1

A Qualitative Overview of MHD

The neglected borderland between two branches of knowledge is often that which best repays cultivation, or, to use a metaphor of Maxwell's, the greatest benefits may be derived from a cross-fertilisation of the sciences.

Rayleigh (1884)

1.1 What is MHD?

Magnetic fields influence many natural and man-made flows. They are routinely used in industry to heat, pump, stir and levitate liquid metals. There is the terrestrial magnetic field which is maintained by fluid motion in the earth's core, the solar magnetic field which generates sunspots and solar flares, and the galactic magnetic field which is thought to influence the formation of stars from interstellar clouds. The study of these flows is called magnetohydrodynamics (MHD). Formally, MHD is concerned with the mutual interaction of fluid flow and magnetic fields. The fluids in question must be electrically conducting and non-magnetic, which limits us to liquid metals, hot ionised gases (plasmas) and strong electrolytes.

The mutual interaction of a magnetic field, $\mathbf{B}$, and a velocity field, $\mathbf{u}$, arises partially as a result of the laws of Faraday and Ampère, and partially because of the Lorentz force experienced by a current-carrying body. The exact form of this interaction is analysed in detail in the following chapters, but perhaps it is worth stating now, without any form of proof, the nature of this coupling. It is convenient, although somewhat artificial, to split the process into three parts.

(i) The relative movement of a conducting fluid and a magnetic field causes an e.m.f. (of order $|\mathbf{u} \times \mathbf{B}|$) to develop in accordance with Faraday's law of induction. In general, electrical currents will ensue, the current density being of order $\sigma(\mathbf{u} \times \mathbf{B})$, σ being the electrical conductivity.

(ii) These induced currents must, according to Ampère's law, give rise to a second, induced magnetic field. This adds to the original magnetic field and the change is usually such that the fluid appears to 'drag' the magnetic field lines along with it.

(iii) The combined magnetic field (imposed plus induced) interacts with the induced current density, **J**, to give rise to a Lorentz force (per unit volume), $\mathbf{J} \times \mathbf{B}$. This acts on the conductor and is generally directed so as to inhibit the relative movement of the magnetic field and the fluid.

Note that these last two effects have similar consequences. In both cases the relative movement of fluid and field tends to be reduced. Fluids can 'drag' magnetic field lines (effect (ii)) and magnetic fields can pull on conducting fluids (effect (iii)). It is this partial 'freezing together' of the medium and the magnetic field which is the hallmark of MHD.

These effects are, perhaps, more familiar in the context of conventional electrodynamics. Consider a wire loop which is pulled through a magnetic field, as shown in Figure 1.1. As the wire loop is pulled to the right, an e.m.f. of order $|\mathbf{u} \times \mathbf{B}|$ is generated which drives a current as shown (effect (i)). The magnetic field, associated with the induced current perturbs the original magnetic field, and the net result is that the magnetic field lines seem to be dragged along by the wire (effect (ii)). The current also gives rise to a Lorentz force, $\mathbf{J} \times \mathbf{B}$, which acts on the wire in a direction opposite to that of the motion (effect (iii)). Thus it is necessary to provide a force to move the wire. In short, the wire appears to drag the field lines while the magnetic field reacts back on the wire, tending to oppose the relative movement of the two.

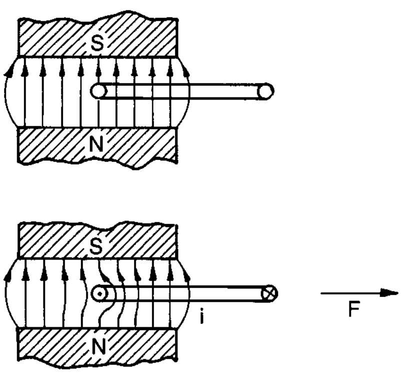

Figure 1.1 Interaction of a magnetic field and a moving wire loop.

Let us consider effect (ii) in a little more detail. As we shall see later, the extent to which a velocity field influences an imposed magnetic field depends on the product of (i) the typical velocity of the motion, (ii) the conductivity of the fluid, and (iii) a characteristic length scale, l, of the motion. Clearly, if the fluid is non-conducting or the velocity negligible there will be no significant induced magnetic field. Conversely, if σ or $\mathbf{u}$ are (in some sense) large, then the induced magnetic field may substantially alter the imposed field. (Consider the wire shown in Figure 1.1. If it is a poor conductor, or moves very slowly, then the induced current, and the associated magnetic field, will be weak.) The reason why l is important is a little less obvious, but may be clarified by the following argument. The e.m.f. generated by a relative movement of the imposed magnetic field and the medium is of order $|\mathbf{u} \times \mathbf{B}|$ and so, by Ohm's law, the induced current density is of the order of $\sigma(|\mathbf{u} \times \mathbf{B}|)$. However, a modest current density spread over a large area can produce a high magnetic field, whereas the same current density spread over a small area induces only a weak magnetic field. It is therefore the product $\sigma \mathbf{u} l$ which determines the ratio of the induced field to the applied magnetic field. In the limit $\sigma \mathbf{u} l \to \infty$ (typical of so-called ideal conductors) the induced and imposed magnetic fields are of the same order. In such cases it turns out that the combined magnetic field behaves as if it were locked into the fluid. Conversely, when $\sigma \mathbf{u} l \to 0$, the imposed magnetic field remains relatively unperturbed. Astrophysical MHD tends to be closer to the first situation, not so much because of the high conductivity of the plasmas involved, but because of the vast characteristic length scale. Liquid-metal MHD, on the other hand, generally lies closer to the second limit, with $\mathbf{u}$ leaving $\mathbf{B}$ unperturbed. Nevertheless, it should be emphasised that effect (iii) is still strong in liquid metals, so that an imposed magnetic field can substantially alter the velocity field.

Perhaps it is worth taking a moment to consider the case of liquid metals in a little more detail. They have a reasonable conductivity ($\sim 10^6 \Omega^{-1} \mathrm{m}^{-1}$), but the velocity involved in a typical laboratory or industrial process is small ($\sim 1\,\mathrm{m/s}$). As a consequence, the induced current densities are generally rather modest (a few Amps per cm^2). When this is combined with a small length-scale ($\sim 0.1\,\mathrm{m}$ in the laboratory), the induced magnetic field is usually found to be negligible by comparison with the imposed field. There is very little 'freezing together' of the fluid and the magnetic field. However, the imposed magnetic field is often strong enough for the Lorentz force, $\mathbf{J} \times \mathbf{B}$, to dominate the motion of the fluid. We tend to think of the coupling as being one way: $\mathbf{B}$ controls $\mathbf{u}$

through the Lorentz force, but **u** does not substantially alter the imposed field, **B**. There are, however, exceptions. Perhaps the most important of these is the earth's dynamo. Here, motion in the liquid-metal core of the earth twists, stretches and intensifies the terrestrial magnetic field, maintaining it against the natural processes of decay. It is the large length-scales which are important here. While the induced current densities are weak, they are spread over a large area and so their combined effect is to induce a substantial magnetic field.

In summary then, the freezing together of the magnetic field and the medium is usually strong in astrophysics, significant in geophysics, weak in metallurgical MHD and utterly negligible in electrolytes. However, the influence of **B** on **u** can be important in all four situations.

1.2 A Brief History of MHD

The laws of magnetism and fluid flow are hardly a twentieth-century innovation, yet MHD became a fully fledged subject only in the late 1930s or early 1940s. The reason, probably, is that there was little incentive for nineteenth century engineers to capitalise on the possibilities offered by MHD. Thus, while there were a few isolated experiments by nineteenth-century physicists such as Faraday (he tried to measure the voltage across the Thames induced by its motion through the earth's magnetic field), the subject languished until the turn of the century. Things started to change, however, when astrophysicists realised just how ubiquitous magnetic fields and plasmas are throughout the universe. This culminated in 1942 with the discovery of the Alfvén wave, a phenomenon which is peculiar to MHD and important in astrophysics. (A magnetic field line can transmit transverse inertial waves, just like a plucked string.) Around the same time, geophysicists began to suspect that the earth's magnetic field was generated by dynamo action within the liquid-metal of its core, an hypothesis first put forward in 1919 by Larmor in the context of the sun's magnetic field. A period of intense research followed and continues to this day.

Plasma physicists, on the other hand, acquired an interest in MHD in the 1950s as the quest for controlled thermonuclear fusion gathered pace. They were particularly interested in the stability, or lack of stability, of plasmas confined by magnetic fields, and great advances in stability theory were made as a result.

The development of MHD in engineering was slower and did not really get going until the 1960s. However, there was some early pioneering work by the engineer J. Hartmann, who invented the electromagnetic pump in 1918. Hartmann also undertook a systematic theoretical and experimental investigation of the flow of mercury in a homogeneous magnetic field. In the introduction to the 1937 paper describing his researches he observed:

> The invention [his pump] is, as will be seen, no very ingenious one, the principle utilised being borrowed directly from a well-known apparatus for measuring strong magnetic fields. Neither does the device represent a particularly effective pump, the efficiency being extremely low due mainly to the large resistivity of mercury and still more to the contact resistance between the electrodes and the mercury. In spite hereof considerable interest was in the course of time bestowed on the apparatus, firstly because of a good many practical applications in cases where the efficiency is of small moment and then, during later years, owing to its inspiring nature. As a matter of fact, the study of the pump revealed to the author what he considered a new field of investigation, that of flow of a conducting liquid in a magnetic field, a field for which the name Hg-dynamics was suggested.

The name, of course, did not stick, but we may regard Hartmann as the father of liquid-metal MHD, and indeed the term 'Hartmann flow' is now used to describe duct flows in the presence of a magnetic field. Despite Hartmann's early researches, it was only in the early 1960s that MHD began to be exploited in engineering. The impetus for change came largely as a result of three technological innovations: (i) fast-breeder reactors use liquid sodium as a coolant and this needs to be pumped; (ii) controlled thermonuclear fusion requires that the hot plasma be confined away from material surfaces by magnetic forces; and (iii) MHD power generation, in which ionised gas is propelled through a magnetic field, was thought to offer the prospect of improved power station efficiencies. This last innovation turned out to be quite impracticable, and its failure was rather widely publicised in the scientific community. However, as the interest in power generation declined, research into metallurgical MHD took off. Two decades later, magnetic fields are routinely used to heat, pump, stir and levitate liquid metals in the metallurgical industries. The key point is that the Lorentz force provides a non-intrusive means of controlling the flow of metals. With constant commercial pressure to produce cheaper, better and more consistent materials, MHD provides a unique means of exercising greater control over casting and refining processes.

1.3 From Electrodynamics to MHD: A Simple Experiment

Now the only difference between MHD and conventional electrodynamics lies in the fluidity of the conductor. This makes the interaction between **u** and **B** more subtle and difficult to quantify. Nevertheless, many of the important features of MHD are latent in electrodynamics and can be exposed by simple laboratory experiments. An elementary grasp of electromagnetism is then all that is required to understand the phenomena. Just such an experiment is described below. First, however, we shall discuss those features of MHD which the experiment is intended to illustrate.

1.3.1 Some important parameters in electrodynamics and MHD

Let us introduce some notation. Let μ be the permeability of free space, σ and ρ denote the electrical conductivity and density of the conducting medium, respectively, and l be a characteristic length scale. Three important parameters in MHD are:

$$\text{Magnetic Reynolds number,} \quad R_m = \mu\sigma u l \tag{1.1}$$

$$\text{Alfvèn velocity,} \quad v_a = B/\sqrt{\rho\mu} \tag{1.2}$$

$$\text{Magnetic damping time,} \quad \tau = \left[\sigma B^2/\rho\right]^{-1} \tag{1.3}$$

The first of these parameters may be considered as a dimensionless measure of the conductivity, while the second and third quantities have the dimensions of speed and time, respectively, as their names suggest.

Now we have already hinted that magnetic fields behave very differently depending on the conductivity of the medium. In fact, it turns out to be R_m, rather than σ, which is important. Where R_m is large, the magnetic field lines act rather like elastic bands frozen into the conducting medium. This has two consequences. First, the magnetic flux passing through any closed material loop (a loop always composed of the same material particles) tends to be conserved during the motion of the fluid. This is indicated in Figure 1.1. Second, as we shall see, small disturbances of the medium tend to result in near-elastic oscillations, with the magnetic field providing the restoring force for the vibration. In a fluid, this results in Alfvén waves, which turn out to have a frequency of $\omega \sim v_a/l$

When R_m is small, on the other hand, **u** has little influence on **B**, the induced field being negligible by comparison with the imposed field. The magnetic field then behaves quite differently. We shall see that it is dissipative in nature, rather than elastic, damping mechanical motion by converting kinetic energy into heat via Joule dissipation. The relevant time scale is now the damping time, τ, rather than l/v_a.

All of this is dealt with more fully in Chapters 4–6. The purpose of this chapter is to show that a familiar high-school experiment is sufficient to expose these two very different types of behaviour, and to highlight the important rôles played by R_m, v_a and τ.

1.3.2 A brief reminder of the laws of electrodynamics

Let us start with a reminder of the elementary laws of electromagnetism. (A more detailed discussion of these laws is given in Chapter 2.) The laws which concern us here are those of Ohm, Faraday and Ampère. We start with Ohm's law (Figure 1.2(i)).

This is an empirical law which, for stationary conductors, takes the form $\mathbf{J} = \sigma\mathbf{E}$, where **E** is the electric field and **J** the current density. We interpret this as **J** being proportional to the Coulomb force $f = q\mathbf{E}$ which acts on the free charge carriers, q being their charge. If, however, the conductor is moving in a magnetic field with velocity **u**, the free charges will experience an additional force, $q\mathbf{u} \times \mathbf{B}$, and Ohm's law becomes

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{1.4}$$

The quantity $\mathbf{E} + \mathbf{u} \times \mathbf{B}$, which is the total electromagnetic force per unit charge, arises frequently in electrodynamics and it is convenient to give it a label. We use

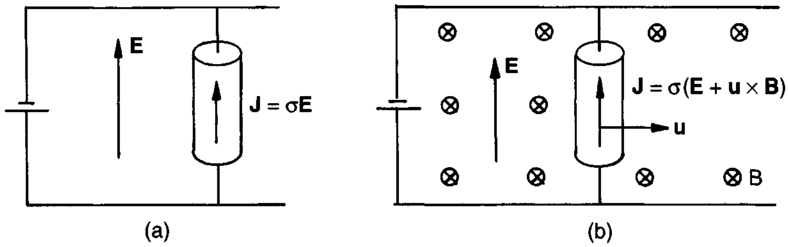

Figure 1.2 (i) Ohm's law in stationary and moving conductors.

$$\mathbf{E_r} = \mathbf{E} + \mathbf{u} \times \mathbf{B} = \mathbf{f}/q$$

Formally, $\mathbf{E}_r$ is the electric field measured in a frame of reference moving with velocity $\mathbf{u}$ relative to the laboratory frame (see Chapter 2). However, for the present purposes it is more useful to think of $\mathbf{E}_r$ as $\mathbf{f}/q$. Some authors refer to $\mathbf{E}_r$ as the *effective electric field.* In terms of $\mathbf{E}_r$, (1.4) becomes $\mathbf{J} = \sigma\mathbf{E}_r$.

Faraday's law (Figure 1.2 (ii)) tells us about the e.m.f. which is generated in a conductor as a result of: (i) a time-dependent magnetic field; or (ii) the motion of a conductor within a magnetic field. In either case Faraday's law may be written as

$$\text{emf} = \oint_C \mathbf{E}_r \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{1.5}$$

Here C is a closed curve composed of line elements $d\mathbf{l}$. The curve may be fixed in space, or else move with the conducting medium (if the medium does indeed move). S is any surface which spans C. (We use the right-hand screw convention to define the positive directions of $d\mathbf{l}$ and $d\mathbf{S}$.) The subscript on $\mathbf{E_r}$ indicates that we must use the 'effective' electric field for each line element $d\mathbf{l}$:

$$\mathbf{E}_r = \mathbf{E} + \mathbf{u} \times \mathbf{B} \tag{1.6}$$

where $\mathbf{E}$, $\mathbf{u}$ and $\mathbf{B}$ are measured in the laboratory frame and $\mathbf{u}$ is the velocity of the line element $d\mathbf{l}$.

Next, we need Ampère's law (Figure 1.3). This (in a round-about way) tells us about the magnetic field associated with a given distribution of current, $\mathbf{J}$. If C is a closed curve drawn in space, and S is any surface spanning that curve, then Ampère's circuital law states that

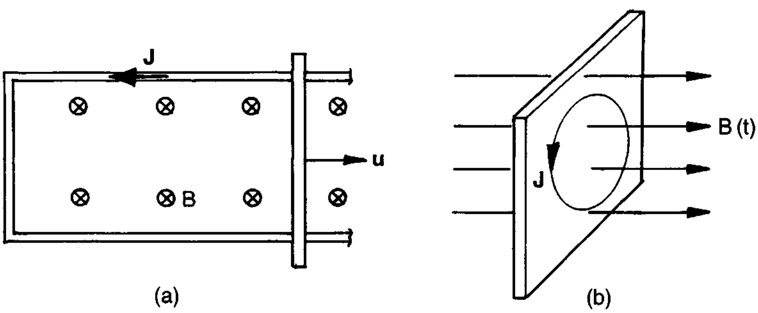

Figure 1.2 (ii) Faraday's law (a) e.m.f. generated by movement of a conductor; (b) e.m.f. generated by a time-dependent magnetic field.

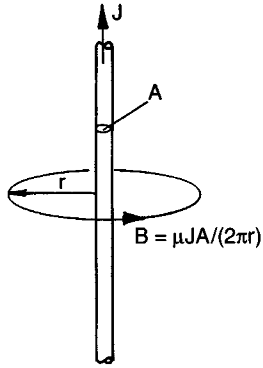

Figure 1.3 Ampère's law applied to a wire.

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu \int_S \mathbf{J} \cdot d\mathbf{S} \tag{1.7}$$

Finally, there is the Lorentz force, **F**. This acts on all conductors carrying a current in a magnetic field. It has its origins in the force acting on individual charge carriers, $\mathbf{f} = q(\mathbf{u} \times \mathbf{B})$ and it is easy to show that the force per unit volume of the conductor is given by

$$\mathbf{F} = \mathbf{J} \times \mathbf{B} \tag{1.8}$$

1.3.3 A familiar high-school experiment

We now turn to the laboratory experiment. Consider the apparatus illustrated in Figure 1.4. This is frequently used to illustrate Faraday's law of induction. It consists of a horizontal, rectangular circuit sitting in a vertical magnetic field, $\mathbf{B}_0$. The circuit is composed of a frictionless, conducting slide which is free to move horizontally between two rails. We take the rails and slide to have a common thickness Δ and to be made from the same material. To simplify matters, we shall also suppose that the depth of the apparatus is much greater than its lateral dimensions, L and W, so that we may treat the problem as two-dimensional. Also, we take Δ to be much smaller than L or W.

We now show that, if the slide is given a tap, and it has a high conductivity, it simply vibrates as if held in place by a (magnetic) spring. On the other hand, if the conductivity is low, it moves forward as if immersed in treacle, slowing down on a time scale of τ. Suppose that, at $t = 0$, the

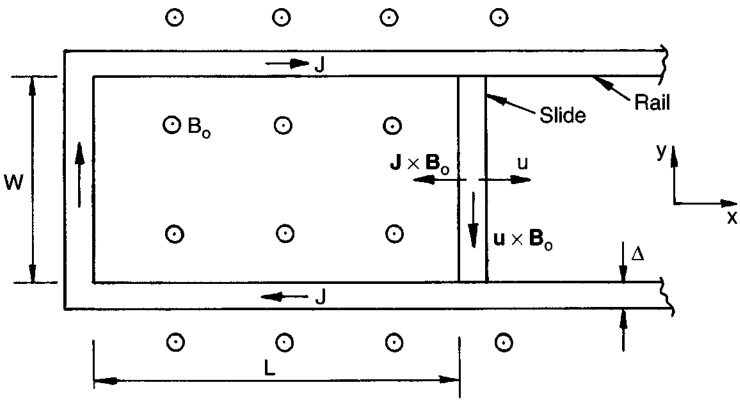

Figure 1.4 A simple experiment for illustrating MHD phenomena.

slide is given a forward motion, **u**. This movement of the slide will induce a current density, **J**, as shown. This, in turn, produces an induced field $\mathbf{B}_i$ which is negligible outside the closed current path but is finite and uniform within the current loop. It may be shown, from Ampère's law, that $\mathbf{B}_i$ is directed downward and has a magnitude and direction given by

$$\mathbf{B}_i = -(\mu \Delta J)\hat{\mathbf{e}}_z \tag{1.9}$$

Note that the direction of $\mathbf{B}_i$ is such as to try to maintain a constant flux in the current loop (Lenz's law) (Figure 1.5). Next we combine (1.4) and (1.5) to give

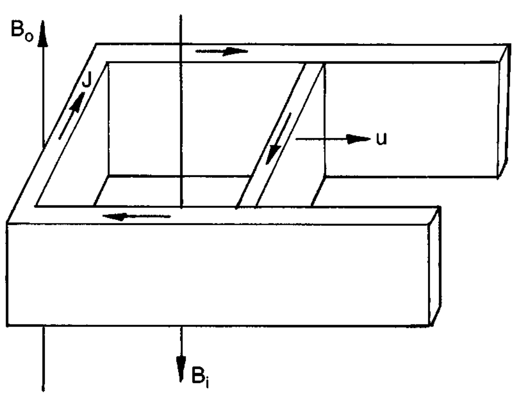

Figure 1.5 Direction of the magnetic field induced by current in the slide.

$$\frac{1}{\sigma}\oint_C \mathbf{J}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} \tag{1.10}$$

where C is the material circuit comprising the slide and the return path for $\mathbf{J}$. This yields

$$\frac{d\Phi}{dt} = \frac{d}{dt}[LW(B_0 - \mu\Delta J)] = 2J(L+W)/\sigma \tag{1.11}$$

Here $\Phi = (B_0 - \mu\Delta J)LW$ is the flux through the circuit (see Figure 1.6). Finally, the Lorentz force (per unit depth) acting on the slide is

$$\mathbf{F} = -J(B_0 - \mu\Delta J/2)\Delta W\hat{\mathbf{e}}_x \tag{1.12}$$

where the expression in parentheses represents the average field within the slide (Figure 1.7). The equation of motion for the slide is therefore

$$\rho\frac{d^2L}{dt^2} = \rho\frac{du}{dt} = -J(B_0 - \mu\Delta J/2) \tag{1.13}$$

where ρ is the density of the metal.

Equations (1.11) and (1.13) are sufficient to determine the two unknown functions $L(t)$ and $J(t)$. Let us introduce some simplifying notation: $B_I = \mu\Delta J$, $l = \Delta W/L$, $T = \mu\sigma\Delta W$ and $R_m = \mu\sigma ul$. Evidently, B_I is the magnitude of the induced field and T is a measure of the conductivity, σ, which happens to have the dimensions of time. Our two equations may be rewritten as

$$\frac{d}{dt}[L(B_0 - B_I)] = \frac{2(L+W)B_I}{T} \tag{1.14}$$

and

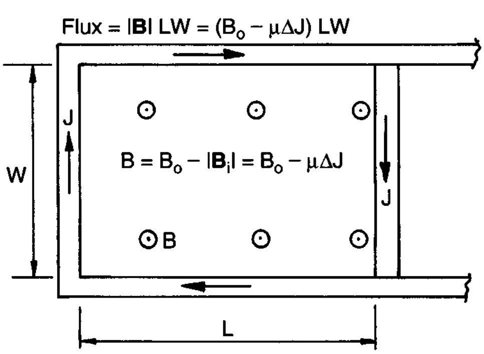

Figure 1.6 Relationship between flux and current.

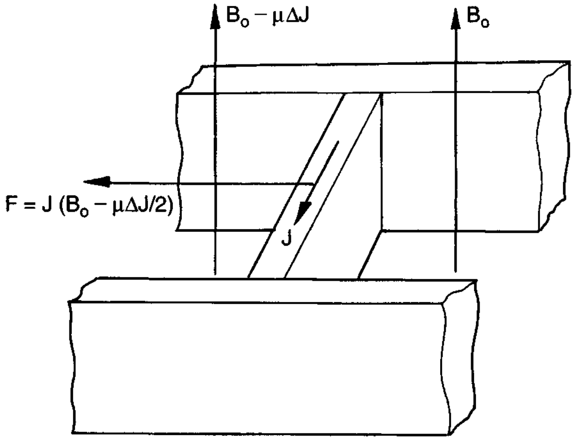

Figure 1.7 Forces acting on the slide.

$$2\rho\mu\Delta\frac{d^2L}{dt^2} = 2\rho\mu\Delta\frac{du}{dt} = (B_0 - B_I)^2 - B_0^2 \qquad (1.15)$$

Now we might anticipate that the solutions of (1.14) and (1.15) will depend on the conductivity of the apparatus as represented by T, and so we consider two extreme cases:

1. high conductivity limit; $\frac{u}{L} \gg \frac{1}{T}$ $(R_m = \mu\sigma ul \gg 1)$
2. low conductivity limit; $\frac{u}{L} \ll \frac{1}{T}$ $(R_m = \mu\sigma ul \ll 1)$

In the high conductivity limit, the right-hand side of (1.14) may be neglected and so the flux Φ linking the current path is conserved during the motion. In such cases we may look for solutions of (1.15) of the form $L = L_0 + \eta$, where η is an infinitesimal change of L and $L_0 = \Phi/B_0W$. Multiplying through (1.15) by L^2W, noting that Φ is constant and equal to L_0B_0W, and retaining only leading order terms in η, yields

$$\frac{d^2\eta}{dt^2} + \frac{B_0^2}{\rho\mu\Delta L_0}\eta = 0 \qquad (1.16)$$

Thus, when the magnetic Reynolds number is high, the slide oscillates in an elastic manner, with an angular frequency of $\omega \sim v_a/\sqrt{\Delta L_0}$, v_a, being the Alfvén velocity. In short, if we tap the slide it will vibrate (Figure 1.8). It seems to be held in place by the magnetic field.

Now consider the low conductivity limit, $R_m \ll 1$ (Figure 1.9). In this case the induction equation (1.14) tells us that $B_I \ll B_0$ and so the left-

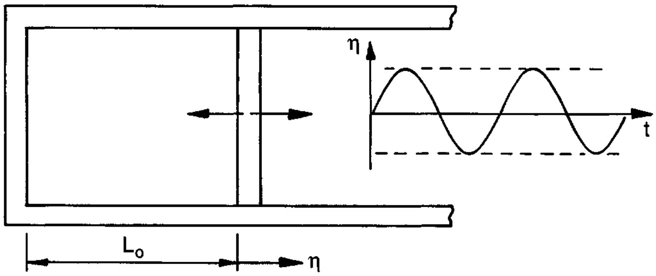

Figure 1.8 Oscillation of the slide when $R_m \gg 1$.

hand side of (1.14) reduces to uB_0. Substituting for B_I (in terms of u) in the equation of motion (1.15) yields

$$\frac{du}{dt} + \frac{W}{2(L+W)}\left(\frac{\sigma B_0^2}{\rho}\right)u = 0 \tag{1.17}$$

Again we look for solutions of the form $L_0 + \eta$, with $\eta \ll L_0$ and $L_0 = L(t=0)$. This time u declines exponentially on a time scale of $\tau = (\sigma B_0^2/\rho)^{-1}$, the magnetic damping time. The magnetic field now appears to play a dissipative rôle. Indeed, it is not difficult to show that

$$\frac{dE}{dt} = -\int (J^2/\sigma)dV \tag{1.18}$$

where the volume integral is taken over the entire conductor and E is the kinetic energy of the slide. Thus the mechanical energy of the slide is lost to heat via Ohmic dissipation.

Let us summarise our findings. When $R_m \gg 1$, and the slide is abruptly displaced from its equilibrium position, it oscillates in an elastic manner at a frequency proportional to the Alfvén velocity. During the oscillation

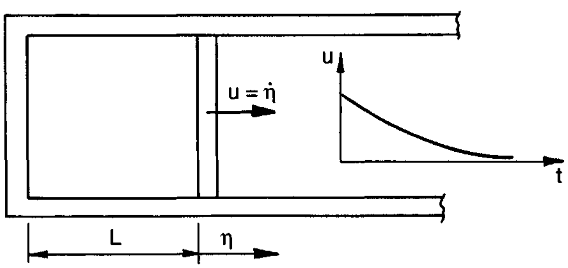

Figure 1.9 Motion of the slide when $R_m \ll 1$.

the magnetic flux trapped between the slide and the rails remains constant. If $R_m \ll 1$, on the other hand, and the slide is given a push, it moves forward as if it were immersed in treacle. Its kinetic energy decays exponentially on a time scale of $\tau = (\sigma B_0^2/\rho)^{-1}$, the energy being lost to heat via Ohmic dissipation. Also, when R_m is small, the induced magnetic field is negligible.

We shall see that precisely the same behaviour occurs in fluids. The counterpart of the vibration is an Alfvén wave (Figure 1.10), which is a common feature of astrophysical MHD. In liquid-metal MHD, on the other hand, the primary rôle of **B** is to dissipate mechanical energy on a time scale of τ.

We have yet to explain these two types of behaviour. Consider first the high R_m case. Here the key equation is Faraday's law (1.10),

$$\frac{1}{\sigma}\oint_C \mathbf{J}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}$$

As $\sigma \to \infty$, the flux, Φ, enclosed by the slide and rails must be conserved. If the slide is pushed forward, $J = B_I/\mu\Delta$ must rise to conserve Φ. The Lorentz force therefore increases until the slide is halted. At this point the Lorentz force $\mathbf{J} \times \mathbf{B}$ is finite but **u** is zero and so the slide starts to return. The induced field B_I, and hence J, now falls to maintain the magnetic flux. Eventually the slide returns to its equilibrium position and the Lorentz force falls to zero. However, the inertia of the slide carries it over its neutral point and the whole process now begins in reverse. It is the conservation of flux, combined with the inertia of the conductor,

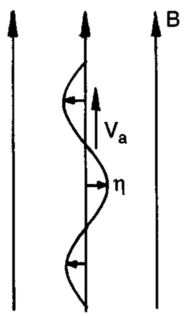

Figure 1.10 Alfvén waves. A magnetic field behaves like a plucked string, transmitting a transverse inertial wave with a phase velocity of v_a.

which leads to oscillations in this experiment, and to Alfvén waves in plasmas (Figure 1.11).

Now consider the case where $R_m \ll 1$. It is Ohm's law which plays the critical rôle here. The high resistivity of the circuit means that the currents, and hence induced field, are small. We may consider $\mathbf{B}$ to be approximately equal to the imposed field, $\mathbf{B}_0$. Since $\mathbf{B}$ is now almost constant, the electric field must be irrotational

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \approx 0$$

Ohm's law and the Lorentz force per unit volume now simplify to

$$\mathbf{J} = \sigma[-\nabla V + \mathbf{u} \times \mathbf{B}_0], \qquad \mathbf{F} = \mathbf{J} \times \mathbf{B}_0 \tag{1.19a,b}$$

where V is the electrostatic potential. Integrating Ohm's law around the closed current loop eliminates V and yields a simple relationship between u and J:

$$2J(L + W) = \sigma W B_0 u$$

The Lorentz force per unit mass becomes

$$\frac{\mathbf{F}}{\rho} = -\frac{W}{2(L+W)}\left(\frac{\sigma B_0^2}{\rho}\right)\mathbf{u} \sim -\frac{\mathbf{u}}{\tau}$$

from which

$$\frac{d\mathbf{u}}{dt} \sim -\frac{\mathbf{u}}{\tau} \tag{1.20}$$

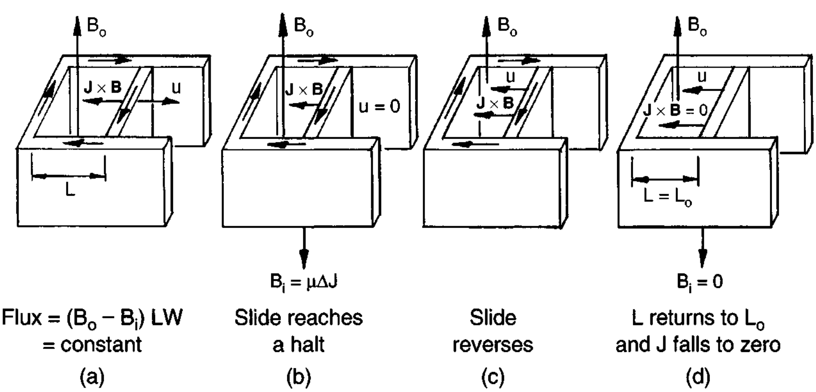

Figure 1.11 Mechanism for oscillation of the slide.

Thus the slide slows down exponentially on a time scale of τ. The rôle of the induced current here is quite different to the high R_m case. The fact that J creates an induced field is irrelevant. It is the contribution of J to the Lorentz force $\mathbf{J} \times \mathbf{B}_0$ which is important. This always acts to retard the motion. As we shall see, the two equations $\mathbf{J} = \sigma[-\nabla V + \mathbf{u} \times \mathbf{B}_0]$ and $\mathbf{F} = \mathbf{J} \times \mathbf{B}_0$ are the hallmark of low-R_m MHD.

This familiar high-school experiment encapsulates many of the phenomena which will be explored in the subsequent chapters. The main difference is that fluids have, of course, none of the rigidity of electrodynamic machines, and so they behave in more subtle and complex ways. Yet it is precisely this subtlety which makes MHD so intriguing.

1.3.4 A summary of the key results for MHD

1. When the medium is highly conducting ($R_m \gg 1$), Faraday's law tells us that the flux through any closed material loop is conserved. When the material loop contracts or expands, currents flow so as to keep the flux constant. These currents lead to a Lorentz force which tends to oppose the contraction or expansion of the loop. The result is an elastic oscillation with a characteristic frequency of $\sim v_a/l$, v_a being the Alfvén velocity.
2. When the medium is a poor conductor (($R_m \ll 1$), the magnetic field induced by motion is negligible by comparison with the imposed field, B_0. The Lorentz force and Ohm's law simplify to

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}_0, \qquad \mathbf{J} = \sigma[-\nabla V + \mathbf{u} \times \mathbf{B}_0]$$

   The Lorentz force is now dissipative in nature, converting mechanical energy into heat on a time scale of the magnetic damping time, τ.

Statements 1 and 2 are, in effect, a summary of Chapters 4–6.

1.4 Some Simple Applications of MHD

We close this introductory chapter with a brief overview of the scope of MHD, and of this book. In fact, MHD operates on every scale, from the vast to the minute. For example, magnetic fields pervade interstellar space and aid the formation of stars by removing excess angular momentum from collapsing interstellar clouds. Closer to home, sunspots and solar flares are magnetic in origin, sunspots being caused by buoyant

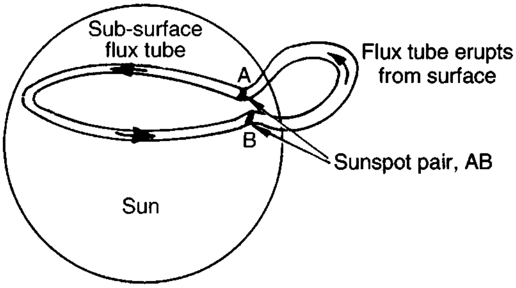

Figure 1.12 Sunspot formation.

magnetic flux tubes, perhaps 10^4 km in diameter and 10^5 km long, erupting from the surface of the sun (Figure 1.12). Sunspots are discussed in Chapter 4.

Back on earth, the terrestrial magnetic field is now known to be maintained by fluid motion in the core of the earth (Figure 1.13). This process, called dynamo action, is reviewed in Chapter 6.

MHD is also an intrinsic part of controlled thermonuclear fusion. Here plasma temperatures of around 10^8 K must be maintained, and magnetic forces are used to confine the hot plasma away from the reactor walls. A simple example of a confinement scheme is shown in Figure 1.14. Unfortunately, such schemes are prone to hydrodynamic instabilities, the nature of which is discussed in Chapter 6.

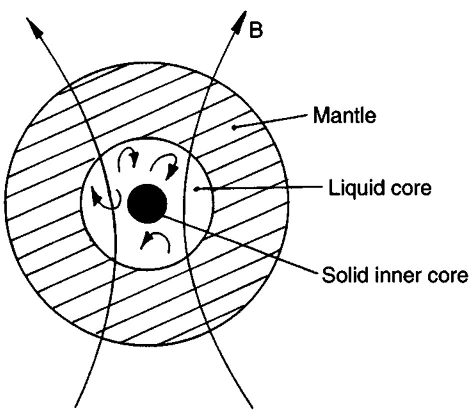

Figure 1.13 Motion in the earth's core maintains the terrestrial magnetic field.

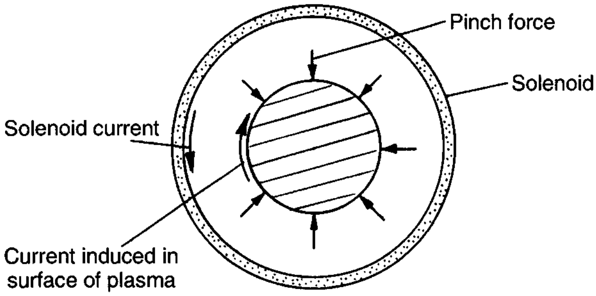

Figure 1.14 Plasma confinement. A current in the solenoid which surrounds the plasma induces an opposite current in the surface of the plasma and the resulting Lorentz force pinches radially inward.

In the metallurgical industries, magnetic fields are routinely used to heat, pump, stir and levitate liquid metals. Perhaps the earliest application of MHD is the electromagnetic pump (Figure 1.15). This simple device consists of mutually perpendicular magnetic and electric fields arranged normal to the axis of a duct. Provided the duct is filled with a conducting liquid, so that currents can flow, the resulting Lorentz force provides the necessary pumping action. First proposed back in 1832, the electromagnetic pump has found its ideal application in fast-breeder nuclear reactors, where it is used to pump liquid sodium coolant through the reactor core.

Perhaps the most widespread application of MHD in engineering is the use of electromagnetic stirring. A simple example is shown in Figure 1.16. Here the liquid metal which is to be stirred is placed in a rotating magnetic field. In effect, we have an induction motor, with the liquid metal taking the place of the rotor. This is routinely used in casting operations to homogenise the liquid zone of a partially solidified ingot. The resulting motion has a profound influence on the solidification process, ensuring good mixing of the alloying elements and the continual fragmentation of

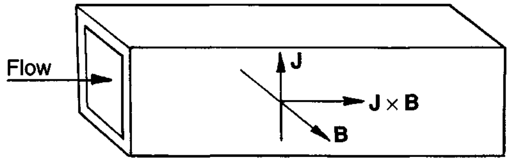

Figure 1.15 The electromagnetic pump.

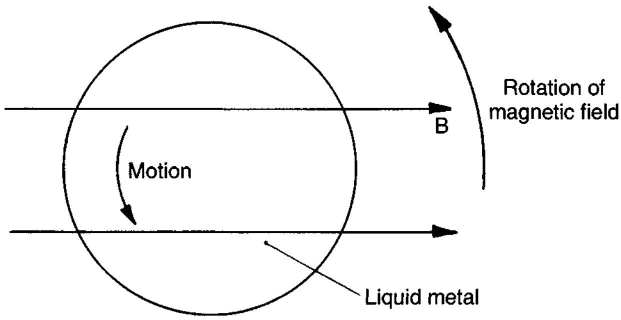

Figure 1.16 Magnetic stirring of an ingot.

the snowflake-like crystals which form in the melt. The result is a fine-structured, homogeneous ingot. This is discussed in detail in Chapter 8.

Perversely, in yet other casting operations, magnetic fields are used to dampen the motion of liquid metal. Here we take advantage of the ability of a static magnetic field to convert kinetic energy into heat via Joule dissipation (as discussed in the last section). A typical example is shown in Figure 1.17, in which an intense, static magnetic field is imposed on a casting mould. Such a device is used when the fluid motion within the mould has become so violent that the free surface of the liquid is disturbed, causing oxides and other pollutants to be entrained into the bulk. The use of magnetic damping promotes a more quiescent process, thus minimising contamination. The damping of jets and vortices is discussed in Chapters 5 and 9.

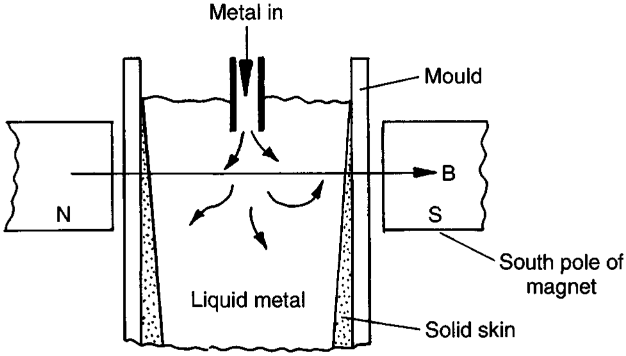

Figure 1.17 Magnetic damping of motion during casting.

Another common application of MHD in metallurgy is magnetic levitation or confinement. This relies on the fact that a high-frequency induction coil repels conducting material by inducing opposing currents in any adjacent conductor (opposite currents repel each other). Thus a 'basket' formed from a high-frequency induction coil can be used to levitate and melt highly reactive metals, or a high-frequency solenoid can be used to form a non-contact magnetic valve which modulates and guides a liquid metal jet (Figure 1.18). Such applications are discussed in Chapter 12.

MHD is also important in electrolysis, particularly in those electrolysis cells used to reduce aluminium oxide to aluminium. These cells consist of broad but shallow layers of electrolyte and liquid aluminium, with the electrolyte lying on top. A large current (perhaps 200 kAmps) passes vertically downward through the two layers, continually reducing the oxide to metal. The process is highly energy intensive, largely because of the high electrical resistance of the electrolyte. For example, in the USA, around 3% of all generated electricity is used for aluminium production. It has long been known that stray magnetic fields can destabilise the interface between the electrolyte and aluminium, in effect through the generation of interfacial gravity waves (Figure 1.19). In order to avoid this instability, the electrolyte layer must be maintained at a depth above some critical threshold, and this carries with it a severe energy penalty. This instability turns out to involve a rather subtle mechanism, in which interfacial oscillations absorb energy from the ambient magnetic field,

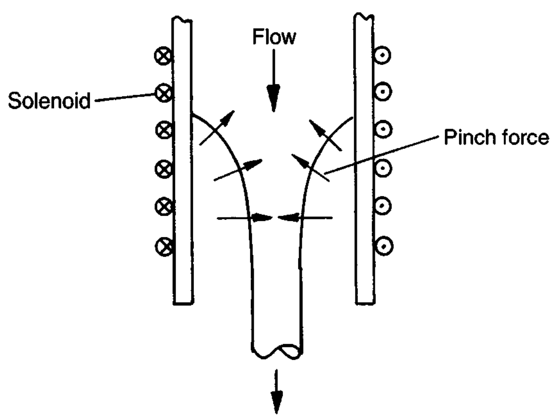

Figure 1.18 An electromagnetic valve.

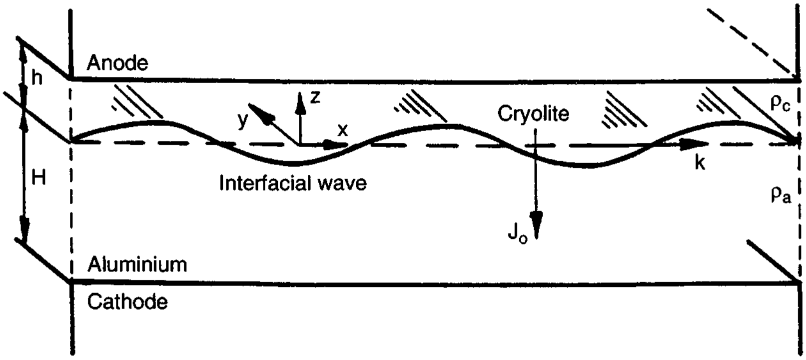

Figure 1.19 Instabilities in an aluminium reduction cell.

converting it into kinetic energy. The stability of aluminium reduction cells is discussed in Chapter 11.

There are many other applications of MHD in engineering and metallurgy which, in the interests of brevity, we have not described here. These include electromagnetic (non-contact) casting of aluminium, vacuum-arc remelting of titanium and nickel-based super alloys (a process which resembles a gigantic electric welding rod – see Chapter 10), electromagnetic removal of non-metallic inclusions from melts, electromagnetic launchers (which have the same geometry as Figure 1.4, but where the slide is now a projectile and current is forced down the rails accelerating the slide) and the so-called 'cold-crucible' induction melting process, in which the melt is protected from the crucible walls by a thin solid crust of its own material. This latter technology is currently finding favour in the nuclear industry, where it is used to vitrify highly active nuclear waste.

All-in-all, it would seem that MHD has now found a substantial and permanent place in the world of materials processing. However, it would be wrong to pretend that every engineering venture in MHD has been a success, and so we end this section on a lighter note, describing one of MHD's less notable developments: that of MHD propulsion for military submarines.

Stealth is all important in the military arena and so, in an attempt to eliminate the detectable (and therefore unwanted) cavitation noise associated with propellers, MHD pumps were once proposed as a propulsion mechanism for submarines. The idea is that sea-water is drawn into ducts at the front of the submarine, passed through MHD pumps within the submarine hull, and then expelled at the rear of the vessel in the form of high-speed jets. It is an appealing idea, dating back to the 1960s, and in principle it works, as demonstrated recently in Japan by the surface ship *Yamato*. Indeed, this idea has even found its way into popular fiction! The concept found renewed favour with the military authorities in the 1980s (the armaments race was at fever pitch) and serious design work commenced. Unfortunately, however, there is a catch. It turns out that the conductivity of sea-water is so poor that the efficiency of such a device is, at best, a few per cent, nearly all of the energy going to heat the water. Worse still, the magnetic field required to produce a respectable thrust is massive, at the very limits of the most powerful superconducting magnets. So, while in principle it is possible to eliminate propeller cavitation, in the process a (highly detectable) magnetic signature is generated, to say nothing of the thermal and chemical signatures induced by electrolysis in the ducts. To locate an MHD submarine, therefore, you simply have to borrow a Gauss meter, buy a thermometer, invest in litmus paper, or just follow the trail of dead fish!

Submarine propulsion apart, engineering MHD has scored some notable successes in recent years, particularly in its application to metallurgy. It is this which forms the basis of Part B of this text.

Suggested Reading

J A Shercliff: *A textbook of magnetohydrodynamics, 1965,* Pergamon Press. (Chapter 1 gives a brief history of MHD.)

Examples

1.1 A bar of small but finite conductivity slides at a constant velocity u along conducting rails in a region of uniform magnetic field. The resistance in the circuit is R and the inductance is negligible. Calculate: (i) the current I flowing in the circuit; (ii) the power required to move the bar; and (iii) the Ohmic losses in the circuit.

1.2 A square metal bar of length l and mass m slides without friction down parallel conducting rails of negligible resistance. The rails are connected to each other at the bottom by a resistanceless rail parallel to the bar, so that the bar and rails form a closed loop. The plane of the rails makes an angle θ with the horizontal, and a uniform vertical field, B, exists throughout the region. The bar has a small but finite conductivity and has a resistance of R. Show that the bar acquires a steady velocity of $u = mgR\sin\theta/(Bl\cos\theta)^2$.

1.3 A steel rod is 0.5 m long and has a diameter of 1 cm. It has a density and conductivity of $7 \times 10^3\,\text{kg/m}^3$ and 10^6 mho/m, respectively. It lies horizontally with its ends on two parallel rails, 0.5 m apart. The rails are perfectly conducting and are inclined at an angle of 15° to the horizontal. The rod slides up the rails with a coefficient of friction of 0.25, propelled by a battery which maintains a constant voltage difference of 2 V between the rails. There is a uniform, unperturbed vertical magnetic field of 0.75 T. Find the velocity of the bar when travelling steadily.

1.4 When Faraday's and Ohm's laws are combined, we obtain (1.10). Consider an isolated flux tube sitting in a perfectly conducting fluid, and let C_m be a material curve (a curve always composed of the same material) which at some initial instant encircles the flux tube, lying on the surface of the tube. Show that the flux enclosed by C_m will remain constant as the flow evolves, and that this is true of each and every curve enclosing the tube at $t = 0$. This suggests that the tube itself moves with the fluid, as if frozen into the medium. Now suppose that the diameter of the flux tube is very small. What does this tell us about magnetic field lines in a perfectly conducting fluid?

1.5 Consider a two-dimensional flow consisting of an (initially) thin jet propagating in the x-direction and sitting in a uniform magnetic field which points in the y-direction. The magnetic Reynolds number is low. Show that the Lorentz force (per unit volume) acting on the fluid is $-\sigma u_x B^2 \hat{\mathbf{e}}_x$. Now consider a fluid particle sitting on the axis of the jet. It has an axial acceleration of $u_x(\partial u_x/\partial x)$. Show that the jet is annihilated within a finite distance of $L \sim u_0\tau$, where u_0 is the initial value of u_x (τ is the magnetic damping time).

1.6 Calculate the magnetic Reynolds number for motion in the core of the earth, using the radius of the core, $R_c = 3500$ km as the characteristic length-scale and $u \sim 2 \times 10^{-4}$ m/s as a typical velocity. Take the conductivity of iron as 0.5×10^6 mho/m. Now calculate the magnetic Reynolds number for motion in the outer regions of the sun taking $l \sim 10^3$ km, $u \sim 1$ km/s and $\sigma = 10^4$ mho/m. Explain why it is difficult to model solar and geo-dynamos using scaled laboratory experiments with liquid metals.

1.7 Magnetic forces are sometimes used to levitate objects. For example, if a metal object is situated near a coil carrying an alternating current

I, eddy currents will flow in the object and there will result a repulsive force. Show that the force in the x-direction is $\frac{1}{2}I^2(\partial L/\partial x)$ if the object is allowed to move in the x-direction (L is the effective inductance of the coil).