Appendix 4

MHD Turbulence at Low R_m

We have commented on MHD turbulence in a number of places in this book. In Chapter 7 we discussed the decay laws for freely evolving MHD turbulence, while in Chapter 9 we examined the influence of an intense magnetic field on isolated vortices. The purpose of this appendix is to pull together these various threads and to produce a coherent picture of MHD turbulence at low R_m. We shall restrict ourselves to statistically homogeneous turbulence, so that the influence of boundaries may be ignored. We shall assume that there is no mean motion.

We are interested in the evolution of turbulence in a uniform, imposed magnetic field, $\mathbf{B}_0$. For simplicity, the initial conditions are taken to be statistically isotropic and R_m is assumed to be small. This latter condition implies that the induced magnetic field is negligible by comparison with the imposed field.

The nature of MHD turbulence depends crucially on the initial value of the interaction parameter, $N = \sigma B_0^2 l / \rho u$, where l is the integral scale of the turbulence and u is a typical velocity fluctuation, say $u = \left(\langle u_x^2 \rangle\right)^{1/2}$. When N is initially small, $\mathbf{J} \times \mathbf{B}_0$ is negligible by comparison with inertia and the turbulence evolves as discussed in Section 1.5 of Chapter 7. We then have conventional, decaying turbulence. This is governed by two equations:

$$\frac{du^2}{dt} = -\alpha \frac{u^3}{l}$$

$$\int r^2 \langle \mathbf{u} \cdot \mathbf{u}' \rangle d\mathbf{r} = \text{constant}$$

The first of these describes the rate at which energy is lost by the large eddies to the energy cascade, the large eddies breaking up on a timescale of l/u, the eddy turn-over time. The coefficient α is found experimentally to lie in the range $a = 1.0 \to 1.2$. The second equation represents the conservation of Loitsyansky's integral and implies that

$$u^2 l^5 = \text{constant}$$

These equations may be combined to yield Kolmogorov's decay laws

$$u^2 = u_0^2[1 + (7\alpha/10)(u_0 t/l_0)]^{-10/7}, \qquad l = l_0[1 + (7\alpha/10)(u_0 t/l_0)]^{2/7}$$

where u_0 and l_0 are the initial values of u and l.

When N is very large, on the other hand, the turbulence is governed by the linearised equation of motion

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla(P) + \mathbf{J} \times \mathbf{B}_0$$

(Recall that $\mathbf{J}$ is linear in $\mathbf{u}$.) In view of the linearity of this equation, we might regard the turbulence as an ensemble of independent eddies. Some of the eddies will have an axis of rotation which is more or less aligned with $\mathbf{B}_0$. Some will be non-aligned. These eddies will evolve in a manner not unlike those described in Section 4 of Chapter 9. Vortices whose axes of rotation are initially aligned with $\mathbf{B}_0$ will develop into long, columnar structures, with characteristic length scales, $l_{//}$ and $l_\perp$, evolving as $l_\perp \sim l_0$ and $l_{//} \sim l_0(t/\tau)^{1/2}$, where τ is the Joule damping time, $(\sigma \mathbf{B}_0^2/\rho)^{-1}$. The energy of such an eddy declines as $u^2 \sim u_0^2(t/\tau)^{-1/2}$. Vortices which are initially perpendicular to $\mathbf{B}_0$, on the other hand, will develop into sheet-like structures consisting of thin, interwoven layers of oppositely signed vorticity (see Chapter 9, Section 4). The dominant velocity component in these sheets is $\mathbf{u}_{//}$. Like the columns, we find $l_\perp \sim l_0$, $l_{//} \sim l_0(t/\tau)^{1/2}$ and $u^2 \sim u_0^2(t/\tau)^{-1/2}$. Thus, as long as N remains large, we expect two distinct types of structures to emerge from isotropic initial conditions: columns and sheets. We might anticipate that more sheets than columns will develop since relatively few vortices will have their axis of rotation aligned with $\mathbf{B}_0$ at $t = 0$. If this is the case, we might expect $\mathbf{u}_{//}^2$ gradually to exceed $\mathbf{u}_\perp^2$ as the anisotropy develops. In fact, this is exactly what is observed in numerical experiments, with $\mathbf{u}_{//}^2 \sim 2\mathbf{u}_\perp^2$ at large times.

The energy equation governing high-N turbulence is the Joule dissipation equation (5.7)

$$\frac{d}{dt}\left[\frac{1}{2}\rho < u^2 >\right] = - < J^2 > /\sigma$$

Using (5.6b) to estimate $\langle J^2 \rangle$, this yields

$$\frac{du^2}{dt} \sim -\frac{1}{\tau}\left(\frac{l_\perp}{l_{//}}\right)^2 u^2$$

which we might rewrite as

$$\frac{du^2}{dt} = -\beta\left(\frac{l_\perp}{l_{//}}\right)^2 \frac{u^2}{\tau}$$

where β is a coefficient of order unity. (For isotropic turbulence it is possible to show that $\beta = \frac{2}{3}$.) In addition, conservation of the Loitsyansky integral for MHD turbulence yields (see equation 7.49)

$$u^2 l_{//} l_\perp^4 = \text{constant}$$

Moreover, when N is large, we know from our analysis of individual vortices that $l_\perp = \text{constant}$ on a time scale of τ. It follows that the two expressions above may be combined to yield

$$u^2 = u_0^2[1 + 2\beta t/\tau]^{-1/2}$$
$$l_{//} = l_0[1 + 2\beta t/\tau]^{1/2}$$
$$l_\perp = l_0$$

Let us now consider the case where N takes some intermediate value, perhaps of order unity. The energy equation must now combine the influence of the energy cascade and Joule dissipation. From (7.48) we have

$$\boxed{\frac{du^2}{dt} = -\alpha\frac{u^3}{l_\perp} - \beta\left(\frac{l_\perp}{l_{//}}\right)^2 \frac{u^2}{\tau}} \tag{A4.1}$$

Moreover, we still have conservation of Loitsyansky's integral for MHD turbulence:

$$\boxed{u^2 l_{//} l_\perp^4 = \text{constant}} \tag{A4.2}$$

Unfortunately, these two equations are insufficient to determine the three unknowns $l_{//}$, $l_\perp$ and u. Let us introduce a third, heuristic equation for $l_{//}/l_\perp$. When N is small we have $l_{//} \sim l_\perp$ on a timescale of l/u. Conversely, when N is large, we have $l_{//}/l_\perp \sim (1 + 2\beta t/\tau)^{1/2}$. Both extremes ($N \gg 1$ and $N \ll 1$) satisfy

$$\frac{d}{dt}\left(\frac{l_{//}}{l_\perp}\right)^2 = \frac{2\beta}{\tau} \tag{A4.3}$$

Let us suppose that the heuristic equation (A4.3) also applies for intermediate values of N. Then (A4.1)–(A4.3) provides a closed system of equations for u, $l_{//}$ and $l_\perp$. For simplicity we shall take $\alpha = \beta = 1.0$ and $l_{//} = l_\perp = l_0$ at $t = 0$. The general solution to these equations is then

$$\boxed{u^2/u_0^2 = \hat{t}^{-1/2}\left[1 + (7/15)\left(\hat{t}^{3/4} - 1\right)N_0^{-1}\right]^{-10/7}} \tag{A4.4}$$

$$\boxed{l_\perp/l_0 = \left[1 + (7/15)\left(\hat{t}^{3/4} - 1\right)N_0^{-1}\right]^{2/7}} \tag{A4.5}$$

$$\boxed{l_{//}/l_0 = \hat{t}^{1/2}\left[1 + (7/15)\left(\hat{t}^{3/4} - 1\right)N_0^{-1}\right]^{2/7}} \tag{A4.6}$$

where N_0 is the initial value of N and $\hat{t} = 1 + 2(t/\tau)$. The high- and low-N results given above are special cases of (A4.4)–(A4.6). Note that, in general, u^2, $l_{//}$ and $l_\perp$ do not obey simple power laws. However, for the special case of $N_0 = 7/15$ we have,

$$u^2/u_0^2 = \hat{t}^{-11/7}, \qquad l_{//}/l_0 = \hat{t}^{5/7}$$

The dependence $u^2 \sim t^{-11/7}$ is not far out of line with the experimental data for $N_0 \sim 1$, which suggests $u^2 \sim t^{-1.6}$.