4

Kinematics of MHD: Advection and Diffusion of a Magnetic Field

> We adopt the suggestion of Ampère, and use the term 'Kinematics' for the purely geometrical science of motion in the abstract. Keeping in view the properties of language, and following the example of most logical writers, we employ the term 'dynamics' in its true sense as the science which treats the action of force.
>
> *Kelvin (1879), preface to Natural Philosophy*

We now consider one half of the coupling between **B** and **u**. Specifically, we look at the influence of **u** on **B** without worrying about the origin of the velocity field or the back reaction of the Lorentz forces on the fluid. In effect, we take **u** to be prescribed, forget about the Navier–Stokes equation, and focus on the interaction of **u** with Maxwell's equations. This is referred to as the *kinematics of MHD*.

4.1 The Analogy to Vorticity

In Chapter 2 we showed that Maxwell's equations lead to the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \lambda \nabla^2 \mathbf{B} \tag{4.1}$$

where $\lambda = (\mu\sigma)^{-1}$. Compare this with the transport equation for vorticity,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega} \tag{4.2}$$

There appears to be an exact analogy. In fact, the analogy is not perfect because $\boldsymbol{\omega}$ is functionally related to **u** in a way that **B** is not. Nevertheless, this does not stop us from borrowing many of the theorems of classical vortex dynamics and re-interpreting them in terms of MHD, with **B** playing the rôle of $\boldsymbol{\omega}$. For example, **B** is advected by **u** and diffused by λ, and in the limit $\lambda \to 0$, the counterpart of Helmholtz's first law is that **B** is frozen into the fluid.

4.2 Diffusion of a Magnetic Field

When $\mathbf{u} = 0$ we have

$$\frac{\partial \mathbf{B}}{\partial t} = \lambda \nabla^2 \mathbf{B} \tag{4.3}$$

which may be compared with the diffusion equation for heat,

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \tag{4.4}$$

It appears that, like heat, magnetic fields will diffuse through a medium at a finite rate. We cannot suddenly 'impose' a distribution of **B** throughout a conductor. All we can do is specify values at the boundaries and wait for it to diffuse inward. For example, suppose we have a semi-infinite region of conducting material occupying $y > 0$, and at $t = 0$ we apply a magnetic field $B_0 \hat{\mathbf{e}}_x$ at the surface $y = 0$. Then **B** will diffuse into the conductor in precisely the same way as heat or vorticity diffuses. In fact, to find the distribution of **B** at any instant we may simply lift the solution directly from the equivalent thermal problem. Such diffusion problems were discussed in Section 3.3, where we found that T (or ω) diffuses a distance $l \sim \sqrt{\alpha t}$, (or $\sqrt{\nu t}$) in a time t. By implication, **B** diffuses a distance of order $\sqrt{\lambda t}$ in the same time.

Example: Extinction of a magnetic field

Consider a long conducting cylinder which, at $t = 0$, contains a uniform axial magnetic field, B_0. The field outside the cylinder is zero. The axial field inside the cylinder will decay according to the diffusion equation

$$\frac{\partial B}{\partial t} = \lambda \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial B}{\partial r}\right)$$

subject to $B = 0$ at $r = R$ and $B = B_0$ at $t = 0$. Show that a Fourier–Bessel series of the form

$$B = \sum_{n=1}^{\infty} A_n J_0(\gamma_n r/R) \exp\left(-\gamma_n^2 \lambda t/R^2\right)$$

is a possible solution, where J_0 is the usual Bessel function, γ_n are the zeros of J_0, and A_n represents a set of amplitudes. Deduce that the field decays on a time scale of $R^2/(5.75\lambda)$.

4.3 Advection in Ideal Conductors: Alfvén's Theorem

4.3.1 Alfvén's theorem

We now consider the other extreme, where there is no diffusion ($\lambda = 0$) but $\mathbf{u}$ is finite. This applies to conducting fluids with a very high conductivity (ideal conductors). Consider the similarity between

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \tag{4.5}$$

and the vorticity equation for an inviscid fluid,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega})$$

We might anticipate, therefore, that Helmholtz's first law and Kelvin's theorem (which is, in effect, Helmholtz's second law) have their analogues in MHD. This turns out to be so. The equivalent theorems are:

Theorem I: (analogue of Helmholtz's first law)	The fluid elements that lie on a magnetic field line at some initial instant continue to lie on that field line for all time, i.e. the field lines are frozen into the fluid.
Theorem II: (analogue of Kelvin's theorem)	The magnetic flux linking any loop moving with the fluid is constant.

These two results are collectively known as Alfvén's theorem. In fact, Theorem II is a direct consequence of the generalised version of Faraday's law, which was introduced in Section 2.7.4. Moreover, the first theorem may be proved in precisely the same manner as Helmholtz's first law, the proof relying on the analogy between (4.5) in the form

$$\frac{D\mathbf{B}}{Dt} = (\mathbf{B} \cdot \nabla)\mathbf{u} \tag{4.6}$$

and equation (3.21) for a material line element

$$\frac{D}{Dt}(d\mathbf{l}) = ((d\mathbf{l}) \cdot \nabla)\mathbf{u}$$

The 'frozen-in' nature of magnetic fields is of crucial importance in astrophysics, where R_m is usually very high. For example, one might ask: why do many stars possess magnetic fields of the order of 10 or 1000 Gauss? The answer, possibly, is that there exists a weak galactic field of $\sim 10^{-6}$ Gauss. As a star starts to form due to the gravitational collapse of an interstellar cloud, the galactic field, which is trapped in the plasma, becomes concentrated by the inward radial movement. A simple estimate of the increase in **B** due to this mechanism can be obtained if we assume the cloud remains spherical, of radius r, during the collapse. Two invariants of the cloud are its mass, $M \propto \rho r^3$, and the flux of the galactic field which traverses the cloud, $\Phi \propto Br^2$. It follows that during the collapse, $B \propto \rho^{2/3}$ which suggest $(B_{\text{star}}/B_{\text{gal}}) \sim (\rho_{\text{star}}/\rho_{\text{gal}})^{2/3}$. Actually, this overestimates B_{star} somewhat, possibly because the collapse is not spherical, and possibly because there is some turbulent diffusion of **B**, despite the high value of R_m.

Now the analogy between **B** and $\boldsymbol{\omega}$ can be pushed even further. For example, our experience with vorticity suggests that, in three-dimensions, we can stretch the magnetic field lines (or flux tubes) leading to an intensification of **B**. That is, the left of (4.6) represents the material advection of the magnetic field, so that when $(\mathbf{B} \cdot \nabla)\mathbf{u} = 0$ (as would be the case in certain two-dimensional flows) the magnetic field is passively advected. However, in three-dimensional flows $(\mathbf{B} \cdot \nabla)\mathbf{u}$ need not be zero and, because of the analogy with vortex tubes, we would expect this to lead to a rise in **B** whenever the flux tubes are stretched by the flow (see Section 3.3). In fact, this turns out to be true, as it must because the mathematics in the two cases are formally identical. However, the physical interpretation of this process of intensification is different in the two situations. In vortex dynamics it is a direct consequence of the conservation of angular momentum. In MHD, however, it follows from a combination of the conservation of mass, $\rho \delta V = \rho \delta A dl$, and flux, $\Phi = B\delta A$, applied to an element of a flux tube, as shown in Figure 4.1. If the flux

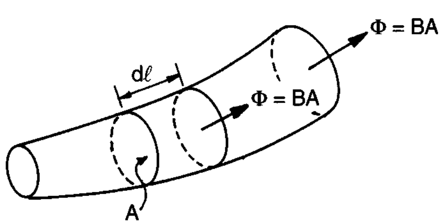


Figure 4.1 Stretching of a flux tube intensifies B.

tube is stretched, δA decreases and so B rises to conserve flux. This is the basis of dynamo theory in MHD, whereby magnetic fields are intensified by continually stretching the flux tubes.

4.3.2 An aside: sunspots

As an illustration of the 'frozen-in' behaviour of magnetic fields, and of flux-tube stretching, we shall describe here the phenomenon of sunspots. We give only a qualitative description, but the interested reader will find more details in the suggested reading list at the end of this chapter.

If you look at the sun through darkened glass it is possible to discern small dark spots on its surface. These come and go, with a typical lifetime of several days. These spots (sunspots) typically appear in pairs and are concentrated around the equatorial plane. The spots have a diameter of $\sim 10^4$ km, which is around the same size as the earth! To understand how they arise, you must first know a little bit about the structure of the sun.

The surface of the sun is not uniformly bright, but rather has a granular appearance. This is because the outer layer of the sun is in a state of convective turbulence. This *convective layer* has a thickness of 2×10^5 km (the radius of the sun is 7×10^5 km) and consists of a continually evolving pattern of convection cells, rather like those seen in Bénard convection (Figure 4.2). The cells nearest the surface are about 10^3 km across. Where hot fluid rises to the surface, the sun appears bright, while the cooler fluid, which falls at the junction of adjacent cells, appears dark. A typical convective velocity is around 1 km/s and estimates of Re and R_m are, Re $\sim 10^{11}$, $R_m \sim 10^8$, i.e. very large!

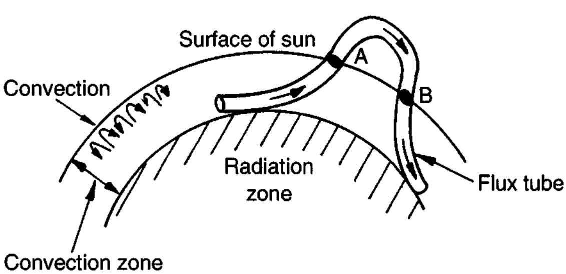


Figure 4.2 Schematic representation of the formation of sunspots. A buoyant flux tube erupts through the surface of the sun. Sunspots form at A and B where the magnetic field suppresses the turbulence, cooling the surface.

Now the sun has an average surface magnetic field of a few Gauss, rather like that of the earth. Because R_m is large, this dipole field tends to be frozen into the fluid in the convective zone. Large-scale differential rotation in this zone stretches and intensifies this field until large field strengths (perhaps 10^3 Gauss) build up in azimuthal flux tubes of varying diameter. The pressure inside these flux tubes is significantly less than the ambient pressure in the convective zone, essentially because the Lorentz forces in a flux tube point radially outward. The density inside the tubes is correspondingly less, and so the tubes experience a buoyancy force which tends to propel them towards the surface. This force is strongest in the thick tubes, parts of which become convectively unstable and drift upwards, with a rise time of perhaps a month. Periodically then, flux tubes of diameter $\sim 10^4$ km burst through the surface into the sun's atmosphere (Figure 4.2). Sunspots are the footpoint areas where the tubes pierce the surface (A and B in Figure 4.2). These footpoints appear dark because the intense magnetic field in the flux tubes ($\sim$3000 Gauss) locally suppresses the fluid motion and convective heat transfer, thus cooling the surface.

This entire phenomenon relies on the magnetic field being (partially) frozen into the fluid. It is this which allows intense fields to form in the first place, and which ensures that the buoyant fluid at the core of a flux tube carries the tube with it as it moves upward. We shall return to this topic in Chapter 6, where we shall see that sun spots are often accompanied by other magnetic phenomena, such as solar flares.

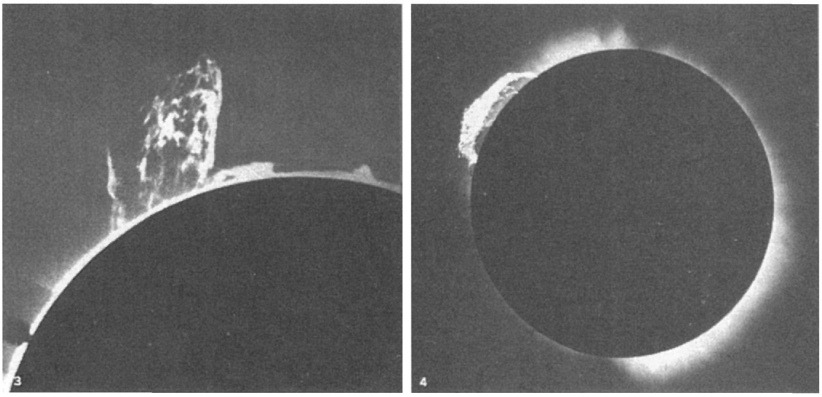


Figure 4.3 Magnetic activity in the solar atmosphere (Encyclopaedia Britannica).

4.4 Magnetic Helicity

We can take the analogy between $\boldsymbol{\omega}$ and $\mathbf{B}$ yet further. In Section 3.4 we saw that the helicity

$$h = \int_{V_\omega} \mathbf{u} \cdot \boldsymbol{\omega} dV$$

is conserved in an inviscid flow. Moreover, this is a direct consequence of the conservation of vortex-line topology which is enforced by Helmholtz's laws. We would expect, therefore, that the magnetic helicity

$$h_m = \int_{V_B} \mathbf{A} \cdot \mathbf{B} dV \tag{4.7}$$

will be conserved as a consequence of Alfvén's theorem. ($\mathbf{A}$ is the vector potential defined by (2.19).) This is readily confirmed. First we uncurl (4.5) to give

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \times \mathbf{B} + \nabla \phi \tag{4.8}$$

where ϕ is a scalar defined by the divergence of (4.8). From (4.8) and (4.5) we have

$$\frac{\partial}{\partial t}(\mathbf{A} \cdot \mathbf{B}) = \nabla \cdot (\phi \mathbf{B}) + \mathbf{A} \cdot [\nabla \times (\mathbf{u} \times \mathbf{B})]$$

which, with the help of the vector relationship

$$\nabla \cdot [\mathbf{A} \times (\mathbf{B} \times \mathbf{u})] = \nabla \cdot [(\mathbf{A} \cdot \mathbf{u})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{u}] = \mathbf{A} \cdot \nabla \times [\mathbf{u} \times \mathbf{B}]$$

becomes

$$\frac{D}{Dt}(\mathbf{A} \cdot \mathbf{B}) = \nabla \cdot [(\phi + \mathbf{A} \cdot \mathbf{u})\mathbf{B}] \tag{4.9}$$

We now integrate (4.9) over a material volume V_B which always consists of the same fluid particles (each of volume δV) and for which $\mathbf{B} \cdot d\mathbf{S} = 0$. Remembering that $D(\delta V)Dt = 0$ in an incompressible fluid, we obtain,

$$\frac{d}{dt} \int_{V_B} (\mathbf{A} \cdot \mathbf{B}) dV = 0$$

as required. As with the helicity of a vorticity field, this conservation law is topological in nature. It stems from the fact that interlinked flux tubes in an ideal conductor remain linked for all time, conserving their relative topology as well as their individual fluxes (see Section 3.4). Finally, we note that minimising magnetic energy subject to the conservation of helicity leads to the field $\nabla \times \mathbf{B} = \alpha \mathbf{B}$ which, as noted in Chapter 2, is called a force-free field.

There is one other topological invariant of ideal (diffusionless) MHD. This is called the cross-helicity, and is defined as

$$\int_{V_B} \mathbf{B} \cdot \mathbf{u} dV$$

Cross-helicity is conserved whenever λ and ν are zero. It represents the degree of linkage of the vortex lines and **B**-lines. We shall not pause here to prove the conservation of cross-helicity, but leave it as an exercise for the reader.

4.5 Advection plus Diffusion

We now consider the combined effects of diffusion and advection. For simplicity we focus on two-dimensional flows in which there is no flux-tube stretching. In such cases it is convenient to work with the vector potential **A**, rather than **B**. Suppose that $\mathbf{u} = \left(\partial\psi/\partial y, -\partial\psi/\partial x, 0\right)$ and $\mathbf{B} = \left(\partial A/\partial y, -\partial A/\partial x, 0\right)$ where ψ is the streamfunction for **u**, $\mathbf{u} = \nabla\times(\psi\hat{\mathbf{e}}_z)$, and A is the analogous *flux function* for **B**, $\mathbf{B} = \nabla \times (A\hat{\mathbf{e}}_z)$. Then the induction equation (4.1) becomes $\partial \mathbf{A}/\partial t = \mathbf{u} \times \mathbf{B} + \lambda \nabla^2 \mathbf{A}$, from which

$$\frac{DA}{Dt} = \lambda \nabla^2 A \tag{4.10}$$

Note that the contours of constant A represent magnetic field lines. Also, as noted in Section 3.8, $R_m = \mu\sigma ul = ul/\lambda$ is a measure of the relative strengths of advection and diffusion.

4.5.1 Field sweeping

Now A is transported just like heat, c.f. (4.4). Let us start, therefore, with a problem which is analogous to a heated wire in a cross flow, as this example was discussed at some length in Chapter 3, Section 3. The equivalent MHD problem is sketched in Figure 4.4.

We have a thin wire carrying a current I (directed into the page) which sits in a uniform cross flow, **u**. The magnetic field lines surrounding the

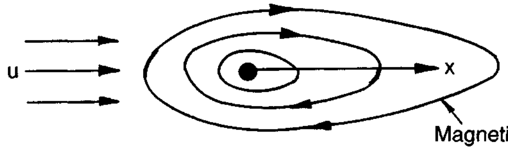


Figure 4.4 Magnetic field induced by a current-carrying wire in a cross flow.

wire are swept downstream by **u**, just like the isotherms in Figure 3.11. In the steady state (4.10) can be written as

$$u\frac{\partial A}{\partial x} = \lambda \nabla^2 A \tag{4.11}$$

Now there is no natural length scale for this problem. (The wire is considered to be vanishingly thin.) The only way of constructing a magnetic Reynolds number is to use $r = (x^2 + y^2)^{1/2}$ as the characteristic length-scale. Thus, $R_m = \mu\sigma ur = ur/\lambda$. Near the wire, therefore, we will have a diffusion-dominated regime ($R_m \ll 1$), while at large distances from the wire ($R_m \gg 1$) advection of **B** will dominate. It turns out that equation (4.11) may be solved by looking for solutions of the form $A = f(x, y)\exp(ux/2\lambda)$. This yields $\nabla^2 f = (u/2\lambda)^2 f$, and the solution for A is thus

$$A = CK_0(ur/2\lambda)\exp(ux/2\lambda)$$

where K_0 is the zero-order Bessel function normally denoted by K. The constant C may be determined by matching this expression to the diffusion-dominated solution $A = (\mu I/2\pi)\ln(r)$ at $r \to 0$. This gives $C = \mu I/2\pi$. The shape of the field lines is as shown in Figure 4.4. They are identical to the isotherms in the analogous thermal problem.

4.5.2 Flux expulsion

We now consider another example of combined advection and diffusion. This is a phenomenon called flux expulsion which, from the mathematical point of view, is nothing more than the Prandtl–Batchelor theorem applied to **A** rather than $\boldsymbol{\omega}$. Suppose that we have a steady flow consisting of a region of closed streamlines of size l, and that $R_m = ul/\lambda$ is large. Then we may show that any magnetic field which lies within that region is gradually expelled (Figure 4.5). The proof is essentially the same as that for the Prandtl–Batchelor theorem. In brief, the argument goes as follows. We have seen that A satisfies an advection

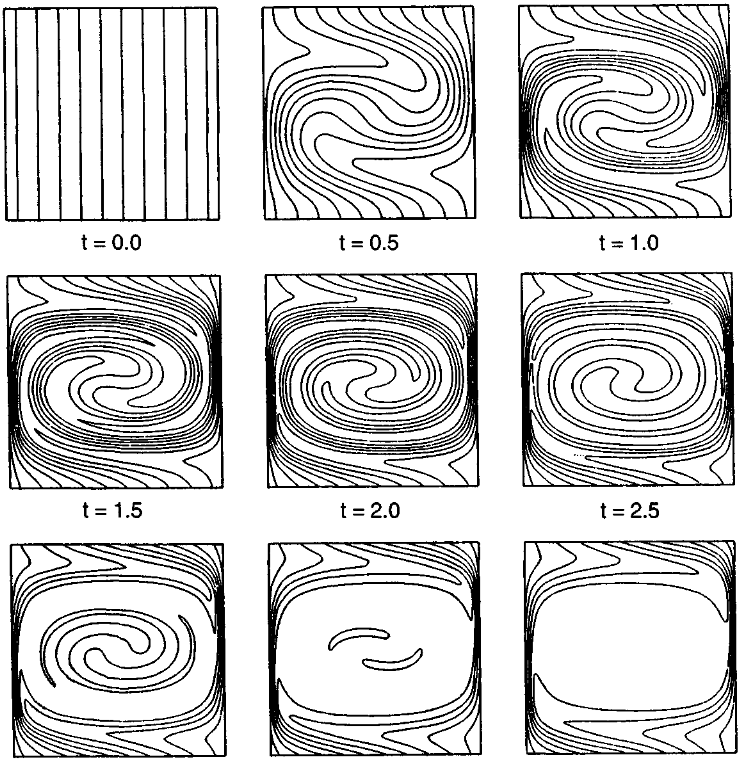


Figure 4.5 An example of flux expulsion in a square at $R_m = 100$ (based on computations by N O Weiss). The figures show the distortion of an initially uniform field by a clockwise eddy. (From H K Moffat, *Magnetic field generation in electrically conducting fluids*. CUP, 1978, with permission.)

diffusion equation, just like vorticity. When R_m is large we find A is almost constant along the streamlines. However, a small but finite diffusion slowly eradicates cross-stream gradients in A until it is perfectly uniform, giving $\mathbf{B} = 0$. We now work through the details, starting with the high R_m equation

$$\frac{DA}{Dt} \approx 0$$

In the steady state this simplifies to $\mathbf{u} \cdot \nabla A = 0$, which in turn implies $A = A(\psi)$. That is, A is constant along the streamlines so that $\mathbf{B}$ and $\mathbf{u}$ are co-linear. Now suppose that λ is small but finite. The steady version of (4.10)

$$\mathbf{u} \cdot \nabla A = \lambda \nabla^2 A$$

yields the integral equation

$$I = \lambda \int_{V_\psi} \nabla^2 A \, dV = 0 \tag{4.12}$$

where V_ψ is the volume enclosed by a closed streamline. Now (4.12) must hold true for any finite value of λ, and in particular it remains valid when λ is very small, so that $A \approx A(\psi)$. Let us now explore the consequences of the integral constraint (4.12) for our high-*Rm* flow. We have, using Gauss's theorem,

$$I = \lambda \int_{V_\psi} \nabla^2 A \, dV = \lambda \oint_{S_\psi} \nabla A \cdot d\mathbf{S} = \lambda A'(\psi) \oint_{S_\psi} \nabla \psi \cdot d\mathbf{S}$$

where $A'(\psi)$ is the cross-stream gradient of A, which is constant on the surface S_ψ. However, the integral on the right is readily evaluated. We use Gauss's and Stokes' theorems as follows:

$$\oint_{S_\psi} \nabla \psi \cdot d\mathbf{S} = \int_{V_\psi} \nabla^2 \psi dV = -\int_{V_\psi} \omega dV = -\oint_{C_\psi} \mathbf{u} \cdot d\mathbf{l}$$

Here, C_ψ is the streamline which defines S_ψ. It follows that our integral constraint may be rewritten as

$$I = -\lambda A'(\psi) \oint_{C_\psi} \mathbf{u} \cdot dl = 0 \tag{4.13}$$

Again it is emphasised that this holds true no matter how small we make λ. It is only necessary that λ be finite. Now it follows from (4.13) that $A'(\psi) = 0$, since the line integral cannot be zero. We conclude, therefore, that in a region of closed streamlines with a high value of R_m, the flux function is constant. It follows that $\mathbf{B} = 0$. This phenomenon is known as flux expulsion.

An example of this process is shown below. A magnetic field $\mathbf{B} = B_0 \hat{\mathbf{e}}_y$ pervades a conducting fluid, and a region of this fluid, $r < R$, is in a state of rigid body rotation, the remainder being quiescent. This local rotation distorts $\mathbf{B}$ and the distortion is readily calculated. Let Ω be the angular velocity of the fluid. In the steady state, (4.10) gives us

$$\frac{\Omega}{\lambda} \frac{\partial A}{\partial \theta} = \nabla^2 A, \qquad 0 < r < R$$

with $\nabla^2 A = 0$ for $r > R$. It is natural to look for solutions of the form $A = f(r)\exp(j\theta)$, where we extract only the real part of A. This yields

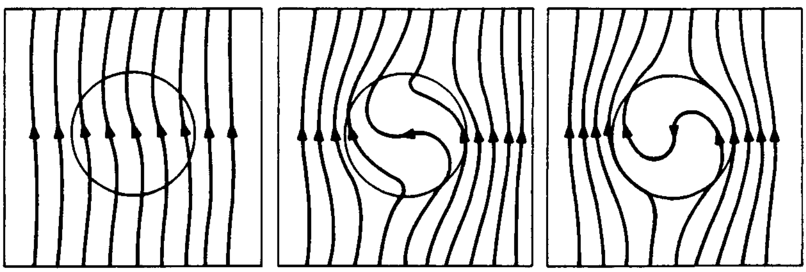
$$\frac{j\Omega}{\lambda}f(r) = \left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{1}{r^2}\right)f, \qquad 0 < r < R$$

The solution for f is then

$$f = -B_0 r + C/r, \qquad r > R$$
$$f = DJ_1(pr), \qquad 0 < r < R$$

where C and D are constants, J_1 is the usual first-order Bessel function, and $p = (1 - j)(\Omega/2\lambda)^{1/2}$. The unknown constants can be evaluated from the condition that $\mathbf{B}$ is continuous at $r = R$. In the limit of $R_m \to \infty$, we find that $A = 0$ inside $r = R$ and $A = -B_0(r - R^2/r)\cos\theta$ for $r > R$.

The flux function, A, is then identical to the streamlines of an irrotational flow past a cylinder. The shape of the magnetic field lines for different values of $R_m = \Omega R^2/\lambda$ are shown in Figure 4.6. As R_m increases the distortion of the field becomes greater, and this twisting of the **B**-lines, combined with cross-stream diffusion, gradually eradicates **B** from the rotating fluid. This form of flux expulsion is related to the *skin effect* in electrical engineering. Suppose we change the frame of reference and rotate with the fluid. Then the problem is that of a magnetic field rotating around a stationary conductor. In such a case it is well known that the field will penetrate only a finite distance, $\delta = \sqrt{2\lambda/\Omega}$, into the conductor. This distance is known as the skin depth. As $\Omega \to \infty$ the field is excluded from the interior of the conductor.



(a) $R_m = 1$ (b) $R_m = 10$ (c) $R_m = 25$

Figure 4.6 Distortion and expulsion of a magnetic field by differential rotation. (From H K Moffatt, *Magnetic field generation in electrically conducting fluids*. CUP, 1978, with permission.)

4.5.3 Azimuthal field generation by differential rotation

Our penultimate example of combined advection and diffusion is axisymmetric rather than planar. It is mainly of interest to astrophysics and concerns a rotating fluid permeated by a magnetic field. It turns out that stars do not always rotate as a rigid body. Our own sun, for example, exhibits a variation of rotation with latitude. Consider a non-uniformly rotating star possessing a poloidal magnetic field, i.e. a field of the form $\mathbf{B}_p(r, z) = (B_r, 0, B_z)$ in (r, θ, z) coordinates. Suppose the sun rotates faster at the equator than at its poles, $\mathbf{u} = (0, \Omega(z)r, 0)$, then, by Alfvén's theorem, the poloidal field lines will tend to be advected, as shown in Figure 4.7. The field lines will bow out until such time as the diffusion created by the distortion is large enough to counter the effects of field sweeping. This is readily seen from the azimuthal component of the steady induction equation

$$\frac{\partial \mathbf{B}_\theta}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}_p) + \lambda \nabla^2 \mathbf{B}_\theta = 0$$

where $\mathbf{B}_\mathrm{p}$ is the poloidal magnetic field. The source of the azimuthal field is the term $\nabla \times (\mathbf{u} \times \mathbf{B}_p)$, which may be rewritten as $r(\mathbf{B}_p \cdot \nabla)\Omega$, showing the rôle played by $\Omega(z)$ in generating the azimuthal field. Note that, if λ is very small, then extremely large azimuthal fields may be generated by this mechanism, of order $R_m|\mathbf{B}_p|$. This is a key process in many theories relating to solar MHD, such as the origin of sunspots.

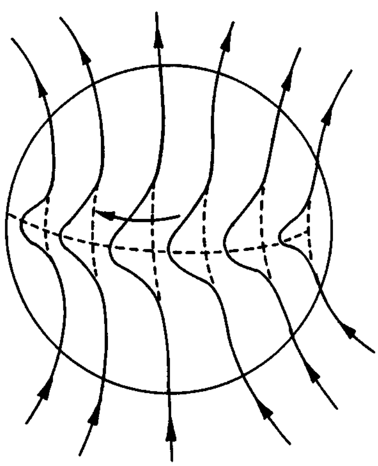


Figure 4.7 Distortion of the magnetic field lines by differential rotation.

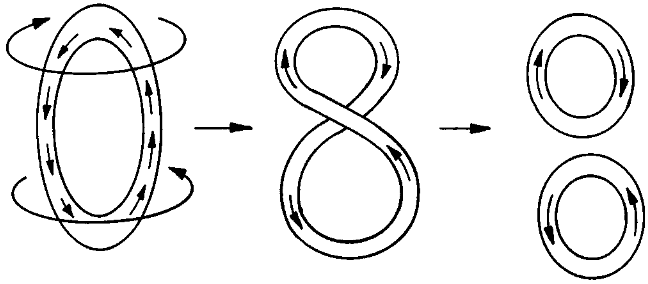


Figure 4.8 Severing of a flux tube.

4.5.4 Magnetic reconnection

Finally, we consider the rôle played by a small but finite diffusivity in the reconnection of magnetic flux tubes. Consider a flux tube in the form of a ring which at $t = 0$ sits in a differentially rotating fluid, as shown in Figure 4.8. When the two branches of the tube come into contact, the field lines locally compress and the gradients in $\mathbf{B}$ become large. Eventually the gradients become so large that, despite the smallness of λ, significant diffusion sets in. The result is that the magnetic field lines reconnect, forming two smaller flux tubes. This kind of process is very important in solar MHD, particularly the theory of solar flares, where the nominal value of R_m is very large, yet flux-tube reconnections are an important part of the origin of flares.

Suggested Reading

J A Shercliff, *A textbook of magnetohydrodynamics*, 1965, Pergamon Press. (Chapter 3)
R Moreau, *Magnetohydrodynamics*, 1990, Kluwer Acad. Pub. (Chapter 2)
P H Roberts, *An Introduction to magnetohydrodynamics*, 1967, Longmans. (Chapter 2)
H K Moffatt, *Magnetic field generation in electrically conducting fluids*, 1978, Cambridge University Press. (Chapter 3 for kinematics, Chapter 5 for sunspots.)

Examples

4.1 A semi-infinite region of conducting material is subject to mutually perpendicular electric and magnetic fields of frequency ω at, and parallel to, its plane boundary. There are no fields deep inside the stationary conductor. Derive expressions for the variation of amplitude and phase of the magnetic field as a function of distance from the surface.

4.2 A perfectly conducting fluid undergoes an axisymmetric motion and contains an azimuthal magnetic field $\mathbf{B}_\theta$. Show that B_θ/r is conserved by each fluid element.

4.3 An electromagnetic flow meter consists of a circular pipe under a uniform transverse magnetic field. The voltage induced by the fluid motion between electrodes, placed at the ends of a diameter of the pipe perpendicular to the field, is used to indicate the flow rate. The pipe walls are insulated and the flow axisymmetric. Show that the induced voltage depends only on the total flow rate and not on the velocity profile.

4.4 A perfectly conducting, incompressible fluid is deforming in such a way that the magnetic field lines are being stretched with a rate of strain S. Show that the magnetic energy rises at a rate SB^2/μ per unit volume.

4.5 Fluid flows with uniform velocity past an insulated, thin flat plate containing a steady current sheet orientated perpendicular to the velocity. The intensity of the current sheet varies sinusoidally with the streamwise coordinate, and the electric field in the fluid is zero. Find the form of the magnetic field and show that it is confined to boundary layers when R_m is large.