

# Probabilistic models - Bayesian Methods

Mengye Ren

NYU

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- Midterm next week Oct 24 4:55 - 6:55 PM (same as lecture).
- Send me your course project group by Oct 20 (NetID + NYU ID, Email title contains **CSCI-GA 2565**).
- Course project 2 page proposal due Oct 27.

# Overview

# Why probabilistic modeling?

- A unified framework that covers many models, e.g., linear regression, logistic regression
- Learning as **statistical inference**
- Principled ways to incorporate your belief on the data generating distribution (inductive biases)

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  - **Conditional:**  $p(y | x)$
  - **Generative:**  $p(x, y)$
- How to estimate the parameters of our model? Maximum likelihood estimation.
- Compare and contrast conditional and generative models.



## Conditional models

# Linear regression

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**Examples:**

- Predicting house price given location, condition, build year etc.
- Predicting medical cost of a person given age, sex, region, BMI etc.
- Predicting age of a person based on their photos.

# Problem setup

**Data** Training examples  $\mathcal{D} = \{(x^{(n)}, y^{(n)})\}_{n=1}^N$ , where  $x \in \mathbb{R}^d$  and  $y \in \mathbb{R}$ .

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$$h(x) = \sum_{i=0}^d \theta_i x_i = \theta^T x, \quad (1)$$

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Note that

- We incorporate the **bias term** (also called the intercept term) into  $x$  (i.e.  $x_0 = 1$ ).
- We use superscript to denote the example id and subscript to denote the dimension id.

# Parameter estimation

**Loss function** We estimate  $\theta$  by minimizing the **squared loss** (the least square method):

$$J(\theta) = \frac{1}{N} \sum_{n=1}^N \left( y^{(n)} - \theta^T x^{(n)} \right)^2. \quad (\text{empirical risk}) \quad (2)$$

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- Let  $X \in \mathbb{R}^{N \times d}$  be the **design matrix** whose rows are input features.
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## Review questions

- Derive the solution for linear regression.
- What if  $X^T X$  is not invertible?

We've seen

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- Why squared loss is a reasonable choice for regression problems?
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Next,

- Derive linear regression from a probabilistic modeling perspective.

# Assumptions in linear regression

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- $x$  and  $y$  are related through a linear function:

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In practice, we maximize the **log likelihood**  $\ell(\theta)$ , or equivalently, minimize the negative log likelihood (NLL).



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However,

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Next,

- Derive **logistic regression** for classification.

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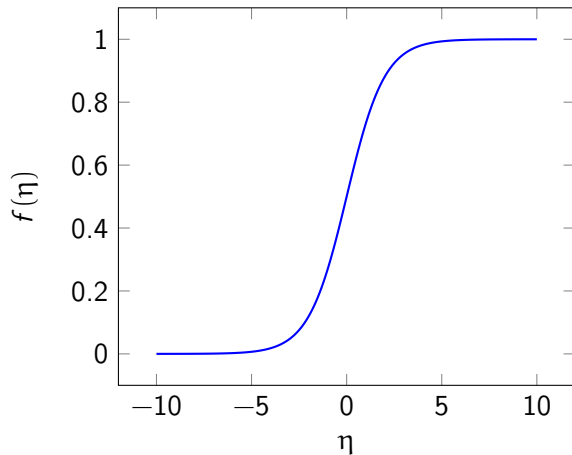
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- What is the mean of  $Y | X = x$ ?  $h(x)$ . (Think how we parameterize the mean in linear regression)
- Need a function  $f$  to map the linear predictor  $\theta^T x$  in  $\mathbb{R}$  to  $(0, 1)$ :

$$f(\eta) = \frac{1}{1 + e^{-\eta}} \quad \text{logistic function} \quad (17)$$

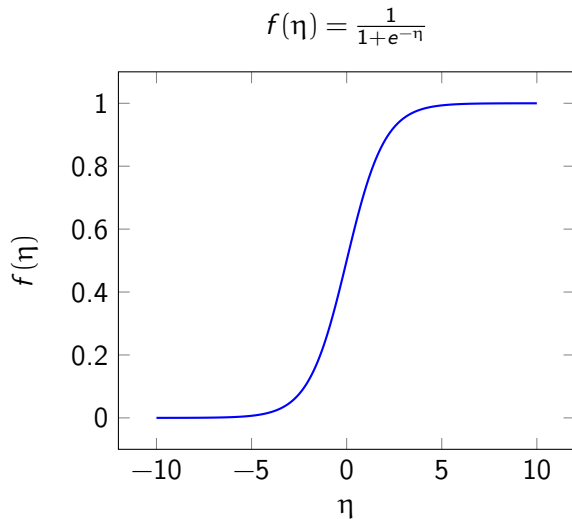
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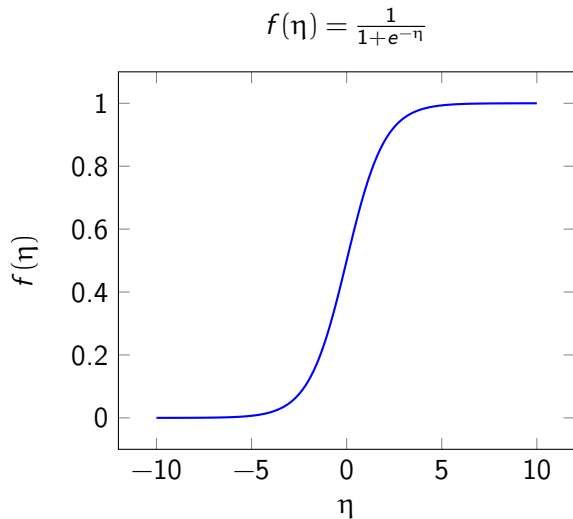
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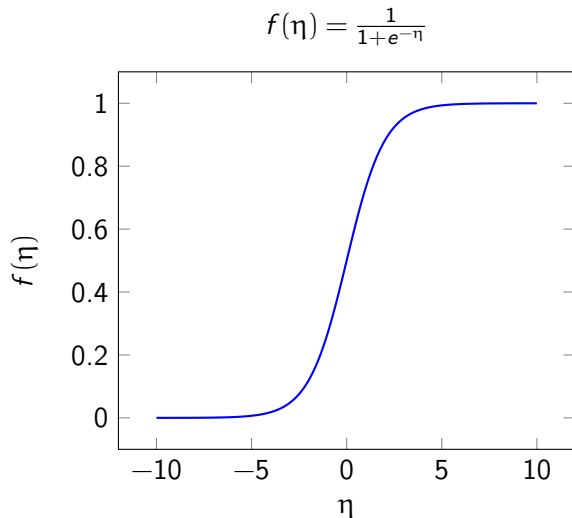


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- **Exercise:** show that the **log odds** is

$$\log \frac{p(y = 1 | x)}{p(y = 0 | x)} = \theta^T x. \quad (18)$$

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- How do we extend it to multiclass classification? (more on this later)



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- Closed-form solutions are not available.
- But, the likelihood is concave—[gradient ascent](#) gives us the unique optimal solution.

$$\theta := \theta + \alpha \nabla_{\theta} \ell(\theta). \quad (22)$$

# Gradient descent for logistic regression

## Math review: Chain rule

If  $z$  depends on  $y$  which itself depends on  $x$ , e.g.,  $z = (y(x))^2$ , then  $\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$ .

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$$= \left( \frac{y^{(n)}}{f^n} - \frac{1 - y^{(n)}}{1 - f^n} \right) \frac{\partial f^n}{\partial \theta_i} \quad \frac{d}{dx} \ln x = \frac{1}{x} \quad (24)$$

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# Gradient descent for logistic regression

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If  $z$  depends on  $y$  which itself depends on  $x$ , e.g.,  $z = (y(x))^2$ , then  $\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$ .

Likelihood for a single example:  $\ell^n = y^{(n)} \log f(\theta^T x^{(n)}) + (1 - y^{(n)}) \log(1 - f(\theta^T x^{(n)}))$ .

$$\frac{\partial \ell^n}{\partial \theta_i} = \frac{\partial \ell^n}{\partial f^n} \frac{\partial f^n}{\partial \theta_i} \quad (23)$$

$$= \left( \frac{y^{(n)}}{f^n} - \frac{1 - y^{(n)}}{1 - f^n} \right) \frac{\partial f^n}{\partial \theta_i} \quad \frac{d}{dx} \ln x = \frac{1}{x} \quad (24)$$

$$= \left( \frac{y^{(n)}}{f^n} - \frac{1 - y^{(n)}}{1 - f^n} \right) \left( f^n(1 - f^n) x_i^{(n)} \right) \quad \text{Exercise: apply chain rule to } \frac{\partial f^n}{\partial \theta_i} \quad (25)$$

$$= (y^{(n)} - f^n) x_i^{(n)} \quad \text{simplify by algebra} \quad (26)$$

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The full gradient is thus  $\frac{\partial \ell}{\partial \theta_i} = \sum_{n=1}^N (y^{(n)} - f(\theta^T x^{(n)})) x_i^{(n)}$ .

## A closer look at the gradient

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- Does this look familiar?
- Our derivation for linear regression and logistic regression are quite similar...
- Next, a more general family of models.

# Compare linear regression and logistic regression

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Transfer function $f(\theta^T x)$	identity	logistic

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Mean $\mathbb{E}(Y   X = x; \theta)$	$f(\theta^T x)$	$f(\theta^T x)$

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- $x$  enters through a linear function.
- The main difference between the formulations is due to different conditional distributions.
- Can we generalize the idea to handle other output types, e.g., positive integers?

# Construct a generalized regression model

**Task:** Given  $x$ , predict  $p(y | x)$

**Modeling:**

- Choose a parametric family of distributions  $p(y; \theta)$  with parameters  $\theta \in \Theta$
- Choose a transfer function that maps a linear predictor in  $\mathbb{R}$  to  $\Theta$

$$\underbrace{x}_{\in \mathbb{R}^d} \mapsto \underbrace{w^T x}_{\in \mathbb{R}} \mapsto \underbrace{f(w^T x)}_{\in \Theta} = \theta, \quad (28)$$

**Learning:** MLE:  $\hat{\theta} \in \arg \max_{\theta} \log p(\mathcal{D}; \theta)$

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## Example: Construct Poisson regression

Say we want to predict the number of people entering a restaurant in New York during lunch time.

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### Math review: Poisson distribution

Given a random variable  $Y \in 0, 1, 2, \dots$  following  $\text{Poisson}(\lambda)$ , we have

$$p(Y = k; \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad (29)$$

where  $\lambda > 0$  and  $\mathbb{E}[Y] = \lambda$ .

The Poisson distribution is usually used to model the number of events occurring during a fixed period of time.

## Example: Construct Poisson regression

We've decided that  $Y | X = x \sim \text{Poisson}(\eta)$ , what should be the transfer function  $f$ ?  
 $x$  enters linearly:

$$x \mapsto \underbrace{w^T x}_{\mathbb{R}} \mapsto \lambda = \underbrace{f(w^T x)}_{(0, \infty)}$$

Standard approach is to take

$$f(w^T x) = \exp(w^T x).$$

Likelihood of the full dataset  $\mathcal{D} = \{(x_1, y_1), \dots, (x_n, y_n)\}$ :

$$\log p(y_i; \lambda_i) = [y_i \log \lambda_i - \lambda_i - \log(y_i!)] \quad (30)$$

$$\log p(\mathcal{D}; w) = \sum_{i=1}^n [y_i \log [\exp(w^T x_i)] - \exp(w^T x_i) - \log(y_i!)] \quad (31)$$

$$= \sum_{i=1}^n [y_i w^T x_i - \exp(w^T x_i) - \log(y_i!)] \quad (32)$$



# Multinomial Logistic Regression

- Say we want to get the predicted categorical distribution for a given  $x \in \mathbb{R}^d$ .
- First compute the scores ( $\in \mathbb{R}^k$ ) and then their softmax:

$$x \mapsto (\langle w_1, x \rangle, \dots, \langle w_k, x \rangle) \mapsto \theta = \left( \frac{\exp(w_1^T x)}{\sum_{i=1}^k \exp(w_i^T x)}, \dots, \frac{\exp(w_k^T x)}{\sum_{i=1}^k \exp(w_i^T x)} \right)$$

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- We can write the conditional probability for any  $y \in \{1, \dots, k\}$  as

$$p(y | x; w) = \frac{\exp(w_y^T x)}{\sum_{i=1}^k \exp(w_i^T x)}.$$

Recipe for constructing a conditional distribution for prediction:

- ① Define input and output space (as for any other model).
- ② Choose the output distribution  $p(y | x; \theta)$  based on the task
- ③ Choose the transfer function that maps  $w^T x$  to a  $\Theta$ .
- ④ (The formal family is called “generalized linear models”.)

Learning:

- Fit the model by maximum likelihood estimation.
- Closed solutions do not exist in general, so we use gradient ascent.

# Generative models

We've seen

- Model the conditional distribution  $p(y | x; \theta)$  using generalized linear models.
- (Previously) Directly map  $x$  to  $y$ , e.g., perceptron.

Next,

- Model the **joint distribution**  $p(x, y; \theta)$ .
- Predict the label for  $x$  as  $\arg \max_{y \in \mathcal{Y}} p(x, y; \theta)$ .

# Generative modeling through the Bayes rule

Training:

$$p(x, y) \tag{33}$$

(35)

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$$p(y | x) = \frac{p(x | y)p(y)}{p(x)} \quad \text{Bayes rule} \quad (34)$$

$$\arg \max_y p(y | x) = \arg \max_y p(x | y)p(y) \quad (35)$$

# Naive Bayes (NB) models

Let's consider binary text classification (e.g., fake vs genuine review) as a motivating example.

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**Bag-of-words** representation of a document

- ["machine", "learning", "is", "fun", "."]
- $x_i \in \{0, 1\}$ : whether the  $i$ -th word in our vocabulary exists in the input

$$\mathbf{x} = [x_1, x_2, \dots, x_d] \quad \text{where } d = \text{vocabulary size} \quad (36)$$

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What's the probability of a document  $x$ ?

$$p(x | y) = p(x_1, \dots, x_d | y) \quad (37)$$

$$= p(x_1 | y) p(x_2 | y, x_1) p(x_3 | y, x_2, x_1) \dots p(x_d | y, x_{d-1}, \dots, x_1) \quad \text{chain rule} \quad (38)$$

$$= \prod_{i=1}^d p(x_i | y, x_{<i}) \quad (39)$$

# Naive Bayes assumption

**Challenge:**  $p(x_i | y, x_{<i})$  is hard to model (and estimate), especially for large  $i$ .



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Solution:

### Naive Bayes assumption

Features are **conditionally independent** given the label:

$$p(x | y) = \prod_{i=1}^d p(x_i | y). \quad (40)$$

A strong assumption in general, but works well in practice.

## Parametrize $p(x_i | y)$ and $p(y)$

For binary  $x_i$ , assume  $p(x_i | y)$  follows Bernoulli distributions.

$$p(x_i = 1 | y = 1) = \theta_{i,1}, \quad p(x_i = 1 | y = 0) = \theta_{i,0}. \quad (41)$$

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Thus,

$$p(x, y) = p(x | y)p(y) \quad (43)$$

$$= p(y) \prod_{i=1}^d p(x_i | y) \quad \text{NB assumption} \quad (44)$$

$$= p(y) \prod_{i=1}^d \theta_{i,y} \mathbb{I}\{x_i = 1\} + (1 - \theta_{i,y}) \mathbb{I}\{x_i = 0\} \quad (45)$$

Indicator function  $\mathbb{I}\{\text{condition}\}$  evaluates to 1 if “condition” is true and 0 otherwise.

## MLE for our NB model

We maximize the likelihood of the data  $\prod_{n=1}^N p_{\theta}(x^{(n)}, y^{(n)})$  (as opposed to the *conditional* likelihood we've seen before).

(48)

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(46)

(48)

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$$= \sum_{n=1}^N \mathbb{I} \{y^{(n)} = 1 \wedge x_j^{(n)} = 1\} \frac{1}{\theta_{j,1}} + \mathbb{I} \{y^{(n)} = 1 \wedge x_j^{(n)} = 0\} \frac{1}{1 - \theta_{j,1}} \quad \text{ignore } y^{(n)} = 0 \quad (48)$$



## MLE solution for our NB model

Set  $\frac{\partial}{\partial \theta_{j,1}} \ell$  to zero:

$$\theta_{j,1} = \frac{\sum_{n=1}^N \mathbb{I} \left\{ y^{(n)} = 1 \wedge x_j^{(n)} = 1 \right\}}{\sum_{n=1}^N \mathbb{I} \left\{ y^{(n)} = 1 \right\}} \quad (49)$$

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In practice, count words:

$$\frac{\text{number of fake reviews containing "absolutely"}}{\text{number of fake reviews}}$$

Exercise: show that

$$\theta_{j,0} = \frac{\sum_{n=1}^N \mathbb{I}\{y^{(n)} = 0 \wedge x_j^{(n)} = 1\}}{\sum_{n=1}^N \mathbb{I}\{y^{(n)} = 0\}} \quad (50)$$

$$\theta_0 = \frac{\sum_{n=1}^N \mathbb{I}\{y^{(n)} = 1\}}{N} \quad (51)$$

NB assumption: **conditionally independent** features given the label

Recipe for learning a NB model:

- 1 Choose  $p(x_i | y)$ , e.g., Bernoulli distribution for binary  $x_i$ .
- 2 Choose  $p(y)$ , often a categorical distribution.
- 3 Estimate parameters by MLE (same as the strategy for conditional models) .

Next, NB with continuous features.

## NB with continuous inputs

Let's consider a multiclass classification task with continuous inputs.

$$p(x_i | y) \sim \mathcal{N}(\mu_{i,y}, \sigma_{i,y}^2) \quad (52)$$

$$p(y = k) = \theta_k \quad (53)$$

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Likelihood of the data:

$$p(\mathcal{D}) = \prod_{n=1}^N p(y^{(n)}) \prod_{i=1}^d p(x_i^{(n)} | y^{(n)}) \quad (54)$$

$$= \prod_{n=1}^N \theta_{y^{(n)}} \prod_{i=1}^d \frac{1}{\sqrt{2\pi}\sigma_{i,y^{(n)}}} \exp\left(-\frac{1}{2\sigma_{i,y^{(n)}}^2} \left(x_i^{(n)} - \mu_{i,y^{(n)}}\right)^2\right) \quad (55)$$

# MLE for Gaussian NB

Log likelihood:

$$\ell = \sum_{n=1}^N \log \theta_{y^{(n)}} + \sum_{n=1}^N \sum_{i=1}^d \log \frac{1}{\sqrt{2\pi}\sigma_{i,y^{(n)}}} - \frac{1}{2\sigma_{i,y^{(n)}}^2} \left( x_i^{(n)} - \mu_{i,y^{(n)}} \right)^2 \quad (56)$$

(58)

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$$\frac{\partial}{\partial \mu_{j,k}} \ell = \frac{\partial}{\partial \mu_{j,k}} \sum_{n: y^{(n)}=k} -\frac{1}{2\sigma_{j,k}^2} \left( x_j^{(n)} - \mu_{j,k} \right)^2 \quad \text{ignore irrelevant terms} \quad (57)$$

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Set  $\frac{\partial}{\partial \mu_{j,k}} \ell$  to zero:

$$\mu_{j,k} = \frac{\sum_{n:y^{(n)}=k} x_j^{(n)}}{\sum_{n:y^{(n)}=k} 1} \quad (59)$$

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Set  $\frac{\partial}{\partial \mu_{j,k}} \ell$  to zero:

$$\mu_{j,k} = \frac{\sum_{n:y^{(n)}=k} x_j^{(n)}}{\sum_{n:y^{(n)}=k} 1} = \text{sample mean of } x_j \text{ in class } k \quad (59)$$

Show that

$$\sigma_{j,k}^2 = \frac{\sum_{n:y^{(n)}=k} \left(x_j^{(n)} - \mu_{j,k}\right)^2}{\sum_{n:y^{(n)}=k} 1} = \text{sample variance of } x_j \text{ in class } k \quad (60)$$

$$\theta_k = \frac{\sum_{n:y^{(n)}=k} 1}{N} \quad (\text{class prior}) \quad (61)$$

## Decision boundary of the Gaussian NB model

Is the Gaussian NB model a linear classifier?

(66)

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$$= \log \frac{\theta_0}{1-\theta_0} + \sum_{i=1}^d \left( \log \sqrt{\frac{\sigma_{i,0}^2}{\sigma_{i,1}^2}} + \left( \frac{(x_i - \mu_{i,0})^2}{2\sigma_{i,0}^2} - \frac{(x_i - \mu_{i,1})^2}{2\sigma_{i,1}^2} \right) \right)$$

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## Decision boundary of the Gaussian NB model

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$$\text{assume that } \sigma_{i,0} = \sigma_{i,1} = \sigma_i, \quad (\theta_0 = 0.5) \quad (64)$$

$$= \sum_{i=1}^d \frac{1}{2\sigma_i^2} \left( (x_i - \mu_{i,0})^2 - (x_i - \mu_{i,1})^2 \right) \quad (65)$$

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## Decision boundary of the Gaussian NB model

Assuming the variance of each feature is the same for both classes, we have

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$$\theta_i = \frac{\mu_{i,1} - \mu_{i,0}}{\sigma_i^2} \quad \text{for } i \in [1, d] \quad (70)$$

$$\theta_0 = \sum_{i=1}^d \frac{\mu_{i,0}^2 - \mu_{i,1}^2}{2\sigma_i^2} \quad \text{bias term} \quad (71)$$

# Naive Bayes vs logistic regression

	logistic regression	Gaussian naive Bayes
model type	conditional/discriminative	generative
parametrization	$p(y   x)$	$p(x   y), p(y)$
assumptions on $Y$	Bernoulli	Bernoulli
assumptions on $X$	—	Gaussian
decision boundary	$\theta_{\text{LR}}^T x$	$\theta_{\text{GNB}}^T x$

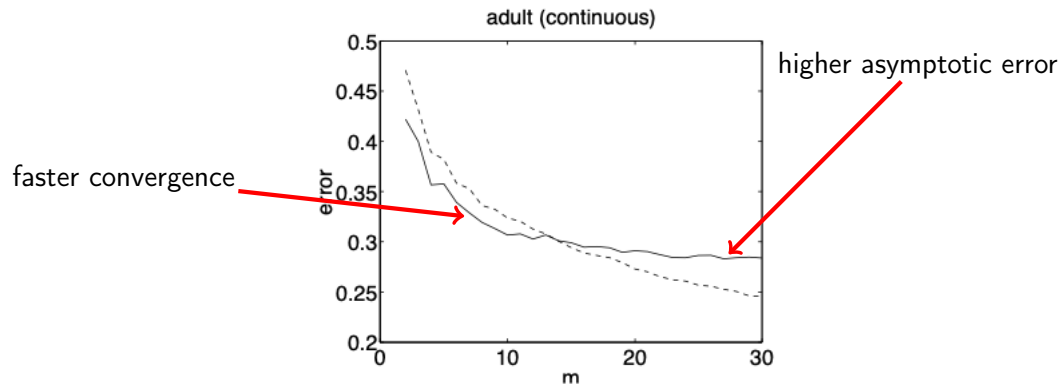
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Given the same training data, is  $\theta_{\text{LR}} = \theta_{\text{GNB}}$ ?

# Generative vs discriminative classifiers

Ng, A. and Jordan, M. (2002). [On discriminative versus generative classifiers: A comparison of logistic regression and naive Bayes](#). In Advances in Neural Information Processing Systems 14.



Solid line: naive Bayes; dashed line: logistic regression.

# Naive Bayes vs logistic regression

Logistic regression and Gaussian naive Bayes converge to the same classifier asymptotically, assuming the GNB assumption holds.

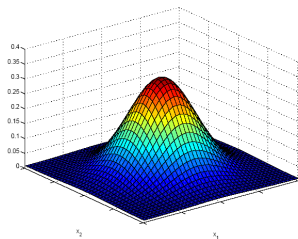
- Data points are generated from Gaussian distributions for each class
- Each dimension is independently generated
- Shared variance

What if the GNB assumption is not true?

# Multivariate Gaussian Distribution

- $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ , a Gaussian (or normal) distribution defined as

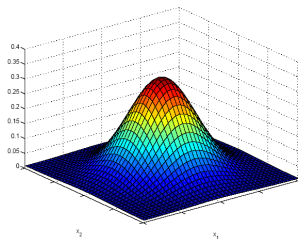
$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} \exp \left[ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right]$$



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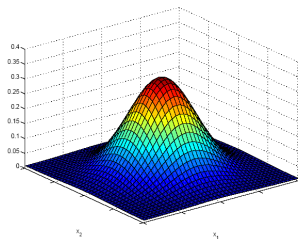
- Mahalanobis distance  $(x - \mu_k)^T \Sigma^{-1} (x - \mu_k)$  measures the distance from  $x$  to  $\mu$  in terms of  $\Sigma$



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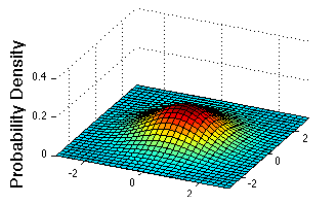
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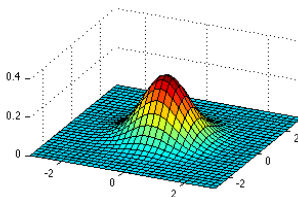
- Mahalanobis distance  $(x - \mu_k)^T \Sigma^{-1} (x - \mu_k)$  measures the distance from  $x$  to  $\mu$  in terms of  $\Sigma$
- It normalizes for difference in variances and correlations

# Bivariate Normal

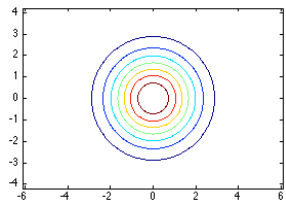
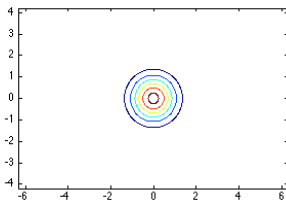
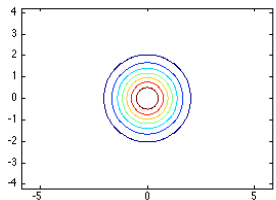
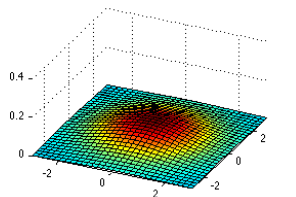
$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



$$\Sigma = 0.5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

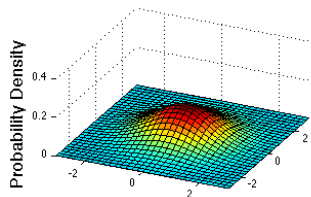


$$\Sigma = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

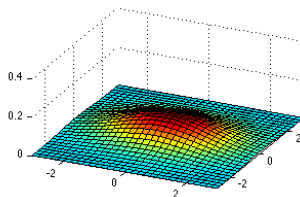


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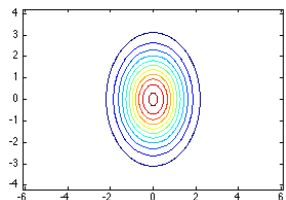
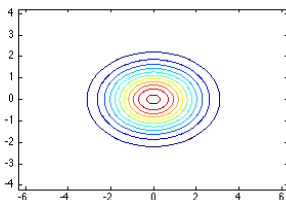
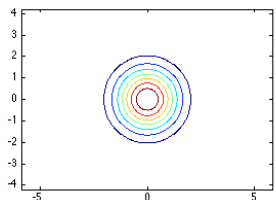
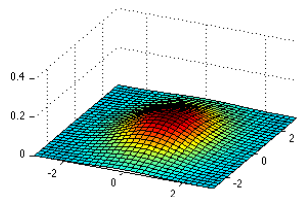
$$\text{var}(x_1) = \text{var}(x_2)$$



$$\text{var}(x_1) > \text{var}(x_2)$$

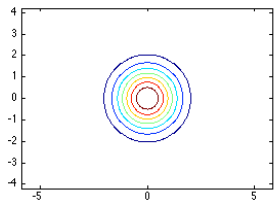
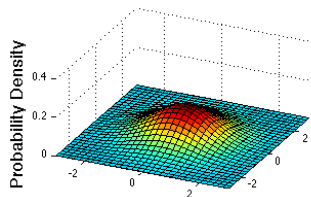


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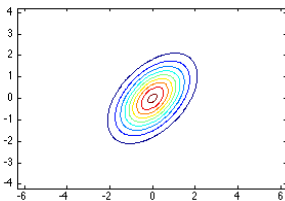
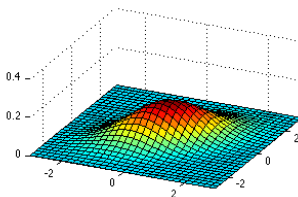


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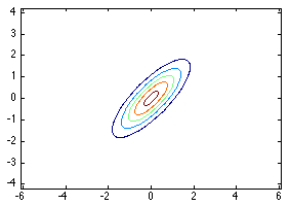
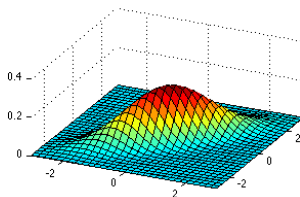
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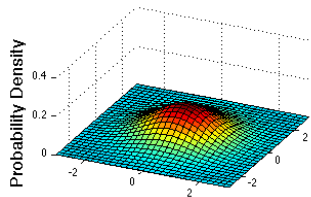


$$\Sigma = \begin{pmatrix} 1 & 0.8 \\ 0.8 & 1 \end{pmatrix}$$

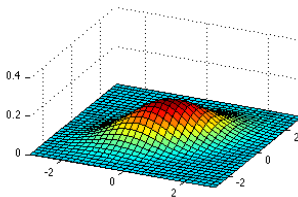


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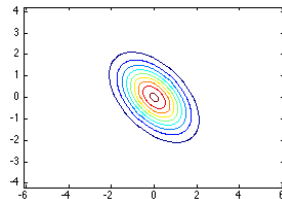
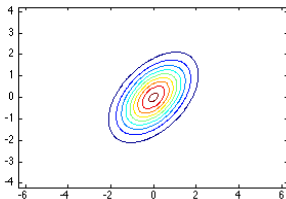
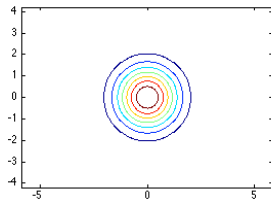
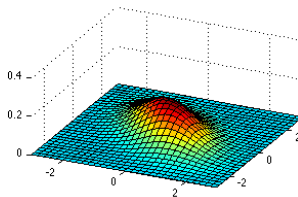
$$\text{Cov}(x_1, x_2) = 0$$



$$\text{Cov}(x_1, x_2) > 0$$



$$\text{Cov}(x_1, x_2) < 0$$



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- Gaussian Bayes Classifier in its general form assumes that  $p(\mathbf{x}|y)$  is distributed according to a multivariate normal (Gaussian) distribution
- Multivariate Gaussian distribution:

$$p(\mathbf{x}|t = k) = \frac{1}{(2\pi)^{d/2}|\Sigma_k|^{1/2}} \exp \left[ -\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_k)^T \Sigma_k^{-1}(\mathbf{x} - \boldsymbol{\mu}_k) \right]$$

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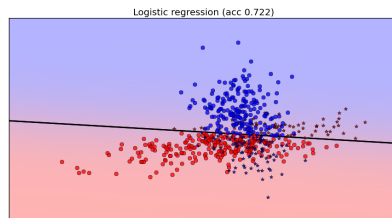
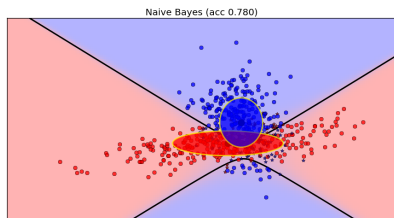
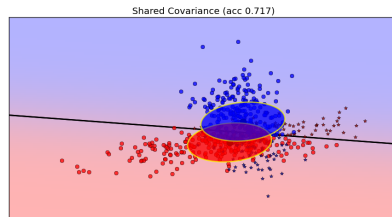
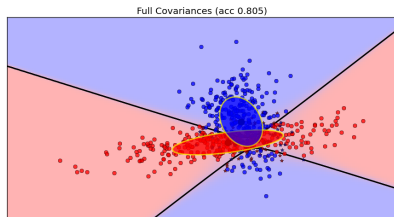
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- $\Sigma_k$  has  $\mathcal{O}(d^2)$  parameters - could be hard to estimate



# Example



# Gaussian Bayes Binary Classifier Cases

Different cases on the covariance matrix:

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GBC vs. Logistic Regression:

- If data is truly Gaussian distributed, then shared covariance = logistic regression.
- But logistic regression can learn other distributions.

# Summary

- Probabilistic framework of using maximum likelihood as a more principled way to derive loss functions.
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- Generative models the joint distribution, and may lead to more assumption on the data.
- When there is very few data point, it may be hard to model the distribution.
- Is there an equivalent “regularization” in a probabilistic framework?

## Bayesian ML: Classical Statistics

---

# Parametric Family of Densities

- A **parametric family of densities** is a set

$$\{p(y \mid \theta) : \theta \in \Theta\},$$

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- $\theta$  is a **parameter** in a [finite dimensional] **parameter space**  $\Theta$ .
- This is the common starting point for a treatment of classical or Bayesian statistics.
- In this lecture, whenever we say “density”, we could replace it with “mass function.” (and replace integrals with sums).



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- Statistics is about how to get by with  $\mathcal{D}$  in place of  $\theta$ .

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  - **Efficiency:** (Roughly speaking)  $\hat{\theta}_n$  is as accurate as we can get from a sample of size  $n$ .
- **Maximum likelihood estimators** are consistent and efficient under reasonable conditions.

# Example of Point Estimation: Coin Flipping

- Parametric family of mass functions:

$$p(\text{Heads} \mid \theta) = \theta,$$

for  $\theta \in \Theta = (0, 1)$ .

# Coin Flipping: MLE

- Data  $\mathcal{D} = (H, H, T, T, T, T, T, H, \dots, T)$ , assumed i.i.d. flips.
  - $n_h$ : number of heads
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$$L_{\mathcal{D}}(\theta) = p(\mathcal{D} \mid \theta) = \theta^{n_h} (1 - \theta)^{n_t}$$

- As usual, it is easier to maximize the log-likelihood function:

$$\begin{aligned}\hat{\theta}_{\text{MLE}} &= \arg \max_{\theta \in \Theta} \log L_{\mathcal{D}}(\theta) \\ &= \arg \max_{\theta \in \Theta} [n_h \log \theta + n_t \log(1 - \theta)]\end{aligned}$$



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# Coin Flipping: MLE

- Data  $\mathcal{D} = (H, H, T, T, T, T, T, H, \dots, T)$ , assumed i.i.d. flips.
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# Bayesian Statistics: Introduction

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- The prior reflects our belief about  $\theta$ , **before seeing any data**.

# A Bayesian Model

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- Putting the pieces together, we get a joint density on  $\theta$  and  $\mathcal{D}$ :

$$p(\mathcal{D}, \theta) = p(\mathcal{D} \mid \theta)p(\theta).$$

# The Posterior Distribution

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- Maximum a posteriori: Find  $\hat{\theta}_{MAP}$  Maximize the posterior distribution.

# Coin Flipping: Bayesian Model

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- One convenient choice would be a distribution from the Beta family

# Coin Flipping: Beta Prior

- Prior:

$$\begin{aligned}\theta &\sim \text{Beta}(\alpha, \beta) \\ p(\theta) &\propto \theta^{\alpha-1} (1-\theta)^{\beta-1}\end{aligned}$$

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Figure by Horas based on the work of Krishnavedala (Own work) [Public domain], via Wikimedia Commons  
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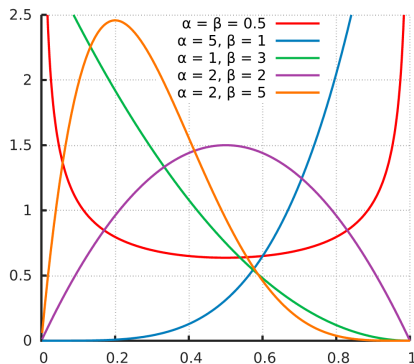


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- Mode of Beta distribution:

$$\arg \max_{\theta} p(\theta) = \frac{h-1}{h+t-2}$$

for  $h, t > 1$ .

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- Interpretation:

- Prior initializes our counts with  $h$  heads and  $t$  tails.
- Posterior increments counts by observed  $n_h$  and  $n_t$ .

## Sidebar: Conjugate Priors

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A family of distributions  $\pi$  is **conjugate to** parametric model  $P$  if for any prior in  $\pi$ , the posterior is always in  $\pi$ .

- The beta family is conjugate to the coin-flipping (i.e. Bernoulli) model.

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- Suppose we have a coin, possibly biased (**parametric probability model**):

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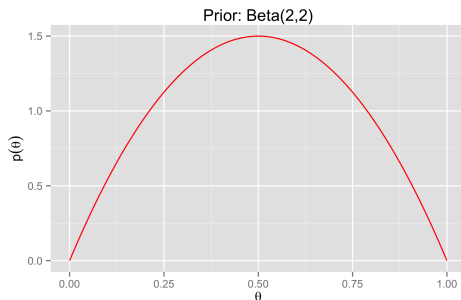


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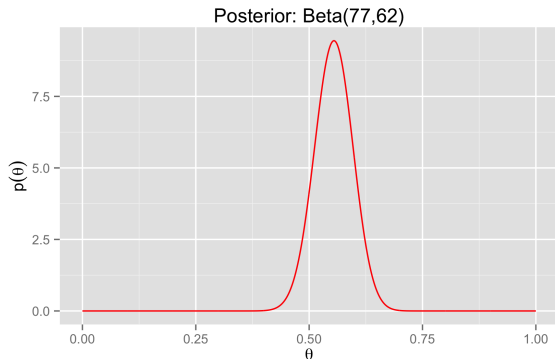
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- **Posterior distribution:**  $\theta \mid \mathcal{D} \sim \text{Beta}(77, 62)$ :



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  - **maximum a posteriori (MAP) estimate**  $\hat{\theta} = \arg \max_{\theta} p(\theta \mid \mathcal{D})$ 
    - Note: this is the **mode** of the posterior distribution

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- Select a point estimate using **Bayesian decision theory**:
  - Choose a loss function.
  - Find action **minimizing expected risk w.r.t. posterior**