2

The Governing Equations of Electrodynamics

> From a long view of history of mankind – seen from, say, ten thousand years from now – there can be little doubt that the most significant event of the 19th Century will be judged as Maxwell's discovery of the laws of electrodynamics.
>
> *R P Feynman (1964)*

We are concerned here with conducting, non-magnetic materials. For simplicity, we shall assume that all material properties, such as the conductivity, σ, are spatially uniform, and that the medium is incompressible. The topics which concern us are the Lorentz force, Ohm's law, Ampère's law and Faraday's law. We shall examine these one at a time.

2.1 The Electric Field and the Lorentz Force

A particle moving with velocity $\mathbf{u}$ and carrying a charge q is, in general, subject to three electromagnetic forces:

$$\mathbf{f} = q\mathbf{E}_s + q\mathbf{E}_i + q\mathbf{u} \times \mathbf{B} \tag{2.1}$$

The first is the electrostatic force, or Coulomb force, which arises from the mutual repulsion or attraction of electric charges ($\mathbf{E}_s$ is the electrostatic field). The second is the force which the charge experiences in the presence of a time-varying magnetic field, $\mathbf{E}_i$ being the electric field induced by the changing magnetic field. The third contribution is the Lorentz force which arises from the motion of the charge in a magnetic field. Now Coulomb's law tells us that $\mathbf{E}_s$ is irrotational, and Gauss's law fixes the divergence of $\mathbf{E}_s$. Together these laws yield

$$\nabla \cdot \mathbf{E}_s = \rho_e/\varepsilon_0, \qquad \nabla \times \mathbf{E}_s = 0 \tag{2.2a,b}$$

Here ρ_e is the total charge density (free charges plus bound charges) and ε_0 is the permittivity of free space. In view of (2.2b), we may introduce the electrostatic potential, V, defined by $\mathbf{E}_s = -\nabla V$. It follows from (2.2a) that $\nabla^2 V = -\rho_e/\varepsilon_0$.

The induced electric field, on the other hand, has zero divergence, while its curl is finite and governed by Faraday's law.

$$\nabla \cdot \mathbf{E}_i = 0, \qquad \nabla \times \mathbf{E}_i = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.3}$$

It is convenient to define the total electric field as $\mathbf{E} = \mathbf{E}_s + \mathbf{E}_i$, and so we have

$$\begin{array}{cc} \nabla \cdot \mathbf{E} = \rho_e/\varepsilon_0, & \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \text{(Gauss's law)} & \text{(Faraday's law)} \end{array} \tag{2.4}$$

$$\begin{array}{l} \mathbf{f} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \\ \text{(electrostatic force plus Lorentz force)} \end{array} \tag{2.5}$$

Equations (2.4) uniquely determine the electric field since the requirements are that the divergence and curl of the field be known (and suitable boundary conditions are specified). It is customary to use equation (2.5) to define the electric field **E** and the magnetic field **B**. Thus, for example, the electric field **E** is the force per unit charge on a small test charge *at rest* in the observer's frame of reference.

Due attention must be given to frames of reference. Suppose that in the laboratory frame there is an electric field and a magnetic field. The electric field, **E**, is defined by the force per unit charge on a charge at rest in that frame. If the charge is moving, the force due to the electric field is still given by $\mathbf{f} = q\mathbf{E}$ but the additional force $q\mathbf{u} \times \mathbf{B}$ appears, which is used to define **B**. If, however, we use a frame of reference in which the charge is instantaneously *at rest* (but moving with velocity **u** relative to the laboratory frame), then the force on the charge can only be interpreted as due to an electric field, say $\mathbf{E}_r$ (the subscript r indicates 'relative to a moving frame'). Newton's second law then gives, for the two frames, $\mathbf{f} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$ and $\mathbf{f}_r = q\mathbf{E}_r$. However, Newtonian relativity (which is all that is required for MHD) tells us that $\mathbf{f} = \mathbf{f}_r$. It follows that the electric fields in the two frames are related by

$$\mathbf{E}_r = \mathbf{E} + \mathbf{u} \times \mathbf{B} \tag{2.6}$$

The magnetic fields **B** and $\mathbf{B}_r$ are equal.

We close this section by noting that **B** is a pseudo-vector and not a true vector. That is to say, the sense of **B** is somewhat arbitrary, to the extent that **B** reverses direction if we move from a right-handed coordinate system (the usual convention) to a left-handed one. This may be seen as follows. Suppose we transform our coordinate system according to $\mathbf{x} \to \mathbf{x}' = -\mathbf{x}$. (This is referred to as an inversion of the coordinates, or else as a reflection about the origin.) We have moved from a right-handed coordinate system to a left-handed one in which $x' = -x$, $y' = -y$,

$z' = -z$, $\mathbf{i}' = -\mathbf{i}$, $\mathbf{j}' = -\mathbf{j}$ and $\mathbf{k}' = -\mathbf{k}$. Now the components of a true vector, such as force, $\mathbf{f}$, or velocity, $\mathbf{u}$, transform like $f_x' = -f_x$ etc., which leaves the physical direction of the vector unchanged since

$$\mathbf{f} = f_x \mathbf{i}_x + f_y \mathbf{i}_y + f_z \mathbf{i}_z = (-f_x')(-\mathbf{i}_x') + (-f_y')(-\mathbf{i}_y') + (-f_z')(-\mathbf{i}_z')$$

Thus, after an inversion of the coordinates, a true vector (such as $\mathbf{f}$ or $\mathbf{u}$) has the same magnitude and direction as before, although the numerical values of its components change sign. Now consider the definition of $\mathbf{B}$: $\mathbf{f} = q(\mathbf{u} \times \mathbf{B})$. Under an inversion of the coordinates the components of $\mathbf{u}$ and $\mathbf{f}$ both change sign and so those of $\mathbf{B}$ cannot. Thus the magnetic field transforms according to $B_x' = B_x$, etc. By implication, the physical direction of $\mathbf{B}$ reverses. (Such vectors are called pseudo-vectors.) So, if one morning we all agreed to change convention from a right-handed coordinate system to a left-handed one, all the magnetic field lines would reverse direction! The fact that $\mathbf{B}$ is a pseudo-vector is important in dynamo theory.

2.2 Ohm's Law and the Volumetric Lorentz Force

In a stationary conductor it is found that the current density, $\mathbf{J}$, is proportional to the force experienced by the free charges. This is reflected in the conventional form of Ohm's law, $\mathbf{J} = \sigma \mathbf{E}$. In a conducting fluid the same law applies, only now we must use the electric field measured in a frame moving with the local velocity of the conductor:

$$\mathbf{J} = \sigma \mathbf{E}_r = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{2.7}$$

Note that $\mathbf{u}$ will, in general, vary with position.

Now the Lorentz force (2.5) is important not just because it lies behind Ohm's law, but also because the forces exerted on the free charges are ultimately transmitted to the conductor. In MHD we are less concerned with the forces on individual charges than the bulk force acting on the medium, but this is readily found. If (2.5) is summed over a unit volume of the conductor then $\sum q$ becomes the charge density, ρ_e, and $\sum q\mathbf{u}$ becomes the current density, $\mathbf{J}$. The volumetric version of (2.5) is therefore

$$\mathbf{F} = \rho_e \mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{2.8}$$

where $\mathbf{F}$ is the force per unit volume acting on the conductor. However, in conductors travelling at the sort of speeds we are interested in (much less than the speed of light), the first term in (2.8) is negligible. We may demonstrate this as follows. Conservation of charge requires that

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_e}{\partial t} \tag{2.9}$$

(This simply says that the rate at which charge is decreasing inside a small volume must equal the rate at which charge flows out across the surface of that volume.) By taking the divergence of both sides of (2.7), and using Gauss's law and (2.9), we find that

$$\frac{\partial \rho_e}{\partial t} + \frac{\rho_e}{\tau_e} + \sigma \nabla \cdot (\mathbf{u} \times \mathbf{B}) = 0, \qquad \tau_e = \varepsilon_0 / \sigma$$

The quantity τ_e is called the charge relaxation time, and for a typical conductor has a value of around 10^{-18} s. It is extremely small! To appreciate where its name comes from, consider the situation where $\mathbf{u} = 0$. In this case, $\partial \rho_e / \partial t + \rho_e / \tau_e = 0$ and so

$$\rho_e = \rho_e(0) \exp[-t/\tau_e]$$

Any net charge density which, at $t = 0$, lies in the interior of a conductor will move rapidly to the surface under the action of the electrostatic repulsion forces. It follows that ρ_e is always zero in stationary conductors, except during some minuscule period when a battery, say, is turned on. Now consider the case where $\mathbf{u}$ is non-zero. We are interested in events which take place on a time-scale much longer than τ_e (we exclude events like batteries being turned on) and so we may neglect $\partial \rho_e / \partial t$ by comparison with ρ_e / τ_e. We are left with the pseudo-static equation

$$\rho_e = -\varepsilon_0 \nabla \cdot (\mathbf{u} \times \mathbf{B}) \tag{2.10}$$

Thus, when there is motion, we can sustain a finite charge density in the interior of the conductor. However, it turns out that ρ_e is very small, i.e. too low to produce any significant electric force, $\rho_e \mathbf{E}$. That is, from (2.10) we have $\rho_e \sim \varepsilon_0 u B / l$, while Ohm's law requires $\mathbf{E} \sim \mathbf{J}/\sigma$, and so

$$\rho_e \mathbf{E} \sim [\varepsilon_0 u B / l][J/\sigma] \sim \frac{u \tau_e}{l} JB$$

Here l is a typical length-scale for the flow. Evidently, since $u\tau_e / l \sim 10^{-18}$, the Lorentz force completely dominates (2.8) and we may write

$$\mathbf{F} = \mathbf{J} \times \mathbf{B} \tag{2.11}$$

Note also that (2.10) is equivalent to ignoring $\partial \rho_e / \partial t$ in the charge conservation equation (2.9). That is to say, the charge density is so small that (2.9) simplifies to

$$\nabla \cdot \mathbf{J} = 0 \tag{2.12}$$

2.3 Ampère's Law

The Ampère–Maxwell equation tells us something about the magnetic field generated by a given distribution of current. It is

$$\nabla \times \mathbf{B} = \mu \left[\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right] \tag{2.13}$$

The last term in (2.13) may be unfamiliar. It does not, for example, appear in Ampère's circuital law (1.7). This new term was introduced by Maxwell as a correction to Ampère's law and is called the displacement current. To see why it is necessary, we take the divergence of (2.13). Noting that $\nabla \cdot \nabla \times (\cdot) = 0$ and using Gauss' law, this yields

$$\nabla \cdot \mathbf{J} = -\varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{E} = -\frac{\partial \rho_e}{\partial t}$$

This is just the charge conservation equation which, without the displacement current, would be violated. However, Maxwell's correction is not needed in MHD. That is, we have already noted that $\partial \rho_e / \partial t$ is negligible in conductors, and so we might anticipate that the contribution of $\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ to (2.13) is also small in MHD. This is readily confirmed:

$$\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \sim \frac{\varepsilon_0}{\sigma} \frac{\partial \mathbf{J}}{\partial t} \sim \tau_e \frac{\partial \mathbf{J}}{\partial t} \ll \mathbf{J}$$

We are therefore at liberty to use the pre-Maxwell form of (2.13), which is simply the differential form of Ampère's law:

$$\nabla \times \mathbf{B} = \mu \mathbf{J} \tag{2.14}$$

This is consistent with (2.12), since the divergence of (2.14) yields

$$\nabla \cdot \mathbf{J} = 0$$

Finally, we note that in infinite domains, (2.14) may be inverted using the Biot–Savart law. That is, when the current density is a known function of position, the magnetic field may be calculated directly from

$$\mathbf{B}(\mathbf{x}) = \frac{\mu}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') \times \mathbf{r}}{r^3} d^3\mathbf{x}', \qquad \mathbf{r} = \mathbf{x} - \mathbf{x}' \tag{2.15}$$

This comes from the fact that a small element of material located at $\mathbf{x}'$ and carrying a current density of $\mathbf{J}(\mathbf{x}')$ induces a magnetic field at point $\mathbf{x}$ which is given by (see Figure 2.1)

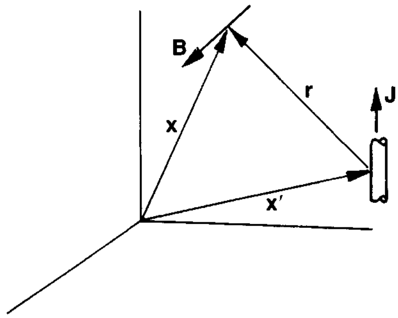

Figure 2.1 Coordinate system used in the Biot–Savart law.

$$d\mathbf{B}(\mathbf{x}) = \frac{\mu}{4\pi}\frac{\mathbf{J}(\mathbf{x}') \times \mathbf{r}}{r^3}d^3\mathbf{x}'$$

Note that (2.15), which is equivalent to (2.14)[1], reveals the true character of Ampère's law. It really tells us about the structure of the magnetic field associated with a given current distribution.

Example: Force-free fields

Magnetic fields of the form $\nabla \times \mathbf{B} = \alpha\mathbf{B}$, $\alpha =$ constant, are known as *force-free* fields, since $\mathbf{J} \times \mathbf{B} = 0$. (More generally, fields of the form $\nabla \times \mathbf{G} = \alpha\mathbf{G}$ are known as *Beltrami* fields.) They are important in plasma MHD where we frequently require the Lorentz force to vanish. Show that, for a force-free field,

$$(\nabla^2 + \alpha^2)\mathbf{B} = 0$$

Deduce that there are no force-free fields, other than $\mathbf{B} = 0$, for which $\mathbf{J}$ is localised in space and $\mathbf{B}$ is everywhere differentiable and $0(x^{-3})$ at infinity.

2.4 Faraday's Law in Differential Form

Faraday's law is sometimes stated in integral form and sometimes in differential form. You have already met both. In Section 2.1 we stated it to be

[1] In fact, (2.15) is a stronger statement than (2.14) as it determines both the divergence and the curl of $\mathbf{B}$.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

This tells us about the electric field induced by a time-varying magnetic field. However, in Chapter 1 we gave the integral version,

$$\text{e.m.f.} = \oint_C \mathbf{E}_r \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{2.16}$$

where $\mathbf{E}_r$ is the electric field measured in a frame of reference moving with $d\mathbf{l}$ (see equation (2.6)). In fact, it is easily seen that (2.16) is a more powerful statement than the differential form of Faraday's law. In words, it states that the e.m.f. around a closed loop is equal to the total rate of change of flux of **B** through that loop. In (2.16), the flux may change because **B** is changing with time, or because the loop is moving uniformly in an inhomogeneous field, or because the loop is changing shape. Whatever the cause, (2.16) gives the induced e.m.f. We shall return to the integral version of Faraday's law in Section 2.7, where we discuss its full significance. In the meantime, we shall show that the differential form of Faraday's law is a special case of (2.16).

Suppose that the loop is rigid and at rest in a laboratory frame. Then the e.m.f. can arise only from a magnetic field which is time-dependent. In this case (2.16) becomes

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \oint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S (\partial \mathbf{B}/\partial t) \cdot d\mathbf{S}$$

Since this is true for any and all (fixed) surfaces, we may equate the integrands in the surface integrals. We then obtain the differential form of Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.17}$$

In this form, Faraday's law becomes one of Maxwell's equations (see Section 2.5). Note, however, that (2.17) is a weaker statement than (2.16). It only tells us about the electric field induced by a time-varying magnetic field.

Now (2.17) ensures that $\partial \mathbf{B}/\partial t$ is solenoidal, since $\nabla \cdot (\nabla \times \mathbf{E}) = 0$. In fact, it transpires that we can make an even stronger statement about **B**. It turns out that **B** is itself solenoidal,

$$\nabla \cdot \mathbf{B} = 0 \tag{2.18}$$

This allows us to introduce another field, **A**, called the vector potential, defined by

$$\nabla \times \mathbf{A} = \mathbf{B}, \qquad \nabla \cdot \mathbf{A} = 0 \tag{2.19a,b}$$

This definition automatically ensures that **B** is solenoidal, since $\nabla \cdot \nabla \times \mathbf{A} = 0$. If we substitute for **A** in Faraday's equation we obtain

$$\nabla \times \mathbf{E} = -\nabla \times (\partial \mathbf{A}/\partial t)$$

from which

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V \tag{2.20}$$

where V is an arbitrary scalar function. However, we also have, from (2.2) and (2.3),

$$\mathbf{E} = \mathbf{E}_i + \mathbf{E}_s, \qquad \nabla \times \mathbf{E}_s = 0, \qquad \nabla \cdot \mathbf{E}_i = 0$$

and so we might anticipate that $\mathbf{E}_i = -\partial \mathbf{A}/\partial t$ and $\mathbf{E}_s = -\nabla V$ where V is now the electrostatic potential. This is readily confirmed by taking the divergence of (2.20) which, given (2.19b), shows that all of the divergence of **E** is captured by ∇V in (2.20), as required by (2.2) and (2.3).

Example: The divergence of **B**

Faraday's law implies that $(\partial/\partial t)(\nabla \cdot \mathbf{B}) = 0$. If this is also true relative to all sets of axes moving uniformly relative to one another, show that $\nabla \cdot \mathbf{B} = 0$.

2.5 The Reduced Form of Maxwell's Equations for MHD

We have mentioned Maxwell's equations several times. When combined with the force law (2.5) and the law of charge conservation (2.9), they embody all that we know about electrodynamics, and so it seems appropriate that, at some point, we should write them down. For materials which are neither magnetic nor dielectric, Maxwell's equations state that:

$$\nabla \cdot \mathbf{E} = \rho_e/\varepsilon_0 \qquad \text{(Gauss' law)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(Solenoidal nature of } \mathbf{B}\text{)}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's law in differential form)}$$

$$\nabla \times \mathbf{B} = \mu\left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \qquad \text{(Ampère–Maxwell equation)}$$

In addition, we have

$$\nabla \cdot \mathbf{J} = -\partial \rho_e / \partial t \quad \text{(charge conservation)},$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

For our purposes these may be simplified considerably. In MHD, the charge density ρ_e plays no significant part. For example, we have seen that the electric force, $q\mathbf{E}$, is minute by comparison with the Lorentz force, and that the contribution of $\partial \rho_e / \partial t$ to the charge conservation equation is also negligible. Apparently ρ_e is significant only in Gauss's law and so we simply drop Gauss's law and ignore ρ_e. Also, we have seen that in MHD the displacement currents are negligible by comparison with the current density, $\mathbf{J}$, and so the Ampère–Maxwell equation reduces to the differential form of Ampère's law. We may now summarise the (pre-Maxwell) form of the electrodynamic equations used in MHD:

Ampère's law plus charge conservation,

$$\boxed{\nabla \times \mathbf{B} = \mu \mathbf{J}}, \quad \boxed{\nabla \cdot \mathbf{J} = 0} \tag{2.21}$$

Faraday's law plus the solenoidal constraint on $\mathbf{B}$,

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}, \quad \boxed{\nabla \cdot \mathbf{B} = 0} \tag{2.22}$$

Ohm's law plus the Lorentz Force,

$$\boxed{\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B})}, \quad \boxed{\mathbf{F} = \mathbf{J} \times \mathbf{B}} \tag{2.23}$$

Equations (2.21)–(2.23) encapsulate all that we need to know about electromagnetism for MHD.

Example 1: A paradox

Although electrostatic forces are of no importance in MHD, they can lead to some unexpected effects in those cases where they are significant, as we now show. Consider a hollow plastic sphere which is mounted on a frictionless spindle and is free to rotate. Charged metal pellets are embedded in the surface of the sphere and a wire loop is placed near its centre, the axis of the loop being parallel to the rotation axis. The loop is connected to a battery, so that a current flows and a dipole-like magnetic field is created. We now ensure that everything is stationary and (somehow) disconnect the battery. The magnetic field declines and so, by Faraday's law, we induce an electric field which is azimuthal, i.e. **E** takes the form of rings which are concentric with the axis of the wire loop. This electric field now acts on the charges to produce a torque on the sphere, causing it to spin up. At the end of the process we have gained some angular momentum in the sphere, but at the cost of the magnetic field. Apparently, we have contravened the principle of conservation of angular momentum! Can you unravel this paradox? (Hint: consult Feynman's 'Lectures on Physics' Vol. 2.)

The earth has a large negative charge on its surface, which gives rise to an average surface electric field of around 100 V/m. It also has a dipole magnetic field, and rotates about an axis which is more-or-less aligned with the magnetic axis. Do you think the rotation rate of the earth changes when the earth's magnetic field reverses (as it occasionally does)?

Example 2: The Poynting vector

Use Faraday's law and Ampère's law to show that

$$\frac{d}{dt}\int_V (\mathbf{B}^2/2\mu)dV = -\int_V \mathbf{J}\cdot\mathbf{E}dV - \oint_S [(\mathbf{E}\times\mathbf{B})/\mu]\cdot d\mathbf{S}$$

Now use Ohm's law to confirm that

$$\int_V \mathbf{J}\cdot\mathbf{E}dV = \frac{1}{\sigma}\int_V \mathbf{J}^2 dV + \int_V (\mathbf{J}\times\mathbf{B})\cdot\mathbf{u}dV$$

Combining the two we obtain

$$\frac{d}{dt}\int (\mathbf{B}^2/2\mu)dV = -\frac{1}{\sigma}\int_V \mathbf{J}^2 dV - \int_V (\mathbf{J}\times\mathbf{B})\cdot\mathbf{u}dV - \oint_S \mathbf{P}\cdot d\mathbf{S}$$

where $\mathbf{P} = (\mathbf{E}\times\mathbf{B})/\mu$ is called the Poynting vector. The integrals on the right represent Joule dissipation, the rate of loss of magnetic energy due to the rate of working of the Lorentz force on the medium, and the rate at which electromagnetic energy flows out through the surface S, the Poynting vector being the electromagnetic energy flux density.

2.6 A Transport Equation for B

If we combine Ohm's law, Faraday's equation and Ampère's law we obtain an expression relating **B** to **u**.

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = -\nabla \times [(\mathbf{J}/\sigma) - \mathbf{u} \times \mathbf{B}] = \nabla \times [\mathbf{u} \times \mathbf{B} - \nabla \times \mathbf{B}/\mu\sigma]$$

Noting that $\nabla \times \nabla \times \mathbf{B} = -\nabla^2 \mathbf{B}$ (since **B** is solenoidal), this simplies to

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \lambda \nabla^2 \mathbf{B}}, \qquad \lambda = (\mu\sigma)^{-1} \tag{2.24}$$

This is sometimes called the induction equation, although, as we shall see, a more descriptive name would be the advection–diffusion equation for **B**. The quantity λ is called the magnetic diffusivity. Like all diffusivities it has the units m^2/s. Equation (2.24) is, in effect, a transport equation for **B**, in the sense that if **u** is known then it dictates the spatial and temporal evolution of **B** from some specified initial condition. We shall spend much of Chapter 4 unpicking the physical implications of (2.24): it is one of the key equations in MHD.

Example: Decay of force-free fields

Show that if, at $t = 0$, there exists a *force-free* field, $\nabla \times \mathbf{B} = \alpha \mathbf{B}$, in a stationary fluid, then that field will decay as $\mathbf{B} \sim \exp(-\lambda\alpha^2 t)$, remaining as a force-free field.

2.7 On the Remarkable Nature of Faraday and of Faraday's Law

We shall now show that the integral version of Faraday's law, (2.16), is a quite remarkable result, encompassing not just one physical law, but two! Moreover, as we shall see, Faraday's law in its most general form embodies many of the key phenomena of MHD. We start, however, with a historical footnote.

2.7.1 An historical footnote

Faraday played a crucial part in the development of MHD for three reasons. First, his law of induction, discovered in 1831, shows that magnetic field lines in a perfectly conducting fluid must move with the fluid, as if frozen into the medium. This result is usually attributed to the 20th century astrophysicist Alfvén, but really it follows directly from Faraday's law. Second, he performed the first experiment in MHD when he tried to measure the voltage induced by the Thames flowing through the earth's magnetic field.[2] Third, he invented magnetic fields!

Prior to the work of Faraday, the scientific and mathematical communities were convinced that the laws of electromagnetism should be formulated in terms of action at a distance. The notion of a field did not exist. For example, Ampère had discovered that two current-carrying wires attract each other, and so, by analogy with Newton's law of gravitational attraction, it seemed natural to try and describe this force in terms of some kind of inverse square law. In this view, nothing of significance exists *between* the wires.

Faraday had a different vision, in which the medium between the wires plays a rôle. In his view, a wire which carries a current introduces a field into the medium surrounding it. This field (the magnetic field) exists whether or not a second wire is present. When the second wire is introduced it experiences a force by virtue of this field. Moreover, in Faraday's view the field is not just some convenient mathematical intermediary. It has real physical significance, possessing energy, momentum and so on.

Of course, it is Faraday's view which now prevails, which is all the more remarkable because Faraday had no formal education and, as a consequence, little mathematical skill. James Clerk Maxwell was greatly impressed by Faraday, and in the preface to his classic treatise on Electricity and Magnetism he wrote:

> Before I began the study of electricity I resolved to read no mathematics on the subject till I had first read through Faraday's Experimental Researches in Electricity. I was aware that there was supposed to be a difference between Faraday's way of conceiving phenomena and that of the mathematicians, so that neither he nor they were satisfied with each other's language...
>
> As I proceeded with the study of Faraday, I perceived that his method of conceiving the phenomena was also a mathematical one, though not exhibited in the conventional form of mathematical symbols...

[2] In Faradays words: 'I made experiments therefore (by favour) at Waterloo bridge, extending a copper wire nine hundred and sixty feet in length upon the parapet of the bridge, and dropping from its extremities other wires with extensive plates of metal attached to them to complete contact with the water. Thus the wire and the water made one conducting circuit; and as the water ebbed and flowed with the tide, I hoped to obtain currents.' (1832)

> For instance, Faraday, in his mind's eye, saw lines of force traversing space where the mathematicians saw centres of force attracting at a distance: Faraday sought the seat of the phenomena in real actions going on in the medium, they were satisfied that they had found it in a power of action at a distance...
>
> When I translated what I considered to be Faraday's ideas into mathematical form, I found that in general the results of the two methods coincided, so that the same phenomena were accounted for, and the same laws of action deduced by both methods, but that Faraday's methods resembled those in which we begin with the whole and arrive at the parts by analysis, while the ordinary mathematical methods were founded on the principle of beginning with the parts and building up the whole by synthesis. I also found that several of the most fertile methods of research discovered by the mathematicians could be expressed much better in terms of the ideas derived by Faraday than in their original form...
>
> If by anything I have written I may assist any student in understanding Faraday's modes of thought and expression, I shall regard it as the accomplishment of one of my principle aims – to communicate to others the same delight which I have found myself in reading Faraday's 'Researches'.
>
> *(1873)*

When Maxwell transcribed Faraday's ideas into mathematical form, correcting Ampère's law in the process, he arrived at the famous laws which now bear his name. Kelvin was similarly taken by Faradays physical insight:

> One of the most brilliant steps made in philosophical exposition of which any instance existed in the history of science was that in which Faraday stated, in three or four words, intensely full of meaning, the law of magnetic attraction or repulsion... Mathematicians were content to investigate the general expression of the resultant force experienced by a globe of soft iron in all such cases; but Faraday, without any mathematics, devined the result of the mathematical investigations. Indeed, the whole language of the magnetic field and 'lines of force' is Faraday's. It must be said for the mathematicians that they greedily accepted it, and have ever since been most zealous in using it to the best advantage.'
>
> *(1872)*

The central rôle played by fields acquires special significance in relativistic mechanics where, because of the finite velocity of propagation of interactions, it is not meaningful to talk of direct interactions of particles (or currents) located at distant points. We can speak only of the field established by one particle and of the subsequent influence of this field on other particles. Of course, Faraday could not have foreseen this! Einstein explicitly noted the important role played by Faraday and Maxwell in his popular introduction to Relativity:

> during the second half of the 19th century, in conjunction with the researches of Faraday and Maxwell, it became more and more clear that the description of electromagnetic processes in terms of fields was vastly superior to a treatment on the basis of the mechanical concepts of material points.... One psychological effect of this immense success was that the field concept, as opposed to the mechanistic framework of classical physics, gradually won greater independence.
>
> *(1916)*

Of course, Faraday's contribution to magnetism did not stop with the introduction of fields. He also discovered electromagnetic induction. In fact, in 1831, in no more than ten full days of research, Faraday unravelled all of the essential features of electromagnetic induction. Even more remarkable, the integral equation now attributed to Faraday encompasses not just one physical law, but two, as we now show. First, however, we need an important kinematic result.

2.7.2 An important kinematic equation

Suppose that $\mathbf{G}$ is a solenoidal field, $\nabla \cdot \mathbf{G} = 0$, and S_m is a surface which is embedded in a conducting medium, i.e. S_m is locked into the medium and moves as the fluid moves. (The subscript m indicates that it is a *material surface*.) Then it may be shown that

$$\boxed{\frac{d}{dt}\int_{S_m} \mathbf{G} \cdot d\mathbf{S} = \int_{S_m} \left[\frac{\partial \mathbf{G}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{G})\right] \cdot d\mathbf{S}} \qquad (2.25a)$$

A formal proof of (2.25) will be given in a moment. First, however, we might try to get a qualitative feel for its origins. The idea behind (2.25a) is the following. The flux of $\mathbf{G}$ through S_m changes for two reasons. First, even if S_m were fixed in space there is a change in flux whenever $\mathbf{G}$ is time-dependent. This is the first term on the right of (2.25a). Second, if the boundary of S_m moves it may expand at points to include additional flux, or perhaps contract at other points to exclude flux. It happens that, in a time δt, the surface adjacent to the line element $d\mathbf{l}$ increases by an amount $d\mathbf{S} = (\mathbf{u} \times d\mathbf{l})\delta t$, and so the increase in flux due to movement of the boundary C_m is

$$\delta \int_{S_m} \mathbf{G} \cdot d\mathbf{S} = \oint_{C_m} \mathbf{G} \cdot (\mathbf{u} \times d\mathbf{l})\delta t = -\oint_{C_m} (\mathbf{u} \times \mathbf{G}) \cdot d\mathbf{l}\,\delta t$$

Using Stoke's theorem, the last line integral may be converted into a surface integral, which accounts for the second term on the right of (2.25a). Of course, we have yet to show that $d\mathbf{S} = (\mathbf{u} \times d\mathbf{l})\delta t$.

The formal proof of (2.25) proceeds as follows. The change in flux through S_m in a time δt is

$$\delta \int_{S_m} \mathbf{G} \cdot d\mathbf{S} = (\delta t) \int_{S_m} (\partial \mathbf{G}/\partial t) \cdot d\mathbf{S} + \oint_{S_m} \mathbf{G} \cdot \delta \mathbf{S}$$

where $\delta\mathbf{S}$ is the element of area swept out by the line element $d\mathbf{l}$ in time δt. However, $\delta\mathbf{S} = d\mathbf{l}' \times d\mathbf{l}$, where $d\mathbf{l}'$ is the infinitesimal displacement of the element $d\mathbf{l}$ in time δt (Figure 2.2). Since $d\mathbf{l}' = \mathbf{u}\delta t$, we have $\delta\mathbf{S} = (\mathbf{u} \times d\mathbf{l})\delta t$ and so

$$\delta \int_{S_m} \mathbf{G} \cdot d\mathbf{S} = (\delta t) \int_{S_m} (\partial \mathbf{G}/\partial t) \cdot d\mathbf{S} - \oint_{C_m} \mathbf{u} \times \mathbf{G} \cdot d\mathbf{l}(\delta t)$$

(We have used the cyclic properties of the scalar triple product to rearrange the terms in the line integral.) Finally, the application of Stoke's theorem to the line integral gets us back to (2.25), and this completes the proof.

Now (2.25) should not be passed over lightly: it is a very useful result. The reason is that often in MHD (or conventional fluid mechanics) we find that certain vector fields obey a transport equation of the form

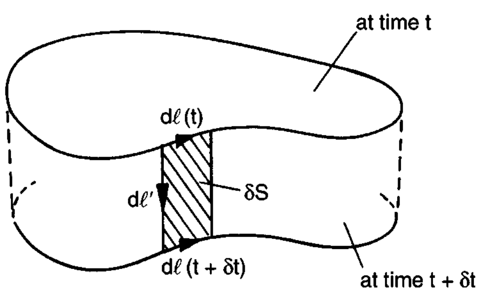

Figure 2.2 Movement of the material surface in a time δt.

$$\frac{\partial \mathbf{G}}{\partial t} = \nabla \times [\mathbf{u} \times \mathbf{G}]$$

This is true of $\nabla \times \mathbf{u}$ in an unforced, inviscid flow (see Chapter 3) and of $\mathbf{B}$ in a perfect conductor (see equation (2.24)). In such cases, (2.25) tells us that the flux of $\mathbf{B}$ (or $\nabla \times \mathbf{u}$) through any material surface, S_m, is conserved as the flow evolves. We shall return to this idea time and again in subsequent chapters.

Note that it is not necessary to invoke the idea of a continuously moving medium and of material surfaces in order to arrive at (2.25). If we consider any curve, C, moving in space with a prescribed velocity, $\mathbf{u}$, then

$$\frac{d}{dt}\int_S \mathbf{G} \cdot d\mathbf{S} = \int_S \left[\frac{\partial \mathbf{G}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{G})\right] \cdot d\mathbf{S} \tag{2.25b}$$

where S is any surface which spans the curve C.

2.7.3 The full significance of Faraday's law

We now return to electrodynamics. Recall that the differential form of Faraday's law is

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \tag{2.26}$$

As noted earlier, this is a weaker statement than the integral version (2.16), since it tells us only about the e.m.f. induced by a time-dependent field. Let us now see if we can *deduce* the more general version of Faraday's law, (2.16), from (2.26).

Suppose we have a curve, C, which deforms in space with a prescribed velocity $\mathbf{u}(\mathbf{x})$. (This could be, but need not be, a material curve.) Then, at each point on the curve, (2.26) gives

$$\nabla \times (\mathbf{E} + \mathbf{u} \times \mathbf{B}) = -\left\{\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B})\right\}$$

We now integrate this over any surface S which spans C and invoke the kinematic equation (2.25b). The result is

$$\oint_C (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

So far we have used only Faraday's law in differential form. We now invoke the idea of the Lorentz force. This tells us that, in a frame of reference moving with velocity $\mathbf{u}$, the electric field is $\mathbf{E}_r = \mathbf{E} + \mathbf{u} \times \mathbf{B}$.

Given that **E** transforms in this way, we may rewrite our integral equation as

$$\oint_C \mathbf{E}_r \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

Note that this applies to any curve C. For example, C may be fixed in space, move with the fluid, or execute some motion quite different to that of the fluid. It does not matter. The final step is to introduce the idea of an e.m.f. We define the e.m.f. to be the closed integral of $\mathbf{E}_r$, from which

$$\text{e.m.f.} = \oint_C \mathbf{E}_r \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{2.27}$$

We have arrived at the integral version of Faraday's law. Note, however, that to get from (2.26) to (2.27) we had to invoke the force law $\mathbf{F} = q(\mathbf{u} \times \mathbf{B})$. Note also that if C and S happen to be material curves and surfaces embedded in a fluid, then (2.27) becomes

$$\text{e.m.f.} = \oint_{C_m} \mathbf{E}_r \cdot d\mathbf{l} = -\frac{d}{dt}\int_{S_m} \mathbf{B} \cdot d\mathbf{S} \tag{2.28}$$

Now it is intriguing that the integral version of Faraday's law describes the e.m.f. generated in two very different situations, i.e. when **E** is induced by a time-dependent magnetic field, and when $\mathbf{E}_r$ is induced (at least in part) by motion of the circuit within a magnetic field. The two extremes are shown in Figure 2.3. If **B** is constant, and the e.m.f. is

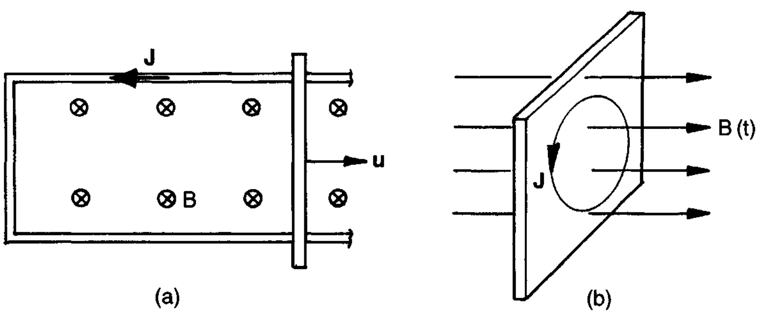

Figure 2.3 An e.m.f. can be generated either by movement of the conducting medium (motional e.m.f.) or else by variation of the magnetic field (transformer e.m.f.).

due solely to movement of the circuit, then $\oint \mathbf{E}_r \cdot d\mathbf{l}$ is called a *motional* e.m.f. If the circuit is fixed and $\mathbf{B}$ is time-dependent, then $\oint \mathbf{E} \cdot d\mathbf{l}$ is termed a *transformer* e.m.f. In either case, however, the e.m.f. is equal to (minus) the rate of change of flux. Now motional e.m.f. is due essentially to the Lorentz force, $q\mathbf{u} \times \mathbf{B}$, while transformer e.m.f. results from the Maxwell equation $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, which is usually regarded as a separate physical law. Yet both are described by the integral equation (2.27). Faraday's law is therefore an extraordinary result. It embodies two quite different phenomena. It seems that it just so happens that motional e.m.f. and transformer e.m.f. can both be described by the same flux rule! (At a deeper level both Maxwell's equations and the Lorentz force can, with some additional assumptions, be deduced from Coulomb's law plus the Lorentz transformation of special relativity, and so it is not just coincidence that Faraday's equation embraces two apparently distinct physical laws. Nevertheless, from a classical viewpoint, it represents a remarkably convenient equation.)

2.7.4 Faraday's law in ideal conductors: Alfvén's theorem

From Ohm's law, $\mathbf{J} = \sigma \mathbf{E}_r$, and (2.28) we have

$$\boxed{\frac{1}{\sigma}\oint_{C_m} \mathbf{J} \cdot d\mathbf{l} = -\frac{d}{dt}\int_{S_m} \mathbf{B} \cdot d\mathbf{S}} \tag{2.29}$$

for any material surface, S_m. Now suppose that $\sigma \to \infty$. Then

$$\frac{d}{dt}\int_{S_m} \mathbf{B} \cdot d\mathbf{S} = 0 \qquad \sigma \to \infty \tag{2.30}$$

We have arrived at a key result in MHD. That is to say, *in a perfect conductor, the flux through any material surface S_m is preserved as the flow evolves*. Now picture an individual flux tube sitting in a perfectly conducting fluid. Such a tube is, by analogy to a stream-tube in fluid mechanics, just an aggregate of magnetic field lines (Figure 2.4). Since $\mathbf{B}$ is solenoidal ($\nabla \cdot \mathbf{B} = 0$), the flux of $\mathbf{B}$ along the tube, Φ, is constant. (This comes from applying Gauss's divergence theorem to a finite portion of the tube.) Now consider a material curve C_m which at some initial instant encircles the flux tube. The flux enclosed by C_m will remain constant as the flow evolves, and this is true of each and every curve enclosing the tube at $t = 0$. This suggests (but does not

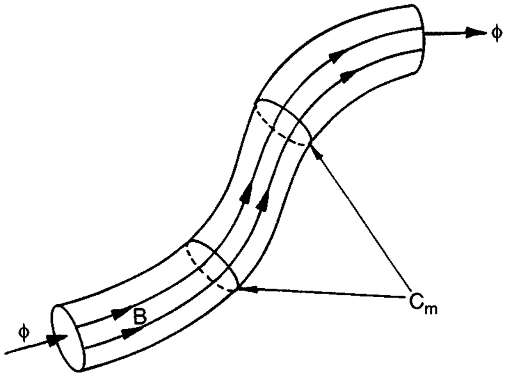

Figure 2.4 A magnetic flux tube.

prove) that the tube itself moves with the fluid, as if frozen into the medium. This, in turn, suggests that every field line moves with the fluid, since we could let the tube have a vanishingly small cross section. We have arrived at Alfvén's theorem (Figure 2.5), which states that:

> magnetic field lines are frozen into a perfectly conducting fluid in the sense that they move with the fluid.

We shall give formal proof of Alfvén's theorem in Chapter 4.

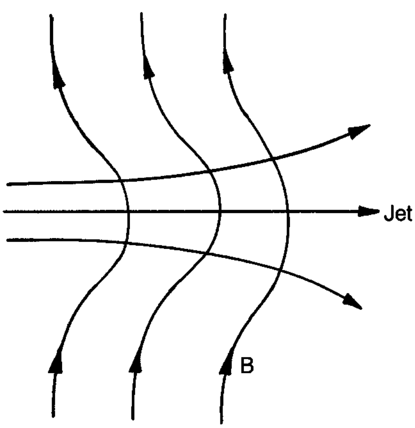

Figure 2.5 An example of Alfvén's theorem. Flow through a magnetic field causes the field lines to bow out.

Suggested Reading

Feynman, Leighton & Sands, *The Feynman lectures on physics, Vol. II, 1964* Addison–Wesley (Chapters 13–18 for an introduction to Maxwell's equations).

P Lorrain & D Corson, *Electromagnetism Principles and Applications*, W H Freeman & Co. (A good all-round text on electricity and magnetism.)

J A Shercliff, *A Textbook of Magnetohydrodynamics, 1965*. Pergamon Press (Chapter 2 for the MHD simplifications of Maxwell's equations).

Examples

2.1 A conducting fluid flows in a uniform magnetic field which is negligibly perturbed by the induced currents. Show that the condition for there to be no net charge distribution in the fluid is that $\mathbf{B} \cdot (\nabla \times \mathbf{u}) = 0$.

2.2 A thin conducting disc of thickness h and diameter d is placed in a uniform alternating magnetic field parallel to the axis of the disc. What is the induced current density as a function of distance from the axis of the disc?

2.3 Show that a coil carrying a steady current, I, tends to orientate itself in a magnetic field in such a way that the total magnetic field linking the coil is a maximum. Also, show that the torque exerted on the coil is $\mathbf{m} \times \mathbf{B}$, where $\mathbf{m}$ is the dipole momentum of the coil. What do you think will happen to a small current loop in a highly conducting fluid which is permeated by a large-scale magnetic field?

2.4 A fluid of small but finite conductivity flows through a tube constructed of insulating material. The velocity is very nearly uniform and equal to u. To measure the velocity of the fluid, a part of the tube is subjected to a uniform transverse magnetic field, B. Two small electrodes which are in contact with the fluid are installed through the tube walls. A voltmeter detects an induced e.m.f. of V. What is the velocity of the fluid?

2.5 Show that it is impossible to construct a generator of electromotive force constant in time operating on the principle of electromagnetic induction.