Numerical Realizations of Galaxies in Cosmological Perturbation Theory

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Abstract

The project uses numerical realizations of cosmological perturbation theory to go from small initial matter fluctuations to galaxies at late times when they are observed. First, Gaussian initial conditions are generated from a given power spectrum of fluctuations (depending on cosmological parameters): this requires using random number generators and mapping their results to Gaussian-distributed random numbers, thus creating a Gaussian random field on a grid in Fourier space. The next step goes from these fluctuations to the final matter fluctuations by evolving via perturbation theory: this requires to compute the linear displacement field in Fourier space, then FFT back to real space to construct source terms for the second-order cosmological solution which obeys a Poisson equation, which is solved by FFT techniques. Once the second-order solution for the displacement field is obtained, it is used to displace the particles on a grid to their final position [1]. The future goal of this project is to use these these particles and convert them to galaxies using consistent galalxy-bias perturbation theory to second order (modifying the Poisson's solutions).

Contents

1 Introduction

Understanding the large-scale structure of the universe is one of the main goals of cosmology. In the last few decades it has become widely accepted that gravitational instability plays a central role in giving rise to the remarkable structures seen in galaxy surveys. Extracting the wealth of information contained in galaxy clustering to learn about cosmology thus requires a quantitative understanding of the dynamics of gravitational instability and application of sophisticated statistical tools that can best be used to test theoretical models against observations.

Even though it is not the scope of this project to discuss any theoretical cosmological models of the expansion of the universe, we just mention that the current explanation for the large-scale structures is that the present distribution of matter on cosmological scales results from the growth of primordial, small, seed fluctuations on an otherwise homogeneous universe amplified by gravitational instability. We don't have observations of these initial fluctuations and they are not deterministic in nature so our goal to is to make statistical physical predictions that might eventually help to the comprehension of the formation of galaxies, cluster of galaxies, etc.

In this project we will consider the most simple type of density fluctuations, **Gaussian fluctuations**. We rally on the proposal that in the early universe the Inflaton Field produced an accelerated expansion and because it was weakly coupled to interactions; theoretical models consider the field was on it is Vacuum State. We consider primordial inhomogeneities to be Gaussian because we assume that the statistical properties of the density field are inherited by the fluctuations of the Inflaton.

In this work we studied **Second order Non-Linear Lagrangian Perturbation Theory (PT) (2LPT)** to model the distribution of particles in the universe. The particles are considered to gravitationally interact solely and we use the seed fluctuations as initial conditions for numerically solving the set of 2LPT equations. The computational techniques will be described in detail along this review and final simulations of structure formation are shown.

2 Dynamics of gravitational instability and formation of structures

In this section we will briefly derive the equations for Lagrangian non-linear perturbation theory. We recommend the reader for a complete derivation and discussion of the model to go over the review [1]. We assume the reader to be familiar with basic notions of GR so we won't get into details of GR cosmological solutions or it's derivations.

Our study of structure formation begins considering a set of particles that interact only gravitationally in an expanding homogeneous universe. Our background spacetime is assumed to obey *Einstein's Gravity* so the manifolds where the fields propagate are described by *Friedman-Robertson-Walker spacetimes*. Generally speaking, when we study Friedmann's equations and arrive to isotropic and homogeneous solutions for the metric it is always assumed that the distributions of matter, photons, dark energy (or any source of Gravity) are described by densities which only depend on time. The effect of this type of sources is that at each slice of time the metric is isotropic and homogeneous and it's explicit time dependence is only through the *radius of the universe* $a(t)$ (also called the scale factor). However, if we want to describe the formation of structures in the Universe we should abandon the assumption that it is maximally symmetric and introduce small perturbations to the distribution of matter. From now on our equations will consider that there is only matter in the universe (no dark energy) defined over a FRW metric. We also consider that we don't need to consider the back-reaction of the small perturbations to the background spacetime so our calculations are valid for any *cosmological solution of GR*.

We define the density fluctuation $\delta(\mathbf{x}, \tau)$ by

$$\rho(\mathbf{x}, \tau) = \bar{\rho}(\tau)(1 + \delta(\mathbf{x}, \tau)) \tag{1}$$

where $\bar{\rho}(\tau)$ is the isotropic and homogeneous matter density and τ is the usual conformal time defined by $dt = a d\tau$.

We start by defining a *Gravity Potential*, $\mathbf{\Phi}$, that is sourced only by density fluctuations $\delta(\mathbf{x}, \tau)$. This potential will be the responsible of making *the particles to move and start forming structures*. Following [1], $\mathbf{\Phi}$ responds to the equation

$$\nabla^2 \mathbf{\Phi} = \frac{3}{2} \Omega_m \mathcal{H}^2 \delta(\mathbf{x}, \tau) \tag{2}$$

where Ω_m is the usual ratio of matter to the critical density, $\rho_{critic} = \frac{3H^2}{8\pi G}$, and $\mathcal{H}^2$ is the conformal expansion rate $\mathcal{H} = Ha$ with H the usual *Hubble "constant"*.

2.1 Lagrangian Perturbation Theory

In areas like Cosmology and Astro-Physics we commonly deal with fields, like density and velocity fields and their equations of motion. However, it is possible to develop non-linear PT in a different framework, the so-called Lagrangian scheme, by following the trajectories of particles or fluid elements (rather than studying the dynamics of the fields). In Lagrangian PT, the object of interest is the displacement field $\mathbf{\Psi}(\mathbf{q})$ which maps the initial particle positions $\mathbf{q}$ into the final particle positions $\mathbf{x}$,

$$\mathbf{x}(\tau) = \mathbf{q} + \mathbf{\Psi}(\mathbf{q}, (\tau)) \tag{3}$$

It can be shown that the equation of motion for particle trajectories $\mathbf{x}$ is

$$\frac{d\mathbf{x}^2}{d\tau^2} + \mathcal{H}\frac{d\mathbf{x}}{d\tau} = -\nabla\mathbf{\Phi}, \tag{4}$$

where $\mathbf{\Phi}$ is our potential sourced only by density perturbations. Taking divergence to this equation we obtain

$$J(\mathbf{q}, (\tau))\nabla. \left[\frac{d\mathbf{\Psi}^2}{d\tau^2} + \mathcal{H}\frac{d\mathbf{\Psi}}{d\tau}\right] = \frac{3}{2}\Omega_m\mathcal{H}^2(J-1), \tag{5}$$

where we have used Poisson equation (2) together with the fact that the density field obeys $\bar{\rho}(\tau)(1+\delta(\mathbf{x})d^3x = \bar{\rho}d^3q$, thus

$$1+\delta(\mathbf{x}) = \frac{1}{Det(\delta_{ij}+\Psi_{i,j})} = \frac{1}{J(\mathbf{q}, (\tau))}, \tag{6}$$

where $\Psi_{i,j} = \partial\Psi_i/\partial q_j$ and $J(\mathbf{q}, (\tau))$ is the Jacobian of the transformation between Eulerian and Lagrangian space. As a final remark, the resulting non-linear equation (5) for $\mathbf{\Psi}$ is then solved perturbatively, expanding about its linear solution.

2.1.1 Linear solutions and the Zel'dovich approximation

The linear solution of (5) is

$$\nabla.\mathbf{\Psi}^{(1)} = -D_1(\tau)\delta(\mathbf{q}), \tag{7}$$

where the $\delta(\mathbf{q})$ denotes the density field by the initial conditions and $D_1(\tau)$ is the *linear growth factor* which obeys the equation

$$\frac{dD_1(\tau)^2}{d\tau^2} + \mathcal{H}\frac{dD_1(\tau)}{d\tau} = \frac{3}{2}\Omega_m\mathcal{H}^2D_1(\tau) \tag{8}$$

It is important to underscore that $\delta(\mathbf{q},\tau) = D_1(\tau)\delta(\mathbf{q},\tau = 0)$ so we highlight that we have decoupled the *time evolution* of $\mathbf{\Psi}$ from *it's spatial derivatives*! In other words, this is very useful because once we have solved the ODE for $D_1(\tau)$ the only thing we are required to do to get Ψ at a different time is just rescale the source $\delta(\mathbf{x},\tau = 0)$ and solve (7) for this rescaled source.

Then, we say that *Zel'dovich approximation* (hereafter ZA) consists in using the linear displacement field (Ψ to first order in perturbation theory) as an approximate solution for the dynamical equations.

2.1.2 Second Order Lagrangian Perturbation Theory

We now move into Second-order Lagrangian PT for the displacement field (2LPT) and we assume that to this order in perturbation theory the displacement field is irrotational. One way to understand the necessity of a higher order solution is to recall that the Lagrangian picture is intrinsically non-linear in the density field (see equation (6)), and a small perturbation in Lagrangian fluid element paths carries a considerable amount of non-linear information about the corresponding Eulerian density field. The solution reads

$$\nabla.\mathbf{\Psi}^{(2)} = \frac{1}{2}D_2(\tau)\sum_{i\neq j}\left(\Psi^{(1)}_{i,i}\Psi^{(1)}_{j,j} - \Psi^{(1)}_{i,j}\Psi^{(1)}_{j,i}\right), \tag{9}$$

where $D_2(\tau)$ denotes the second-order growth factor which, which obeys

$$D_2(\tau) \simeq -\frac{3}{7}D_1(\tau) \tag{10}$$

The reason for the improvement of 2LPT over ZA is in fact not surprising. The solution of (5)) to second order describes the correction to the ZA displacement due to gravitational tidal effects, that is, it takes into account the fact that gravitational instability is non-local.

Now, we take advantage of the useful fact that we consider our Eulerian fields to be irrotational, so the can be expressed as gradients of Scalar Potentials. We introduce the scalar fields $\phi^{(1)}$ and $\phi^{(2)}$ which are determined by

$$\nabla^2\phi^{(1)}(\mathbf{q}) = \delta(\mathbf{q}) \tag{11}$$

$$\nabla^2\phi^{(2)}(\mathbf{q}) = \sum_{i\neq j}\left(\phi^{(1)}_{,ii}\phi^{(1)}_{,jj} - \left(\phi^{(1)}_{,ij}\right)^2\right) \tag{12}$$

Then, the displacement fields as a function of the scalar fields are

$$\begin{aligned} \mathbf{\Psi}^{(1)} &= \nabla\phi^{(1)} \\ \mathbf{\Psi}^{(2)} &= \frac{1}{2}\nabla\phi^{(2)} \end{aligned} \tag{13}$$

So, the final positions for the particles according 2LPT are

$$\begin{aligned} \mathbf{x}(\tau) &= \mathbf{q} + \mathbf{\Psi}(\mathbf{q}, (\tau)) \\ \mathbf{x}(\tau) &= \mathbf{q} - D_1\nabla\phi^{(1)} - \frac{3}{14}D_1\nabla\phi^{(2)} \end{aligned} \tag{14}$$

We conclude this section mentioning that we got the positions (14) for a specific value of $D_1(\tau)$ associated to *redshift* $z = 0$ (in Cosmology redshift zero means the current time, the present). In other words, our result is a *picture* of the formation of the galaxies because we have the final positions of the particles at a certain instant of time but not for *every* instant of time. Future work might be to solve (8) to get the growth factor and perform a simulation in space and time of the formation of galaxies or cluster of galaxies!

3 Random Fields, Power Spectrum

As we said in the Introduction, we will follow a statistical approach to characterize the properties of our density fluctuations. Then, we consider relevant to mention the most important features of **Gaussian random fields**.

3.1 Gaussian Numbers

By definition, a uniform random sequence of numbers between 0 and 1 can be thought as a sequence of numbers that all have the same probability to appear. Then, we say that a uniform random variable x is defined by a uniform probability distribution function (PDF). Our goal is to map random real numbers r uniformly distributed in the range $0 < r < 1$ into a sequence with a Gaussian PDF.

Suppose we want numbers distributed according to some PDF, $P(x)$. All we want to do is to map uniform numbers n into $x's$ with $P(x)$. Observe that

$$u(x) = \int^{x} P(x')dx', \tag{15}$$

has a PDF given by conservation law:

$$P_u(u)du = P(x)dx, \tag{16}$$

but because of (15), $\frac{du}{dx} = P(x)$, which implies that

$$P_u = 1 \Rightarrow UNIFORM! \tag{17}$$

So the idea is to generate a uniform random number u, and then invert the cumulative PDF to get x. What we will do next is apply this idea to generate Gaussian random numbers.

For a joint Gaussian of 2 independent variables we have:

$$P_{(x,y)} = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \tag{18}$$

Changing variables to polar coordinates factorizes nicely into two invertible PDF's.

$$P_{(x,y)}dxdy = P_{(r,\theta)}drd\theta = \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} rdrd\theta \tag{19}$$

So, we have $P_{(r,\theta)} = P_{(r)}P_{(\theta)}$ where

$$\begin{aligned} P_{(\theta)} &= \frac{1}{2\pi} \\ P_{(r)} &= \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \end{aligned} \tag{20}$$

This says that θ is uniform between 0 and 2π and for r we have

$$v = \frac{1}{\sigma^2} \int^r dr' r' e^{-\frac{r'^2}{2\sigma^2}} = 1 - e^{-\frac{r^2}{2\sigma^2}} \tag{21}$$

Inverting this relation we get $r = \sqrt{-2\sigma^2 \ln(1-v)}$. So, the trick is to generate two uniform numbers u, $v \in (0,1)$ and then do

$$\begin{aligned} \theta &= 2\pi u \\ r &= \sqrt{-2\sigma^2 \ln(1-v)} \end{aligned} \tag{22}$$

and in Cartesian coordinates we get

$$
\begin{aligned}
x &= r\cos(\theta) \\
y &= r\sin(\theta)
\end{aligned}
\tag{23}
$$

Finally, in agreement with our PDF (18) we constructed x and y as Gaussian random variables as function of two uniform random numbers!

3.2 Gaussian Fields

We now turn our attention to the properties of the so called **Gaussian Fields**. Their importance in Physics can not be overestimated because in active areas such as Condensed Matter Physics, QFT, Cosmology, Statistical Physics (and many more) where we are usually interested in calculating correlating functions of free fields, we will necessarily use Gaussian Fields.

The complete study of features of Gaussian Fields totally exceeds the scope of this work. We will introduce a few tools and then explain how to apply them in our particular problem of galaxies' formation simulation. Let's Φ be our Gaussian field. We start our discussion mentioning that for a Gaussian random field, everything is determined by it's two-point correlator function:

$$
< \Phi(\bar{x})\Phi(\bar{y}) >= \xi_\Phi(\bar{x},\bar{y}) \tag{24}
$$

A random field is said to be statistically homogeneous and isotropic if it's two point correlator does only depend on the distance between the two insertion points.

$$
\xi_\Phi(\bar{x},\bar{y}) = \xi_\Phi(|\bar{x}-\bar{y}|) \tag{25}
$$

Translation invariance of correlation functions immediately suggests working in Fourier space (since plane waves are eigenfunctions of gradient operator). While in real space we see by equation (25) that the field at different points is correlated, we see that in Fourier Space different Fourier modes (different $\bar{k}$) are uncorrelated!

We define the usual Fourier transform $\Phi(\bar{k})$

$$
\Phi(\bar{k}) = \int \frac{dx^3}{(2\pi)^3} e^{i\bar{k}\bar{x}}\Phi(\bar{x}) \tag{26}
$$

Using this expression, the two point function for two Fourier modes, $< \Phi(\bar{k_1}), \Phi(\bar{k_2}) >$, can be written as

$$
< \Phi(\bar{k_1})\Phi(\bar{k_2}) >= P_\Phi(\bar{k_1})\delta(\bar{k_1}+\bar{k_2}), \tag{27}
$$

First, me mention that the Dirac Delta $\delta(\bar{k_1}+\bar{k_2})$ physically means Momentum conservation. Second, we state that for reaching (28) we defined the **Power Spectrum**, $P_\Phi(\bar{k})$, as the Fourier transform of the two-point correlation function,

$$P_\Phi(\bar{k}) = \int \frac{dx^3}{(2\pi)^3} e^{i\bar{k}\bar{x}} \xi_\Phi(\bar{x}) \tag{28}$$

Because by assumption our field is statistically homogeneous and isotropic the power spectrum depends only on the module of the wave vector, $\bar{k}$ so to simplify notation we will write $P(k)$.

Just as a matter of completeness, we mention that the power spectrum is a well-defined quantity for almost all homogeneous random fields. But this concept becomes, however, extremely fruitful when one considers a Gaussian field because we know that the *n-point functions* can be written as sum of products of two point correlators. Not being too mathematically formal, we say that in general

$$\begin{aligned} &< \Phi(\bar{k_1})...\Phi(\bar{k_{2n+1}}) > = 0 \\ &< \Phi(\bar{k_1})...\Phi(\bar{k_{2n}}) > = \sum_{all\, pairs} \prod_{(i,j)} < \Phi(\bar{k_i})\Phi(\bar{k_j}) > \end{aligned} \tag{29}$$

From the R.H.S of this equation we notice that all those pairs of correlation functions are indeed proportional to the power spectrum. This is just the famous **Wick theorem**, a fundamental theorem for classic and quantum field theories.

What we want to stress is that the statistical properties of the random variables, which in our particular case turns out to be the density fluctuation, $\delta(\bar{k})$, are then entirely determined by the shape and normalization of $P(k)$. So, a specific cosmological model will eventually be determined, for example, by the power spectrum in the linear regime and Ω_m (if we are not considering Dark Energy, as in our case).

As we have studied the basic concepts of Gaussian Fields we are now finally able to construct $\Phi(\bar{x})$. We start constructing it's Fourier transform

$$\Phi(\bar{k}) = A(\bar{k}) + iB(\bar{k}), \tag{30}$$

where $A(\bar{k})$ and $B(\bar{k})$ are real random Gaussian variables. Because we are interested in studying density fluctuations, we need our field $\Phi(\bar{x})$ to be real so we impose that

$$\Phi(\bar{k}) = \Phi(-\bar{k})^*, \tag{31}$$

this condition inputs some constraints to A and B

$$A(\bar{k}) = A(\bar{-k}) \Rightarrow even$$
$$B(\bar{k}) = -B(\bar{-k}) \Rightarrow odd \tag{32}$$

Now, let's see how we generate the coefficients $A(\bar{k})$ and $B(\bar{k})$. We have addressed the problem in such a way that these two are random Gaussian variables independent for each wave vector $\bar{k}$ and we will create them using a pair of uniform independent variables u and v. Taking into account the equations (22) and (23), for each $\bar{k}$, we set the Fourier coefficients to be

$$A(\bar{k}) = \sqrt{\frac{P(k)}{2}} r_{\bar{k}} \cos(\theta_{\bar{k}})$$
$$B(\bar{k}) = \sqrt{\frac{P(k)}{2}} r_{\bar{k}} \sin(\theta_{\bar{k}}), \tag{33}$$

with

$$\theta_{\bar{k}} = 2\pi u_{\bar{k}}$$
$$r_{\bar{k}} = \sqrt{-2\ln(1 - v_{\bar{k}})} \tag{34}$$

This means that for each $\bar{k}$, we generate two uniform numbers $u_{\bar{k}}$ and $v_{\bar{k}}$, find $\theta_{\bar{k}}$ and $r_{\bar{k}}$, and then use (33) to build the Fourier coefficients, scaling things using the value of the power spectrum for that particular mode. Once we have the Fourier coefficients (constrained to the condition (32)), we Fourier transform back to get our Gaussian field $\Phi(\bar{x})$.

Before moving to write our field in Real space, we mention that as A and B are Gaussian independent numbers for each wave vector, there are some mathematical properties they have which are also satisfied by our code to order of machine precision

$$< A(\bar{k}) > =< B(\bar{k}) >=< A(\bar{k})B(\bar{k}) >= 0$$
$$< A(\bar{k})^2 >+ < B(\bar{k})^2 >= \frac{P(k)}{2}, \tag{35}$$

Finally, we get $\Phi(\bar{x})$

$$\Phi(\bar{r}) = \int \frac{dk^3}{(2\pi)^3} e^{-i\bar{k}\bar{x}} \Phi(\bar{x}) \tag{36}$$

Because our solution to the problem will be numerical, we will define a $3D$ grid, a *cube of length L divided in N slices.* As we are interested in periodic boundary conditions we will define the vector of integer numbers (N is even) $\bar{m}$ with components $m_i \in (\frac{-N}{2}, \frac{N}{2})$. The wave vectors are

$$\bar{k} = \frac{2\pi}{L}\bar{m} \tag{37}$$

4 Numerical Realization of Galaxies

In the previous sections we have studied the physics of gravitation instability which leads to the formation of structures and we learnt how to construct Gaussian random fields. In this section we apply these concepts and tools to perform the numerical simulation of the formation of structures.

4.1 Constructing the initial conditions

In order to get the numerical solution to first order in perturbations, the Zel'dovich approximation (see equation (7)), we first need to construct the density fluctuation $\delta(\mathbf{q})$ as a Gaussian field.

The input of our code is the power spectrum, $P(k)$, normalized to redshift $z = 0$. In the next image we show the input data of our code

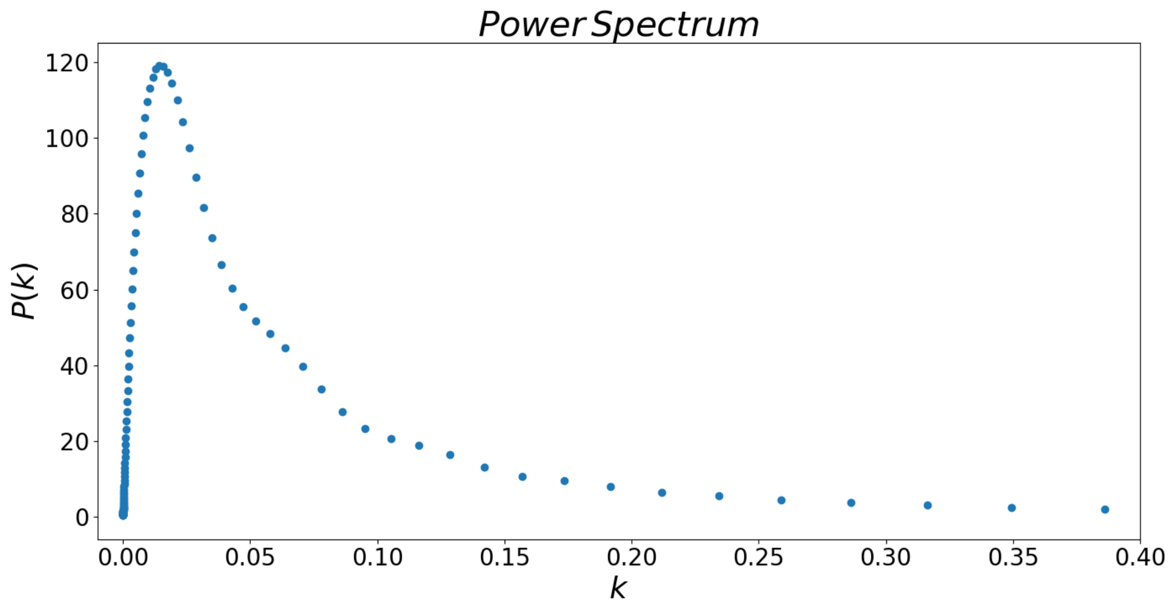

Figure 1: Power Spectrum at $z = 0$.

We constructed a $3D$ grid of size $L = 1000Mpc$ and $N = 250$ so each *cell of our Universe* is a cube of $4Mpc^3$. Using this set of parameters we constructed wave vectors using equation (37). Because $N = 250$, each of the components k_i can take 251 possible values.

We remind that $P(k)$ depends only on the module of the vectors $\bar{k}$ so the first thing we did to use the power spectrum was an *interpolation* between the $k's$ of the initial data to the modules of our wave vectors $\bar{k}$. The interpolation determines an approximate value of $P(k)$ to each of the $\bar{k} = \frac{2\pi}{L}\bar{m}$ we generated.

Next, for each of the $\bar{k} = \frac{2\pi}{L}\bar{m}$ we constructed the Fourier coefficients $A(\bar{k})$ and $B(\bar{k})$ considering the equations (33) and imposing the constraints (32) to get a real density field $\delta(\mathbf{q})$.

4.2 Solving the first Laplace equation

Once the interpolation is done and we got the coefficients $A(\bar{k})$ and $B(\bar{k})$, we have our density fluctuation Gaussian field $\delta(\mathbf{k})$ in Fourier space. We continued solving the differential equation (11) which determines the scalar potential $\phi^{(1)}$ directly in Fourier Space. For easy notation, we will use that $k_x[i]$, $k_y[j]$ and $k_z[k]$ means the $i-th$, $j-th$ and $k-th$ components of the wave vector $\bar{k} = (k_x, k_y, k_z)$. Then, when we write, for example, $\phi^{(1)}_{ijk}$ means that we are considering the scalar field $\phi^{(1)}_{k_x[i],k_y[j],k_z[k]}$ in Fourier Space.

Having said that, we solve for $\phi^{(1)}$ in Fourier Space

$$\phi^{(1)}_{ijk} = -\frac{\delta^{(1)}_{ijk}}{k_x[i] * k_x[i] + k_y[j] * k_y[j] + k_z[k] * k_z[k]} \tag{38}$$

We should interpret the equation (38) as the discrete version of Laplace equation in Fourier Space.

4.3 Solving the second Laplace equation

Having solved (11) using it's numerical discrete version (38) allows us to move forward to second order perturbation theory and solve (12). We solved that equation in Fourier space. Our numerical strategy for this part of the code was

i) Calculate the second derivatives of $\phi^{(1)}$ in Fourier Space. In general, $\frac{\partial}{\partial x}$ is equivalent to multiply by ik_x in Fourier Space (same holds for $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$).

ii) Get the second derivatives of $\phi^{(1)}$ in Real Space using *np.fft.ifftn*. This is the *Inverse Fourier transform for n-dimensional objects of Numpy.*

iii) Multiply and sum the second derivatives of $\phi^{(1)}$ in Real space as required in the R.H.S of (12).

iv) Transform the R.H.S of (12) back to Fourier space using *np.fft.fftn* and define this new variable as $\rho^{(2)}_{ijk}$.

v) Use $\rho^{(2)}_{ijk}$ as a source term for the Laplace equation and finally get the scalar potential $\phi^{(2)}_{ijk}$ doing

$$\phi_{ijk}^{(2)} = -\frac{\rho_{ijk}^{(2)}}{k_x[i] * k_x[i] + k_y[j] * k_y[j] + k_z[k] * k_z[k]} \tag{39}$$

4.4 Displacement fields

The most important part of our code has been done because we got the fields $\phi^{(1)}$ and $\phi^{(2)}$ in Fourier space! Our next step was getting the displacement fields $\mathbf{\Psi}^{(1)}$ and $\mathbf{\Psi}^{(2)}$ which are proportional to the gradient of these scalar fields.

i) $\mathbf{\Psi}_{ijk}^{(1)} = i(k_x[i], k_y[j], k_z[k])\phi_{ijk}^{(1)}$. This expression should be understood as multiplying $i\bar{k}$ to our scalar field $\phi^{(1)}$ in Fourier space. Using *np.fft.ifftn* we get $\mathbf{\Psi}^{(1)}$ in Real space.

ii) $\mathbf{\Psi}_{ijk}^{(2)} = \frac{i}{2}(k_x[i], k_y[j], k_z[k])\phi_{ijk}^{(2)}$. Using *np.fft.ifftn* we get $\mathbf{\Psi}^{(2)}$ in Real space.

4.5 Results of our simulation

The goal of our project is to get the positions of the particles under the influence of gravity using equation (14). We recall that our parameters for our grid were $L = 1000Mpc$ and $N = 250$ so each cell is a cube of $4Mpc^3$. We created a uniform grid of 251^3 elements and put a particle in each of the positions of the grid (this is our initial condition for the code). Our final result will be the new positions of the particles considering that they moved under the influence of gravity (this means considering the displacement fields $\mathbf{\Psi}^{(1)}$ and $\mathbf{\Psi}^{(2)}$). As we couldn't construct a $3D$ image of our grid, we show a picture of a slice at a plane $z = constant$ of the initial positions of the particles.

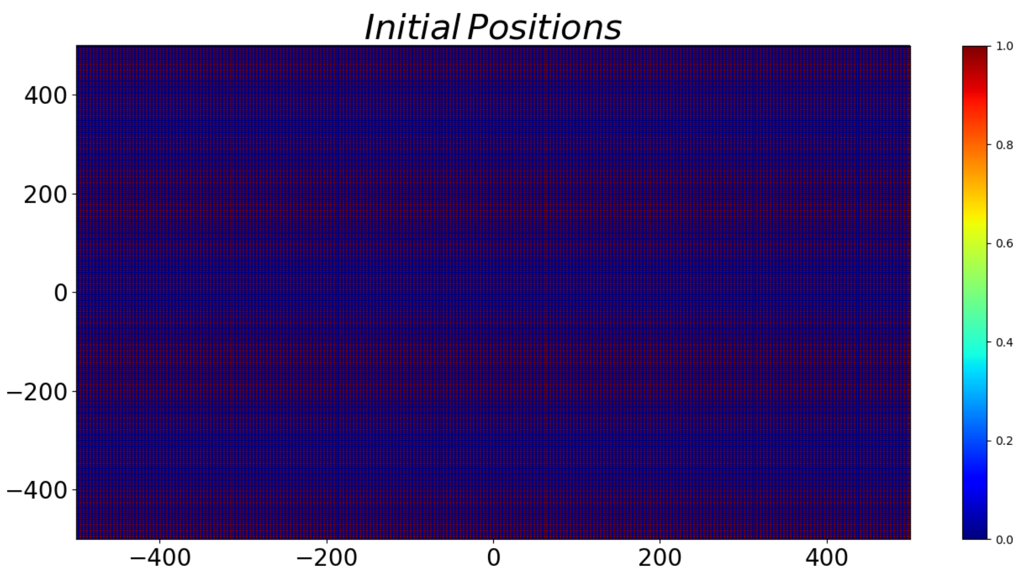

Figure 2: Initial positions of the particles.

From (2) we see that the grid is completely uniform. Once we consider that particles move because of gravitational interaction even tough we don't have the analytical solution to the differential equations a sensible result would be that our particles move from their initial positions a distance of order $4Mpc$.

Our values for the displacements fields were:

i) $\mathbf{\Psi}^{(1)} \simeq 0.5, \mathbf{1}, \mathbf{2}, 4Mpc$. This means that on average, the module of the displacement to first order in perturbation theory is between $1Mpc$ and $2Mpc$.

ii) $\mathbf{\Psi}^{(2)} \simeq 10^{-2}Mpc$. Then, our simulation had as a result that the displacements to second order in perturbation theory are two orders of magnitude smaller than $\mathbf{\Psi}^{(1)}$.

Finally, we show the result of our simulation which are the final positions of the particles

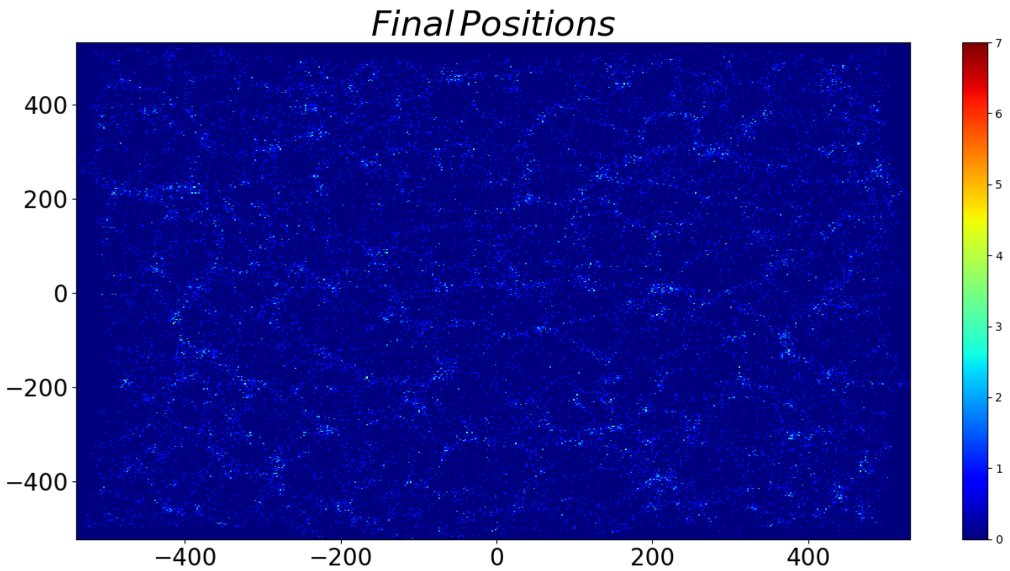

Figure 3: Initial positions of the particles.

We see filaments are being formed in picture (3) and regions were the number of particles is higher. Then, we conclude that our code indeed simulations the formation of structures considering the Gaussian primordial fluctuations $\delta(\mathbf{q})$**.**

5 Summary, conclusions and future steps

In this project we learnt theoretical concepts about Cosmology and used Lagrangian perturbation theory as a model to study the formation of structures in the Universe. Also, we learnt tools of Gaussian probability distribution functions and were able to generate them in our code to represent primordial density fluctuations.

Using FFTs techniques we solved a system of differential equations and finally got the distribution of matter in the Universe for a $3D$ grid of $1000Mpc^3$ with $N = 250$ which determine the size of unite cells to be of order $4Mpc^3$. The result of our code are the displacement fields at redshift $z = 0$ which determine the new positions of the particles due the effect of gravity. For those fields we got displacements of order $\mathbf{\Psi}^{(1)} \simeq 2Mpc$ and $\mathbf{\Psi}^{(2)} \simeq 10^{-2}Mpc$, so we claim that our code is working properly because the displacements to first order in perturbation theory, $\mathbf{\Psi}^{(1)}$, are of the order of the unit cell, and the correction of second order, $\mathbf{\Psi}^{(2)}$, is two order of magnitudes smaller.

Future work lines might be to consider a more realistic model adding extra parameters to the system of equations in order to represent more accurately the formation of galaxies. Great advances towards understanding structure formation would be by solving a differential equation for the *growth factor*. Having $D_1(\tau)$ (not just for $z = 0$) would let us *normalize* the power spectrum $P(k)$ to different redshifts, then, running the code for different $P(k)$ (different power spectrums for every instant of time) would give us a complete *history* of the formation of galaxies!

Finally, as a conclusion of this project we strongly believe it was a great opportunity to apply new knowledge and techniques learnt during *Computational Physics*. Furthermore, it gave us the chance to work independently in a field of research of our interest and gain deeper insight of: *what research can be done, what are the current working lines and what are we able to do using Numerical techniques*. Finally, we are totally convinced that this course was very useful and valuable for our education and future research as a physicist.

References

[1] F. Bernardeau, S. Colombi, E. Gaztañaga, R. Scoccimarro. *Large-scale structure of the Universe and cosmological perturbation theory*, Physics Reports 367 (2002) 1 - 248.