Re has reached ~100 these vortices start to peel off from the rear of the cylinder in a regular, periodic manner (at this point the flow is still laminar). This is called Karman's vortex street. At yet higher values of Re the shed vortices become turbulent, but we still have a discernible vortex street. Finally, at a value of Re $\sim 10^5$, the flow at the rear of the cylinder loses its periodic structure and becomes a turbulent wake. Notice that upstream of the cylinder fluid blobs possess linear momentum but no angular momentum. In the Karman street, however, certain fluid elements (those in the vortices) possess both linear and angular momentum. Moreover, the angular momentum seems to have come from the boundary layer on the cylinder. This leads us to our third and final sub-division in fluid mechanics. In some flows (potential flows) the fluid elements possess only linear momentum. In others (vortical flows) they possess both angular and linear momentum. In order to pursue this idea a little further we need some measure of the rotation of individual fluid elements. This is called vorticity.

So far we have discussed flow fields in terms of the velocity field $\mathbf{u}$. However, there is a closely related quantity, the vorticity, which is defined as $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. From Stokes' theorem we have, for a small disc-like element of fluid (with surface area $d\mathbf{S}$),

$$\boldsymbol{\omega} \cdot d\mathbf{S} = \oint_C \mathbf{u} \cdot d\mathbf{l}$$

We might anticipate, therefore, that $\boldsymbol{\omega}$ is a measure of the angular velocity of a fluid element, and this turns out to be true. In fact, the angular velocity, $\boldsymbol{\Omega}$, of a fluid blob which is passing through point $\mathbf{x}_0$ at time t_0 is just $\boldsymbol{\omega}(\mathbf{x}_0, t_0)/2$. Thus, while $\mathbf{u}$ is related to the linear momentum of fluid elements, $\boldsymbol{\omega}$ is related to the angular momentum of blobs of fluid. Now $\boldsymbol{\omega}$ is a useful quantity because it turns out that, partially as a result of conservation of angular momentum, it cannot be created or destroyed within the interior of a fluid. (At least that is the case in the absence of external forces such as buoyancy or the Lorentz force.)

That is not to say that the vorticity of a fluid particle is constant. Consider the vortices within a Karman vortex street. It turns out that, as they are swept downstream, they grow in size in much the same way that a packet of hot fluid spreads heat by diffusion. Like heat, vorticity can diffuse. In particular, it diffuses between adjacent fluid particles as they sweep through the flow field. However, as with heat, this diffusion does not change the global amount of vorticity (heat) present in an isolated patch of fluid. Thus, as the vortices in the Karman street spread, the intensity of ω in each vortex falls, and it falls in such a way that $\int \omega dA$ is conserved for each vortex.

There is a second way in which the vorticity in a given lump of fluid can change. Consider the ice-skater who spins faster by pulling his or her arms inward. What is true for ice-skaters is true for blobs of fluid. If a spinning fluid blob is stretched by the flow, say from a sphere to cigar shape, it will spin faster, and the corresponding component of $\boldsymbol{\omega}$ increases.

In summary, then, vorticity cannot be created within the interior of a fluid unless there are body forces present, but like heat it spreads by diffusion and can be intensified by the stretching of fluid elements. The way in which we quantify the diffusion and intensification of vorticity will be discussed in the next section. However, for the moment, the important point to grasp is that, like heat, vorticity cannot be created in the interior of a fluid.

So where does the vorticity evident in Figure 3.6 come from? Here the analogy to heat is useful. We shall see that, in the absence of stretching of fluid elements, the governing equation for $\boldsymbol{\omega}$ is identical to that for heat. It is transported by the mean flow (we say it is *advected*) and diffuses outward from regions of intense vorticity. Also, just like heat, it is the boundaries which act as sources of vorticity. In fact, boundary layers are filled with the vorticity which has diffused out from the adjacent surface. (In a pseudo-one-dimensional boundary layer, with velocity $u_x(y)$, the vorticity is $|\omega_z| = \partial u_x/\partial y \sim u/\delta$.) This gives us a new way of thinking about boundary layers: they are diffusion layers for the vorticity generated on a surface. Again there is an analogy to heat. Thermal boundary layers are diffusion layers for the heat which seeps into the fluid from a surface. In both cases the thickness of the boundary layer is fixed by the ratio of: (i) the rate at which heat or vorticity diffuses across the streamlines from the surface, and (ii) the rate at which heat or vorticity is swept downstream by the mean flow. Usually when Re is large, the cross-stream diffusion is slow by comparison with the stream-wise transport of vorticity, and this is why boundary layers are so thin.

We are now in a position to introduce our third and final classification in fluid mechanics. This is the distinction between potential (vorticity free) flows and vortical flows. Consider Figure 3.7(a). This represents classical aerodynamics. There is a boundary layer, which is filled with vorticity, and an external flow. The flow upstream of the aerofoil is (in classical aerodynamics) assumed to be irrotational (free of vorticity), and since the vorticity generated on the surface of the foil is confined to the boundary layer, the entire external flow is irrotational. (This kind of external flow is called a potential flow.)

The problem of computing the external motion is now reduced to solving two kinematic equations: $\nabla \cdot \mathbf{u} = 0$ (conservation of mass) and $\nabla \times \mathbf{u} = 0$. In effect, aerodynamics becomes aerokinematics. However, potential flows (irrotational flows) are extremely rare in nature. In fact, flow over streamlined bodies (plus certain types of water waves) represent the only common examples. Almost all real flows are laden with vorticity: vorticity which has been generated somewhere in a boundary layer and then released into the bulk flow (see Figures 3.6 and 3.7(b)). The rustling of leaves, the blood in our veins, the air in our lungs, the wind blowing down the street, natural convection in a room, and the flows in the oceans and atmosphere are all examples of flows laden with

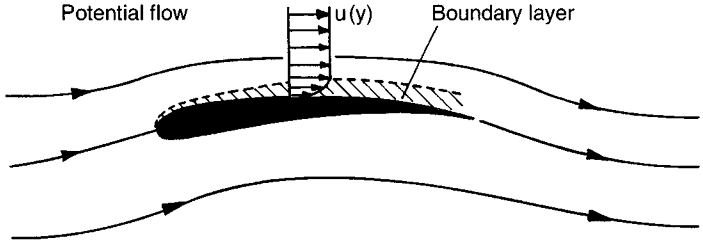
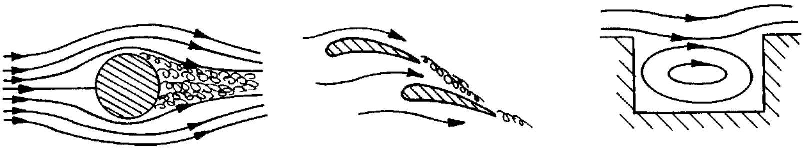


Figure 3.7 (a) Classical aerodynamics (potential flow).



(a) The turbulent wake is full of vorticity

(b) In a turbine the vorticity created by one blade spills out into the next blade

(c) In confined flows the vorticity created at the boundaries slowly seeps into the entire flow, eventually dominating the flow

Figure 3.7 (b) Most real flows are laden with vorticity.

vorticity. So we have two types of flow: potential flows, which are easy to compute but infrequent in nature, and vortical flows, which are very common but much more difficult to understand. The art of quantifying this second category of flow is to track the progress of the vorticity from the boundaries into the bulk of the fluid. Often this arises from wakes or from boundary-layer separation. Sometimes, as in the case of confined flows, it is due to a slow but finite diffusion of vorticity from the boundary into the interior of the flow. In either case, it is the boundaries which generate the vorticity.

To these two classes of flow, potential flow and unforced vortical flow, we should add a third: that of MHD. Here the Lorentz force generates vorticity in the interior of the fluid. On the one hand this makes MHD more difficult to understand, but on the other it makes it more attractive. In MHD we have the opportunity to grab hold of the interior of a fluid and manipulate the flow.

With this brief, qualitative overview of fluid mechanics we now set about quantifying the motion of a fluid. Our starting point is the equation of motion of a fluid blob.

3.1.2 The Navier–Stokes equation

The Navier–Stokes equation is a statement about the changes in linear momentum of a small element of fluid as it progresses through a flow field. Let p be the pressure, τ_{ij} the viscous stresses acting on the fluid, and ν the kinematic viscosity. Then Newton's second law applied to a small blob of fluid of volume δV yields[1]

$$(\rho \delta V)\frac{D\mathbf{u}}{Dt} = -(\nabla p)\delta V + \left[\partial \tau_{ij}/\partial x_j\right]\delta V \tag{3.1}$$

That is to say, the mass of the element, $\rho \delta V$, times its acceleration, $D\mathbf{u}/Dt$, equals the net pressure force acting on the surface of the fluid blob,

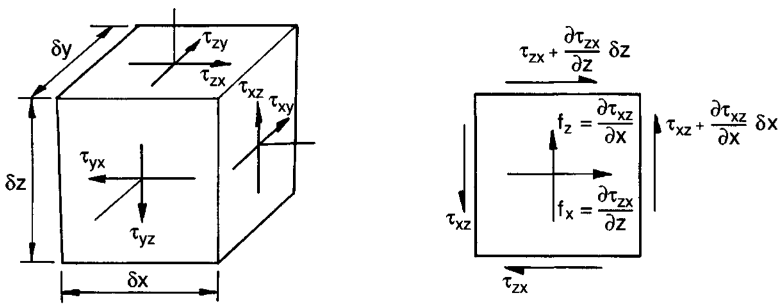
$$\oint (-p)d\mathbf{S} = \int (-\nabla p)dV = -(\nabla p)\delta V$$

plus the net force arising from the viscous stress, τ_{ij}. The last term in (3.1) may be established by considering the forces acting on a small rectangular element $dxdydz$, as indicated in Figure 3.8.

We shall take the fluid to be incompressible so that the conservation of mass, expressed as $\nabla \cdot (\rho \mathbf{u}) = -\partial \rho / \partial t$, reduces to the so-called continuity equation

$$\nabla \cdot \mathbf{u} = 0 \tag{3.2}$$

We also take the fluid to be Newtonian, so that the viscous stresses are given by the constitutive law



(a) Sign convention for stresses (note that $\tau_{ij} = \tau_{ji}$)

(b) Net effective body force, per unit volume, caused by an inbalance in stresses. Only components in th xz plane are shown

Figure 3.8 Stresses acting on a cube of fluid.

[1] Those unfamiliar with tensor notation will find a brief summary in Section 7.1.

$$\tau_{ij} = \rho\nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3.3}$$

where ν is the kinematic viscosity of the fluid. Substituting for τ_{ij} in (3.1) and dividing through by $\rho\delta V$ yields the conventional form of the Navier–Stokes equation

$$\boxed{\frac{D\mathbf{u}}{Dt} = -\nabla(p/\rho) + \nu\nabla^2\mathbf{u}} \tag{3.4}$$

The boundary condition on $\mathbf{u}$ corresponding to (3.4) is that $\mathbf{u} = 0$ on any stationary, solid surface, i.e. the fluid 'sticks' to any solid surface. This is the 'no-slip' condition.

The expression $D(\cdot)/Dt$ represents the convective derivative. It is the rate of change of a quantity associated with a given element of fluid. This should not be confused with $\partial(\cdot)/\partial t$, which is, of course, the rate of change of a quantity at a fixed point in space. For example, DT/Dt is the rate of change of temperature of a fluid lump as it moves around, whereas $\partial T/\partial t$ is the rate of change of temperature at a fixed point (through which a succession of fluid particles will move). It follows that $D\mathbf{u}/Dt$ is the acceleration of a fluid element, which is why it appears on the left of (3.4).

An expression for $D\mathbf{u}/Dt$ may be obtained as follows. Consider a scalar function of position and time, $f(\mathbf{x},t)$. We have $\delta f = (\partial f/\partial t)\delta t + (\partial f/\partial x)\delta x + \ldots$ If we are interested in the change in f following a fluid particle, then $\delta x = u_x\delta t$ etc. and so

$$\frac{D}{Dt}(f) = \frac{\partial}{\partial t}(f) + (\mathbf{u}\cdot\nabla)(f) = \frac{\partial}{\partial t}(f) + u_x\frac{\partial}{\partial x}(f) + \ldots$$

The same expression may be applied to each of the components of the vector field, say $\mathbf{a}$. We write this symbolically as

$$\frac{D}{Dt}(\mathbf{a}) = \frac{\partial}{\partial t}(\mathbf{a}) + (\mathbf{u}\cdot\nabla)(\mathbf{a})$$

which represents three scalar equations, each of the form given above. We now set $\mathbf{a} = \mathbf{u}$, which allows us to rewrite (3.4) in the form

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla(p/\rho) + \nu\nabla^2\mathbf{u} \tag{3.5}$$

Note that, in steady flows (i.e. flows in which $\partial\mathbf{u}/\partial t = 0$), the streamlines represent particle trajectories and the acceleration of a fluid element is $\mathbf{u}\cdot\nabla\mathbf{u}$. The physical origin of this expression becomes clearer when we rewrite $(\mathbf{u} \cdot \nabla)\mathbf{u}$ in terms of curvilinear coordinates attached to a streamline. As noted earlier,

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = V\frac{\partial V}{\partial s}\hat{\mathbf{e}}_t - \frac{V^2}{R}\hat{\mathbf{e}}_n \tag{3.6}$$

(see Chapter 3, Section 1.1). Here $V = |\mathbf{u}|$, $\hat{\mathbf{e}}_t$ and $\hat{\mathbf{e}}_n$ are unit vectors in the tangential and principle normal directions, s is a streamwise coordinate, and R is the local radius of curvature of the streamline. The first expression on the right is the rate of change of speed, $\frac{DV}{Dt}$, while the second is the centripetal acceleration, which is directed toward the centre of curvature of the streamline and is associated with the change in direction of the velocity of a particle.

3.2 Vorticity, Angular Momentum and the Biot–Savart Law

So far we have concentrated on the velocity field, $\mathbf{u}$. However, in common with many other branches of fluid mechanics, in MHD it is often more fruitful to work with the vorticity field defined by

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \tag{3.7}$$

The reason is two-fold. First, the rules governing the evolution of $\boldsymbol{\omega}$ are somewhat simpler than those governing $\mathbf{u}$. For example, pressure gradients appear as a source of linear momentum in (3.5), yet the pressure itself depends on the instantaneous (global) distribution of $\mathbf{u}$. By focusing on vorticity, on the other hand, we may dispense with the pressure field entirely. (The reasons for this will become evident shortly.) The second reason for studying vorticity is that many flows are characterised by localised regions of intense rotation (i.e. vorticity). Smoke rings, dust whirls in the street, trailing vortices on aircraft wings, whirlpools, tidal vortices, tornadoes, hurricanes and the great red spot of Jupiter represent just a few examples!

Let us start by trying to endow $\boldsymbol{\omega}$ with some physical meaning. Consider a small element of fluid in a two-dimensional flow $\mathbf{u}(x, y) = (u_x, u_y, 0)$, $\boldsymbol{\omega} = (0, 0, \omega_z)$. Suppose that, at some instant, the element is circular (a disk) with radius r. Let $\mathbf{u}_0$ be the linear velocity of the centre of the element and Ω be its mean angular velocity, defined as the average rate of rotation of two mutually perpendicular material lines embedded in the element. From Stoke's theorem, or else from the definition of the curl as a line integral per unit area, we have

$$\omega_z \pi r^2 = \int (\nabla \times \mathbf{u}) \cdot d\mathbf{S} = \oint \mathbf{u} \cdot d\mathbf{l} \tag{3.8}$$

We might anticipate that the line integral on the right has a value of $(\Omega r)2\pi r$. If this were the case, then

$$\omega_z = 2\Omega \tag{3.9}$$

In fact exact analysis confirms that this is so: the anti-clockwise rotation rate of a short line element, dx, orientated parallel to the x-axis is $\partial u_y/\partial x$, while the rotation rate of a line element, dy, parallel to the y-axis is $-\partial u_x/\partial y$, giving $\Omega = (\partial u_y/\partial x - \partial u_x/\partial y)/2 = \omega_z/2$ (Figure 3.9). This confirms equation (3.9). It appears, therefore, that ω_z is twice the angular velocity of the fluid element. This result extends to three dimensions. The vorticity at a particular location is twice the average angular velocity of a blob of fluid passing through that point. In short, $\boldsymbol{\omega}$ is a measure of the local rotation, or spin, of a fluid element.

It should be emphasised, however, that $\boldsymbol{\omega}$ has nothing at all to do with the global rotation of a fluid. Rectilinear flows may possess vorticity, while flows with circular streamlines need not. Consider, for example, the rectilinear shear flow $\mathbf{u}(y) = (\gamma y, 0, 0)$, $\gamma =$ constant. The streamlines are straight and parallel yet the fluid elements rotate at a rate $\omega/2 = -\gamma/2$. This is because vertical line elements, dy, move faster at the top of the element than at the bottom, so they continually rotate towards the horizontal.

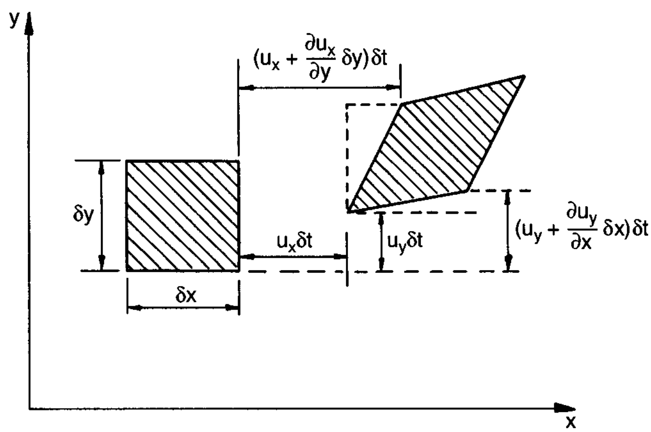


Figure 3.9 Rotation of a fluid element.

Conversely, we can have global rotation of a flow without local rotation of the fluid elements. One example is the so-called free vortex $\mathbf{u}(r) = (0, k/r, 0)$ in (r, θ, z) coordinates. Here k is a constant. It is readily confirmed that $\boldsymbol{\omega} = 0$ in such a vortex.

So far we have been concerned only with kinematics. We now introduce some dynamics. Since we are interested in rotation, it is natural to focus on angular momentum rather than linear momentum. Consider the angular momentum, $\mathbf{H}$, of a small material element that is *instantaneously* spherical. Then

$$\mathbf{H} = \frac{1}{2} I \boldsymbol{\omega}$$

where I is the moment of inertia of the blob. This angular momentum will change at a rate determined by the tangential surface stresses alone. The pressure has no influence on $\mathbf{H}$ at the instant at which the element is spherical since the pressure forces all point radially inward. Therefore, at one particular instant in time, we have

$$\frac{D\mathbf{H}}{Dt} = \nu \mathbf{T}$$

where $\nu\mathbf{T}$ denotes the viscous torque acting on the sphere. Now the convective derivative satisfies the usual rules of differentiation and so we have

$$I \frac{D\boldsymbol{\omega}}{Dt} = -\boldsymbol{\omega} \frac{DI}{Dt} + 2\nu \mathbf{T} \tag{3.10}$$

Evidently, the terms on the right arise from the change in the moment of inertia of a fluid element and the viscous torque, respectively. In cases where viscous stresses are negligible (i.e. outside boundary layers) this simplifies to

$$\frac{D}{Dt}(I\boldsymbol{\omega}) = 0 \tag{3.11}$$

Now (3.10) and (3.11) are not very useful (or even very meaningful) as they stand, since they apply only at the initial instant during which the fluid element is spherical. However, they suggest several results, all of which we shall confirm by rigorous arguments in the next section. First, there is no reference to pressure in (3.10) and (3.11), so that we might anticipate that $\boldsymbol{\omega}$ evolves independently of p. Second, if $\boldsymbol{\omega}$ is initially zero, and the flow is inviscid, then $\boldsymbol{\omega}$ should remain zero in each fluid particle as it is swept around the flow field. This is the basis of potential flow theory in which we set $\boldsymbol{\omega} = 0$ in the upstream fluid, and so we can assume

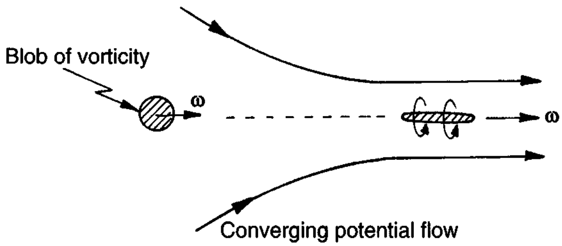


Figure 3.10 Stretching of fluid elements can intensify the vorticity.

that $\boldsymbol{\omega}$ is zero at all points. Third, if I decreases in a fluid element (and $\nu = 0$), then the vorticity of that element should increase. For example, consider a blob of vorticity embedded in an otherwise potential flow field consisting of converging streamlines, as shown in Figure 3.10.

An initially spherical element will be stretched into an ellipsoid by the converging flow. The moment of inertia of the element about an axis parallel to $\boldsymbol{\omega}$ decreases, and consequently $\boldsymbol{\omega}$ must rise to conserve $\mathbf{H}$. It is possible, therefore, to intensify vorticity by stretching fluid blobs. Intense rotation can result from this process, the familiar bath-tub vortex being just one example. We shall see that something very similar happens to magnetic fields. They, too, can be intensified by stretching.

Finally we note that there is an analogy between the differential form of Ampère's law, $\nabla \times \mathbf{B} = \mu\mathbf{J}$, and the definition of vorticity, $\nabla \times \mathbf{u} = \boldsymbol{\omega}$. We can therefore hijack the Biot–Savart law from electromagnetic theory to invert the relationship $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. That is to say, in infinite domains,

$$\mathbf{u} = \frac{1}{4\pi}\int \frac{\boldsymbol{\omega}(\mathbf{x}') \times \mathbf{r}}{r^3} d^3\mathbf{x}', \qquad \mathbf{r} = \mathbf{x} - \mathbf{x}' \tag{3.12}$$

Also, note that, like $\mathbf{u}$ and $\mathbf{B}$, the vorticity field is solenoidal, $\nabla \cdot \boldsymbol{\omega} = 0$, since it is the curl of another vector. Consequently, we may invoke the idea of vortex tubes, which are analogous to magnetic flux tubes or streamtubes.

3.3 Advection and Diffusion of Vorticity

3.3.1 The vorticity equation

We now formally derive the laws governing the evolution of vorticity. We start by writing (3.5) in the form

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times \boldsymbol{\omega} - \nabla(P/\rho + u^2/2) + \nu \nabla^2 \mathbf{u} \tag{3.13}$$

which follows from the identity

$$\nabla(u^2/2) = (\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{u} \times \nabla \times \mathbf{u} = (\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{u} \times \boldsymbol{\omega}$$

Note, in passing, that steady, inviscid flows have the property that $\mathbf{u} \cdot \nabla(P/\rho + u^2/2) = 0$, so that $C = P/\rho + u^2/2$ is constant along a streamline. This is Bernoulli's theorem, C being Bernoulli's function.

We now take the curl of (3.13), noting that the gradient term disappears:

$$\boxed{\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times [\mathbf{u} \times \boldsymbol{\omega}] + \nu \nabla^2 \boldsymbol{\omega}} \tag{3.14}$$

Compare this with (2.24). It appears that $\boldsymbol{\omega}$ and $\mathbf{B}$ obey precisely the same evolution equation! We shall exploit this analogy repeatedly in subsequent chapters. Now since $\mathbf{u}$ and $\boldsymbol{\omega}$ are both solenoidal, we have the vector relationship

$$\nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\boldsymbol{\omega}$$

and so (3.14) may be rewritten as

$$\boxed{\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}} \tag{3.15}$$

Compare this with our angular momentum equation for a blob which is instantaneously spherical:

$$I\frac{D\boldsymbol{\omega}}{Dt} = -\boldsymbol{\omega}\frac{DI}{Dt} + 2\nu \mathbf{T}$$

We might anticipate that the terms on the right of (3.15) represent: (a) the change in the moment of inertia of a fluid element due to stretching of that element; (b) the viscous torque on the element. In other words, the rate of rotation of a fluid blob may increase or decrease due to changes in its moment of inertia, or change because it is spun up or slowed down by viscous stresses.

3.3.2 Advection and diffusion of vorticity: temperature as a prototype

There is another way of looking at (3.15). It may be interpreted as an *advection–diffusion equation* for vorticity. The idea of an advection–diffusion equation is so fundamental to MHD that it is worth dwelling on its significance. Perhaps this is most readily understood in the context of two-dimensional flows, in which $\mathbf{u}(x, y) = (u_x, u_y, 0)$ and $\boldsymbol{\omega}(x, y) = (0, 0, \omega_z)$. The first term on the right of (3.15) now vanishes to yield

$$\frac{D\omega_z}{Dt} = \nu\nabla^2\omega_z \tag{3.16}$$

Compare this with the equation governing the temperature, T, in a fluid,

$$\frac{DT}{Dt} = \alpha\nabla^2 T \tag{3.17}$$

where α is the thermal diffusivity. This is the advection–diffusion equation for heat. In some ways (3.17) represents the prototype advection–diffusion equation and we shall take a moment to review its properties. When $\mathbf{u}$ is zero, we have, in effect, a solid: the temperature field evolves according to

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T$$

Heat soaks through material purely by virtue of thermal diffusion (conduction). At the other extreme, if $\mathbf{u}$ is non-zero but the fluid is thermally insulating ($\alpha = 0$), we have

$$\frac{DT}{Dt} = 0$$

As each fluid lump moves around it conserves its heat, and hence temperature. This is referred to as the advection of heat, i.e. the transport of heat by virtue of material movement. In general, though, we have both advection and diffusion of heat. To illustrate the combined effect of these processes, consider the unsteady, two-dimensional distribution of temperature in a uniform cross flow, $(u_x, 0, 0)$. From (3.17) we have

$$\frac{\partial T}{\partial t} + u_x\frac{\partial T}{\partial x} = \alpha\nabla^2 T$$

Suppose that heat is injected into the fluid from a hot wire. When the velocity is low and the conductivity high the isotherms around the wire will be almost circular. When the conductivity is low, however, each fluid element will tend to conserve its heat as it moves. The isotherms will then become elongated, as shown in Figure 3.11.

The relative size of the advection to the diffusion term is given by the Peclet number $P = ul/\alpha$. (Here l is a characteristic length scale.) If the Peclet number is small, then the transfer of heat is diffusion-dominated. When it is large, advection dominates.

Now consider the case of a wire which is being pulsed with electric current to produce a sequence of hot fluid packets. These are swept downstream and grow by diffusion. In Figure 3.12, heat is restricted to the dotted volumes of fluid. Outside these volumes $T = 0$ (or equal to some reference temperature). Note that advection and diffusion represent processes in which heat is redistributed. However, heat cannot be created or destroyed by advection or diffusion. That is, the total amount of heat is conserved. This is most easily seen by integrating (3.17) over a fixed volume in space and then using Gauss's theorem:

$$\frac{d}{dt}\int T dV + \oint (\mathbf{u}T) \cdot d\mathbf{S} = \alpha \oint \nabla T \cdot d\mathbf{S}$$

Now heat per unit mass is directly proportional to T, and so this states that the net rate of change of heat within a fixed volume decreases if heat is advected across the bounding surface but increases if heat is conducted (diffuses) into the volume from the surrounding fluid. We now apply this equation to a volume which encloses one of the dotted volumes shown in Figure 3.12 (where T is zero at the boundary). We obtain

$$\frac{d}{dt}\int T dV = 0$$

Heat is conserved within each of the dotted volumes as it is swept downstream.

From (3.16) we see that the vorticity in a two-dimensional flow is advected and diffused just like heat. The analogue of the diffusion coefficient is ν and the Reynolds number, ul/ν, now plays the rôle of the Peclet

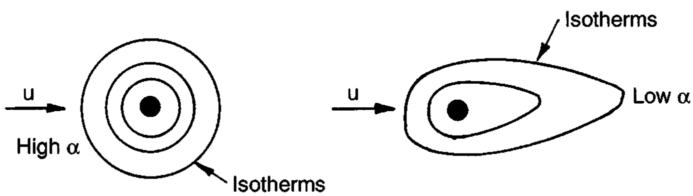


Figure 3.11 Advection and diffusion of heat from a hot wire.

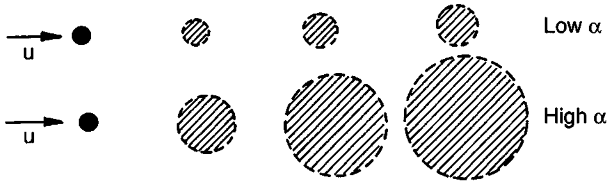


Figure 3.12 Advection and diffusion of heat from a pulsed wire.

number. In other words, vorticity is advected by **u** and diffused by the viscous stresses. Moreover, just like heat, vorticity cannot be created or destroyed within the interior of the flow. The net vorticity within a volume V can change only if vorticity is advected in or out of the volume, or else diffused across the boundary. In the absence of surface effects, global vorticity is conserved. A simple example of this (analogous to the blobs of heat above) are the vortices in the Karman vortex street behind a cylinder (Figure 3.13). The vortices are advected by the velocity and spread by diffusion, but the total vorticity within each eddy remains constant as it moves downstream.

A simple illustration of the diffusion of vorticity is given by the following example (Figure 3.14). Suppose that a plate of infinite length is immersed in a still fluid. At time $t = 0$ it suddenly acquires a constant velocity V in its own plane. We want to find the subsequent motion, $\mathbf{u}(y,t)$. Now, by the no-slip condition, the fluid adjacent to the plate sticks to it, and so moving the plate creates a gradient in velocity, which gives rise to vorticity. The plate becomes a source of vorticity, which subsequently diffuses into the fluid. Since u_x is not a function of x, the continuity equation gives $\partial u_y/\partial y = 0$, and since u_y is zero at the plate, $u_y = 0$ everywhere. The vorticity equation (3.16) then becomes

$$\frac{\partial \omega_z}{\partial t} = \nu \frac{\partial^2 \omega_z}{\partial y^2}, \qquad \omega_z = -\frac{\partial u_x}{\partial y} \tag{3.18}$$

This is identical to the equation describing the diffusion of heat from an infinite, heated plate whose surface temperature is suddenly raised from

Figure 3.13 Karman vortices behind a cylinder.

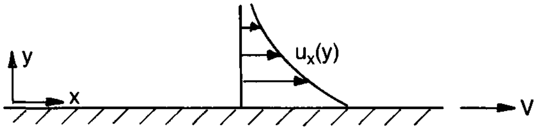


Figure 3.14 Diffusion of vorticity from a plate.

$T = 0$ to $T = T_0$. This sort of diffusion equation can be solved by looking for a similarity solution. To illustrate this, consider first the analogous thermal problem.

We know that heat diffuses a distance $(2\alpha t)^{1/2}$ away from the plate in a time t, and that the temperature distribution has the form

$$T = T_0 f(y/l), \qquad l = (2\alpha t)^{1/2}$$

The quantity l is called the diffusion length. The equation above states that the dimensionless temperature profile, T/T_0, depends only on the dimensionless coordinate y/l. When y is scaled by l in this way, the temperature distribution appears always to have a universal form.

The analogy to heat suggests that we look for a solution of our vorticity equation of the form

$$\omega = \frac{V}{l} f(y/l), \qquad l = (2\nu t)^{1/2}$$

Substituting this into (3.18) reduces our partial differential equation to

$$f'(\eta) + \eta f(\eta) = 0, \qquad \eta = y/l$$

This may be integrated to give

$$\omega_z = \frac{C_1 V}{l} \exp\left[-\eta^2/2\right]$$

To fix the constant of integration, C_1, we need to integrate to find u_x on the surface of the plate. From this we find $C_1^2 = 2/\pi$, and so the vorticity distribution is

$$\omega_z = \frac{V}{(\pi \nu t)^{1/2}} \exp\left[-y^2/(4\nu t)\right]$$

This may now be integrated once more to give the velocity field. However, the details of this solution are perhaps less important than the overall picture. That is, vorticity is created at the surface of the plate by the shear stresses acting on that surface. This vorticity then diffuses into the interior of the fluid in exactly the same way as heat diffuses in from a heated surface. There is no vorticity generation within the interior of the flow. The vorticity is merely redistributed (spread) by virtue of diffusion.

3.3.3 Vortex line stretching

Let us now return to our general vorticity equation (3.15)

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}$$

In three-dimensional flows the first term on the right is non-zero, and it is this additional effect which distinguishes three-dimensional flows from two-dimensional ones. It appears that the vorticity no longer behaves like a temperature field. We have already suggested, by comparing this with our angular momentum equation, that $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$ represents intensification of vorticity by the stretching fluid elements. We shall now confirm that this is indeed the case.

Consider, by way of example, an axisymmetric flow consisting of converging streamlines (in the r–z plane) as well as a swirling component of velocity, u_θ. By writing $\nabla \times \mathbf{u}$ in terms of cylindrical coordinates, we find that, near the axis, the axial component of vorticity is

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)$$

Now consider the axial component of the vorticity equation (3.15) applied near $r = 0$. In addition to the usual advection and diffusion terms we have the expression

$$(\boldsymbol{\omega} \cdot \nabla)\mathbf{u} \sim \omega_z \frac{\partial u_z}{\partial z}$$

This appears on the right of (3.15) and so acts like a source of axial vorticity. In particular, the vorticity, ω_z, intensifies if $\partial u_z / \partial z$ is positive, i.e. the streamlines converge. This is because fluid elements are stretched and elongated on the axis, as shown in Figure 3.15. This leads to a reduction in the axial moment of inertia of the element and so, by conservation of angular momentum, to an increase in ω_z.

More generally, consider a thin tube of vorticity, as shown in Figure 3.16. Let $u_{//}$ be the component of velocity parallel to the vortex tube and s be a coordinate measured along the tube. Then

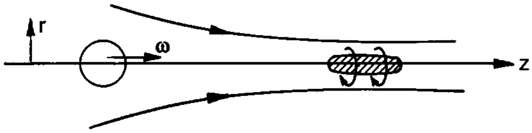


Figure 3.15 Stretching of a material element.

$$|\omega| \frac{du_{//}}{ds} = (\omega \cdot \nabla) u_{//}$$

Now the vortex line is being stretched if the velocity $u_{//}$ at point B is greater than $u_{//}$ at A. That is, the length of the material element AB increases if $du_{//}/ds > 0$. Thus the term $(\omega \cdot \nabla)\mathbf{u}$ represents stretching of the vortex lines. This leads to an intensification of vorticity through conservation of angular momentum, confirming our interpretation of $(\omega \cdot \nabla)\mathbf{u}$ in (3.15).

3.4 Kelvin's Theorem, Helmholtz's Laws and Helicity

We now do something dangerous. We set aside viscosity so that we can discuss the great advances made in inviscid fluid mechanics by the nineteenth century physicists and mathematicians. This is dangerous because, as we shall see, a fluid with no viscosity behaves very differently to a fluid with a small but finite viscosity. To emphasise this, John von Neumann refers to inviscid fluid mechanics as the study of *dry water*!

3.4.1 Kelvin's Theorem and Helmholtz's Laws

We now summarise the classical theorems of inviscid vortex dynamics. We start with the idea of a vortex tube. This is an aggregate of vortex lines, rather like a magnetic flux tube is composed of magnetic field lines. Since $\nabla \cdot \omega = 0$ we have

$$\oint \omega \cdot d\mathbf{S} = 0$$

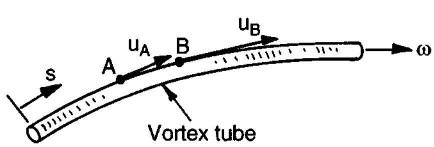


Figure 3.16 Stretching of a tube of vorticity.

It follows that the flux of vorticity, $\Phi = \int \boldsymbol{\omega} \cdot d\mathbf{S}$, is constant along the length of a vortex tube since no flux crosses the side of the tube. A closely related quantity is the circulation, Γ. This is defined as the closed line integral of $\mathbf{u}$:

$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{l} \tag{3.19}$$

If the path C is taken as lying on the surface of a vortex tube, and passing once around it (Figure 3.17), Stoke's theorem tells us that $\Gamma = \Phi$. Γ is sometimes called the strength of the vortex tube.

Kelvin's (1869) theorem is couched in terms of circulation. It says that, if $C_m(t)$ is a closed curve that always consists of the same fluid particles (a material curve), then the circulation

$$\Gamma = \oint_{C_m(t)} \mathbf{u} \cdot d\mathbf{l}$$

is independent of time. Note that this theorem does not hold true if C is fixed in space; C_m must be a material curve moving with the fluid. Nor does it apply if the fluid is subject to a rotational body force, $\mathbf{F}$, such as $\mathbf{J} \times \mathbf{B}$, or for that matter if viscous forces are significant at any point on C_m.

The proof of Kelvin's theorem follows directly from the kinematic equation (2.25).

$$\frac{d}{dt}\int_{S_m} \mathbf{G} \cdot d\mathbf{S} = \int_{S_m} \left[\frac{\partial \mathbf{G}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{G})\right] \cdot d\mathbf{S}$$

If we take $\mathbf{G} = \boldsymbol{\omega}$, invoke the vorticity equation (3.14), and use Stokes' theorem, we have

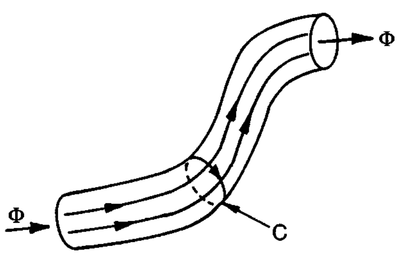


Figure 3.17 A vortex tube.

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\int_{S_m} \boldsymbol{\omega} \cdot d\mathbf{S} = \nu \int_{S_m} \nabla^2 \boldsymbol{\omega} \cdot d\mathbf{S} = 0$$

and Kelvin's theorem is proved.

Helmholtz's laws are closely related to Kelvin's theorem. They were published in 1858 and, like Kelvin's theorem, apply only to inviscid flows. They state that:

(i) the fluid elements that lie on a vortex line at some initial instant continue to lie on that vortex line for all time, i.e. the vortex lines are frozen into the fluid;

(ii) the flux of vorticity

$$\Phi = \int \boldsymbol{\omega} \cdot d\mathbf{S}$$

is the same at all cross sections of a vortex tube and is independent of time.

Consider Helmholtz's first law. In two-dimensional flows it is a trivial consequence of $D\omega_z/Dt = 0$. In three-dimensions, for which

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} \tag{3.20}$$

more work is required. First we need the following result. Let $d\mathbf{l}$ be a short line drawn in the fluid at some instant, and suppose $d\mathbf{l}$ subsequently moves with the fluid. Then the rate of change of $d\mathbf{l}$ is $\mathbf{u}(\mathbf{x} + d\mathbf{l}) - \mathbf{u}(\mathbf{x})$, and so

$$\frac{D}{Dt}(d\mathbf{l}) = \mathbf{u}(\mathbf{x} + d\mathbf{l}) - \mathbf{u}(\mathbf{x})$$

where $\mathbf{x}$ and $\mathbf{x} + d\mathbf{l}$ are the position vectors at the two ends of $d\mathbf{l}$. It follows that

$$\frac{D}{Dt}(d\mathbf{l}) = (d\mathbf{l} \cdot \nabla)\mathbf{u} \tag{3.21}$$

Compare this with (3.20). Evidently, $\boldsymbol{\omega}$ and $d\mathbf{l}$ obey the same equation. Now suppose that at $t = 0$ we have $\boldsymbol{\omega} = \lambda d\mathbf{l}$ then from (3.20) and (3.21) we have $D\lambda/Dt = 0$ at $t = 0$ and so $\boldsymbol{\omega} = \lambda d\mathbf{l}$ for all subsequent times. That is to say, $\boldsymbol{\omega}$ and $d\mathbf{l}$ evolve in identical ways under the influence of $\mathbf{u}$ and so the vortex lines are frozen into the fluid.

Helmholtz's second law is now a trivial consequence of Kelvin's theorem and of $\nabla \cdot \boldsymbol{\omega} = 0$. The fact that the vorticity flux, Φ, is constant along a flux tube follows from the solenoidal nature of $\boldsymbol{\omega}$, and the temporal

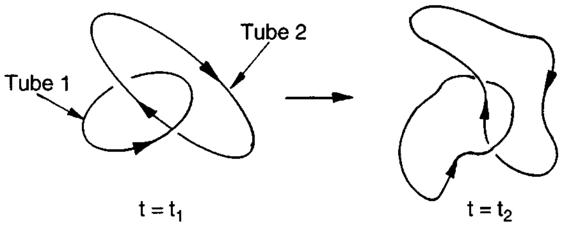


Figure 3.18 Interlinked vortex tubes preserve their topology as they are swept around.

invariance of Φ comes from the fact that a flux tube moves with the fluid and so, from Kelvin's theorem, $\Gamma = \Phi =$ constant. (Here the curve C for Γ lies on the surface of the flux tube.)

Helmholtz's first law, which states that vortex tubes are frozen into an inviscid fluid, has profound consequences for inviscid vortex dynamics. For example, if there exist two interlinked vortex tubes, as shown in Figure 3.18, then as those tubes are swept around they remain linked in the same manner, and the strength of each tube remains constant. Thus the tubes appear to be indestructible and their relative topology is preserved forever. This state of permanence so impressed Kelvin that, in 1867, he developed an atomic theory of matter based on vortices. This rather bizarre theory of the vortex atom has not stood the test of time. However, when, in 1903, the Wright brothers first mastered powered flight, an entirely new incentive for studying vortex dynamics was born. Kelvin's theorem, in particular, plays a central rôle in aerodynamics[2].

3.4.2 Helicity

The conservation of vortex-line topology implied by Helmholtz's laws is captured by an integral invariant called the helicity. This is defined as

$$h = \int_{V_\omega} \mathbf{u} \cdot \boldsymbol{\omega} dV \tag{3.22}$$

where V_ω is a material volume (a volume composed always of the same fluid elements) for which $\boldsymbol{\omega} \cdot d\mathbf{S} = 0$. For example, the surface of V_ω may

[2] Ironically, Kelvin was not a great believer in powered flight. In 1890, on being invited to join the British Aeronautical Society, he is reputed to have said '*I have not the smallest molecule of faith in aerial navigation other than ballooning . . . so you will understand that I would not care to be a member of the society.*'

be composed of vortex lines. A blob of fluid has helicity if its velocity and vorticity are (at least partially) aligned, as indicated in Figure 3.19.

We may confirm that h is an invariant as follows. First we have

$$\frac{D}{Dt}(\mathbf{u}\cdot\boldsymbol{\omega}) = \frac{D\mathbf{u}}{Dt}\cdot\boldsymbol{\omega} + \frac{D\boldsymbol{\omega}}{Dt}\cdot\mathbf{u} = -\nabla(p/\rho)\cdot\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\nabla\mathbf{u})\cdot\mathbf{u}$$

Since $\boldsymbol{\omega}$ is solenoidal, this may be written as

$$\frac{D}{Dt}(\mathbf{u}\cdot\boldsymbol{\omega}) = \nabla\cdot\left[(u^2/2 - p/\rho)\boldsymbol{\omega}\right]$$

Now consider an element of fluid of volume δV. The fluid is incompressible and so $D(\delta V)Dt = 0$. It follows that

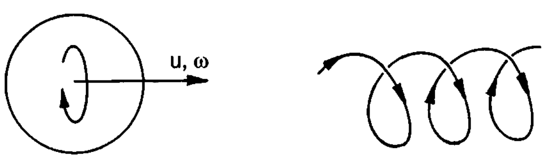
$$\frac{D}{Dt}[(\mathbf{u}\cdot\boldsymbol{\omega})\delta V] = \nabla\cdot\left[(u^2/2 - p/\rho)\boldsymbol{\omega}\right]\delta V$$

from which

$$\frac{d}{dt}\int_{V_\omega}(\mathbf{u}\cdot\boldsymbol{\omega})dV = \oint_{S_\omega}\left[(u^2/2 - p/\rho)\boldsymbol{\omega}\right]\cdot d\mathbf{S} = 0$$

Thus the helicity, h, is indeed conserved. The connection to Helmholtz's laws and vortex-line topology may be established using the following simple example. Suppose that $\boldsymbol{\omega}$ is confined to two thin interlinked vortex tubes, as shown in Figure 3.18, and that V_ω is taken as all space. Then h has two contributions, one from vortex tube 1, which has volume V_1 and flux Φ_1, and another from vortex tube 2. Let these be denoted by h_1 and h_2. Then

$$h_1 = \int_{V_1}(\mathbf{u}\cdot\boldsymbol{\omega})dV = \oint_{C_1}\mathbf{u}\cdot(\Phi_1 d\mathbf{l}) = \Phi_1\oint_{C_1}\mathbf{u}\cdot d\mathbf{l}$$



(a) Alignment of velocity and vorticity gives helicity

(b) Particle trajectories

Figure 3.19 A blob of fluid with helicity.

since $\omega dV = \Phi_1 d\mathbf{l}$ (Figure 3.20). Here C_1 is the closed curve representing tube 1. However, $\oint_{C_1} \mathbf{u} \cdot d\mathbf{l}$ is, from Stoke's theorem, equal to Φ_2. A similar calculation may be made for h_2, and we find that

$$h = h_1 + h_2 = \Phi_1 \oint_{C_1} \mathbf{u} \cdot d\mathbf{l} + \Phi_2 \oint_{C_2} \mathbf{u} \cdot d\mathbf{l} = 2\Phi_1\Phi_2$$

Note that if the sense of direction of $\boldsymbol{\omega}$ in either tube were reversed, h would change sign. Moreover, if the tubes were not linked, then h would be zero. Thus the invariance of h in this simple example stems directly from the conservation of the vortex line topology. More elaborate examples, illustrating the same fact, are readily constructed.

Finally, we note in passing that minimising kinetic energy subject to conservation of global helicity leads to a Beltrami field satisfying $\nabla \times \mathbf{u} = \alpha \mathbf{u}$, α being constant. We shall not pause to prove this result, but we shall make reference to it later.

This ends our discussion of inviscid vortex dynamics. From a mathematical perspective, inviscid fluid mechanics is attractive. The rules of the game are simple and straightforward. Unfortunately, the conclusions are often at odds with reality and so great care must be exercised in using such a theory. The dangers are nicely summarised by Rayleigh:

> The general equations of (inviscid) fluid motion were laid down in quite early days by Euler and Lagrange ... (unfortunately) some of the general propositions so arrived at were found to be in flagrant contradiction with observations, even in cases where at first sight it would not seem that viscosity was likely to be important. Thus a solid body, submerged to a sufficient depth, should experience no resistance to its motion through water. On this principle the screw of a submerged boat would be useless, but, on the other hand, its services would not be needed. It is little wonder that practical men should declare

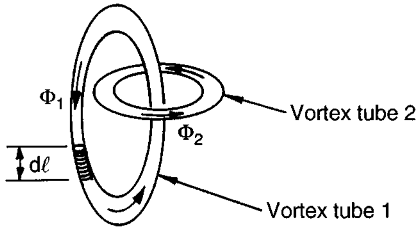


Figure 3.20 The physical interpretation of helicity in terms of flux.

that theoretical hydrodynamics has nothing at all to do with real fluids.

Rayleigh (1914)

Rayleigh was, of course, referring to d'Alembert's paradox. With this warning, we now return to 'real' fluid dynamics.

There are three topics in particular which will be important in our exploration of MHD. The first is the Prandtl–Batchelor theory which says, in effect, that a slow cross-stream diffusion of vorticity can be important even at high Re. The second is the concept of Reynolds' stresses in turbulent motion, and the third is the phenomenon of Ekman pumping, which is a weak secondary flow induced by differential rotation between a viscous fluid and an adjacent solid surface. The Prandtl–Batchelor theorem is important because it has its analogue in MHD (called flux expulsion), Reynolds' stresses and turbulent motion are important because virtually all 'real' MHD is turbulent MHD (Re is invariably very large), and Ekman pumping is important because it dominates the process of magnetic stirring and possibly contributes to the maintenance of the geodynamo.

3.5 The Prandtl–Batchelor Theorem

We now return to viscous flows. We start with the Prandtl–Batchelor theorem which, as we have said, has its analogue in MHD. This theorem is one of the more beautiful results in the theory of two-dimensional viscous flows. It has far-reaching consequences for internal flows. In effect, it states that a laminar motion with high Reynolds number and closed streamlines must have uniform vorticity.

Consider a two-dimensional flow which is steady and has a high Reynolds number. Suppose also that the streamlines are closed. We introduce the streamfunction ψ. The velocity and vorticity are, in terms of ψ,

$$\mathbf{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right), \qquad \omega = -\nabla^2 \psi$$

In two-dimensions, the steady vorticity equation becomes

$$(\mathbf{u} \cdot \nabla)\omega = \nu \nabla^2 \omega$$

If we now take the limit $\nu \to 0$ we obtain

$$(\mathbf{u} \cdot \nabla)\omega = 0$$

The vorticity is therefore constant along the streamlines and so is a function only of ψ, $\omega = \omega(\psi)$. This is all the information which we may obtain from the inviscid equation of motion. Unfortunately, the problem appears to be underdetermined. There are an infinite number of solutions to the equation

$$\nabla^2\psi = -\omega(\psi), \qquad \psi = 0 \text{ on the boundary}$$

each solution corresponding to a different distribution of ω. (Note that $\mathbf{u}\cdot d\mathbf{S} = 0$ at the boundary requires ψ = constant at the boundary and it is usual to take that constant as zero.) So what distribution of ω does nature select? We appear to be missing some information. In cases where the streamlines are open the problem is readily resolved. We must specify the upstream distribution of ω and then track it downstream (Figure 3.21). However, when the streamlines are closed, such as in the cavity in Figure 3.21(b), we have no 'upstream point' at which we can specify $\omega(\psi)$.

Let us go back to the steady, viscous vorticity equation. If we integrate this over the area bounded by some closed streamline then we find, with the help of Gauss's theorem,

$$\nu\int \nabla^2\omega dV = \nu\oint \nabla\omega\cdot d\mathbf{S} = 0$$

This integral constraint must be satisfied for all finite values of ν, no matter how small ν may be. Now if Re is large, then $\omega = \omega(\psi)$ plus some small correction due to the finite value of ν. Consequently

$$\nabla\omega = \omega'(\psi)\nabla\psi + (\text{small correction})$$

where $\omega'(\psi)$ is proportional to the cross-stream gradient of vorticity. Next we substitute back into our integral equation to give

$$\nu\left\{\omega'(\psi)\oint\nabla\psi\cdot d\mathbf{S} + (\text{small correction})\right\} = 0$$

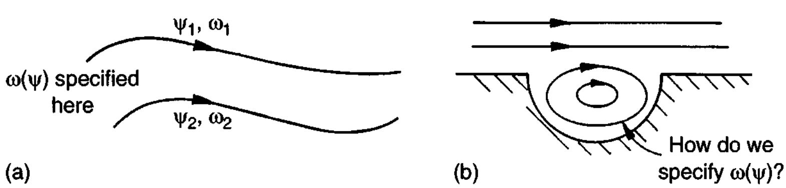


Figure 3.21 The specification of $\omega(\psi)$.

(Note that $\omega'(\psi)$ is constant along a streamline and so may be taken outside the integral.) We now assume ν is very small (Re is large) and throw away the small correction. Invoking Gauss's theorem once again, this time in reverse, we end up with

$$\nu\omega'(\psi)\int \nabla^2\psi dV = -\nu\omega'(\psi)\int \omega dA = 0$$

This expression must be satisfied for all flows with a small but finite value of ν. From Stokes' theorem this can be rewritten in the form

$$\nu\omega'(\psi)\oint_C \mathbf{u}\cdot d\mathbf{l} = 0$$

where C is a streamline. Since ν is finite, the only possibility is that $\omega'(\psi) = 0$. In other words, there is no cross-stream gradient in ω, and so ω is uniform throughout the flow.

We have proved the Prandtl–Batchelor theorem. It states that, for high Reynolds number flows with closed streamlines, the vorticity is uniform throughout the flow, $\omega = \omega_0$, except in the boundary layers (Figure 3.22). (We must exclude the boundary layers since our proof assumed that viscous effects are small and this is clearly not the case in the boundary layers.) In the cavity flow shown below, ω will be constant within the region of closed streamlines (excluding the thin boundary layers). Of all possible vorticity distributions, $\omega(\psi)$, the Prandtl–Batchelor theorem tells us that nature will select the one where vorticity is not only constant along the streamlines, but is also constant *across* the streamlines.

Armed with the Prandtl–Batchelor theorem, it is relatively straightforward to compute flow fields of the type sketched below. One simply solves the equation

$$\nabla^2\psi = -\omega_0, \qquad \psi = 0 \text{ on the boundary}$$

where ω_0 is the (unknown but constant) vorticity in the flow. This yields $\mathbf{u}$ at all points (except in the boundary layer). There is, however, still the

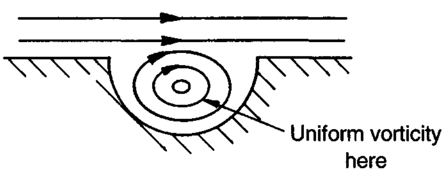


Figure 3.22 Example of the Prandtl–Batchelor theorem.

unknown constant ω_0 to determine. This is usually fixed by solving the boundary layer equations.

Of course, if the Reynolds' number is high enough then the flow will become unsteady and eventually turbulent. The Prandtl–Batchelor theory does not then apply. However, there are many cases where Re is low enough for the motion to be steady and laminar, yet high enough for the Prandtl–Batchelor theorem to work well. One example is the two attached eddies which form in the wake of a cylinder at intermediate Reynolds number. Curiously, even when the flow becomes turbulent, the Prandtl–Batchelor theorem often works surprisingly well when applied to the time-averaged flow. Presumably, this is because the arguments above can be repeated, but with ν now representing an 'eddy viscosity'.

The physical interpretation of the Prandtl–Batchelor theorem is straightforward. Suppose a flow is initiated at, say, $t = 0$. Then, over a short timescale of the order of the eddy turnover-time, the flow adopts a high Reynolds number form: i.e. $\omega = \omega(\psi)$. (Depending on how the flow is initiated, different distributions of $\omega(\psi)$ may appear.) There then begins a slow cross-stream diffusion of vorticity which continues until all the internal gradients in vorticity have been eliminated (except in the boundary layers). This takes a long time, but the flow does not become truly steady until the process is complete.

Examples

1. Starting with the energy equation

$$\frac{DT}{Dt} = \alpha \nabla^2 T$$

show that for laminar, high-Peclet number, closed-streamline flows, the temperature outside the boundary layer is constant. This is the thermal equivalent of the Prandtl–Batchelor theorem. Give a physical interpretation of your result.

2. In two-dimensional MHD flows which have the form $\mathbf{u} = (u_x, u_y, 0)$ and $\mathbf{B} = (B_x, B_y, 0) = \nabla \times (A_z \hat{\mathbf{e}}_z)$, the induction equation (2.24) reduces to

$$\frac{DA_z}{Dt} = \lambda \nabla^2 A_z$$

Where do you think the Prandtl–Batchelor arguments lead here?

3.6 Boundary Layers, Reynolds Stresses and Turbulence Models

3.6.1 Boundary layers

> During the last few years much work has been done in connection with artificial flight. We may hope that before long this may be coordinated and brought into closer relationship with theoretical hydrodynamics. In the meantime one can hardly deny that much of the latter science is out of touch with reality.
>
> *Rayleigh, 1916.*

We have already mentioned boundary layers without really defining what we mean by this term, and so it seems appropriate to review briefly the key aspects of laminar boundary layers. (We leave turbulence to Section 3.6.2.)

The concept of a boundary layer, and of boundary layer separation, was first conceived by the engineer L Prandtl and it revolutionised fluid mechanics. It formed a bridge between the classical 19th century mathematical studies of inviscid fluids and the subject of experimental fluid mechanics, and in doing so it resolved many traditional dilemmas such as d'Alembert's paradox. Prandtl first presented his ideas in 1904 in a short paper crammed with physical insight. Curiously though, it took many years for the full significance of his ideas to be generally appreciated.

Consider a high-Reynolds' number flow over, say, an aerofoil (Figure 3.23). By high Reynolds' number we mean that uL/ν is large where L is a characteristic geometric length scale, say the span of the aerofoil. Since Re is large we might be tempted to solve the inviscid equations of motion,

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla(p/\rho)$$

subject to the inviscid boundary condition $\mathbf{u} \cdot d\mathbf{S} = 0$ on all solid surfaces. This determines the so-called *external problem*.

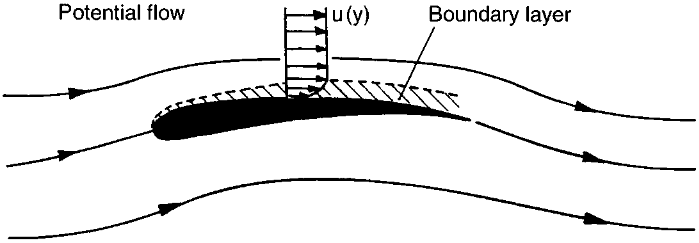


Figure 3.23 Boundary layer near a surface.

Now in reality the fluid satisfies a no-slip boundary condition $\mathbf{u} = 0$ on $d\mathbf{S}$. (We take a frame of reference moving with the aerofoil.) Thus there must exist a region surrounding the aerofoil where the velocity given by the external problem adjusts to zero (Figure 3.23). This region is called the boundary layer, and it is easy to see that such a layer must be thin. The point is that the only mechanical forces available to cause a drop in velocity are viscous shear stresses. Thus the viscous term in the Navier–Stokes equation must be of the same order as the other terms within the boundary layer,

$$\nu\nabla^2\mathbf{u} \sim (\mathbf{u}\cdot\nabla)\mathbf{u}$$

This requires that the transverse length scale, δ, which appears in the Laplacian, is of order

$$\delta \sim (\nu L/u)^{1/2} \sim \mathrm{Re}^{-1/2}$$

which fixes the boundary layer thickness. Since Re is large this implies that

$$\delta \ll L$$

Note that, because the boundary layer is so thin, the pressure within a boundary layer is virtually the same as the pressure immediately outside the layer. (There can be no significant gradient in pressure across a boundary layer since this would imply a significant normal acceleration, which is not possible since the velocity is essentially parallel to the surface.)

Boundary layers occur in other branches of physics; it is not a phenomenon peculiar to velocity fields. In fact, it occurs whenever a small parameter, in this case ν, multiplies a term containing derivatives which are of higher order than the other derivatives appearing in the equation. In the case above, when we throw out $\nu\nabla^2\mathbf{u}$ on the basis that ν is small, our equation drops from second order to first order. There is a corresponding drop in the number of boundary conditions we can meet ($\mathbf{u}\cdot d\mathbf{S} = 0$ rather than $\mathbf{u} = 0$) and so solving the external problem leaves one boundary condition unsatisfied. This is corrected for in a thin transition region (in this case the velocity boundary layer) where the term we had thrown out, i.e. $\nu\nabla^2\mathbf{u}$, is now significant because of the thinness of the transition region. However, we can have other types of boundary layers, such as thermal boundary layers and magnetic boundary layers. In the case of thermal boundary layers the small parameter is α and it multiplies $\nabla^2 T$.

Note that the thickness of a boundary layer is not always $\sim \mathrm{Re}^{-1/2}L$. For example, we shall see later that the force balance within MHD boundary layers is more complicated than that indicated above, and so the estimate $\delta \sim \mathrm{Re}^{-1/2}L$ often needs modifying.

We conclude with one comment. As noted in Section 3.1.1, boundary layers exhibit a phenomenon known as *separation*. That is, when the flow external to the boundary layer decelerates, the pressure gradient causing that deceleration is also imposed on the fluid in the boundary layer. However, this fluid has less kinetic energy than the external flow and so it rapidly comes to a halt and starts to move backward. The fluid within the boundary layer is then ejected into the external flow and a wake is formed, such as the wake at the rear of a cylinder. It is this wake which gives rise to the asymmetric flow over bluff bodies which inviscid theory fails to predict, and which so discredited theoretical hydrodynamics.

3.6.2 Reynolds stresses and turbulence models

We now consider the more elementary aspects of turbulent flow. A more detailed discussion is given in Chapter 7, where we consider the nature of turbulence itself. Here we restrict ourselves to the much simpler problem of characterising the influence of turbulence on the mean flow.

It is an empirical observation that if Re is made large enough (viscosity made small enough) a flow invariably becomes unstable and then turbulent (Figure 3.24). Suppose we have a turbulent flow in which $\boldsymbol{u}$ and p consist of a time-averaged component plus a fluctuating part:

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}', \qquad p = \bar{p} + p'$$

When we time-average the Navier–Stokes equation, new terms arise from the fluctuations in velocity. For example, the x-component of the time-averaged equation of motion is

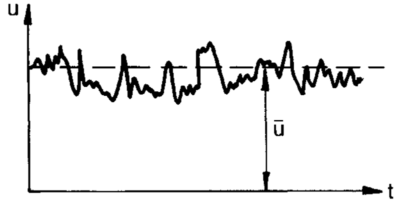


Figure 3.24 Velocity component in a turbulent flow.

$$(\bar{\mathbf{u}} \cdot \nabla)\bar{u}_x = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) + \frac{\partial}{\partial x}\left[2\nu\frac{\partial \bar{u}_x}{\partial x}\right] + \frac{\partial}{\partial y}\left[\nu\left(\frac{\partial \bar{u}_x}{\partial y} + \frac{\partial \bar{u}_y}{\partial x}\right)\right]$$
$$+ \frac{\partial}{\partial z}\left[\nu\left(\frac{\partial \bar{u}_x}{\partial z} + \frac{\partial \bar{u}_z}{\partial x}\right)\right] + \frac{\partial}{\partial x}\left[-\overline{u'_x u'_x}\right] + \frac{\partial}{\partial y}\left[-\overline{u'_x u'_y}\right] + \frac{\partial}{\partial z}\left[-\overline{u'_x u'_z}\right]$$

Here the overbar represents time-averaging. Now the laminar stresses, from Newton's law of viscosity, are given by

$$\sigma_x = 2\rho\nu\frac{\partial u_x}{\partial x}$$
$$\tau_{xy} = \rho\nu\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right]$$
$$\tau_{xz} = \rho\nu\left[\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right]$$

The turbulence seems to have produced additional stresses. These are called Reynolds stresses in honour of Osborne Reynolds' pioneering work on turbulence ($\sim$1883). The stresses are

$$\sigma_x = -\rho\overline{u'_x u'_x}$$
$$\tau_{xy} = -\rho\overline{u'_x u'_y}$$
$$\tau_{xz} = -\rho\overline{u'_x u'_z}$$

We can rewrite the x-component of the time-averaged equation in a more compact way:

$$(\bar{\mathbf{u}} \cdot \nabla)\bar{u}_x = -\frac{\partial}{\partial x}\left[\frac{p}{\rho}\right] + \nu\nabla^2\bar{u}_x + \frac{\partial}{\partial x_i}\left[-\rho\overline{u'_x u'_i}\right]$$

Here the index, i, is summed over x, y and z. Similar expressions may be written for the y and z components. If we wish to make predictions from this equation we need to be able to relate the Reynolds stresses, $-\rho\overline{u'_x u'_i}$, to some quantity which we know about, such as mean velocity gradients of the type $\partial\bar{u}_x/\partial y$. This is the purpose of *turbulence modelling*. Its aim is to recast the time-averaged equations in a form which may be solved, just like the Navier–Stokes equations. In effect, a *turbulence model* provides a means of estimating the Reynolds stresses.

However, it should be emphasised from the outset that 'Reynolds stress modellers' live dangerously. The quest to find a *universal* turbulence model, which may be defended on theoretical grounds, is doomed to failure from the outset. There is no such thing as a universal turbulence model! The best that we can do is construct semi-empirical models, based

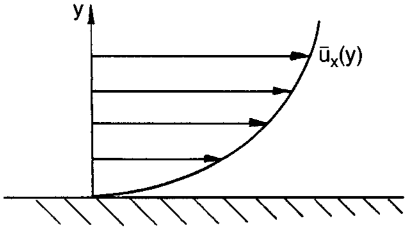


Figure 3.25 Time-averaged velocity in a turbulent flow.

on laboratory tests, and try to apply these models to flows not too different from the laboratory tests on which they were based. For example, Reynolds stress models developed from boundary layer experiments need not work well when applied to rapidly rotating flows.

The reason for this difficulty is the so-called 'closure problem' of turbulence. We can, in principle, derive rigorous equations for $\overline{u_x'u_y'}$ etc. (see Chapter 7). However, this involves quantities of the form $\overline{u_x'u_y'u_z'}$. When an equation for these new quantities is derived, we find that it involves yet more functions such as $\overline{u_x'u_x'u_y'u_z'}$, and so on. There are always more unknowns than equations, and it is impossible to close the set in a rigorous way. This is the price we pay in moving from the instantaneous equations of motion to a statistical (time-averaged) one[3].

This is all bad news since much of fluid mechanics centres around turbulent flows, and quantitative predictions of such flows require a turbulence model. Fortunately, some of the simpler, empirical turbulence models work reasonably well if applied to the appropriate classes of flow.

Historically, the first serious attempt at a theoretical study of turbulent flows was made by Boussinesq around 1877 (6 years before Reynolds' famous pipe experiment). He proposed that the shear-stress strain-rate relationship for time-averaged flows of a one-dimensional nature (Figure 3.25) was of the form

$$\tau_{xy} = (\mu + \mu_t)\frac{\partial \bar{u}_x}{\partial y}$$

(Here μ is the dynamic viscosity, $\mu = \rho\nu$ and should not be confused with the permeability of free space.) Boussinesq termed μ_t an *eddy viscosity*. While μ is a property of the fluid, μ_t will be a property of the *turbulence*. The first attempt to estimate μ_t was due to Prandtl in 1925. He invoked the idea of a *mixing length*, as we shall see shortly.

[3] This is not to say that we cannot make rigorous and useful statements about turbulence. We can! (see Chapter 7.) We cannot, however, produce a rigorous Reynolds stress model.

The idea of an eddy viscosity is not restricted to simple shear flows of the type above, i.e. $\bar{u}_x(y)$. It is common to introduce eddy viscosities into flows of arbitrary complexity. Then,

$$\tau_{xy} = (\mu + \mu_t)\left[\frac{\partial \bar{u}_x}{\partial y} + \frac{\partial \bar{u}_y}{\partial x}\right]$$

$$\tau_{xz} = (\mu + \mu_t)\left[\frac{\partial \bar{u}_x}{\partial z} + \frac{\partial \bar{u}_z}{\partial x}\right]$$

etc. We have, in effect, accounted for the Reynolds stresses by replacing μ in Newton's law of viscosity by $\mu + \mu_t$. Now in virtually all turbulent flows the eddy viscosity is much greater than μ, so the viscous stresses may be dropped, giving

$$\tau_{xy} = -\rho\overline{u'_x u'_y} = \mu_t\left[\frac{\partial \bar{u}_x}{\partial y} + \frac{\partial \bar{u}_y}{\partial x}\right]$$

$$\tau_{xz} = -\rho\overline{u'_x u'_z} = \mu_t\left[\frac{\partial \bar{u}_x}{\partial z} + \frac{\partial \bar{u}_z}{\partial x}\right]$$

We shall refer to these as Boussinesq's equations. The question now is, what is μ_t? Prandtl was struck by the success of the kinetic theory of gases in predicting the properties of ordinary viscosity in which the 'mean free path length' plays a rôle. In fact, the simple kinetic theory of gases leads to the prediction

$$\mu = \frac{1}{3}\rho l V$$

where V is the molecular velocity and l is the mean free path length. Could the same thing be done for the eddy viscosity? In fact, there is an analogy between Newton's law of viscosity and Reynolds stresses. In a laminar flow, layers of fluid which slide over each other experience a mutual shear stress, or drag, because thermally agitated molecules bounce around between the layers exchanging energy and momentum as they do so (Figure 3.26).

For example, a molecule in the slow moving layer at A may move up to B, slowing down the fast moving layer. Conversely, a molecule in the fast moving layer may drop down from C to D, speeding up the lower layer. This is the basic idea which lies behind the expression $\mu = \rho l V/3$. However, just the same sort of thing happens in a turbulent flow, albeit at the macroscopic, rather than molecular, level. Balls of fluid are

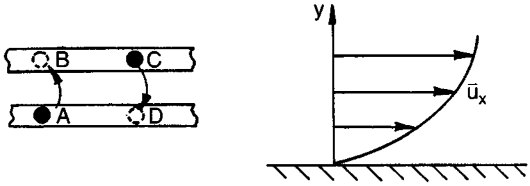


Figure 3.26 Exchange of momentum due to thermal motion of the molecules.

exchanged between the layers due to turbulent fluctuations, and this causes a mixing of momentum across the layers.

This analogy between the transfer of momentum by molecules on the one hand and balls of fluid on the other led Prandtl to propose the relationship

$$\mu_t = \rho l_m V_T$$

where l_m is called the mixing length. (Actually Boussinesq proposed something similar in 1870.) l_m is a measure of the size of the large eddies in the flow. V_T is a measure of u', and indicates the intensity of the turbulence. The more intense the fluctuations, the larger the cross-stream transfer of momentum, and so the larger the eddy viscosity.

To a large extent the equation above is simply a dimensional necessity, since μ_t/ρ has the dimensions of m^2/s. Boussinesq's equations, however, are a little more worrying. At one level we may regard them as simply defining μ_t, and so transferring the problem of estimating τ to one of estimating μ_t, but there is an important assumption here. We are assuming that the eddy viscosity in the xy plane is the same as the eddy viscosity in the xz plane, and so on. This, in turn, requires that the turbulence is statistically isotropic. Still, let us see how far we can get with an eddy viscosity model. We now need to find a way of estimating l_m and V_T. For the particular case of simple shear flows (one-dimensional flows), Prandtl found a way of estimating V_T. This is known as Prandtl's *mixing length theory* (Figure 3.27).

Consider a mean flow, $\bar{u}_x$, which is purely a function of y. Suppose also that the turbulence has components $u'_x, u'_y,\ u'_z$.

$$\bar{\mathbf{u}} = (\bar{u}_x, 0, 0)$$

$$\mathbf{u}' = (u'_x, u'_y, u'_z)$$

Then Prandtl's theory says that, in effect, the fluid at y, with mean velocity $\bar{u}_x(y)$, will, on average, have come from levels $y \pm l$, where l is the

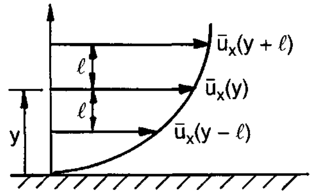


Figure 3.27 Prandtl's mixing length theory.

mixing length. Suppose that, as a fluid lump is thrown from $y+l$ (or $y-l$) to y, it retains its forward momentum, which on average will be $\bar{u}_x(y \pm l)$. Then the mean velocities $\bar{u}_x(y \pm l)$ represent the spread of instantaneous velocities at position y. We can represent this spread by $\bar{u}_x \pm \sqrt{\overline{(u_x')^2}}$. If l is small (unfortunately it is not!), then we have

$$\bar{u}_x(y \pm l) \approx \bar{u}_x(y) \pm l \frac{\partial \bar{u}_x}{\partial y}$$

It follows that

$$\overline{[u_x']^2} \approx l^2 \left[\frac{\partial \bar{u}_x}{\partial y} \right]^2$$

Next we note that there is a strong negative correlation between u_x' and u_y', since a positive u_x' is consistent with fluid coming from $y+l$, requiring a negative u_y'. (If $\partial \bar{u}_x / \partial y$ is negative we expect a positive correlation.) Thus,

$$\overline{u_x' u_y'} = -c_1 \sqrt{\overline{(u_x')^2}} \sqrt{\overline{(u_y')^2}}$$

where c_1 is some constant of order unity (called a correlation coefficient). If u_x' and u_y' are of similar orders of magnitude, we now have

$$\overline{u_x' u_y'} \sim \pm \overline{(u_x')^2} = -c_2 l^2 \left| \frac{\partial \bar{u}_x}{\partial y} \right| \frac{\partial \bar{u}_x}{\partial y}$$

where c_2 is a second constant of order unity. Note the inclusion of the modulus on one of the $\partial \bar{u}_x / \partial y$ terms. This is needed to ensure that the correlation is negative when $\partial \bar{u}_x / \partial y > 0$ and positive when $\partial \bar{u}_x / \partial y < 0$.

We now redefine our mixing length to absorb the unknown constant c_2. We set $l_m^2 = c_2 l^2$, and the end result is

$$\tau_{xy} = -\rho \overline{u_x' u_y'} = \rho l_m^2 \left| \frac{\partial \bar{u}_x}{\partial y} \right| \frac{\partial \bar{u}_x}{\partial y}$$

Compare this with Boussinesq's equation

$$\tau_{xy} = \mu_t \frac{\partial \bar{u}_x}{\partial y} = \rho l_m V_T \frac{\partial \bar{u}_x}{\partial y}$$

Evidently, for this particular sub-class of one-dimensional shear flows, the turbulent velocity scale is

$$V_T = l_m \left| \frac{\partial \bar{u}_x}{\partial y} \right|$$

The eddy viscosity is therefore

$$\mu_t = \rho l_m^2 \left| \frac{\partial \bar{u}_x}{\partial y} \right|$$

This represents what is known as the 'mixing length model' of Prandtl. Conceptually this is a tricky argument which ultimately cannot be justified in any formal way. Besides which, we still need to decide what l_m is, perhaps guided by experiment. Nevertheless, Prandtl's mixing length model appears to work well for one-dimensional shear flows, provided l_m is chosen appropriately. (By shear flows we mean flows like boundary layers, free shear layers and jets.) For flow over a flat plate, it is found that $l_m = ky$, where $k \approx 0.4$ and is known as Karman's constant.

Example: The α-effect in electrodynamics

The process of averaging chaotic or turbulent equations, in the spirit of Reynolds, is not restricted to the Navier–Stokes equation. For example, the heat equation or induction equation can be averaged in a similar way. Suppose we have a highly conducting, turbulent fluid in which $\mathbf{u} = \mathbf{u}_0 + \mathbf{v}$ and $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$ where $\mathbf{u}_0$ and $\mathbf{B}_0$ are steady or slowly varying and $\bar{\mathbf{v}} = 0$, $\bar{\mathbf{b}} = 0$. Show that the averaged induction equation is

$$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times (\mathbf{u}_0 \times \mathbf{B}_0) + \lambda \nabla^2 \mathbf{B}_0 + \nabla \times \overline{(\mathbf{v} \times \mathbf{b})}$$

The quantity $\overline{\mathbf{v} \times \mathbf{b}}$ is the electromagnetic analogue of the Reynolds stress. In some cases it is found that $\overline{\mathbf{v} \times \mathbf{b}} = \alpha \mathbf{B}_0$, where α is the analogue of Boussinesq's eddy diffusivity. This leads to the 'turbulent' induction equation

$$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times (\mathbf{u}_0 \times \mathbf{B}_0) + \alpha \nabla \times \mathbf{B}_0 + \lambda \nabla^2 \mathbf{B}_0$$

In Chapter 6 you will see that the new term, called the α-effect, can give rise to the self-excited generation of a magnetic field.

3.7 Ekman Pumping in Rotating Flows

We now consider the phenomenon of Ekman pumping, which occurs whenever there is differential rotation between a viscous fluid and a solid surface. This turns out to be important in magnetic stirring (see Chapter 8) and in the geo-dynamo (Chapter 6). We start with Karman's solution for laminar flow near the surface of a rotating disk.

Suppose we have an infinite disk rotating in an otherwise still liquid. A boundary layer will develop on the disk due to viscous coupling, and Karman found an exact solution for this boundary layer. Suppose that the disc rotates with angular velocity Ω. Then we might expect a boundary layer thickness to scale as $\delta \sim (\nu/\Omega)^{1/2}$. Karman suggested looking for a solution in polar coordinates in which z (which is normal to the disk) is normalised by δ. Karman's solution is of the form

$$u_r = \Omega r F(\eta), \qquad u_\theta = \Omega r G(\eta), \qquad u_z = \Omega \hat{\delta} H(\eta)$$

Here $\hat{\delta} = (\nu/\Omega)^{1/2}$ and $\eta = z/\hat{\delta}$. If these expressions are substituted into the radial and azimuthal components of the Navier–Stokes equation and the equation of continuity, we obtain three differential equations for three unknown functions F, G and H

$$F^2 + F'H - G^2 = F''$$
$$2FG + HG' = G''$$
$$2F + H' = 0$$

We take z to be measured from the surface of the disk, and so we have the boundary conditions

$$z = 0\colon\ F = 0,\ \ G = 1,\ \ H = 0$$
$$z \to \infty\colon\ F = 0,\ \ G = 0$$

Our three equations can be integrated numerically and the result is shown schematically in Figure 3.28. This represents a flow which is radially outward within a boundary layer of thickness $\delta \sim 4\hat{\delta} = 4(\nu/\Omega)^{1/2}$. The flow pattern in the r–z plane is shown in Figure 3.29. Within the thin boundary layer the fluid is centrifuged radially outward, so that each particle spirals outward to the edge of the disk. Outside the boundary layer u_r and u_θ are both zero, but u_z is non-zero. There is a slow drift of fluid towards the disk, at a rate

$$|u_z| \sim 0.9\Omega\hat{\delta} \sim 0.2\Omega\delta$$

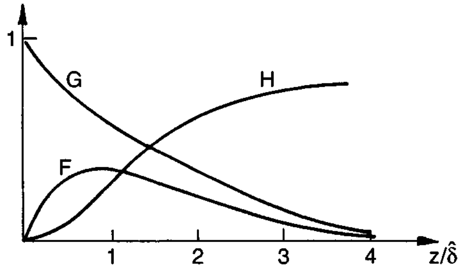


Figure 3.28 Solution of Karman's problem.

Of course, this is required to supply the radial outflow in the boundary layer. We have, in effect, a centrifugal fan.

Suppose now that the disk is stationary but that the fluid rotates like a rigid body ($u_\theta = \Omega r$) in the vicinity of the disk. Near the disk's surface this swirl is attenuated due to viscous drag and so a boundary layer forms. This problem was studied by Bodewadt, who showed that Karman's procedure works as before. It is only necessary to change the boundary conditions. Once again the boundary layer thickness is constant and of the order of $4(\nu/\Omega)^{1/2}$. This time, however, the flow pattern in the r–z plane is reversed (Figure 3.30). Fluid particles spiral radially inward, eventually drifting out of the boundary layer. Outside the boundary layer we have rigid body rotation, $u_\theta = \Omega r$, plus a weak axial flow away from the surface of magnitude

$$u_z \sim 1.4\Omega\hat{\delta} \sim 0.35\Omega\delta$$

The flow in the r–z plane is referred to as a secondary flow, in as much as the primary motion is a swirling flow. The reason for the secondary flow is that, outside the boundary layer, we have the radial force balance

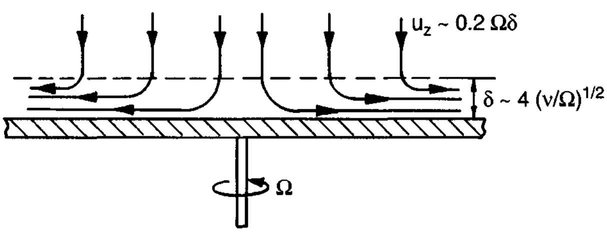


Figure 3.29 Secondary flow in Karman's problem.

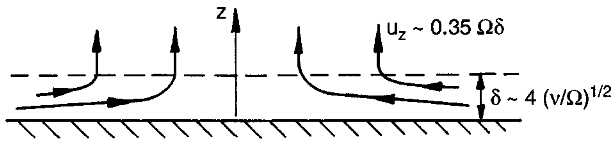


Figure 3.30 Bodewadt's problem (stationary disk, rotating fluid).

$$\frac{\partial p}{\partial r} = \rho \frac{u_\theta^2}{r}$$

That is, the centrifugal force sets up a radial pressure gradient, with a low pressure near the axis. This pressure gradient is imposed throughout the boundary layer on the plate. However, the swirl in this boundary layer is diminished through viscous drag, and so there is a local imbalance between the imposed pressure gradient and the centripetal acceleration. The result is a radial inflow, with the fluid eventually drifting up and out of the boundary layer.

In general then, whenever we have a swirling fluid adjacent to a stationary surface we induce a secondary flow, as sketched in Figure 3.30. This is referred to as Ekman pumping, and the boundary layer is called an Ekman layer.

The axial velocity induced by Ekman pumping is relatively small if the Reynolds number is large:

$$u_z \sim 1.4(\nu\Omega)^{1/2} \ll u_\theta$$

Nevertheless, this weak secondary flow often has profound consequences for the motion as a whole. Consider, for example, the problem of 'spin-down' of a stirred cup of tea. Suppose that, at $t = 0$, the tea is set into a state of (almost) inviscid rotation. Very quickly an Ekman layer will become established on the bottom of the cup, inducing a radial inflow at the base of the vessel. By continuity, this radial flow must eventually drift up and out of the boundary layer, where it is recycled via side layers (called Stewartson layers) on the cylindrical walls of the cup. A secondary flow is established, as shown in Figure 3.31. As each fluid particle passes through the Ekman layer it gives up a significant fraction of its kinetic energy. The tea finally comes to rest when all of the contents of the cup have been flushed through the boundary layer. The existence of the secondary flow is evidenced (in the days before tea-bags!) by the accumulation of tea-leaves at the centre of the cup.

The spin-down time, τ_s, is therefore of the order of the turn-over time of the secondary flow:

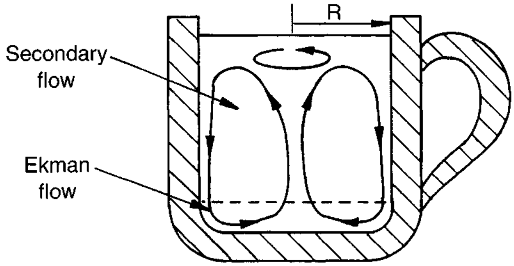


Figure 3.31 Spin-down of a stirred cup of tea.

$$\tau_s \sim R/u_z \sim R/\Omega\delta \sim R/\sqrt{\nu\Omega}$$

If there were no secondary flow, the spin-down time would be controlled by the time taken for the core vorticity to diffuse to the walls:

$$\tau_s^* \sim R^2/\nu$$

Suppose that $R = 3\,\text{cm}$, $\nu = 10^{-6}\,\text{m}^2/\text{s}$ and $\Omega = 1\,\text{s}^{-1}$. Then $\tau_s \sim 30\,\text{s}$, which is about right, whereas $\tau_s^* \sim 15$ min! Evidently, the Ekman layers provide an efficient mechanism for destroying energy.

Now consider a problem more relevant to engineering. Suppose we have a cylindrical vessel in which swirl is induced by rotating the lid (Figure 3.32). If the vessel is much broader than it is deep we might model it as two parallel disks, one rotating and one stationary. If the top disk rotates at a rate Ω, a natural question to ask is: what is the rotation rate in the core flow? It turns out that this is, once again, controlled by the weak secondary flow. The fluid is accelerated by one disk and retarded by the other. Consequently, it will rotate at a rate somewhere between 0 and Ω. It follows that a Bodewadt (or Ekman) layer will form on the lower disk and a Karman layer will form on the upper surface. Fluid will be ejected by the lower boundary layer and then sucked up into the upper layer. The answer to the question of the core rotation rate is now straightforward. The fluid lying outside the two

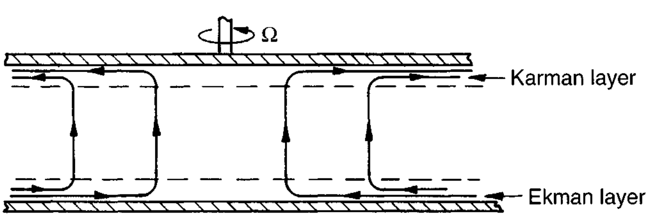


Figure 3.32 Flow between two disks, one of which rotates.

boundary layers will rotate at a rate such that the mass flow out of the Ekman layer balances the mass flow into the Karman layer. If Ω_c is the core rotation rate, then for the fluid leaving the Ekman layer

$$u_z \sim 1.4\Omega_c^{1/2}\nu^{1/2}$$

The fluid entering the Karman layer has velocity

$$u_z \sim 0.9(\Omega - \Omega_c)^{1/2}\nu^{1/2}$$

Equating the two, we find that

$$\Omega_c \sim 0.3\Omega$$

Therefore the bulk of the fluid rotates at approximately one-third of the disk rotation rate. A similar calculation can be performed when an electromagnetic torque induces swirl in a conducting fluid held between two fixed plates (see Chapter 5, Section 5).

Ekman pumping not only dominates confined swirling flows of the type described above, but also plays a key rôle in large-scale geophysical flows. For example, there is some evidence that the solid inner core of the earth rotates at a slightly different rate to the solid outer mantle. Between the two we have liquid iron. If this differential rotation does indeed occur, then we might expect a flow structure such as that shown in Figure 3.33. Ekman layers form on the top and bottom surfaces of the

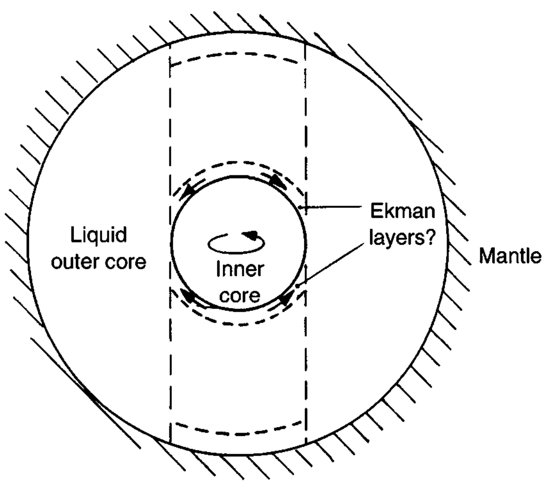


Figure 3.33 Ekman pumping in the core of the earth.

inner core. This kind of differential rotation can cause intense stretching and twisting of magnetic fields and may be a component of the process by which the earth maintains its magnetic field.

Part 2: Incorporating the Lorentz Force

We now incorporate the Lorentz force into the Navier–Stokes equations and consider some of the more elementary and immediate consequences of this.

3.8 The Full Equations of MHD and Key Dimensionless Groups

Let us start by summarising the governing equations of MHD. We have the reduced form of Maxwell's equations

$$\boxed{\nabla \times \mathbf{B} = \mu \mathbf{J}}, \qquad \boxed{\nabla \cdot \mathbf{J} = 0} \tag{3.23}$$

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}, \qquad \boxed{\nabla \cdot \mathbf{B} = 0} \tag{3.24}$$

and the auxiliary expressions

$$\boxed{\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B})}, \qquad \boxed{\mathbf{F} = \mathbf{J} \times \mathbf{B}} \tag{3.25}$$

These combine to give the induction equation

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \lambda \nabla^2 \mathbf{B}}, \qquad \lambda = (\mu\sigma)^{-1} \tag{3.26}$$

On the other hand, Newton's second law gives us

$$\boxed{\frac{D\mathbf{u}}{Dt} = -\nabla(p/\rho) + \nu\nabla^2\mathbf{u} + (\mathbf{J}\times\mathbf{B})/\rho} \tag{3.27}$$

from which we obtain the vorticity equation

$$\boxed{\frac{\partial\boldsymbol{\omega}}{\partial t} = \nabla\times(\mathbf{u}\times\boldsymbol{\omega}) + \nu\nabla^2\boldsymbol{\omega} + \nabla\times(\mathbf{J}\times\mathbf{B})/\rho} \tag{3.28}$$

There are four dimensionless groups which regularly appear in the MHD literature. Three of them represent the relative magnitudes of the different force terms in (3.27). The fourth relates to (3.26). The first is the Reynolds number, $\mathrm{Re} = ul/\nu$, where l is a characteristic length scale of the motion and u is a typical velocity. As in conventional fluid mechanics, this is representative of the ratio of inertia, $(\mathbf{u}\cdot\nabla)\mathbf{u}$, to viscous forces, $\nu\nabla^2\mathbf{u}$. The second dimensionless group is the obscurely named interaction parameter,

$$N = \sigma B^2 l/\rho u = l/u\tau \tag{3.29}$$

where τ is the magnetic damping time $(\sigma B^2/\rho)^{-1}$ which was introduced in Chapter 1. This is relevant in situations where $\mathbf{J}$ is primarily driven by $\mathbf{u}\times\mathbf{B}$ in Ohm's law, and so $|\mathbf{J}| \sim \sigma uB$. In such a case, N represents the ratio of the Lorentz force, $\mathbf{J}\times\mathbf{B}/\rho$, to inertia, $(\mathbf{u}\cdot\nabla)\mathbf{u}$.

The third dimensionless group, called the Hartmann number, is a hybrid of Re and N. It is

$$Ha = (N\mathrm{Re})^{1/2} = Bl(\sigma/\rho\nu)^{1/2} \tag{3.30}$$

Evidently $(Ha)^2$ represents the ratio of the Lorentz force to viscous forces. The final dimensionless group has nothing at all to do with forces. Rather, it is indicative of the relative strengths of advection and diffusion in the induction equation (3.26). This is the magnetic Reynolds number

$$R_m = ul/\lambda = \mu\sigma ul \tag{3.31}$$

which was first introduced in Chapter 1. When R_m is large, diffusion is weak. These various groups are listed in Table 3.1.

Note that the characteristic length scale of the flow, l, need not be known in advance, but rather it may emerge from some internal force balance. The obvious example is a boundary layer, where Re based on the boundary layer thickness is always of the order of unity. That is the whole point about boundary layers: viscous and inertia forces are always

Table 3.1. *Dimensionless groups*

Name	Symbol	Definition	Significance
Reynolds number	Re	ul/ν	Ratio of inertia to shear forces
Interaction Parameter	N	$\sigma B^2 l/\rho u$	Ratio of Lorentz forces to inertia
Hartman number	Ha	$Bl(\sigma/\rho\nu)^{1/2}$	Ratio of Lorentz forces to shear forces
Magnetic Reynolds number	R_m	ul/λ	Ratio of advection to diffusion of **B**

of the same order, no matter how small ν may be. One must always be careful in the choice of length scale when constructing meaningful dimensionless groups. In general, each case must be treated on an individual basis. Nevertheless, dimensionless groups are extremely useful. Often, when they are very large or very small, they allow us to throw out certain terms in the governing equations, thereby greatly simplifying the problem.

3.9 Maxwell Stresses

We conclude this chapter with a discussion of the Lorentz force itself. From Ampère's law we may rewrite the Lorentz force in terms of **B** alone. We start with the vector identity

$$\nabla(\mathbf{B}^2/2) = (\mathbf{B}\cdot\nabla)\mathbf{B} + \mathbf{B}\times\nabla\times\mathbf{B}$$

from which, using $\nabla\times\mathbf{B} = \mu\mathbf{J}$,

$$\mathbf{J}\times\mathbf{B} = (\mathbf{B}\cdot\nabla)(\mathbf{B}/\mu) - \nabla(\mathbf{B}^2/2\mu) \qquad (3.32)$$

The second term on the right of (3.32) acts on the fluid in exactly the same way as the pressure force $-\nabla p$. It is irrotational, and so makes no contribution to the vorticity equation. In flows without a free surface its rôle is simply to augment the fluid pressure. (Its absence from the vorticity equation implies that it cannot influence the flow field.) For this reason, $\mathbf{B}^2/2\mu$ is called the magnetic pressure and in many, if not most, problems it is of no dynamical significance. Which brings us to the first term on the right. We can write the ith component of this force as

$$\mathbf{B}\cdot\nabla(B_i/\mu) = \frac{\partial}{\partial x_j}\left[\frac{B_iB_j}{\mu}\right] \qquad (3.33)$$

where there is an implied summation over the index j. From this we may show that the effect of the body force in (3.33) is exactly equivalent to a distributed set of fictitious stresses, B_iB_j/μ, acting on the surface of fluid elements. One approach is simply to compare (3.33) with the viscous forces in (3.1), making use of Figure 3.8. Alternatively, this can be established by integrating (3.33) over an arbitrary volume V and invoking Gauss's theorem. Since $\nabla \cdot (B_i\mathbf{B}) = \mathbf{B} \cdot \nabla B_i + B_i \nabla \cdot \mathbf{B} = \mathbf{B} \cdot \nabla B_i$, we find

$$\int [\mathbf{B} \cdot \nabla (B_i/\mu)] dV = \oint (B_i/\mu)\mathbf{B} \cdot d\mathbf{S} \tag{3.34}$$

The surface integral on the right of (3.34) is equal to the cumulative effect of the distributed stress system B_iB_j/μ acting over the surface of V. That is to say, the tangential and normal stresses, B_tB_n/μ and B_n^2/μ, acting on the surface element $d\mathbf{S}$ give rise to a force $\mathbf{B}(\mathbf{B} \cdot d\mathbf{S})/\mu$. However, equation (3.34) tells us that this surface stress distribution is, in turn, equivalent to the integrated effect of the volume force $(\mathbf{B} \cdot \nabla)(\mathbf{B}/\mu)$. Since this is true for any volume V, it follows that the body force $(\mathbf{B} \cdot \nabla)(\mathbf{B}/\mu)$ and the stress system B_iB_j/μ are entirely equivalent in their mechanical action.

In summary then, we may replace the Lorentz force, $\mathbf{J} \times \mathbf{B}$, by an imaginary set of stresses

$$\tau_{ij} = \left(B_iB_j/\mu\right) - \left(\mathbf{B}^2/2\mu\right)\delta_{ij} \tag{3.35}$$

where the second term on the right is the magnetic pressure. (The symbol δ_{ij} means: $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$.) These are called the Maxwell stresses, and their utility lies in the fact that we can represent the integrated effect of a distributed body force by surface stresses alone.

Now there is another, perhaps more useful, representation of $\mathbf{J} \times \mathbf{B}$. This comes from replacing $\mathbf{u}$ by $\mathbf{B}$ in (3.6):

$$(\mathbf{B} \cdot \nabla)\mathbf{B} = B\frac{\partial B}{\partial s}\hat{\mathbf{e}}_t - \frac{B^2}{R}\hat{\mathbf{e}}_n \tag{3.36}$$

Here s is now a coordinate measured along a magnetic field line, $\hat{\mathbf{e}}_t$ and $\hat{\mathbf{e}}_n$ are unit vectors in the tangential and principal normal direction, respectively, $B = |\mathbf{B}|$, and R is the local radius of curvature of the field line. It follows that the Lorentz force may be written as

$$\boxed{\mathbf{J} \times \mathbf{B} = \frac{\partial}{\partial s}\left[\frac{B^2}{2\mu}\right]\hat{\mathbf{e}}_t - \frac{B^2}{\mu R}\hat{\mathbf{e}}_n - \nabla(B^2/2\mu)} \tag{3.37}$$

We now have two alternative representations of $\mathbf{J} \times \mathbf{B}$. In cases where the magnetic pressure is unimportant, which is usually the case, we are concerned only with $\mathbf{B} \cdot \nabla(\mathbf{B}/\mu)$. In such situations, $\mathbf{J} \times \mathbf{B}$ may be pictured as the result of the stress system $B_t B_n/\mu$, or else it may be written in the form (3.36). To illustrate the difference, consider a flux tube as shown in Figure 3.34. In the first interpretation, $(\mathbf{B} \cdot \nabla)(\mathbf{B}/\mu)$ arises from *tensile stresses* of $\mathbf{B}^2/\mu$ acting on the ends of the tube (there are no stresses on the sides). These are referred to as Faraday tensions, or Maxwell tensions, in the field lines. In the second interpretation there are *force components* $\mu^{-1}B\partial B/\partial S$ and $B^2/\mu R$ tangential and normal, respectively, to the flux tube at each location.

In a qualitative sense then, we may think of field lines as being in tension and exerting a pseudo-elastic stress on the fluid. This lies at the root of many MHD phenomena, particularly the Alfvén wave. We have already seen in Section 2.7 that, when R_m is high, the magnetic field lines tend to be frozen into a fluid (Figure 3.35). Now we see that the field lines also behave as if they are in tension. Consider what happens then if, at $t = 0$, we try to push a region of highly conductive fluid past a magnetic field. The field lines will be swept along with the fluid, and the resulting curvature of the lines will create a back reaction $B^2/\mu R$ on the fluid. The fluid will eventually come to rest and the Faraday tensions will then reverse the flow. Oscillations (Alfvén waves) may even result. This is effectively what happens in the experiment described in Section 1.3.

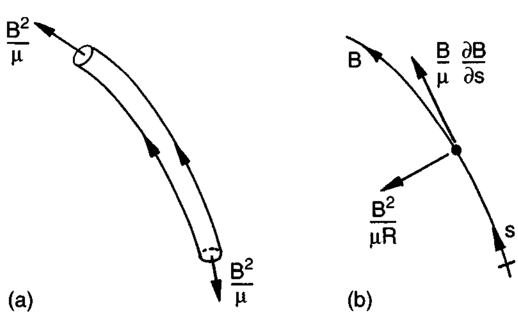


Figure 3.34 The contribution $\mathbf{B} \cdot \nabla(\mathbf{B}/\mu)$ to the Lorentz force may be interpreted either in terms of Faraday tensions, B^2/μ, in the field lines, or else in terms of the forces $(B/\mu)\partial B/\partial S$ and $B^2/\mu R$ acting tangential and normal to the field line, respectively.

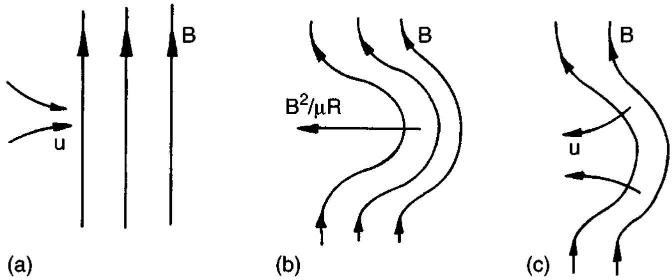


Figure 3.35 Magnetic field lines can behave like elastic bands frozen into the fluid.

Suggested Reading

Feynman, Leighton & Sands, *The Feynman lectures on physics*, vol. II, 1964. Addison–Wesley (Chapters 40, 41 provide an introduction to fluid mechanics).

D J Acheson, *Elementary fluid dynamics*, 1990. Clarendon Press. (Chapter 5 for vortex dynamics, chapter 9 for boundary layers.)

G K Batchelor, *An introduction to fluid mechanics*, 1967. Cambridge University Press. (Chapter 7 for vortex dynamics.)

J A Shercliff, *A textbook of magnetohydrodynamics*, 1965. Pergamon Press. (Chapter 4 for the Lorentz force and Maxwell stresses.)

H Tennekes & J L Lumley, *A first course in turbulence*, 1972. The MIT Press. (Chapter 2 for a discussion of Reynolds stresses.)

Examples

3.1 Show that the circulation is the same around all simple closed circuits enclosing the wing shown in Figure 3.3 (ignore any vorticity in the wake).

3.2 A bathtub vortex forms because random vorticity in the bath becomes aligned with the axis of the vortex by the action of the converging, draining flow. Explain why the vorticity is also intensified by the action of the converging flow.

3.3 A peculiar vortex motion may be observed in rowing. At places where the oar breaks the surface of the water, just previous to being lifted, a pair of small dimples (depressions) appear on the surface. Once the oar is lifted from the water the pair of dimples propagate along the surface. They are the end points of a vortex arc (half of a vortex ring). Explain what is happening.

3.4 Estimate the pressure at the centre of a typical tornado.

3.5 Show that the free surface of a tidal vortex is a hyperbola. You may assume that the velocity distribution is approximately that of a free vortex, $u_\theta \sim \Gamma/r$ where $\Gamma = \text{constant}$ (ignore the singularity at the centre of the vortex).

3.6 Use the Biot–Savart law to calculate the velocity at the centre of a thin vortex ring.