Subgradient Descent

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Slides based on Lecture 3c from David Rosenberg's course material.

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SVM Optimization Problem (no intercept)

- SVM objective function:

$$J(w) = \frac{1}{n}\sum_{i=1}^{n} \max\left(0, 1 - y_i w^T x_i\right) + \lambda \|w\|^2.$$

- Not differentiable... but let's think about gradient descent anyway.
- Hinge loss: $\ell(m) = \max(0, 1-m)$

$$\begin{aligned} \nabla_w J(w) &= \nabla_w \left(\frac{1}{n}\sum_{i=1}^{n} \ell\left(y_i w^T x_i\right) + \lambda \|w\|^2 \right) \\ &= \frac{1}{n}\sum_{i=1}^{n} \nabla_w \ell\left(y_i w^T x_i\right) + 2\lambda w \end{aligned}$$

"Gradient" of SVM Objective

- Derivative of hinge loss $\ell(m) = \max(0, 1-m)$:

$$\ell'(m) = \begin{cases} 0 & m > 1 \\ -1 & m < 1 \\ \text{undefined} & m = 1 \end{cases}$$

- By chain rule, we have

$$\begin{aligned} \nabla_w \ell\left(y_i w^T x_i\right) &= \ell'\left(y_i w^T x_i\right) y_i x_i \\ &= \begin{cases} 0 & y_i w^T x_i > 1 \\ -y_i x_i & y_i w^T x_i < 1 \\ \text{undefined} & y_i w^T x_i = 1 \end{cases} \end{aligned}$$

"Gradient" of SVM Objective

$$
\nabla_w \ell\left(y_i w^T x_i\right) = \begin{cases} 0 & y_i w^T x_i > 1 \\ -y_i x_i & y_i w^T x_i < 1 \\ \text{undefined} & y_i w^T x_i = 1 \end{cases}
$$

So

$$
\begin{aligned}
\nabla_w J(w) &= \nabla_w \left(\frac{1}{n} \sum_{i=1}^{n} \ell\left(y_i w^T x_i\right) + \lambda \|w\|^2 \right) \\
&= \frac{1}{n} \sum_{i=1}^{n} \nabla_w \ell\left(y_i w^T x_i\right) + 2\lambda w \\
&= \begin{cases} \frac{1}{n} \sum_{i : y_i w^T x_i < 1} \left(-y_i x_i\right) + 2\lambda w & \text{all } y_i w^T x_i \neq 1 \\ \text{undefined} & \text{otherwise} \end{cases}
\end{aligned}
$$

Gradient Descent on SVM Objective?

- The gradient of the SVM objective is

$$\nabla_w J(w) = \frac{1}{n} \sum_{i: y_i w^T x_i < 1} (-y_i x_i) + 2\lambda w$$

  when $y_i w^T x_i \neq 1$ for all i, and otherwise is undefined.

Potential arguments for why we shouldn't care about the points of nondifferentiability:

- If we start with a random w, will we ever hit exactly $y_i w^T x_i = 1$?
- If we did, could we perturb the step size by ε to miss such a point?
- Does it even make sense to check $y_i w^T x_i = 1$ with floating point numbers?

However, would gradient descent work if the objective is not differentiable?

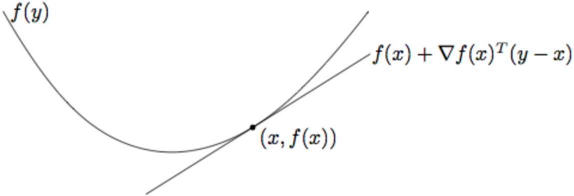
Subgradient

First-Order Condition for Convex, Differentiable Function

- Suppose $f : \mathbf{R}^d \to \mathbf{R}$ is convex and differentiable Then for any $x, y \in \mathbf{R}^d$

$$f(y) \geqslant f(x) + \nabla f(x)^T (y - x)$$

- The linear approximation to f at x is a global underestimator of f:



- This implies that if $\nabla f(x) = 0$ then x is a global minimizer of f.

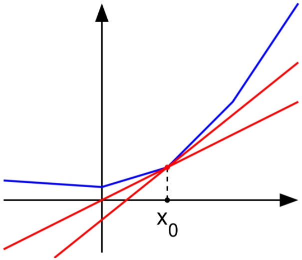
Figure from Boyd & Vandenberghe Fig. 3.2; Proof in Section 3.1.3

Subgradients

Definition

A vector $g \in \mathbb{R}^d$ is a **subgradient** of a *convex* function $f : \mathbb{R}^d \to \mathbb{R}$ at x if for all z,

$$f(z) \geqslant f(x) + g^T(z - x).$$



Blue is a graph of $f(x)$.
Each red line $x \mapsto f(x_0) + g^T(x - x_0)$ is a global lower bound on $f(x)$.

Properties

Definitions

- The set of all subgradients at x is called the **subdifferential**: $\partial f(x)$
- f is **subdifferentiable** at x if $\exists$ at least one subgradient at x.

For convex functions:

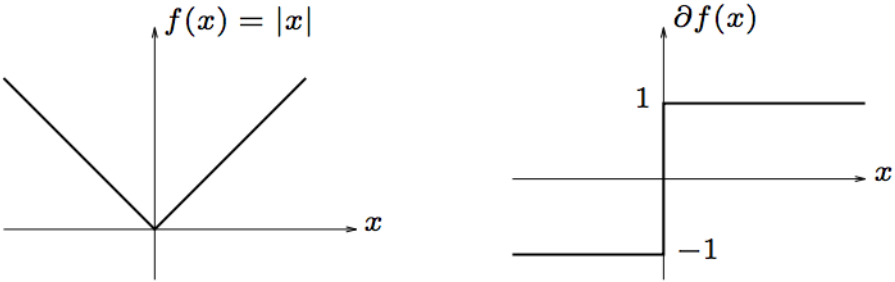
- f is differentiable at x iff $\partial f(x) = \{\nabla f(x)\}$.
- Subdifferential is always non-empty ($\partial f(x) = \emptyset \implies f$ is not convex)
- x is the global optimum iff $0 \in \partial f(x)$.

For non-convex functions:

- The subdifferential may be an empty set (no global underestimator).

Subdifferential of Absolute Value

- Consider $f(x) = |x|$

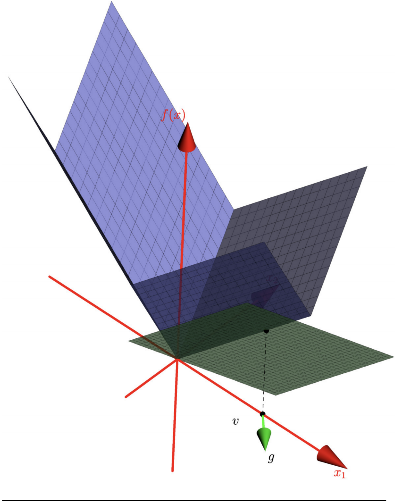


- Plot on right shows $\{(x, g) \mid x \in \mathbf{R},\ g \in \partial f(x)\}$

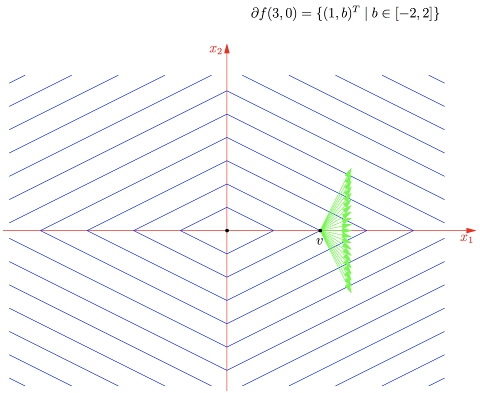
Boyd EE364b: Subgradients Slides

Subgradients of $f(x_1, x_2) = |x_1| + 2|x_2|$

- Let's find the subdifferential of $f(x_1, x_2) = |x_1| + 2|x_2|$ at $(3, 0)$.
- First coordinate of subgradient must be 1, from $|x_1|$ part (at $x_1 = 3$).
- Second coordinate of subgradient can be anything in $[-2, 2]$.
- So graph of $h(x_1, x_2) = f(3, 0) + g^T(x_1 - 3, x_2 - 0)$ is a global underestimate of $f(x_1, x_2)$, for any $g = (g_1, g_2)$, where $g_1 = 1$ and $g_2 \in [-2, 2]$.



Subdifferential on Contour Plot



Contour plot of $f(x_1, x_2) = |x_1| + 2|x_2|$, with set of subgradients at $(3,0)$. .

Plot courtesy of Brett Bernstein.

Basic Rules for Calculating Subdifferential

- Non-negative scaling: $\partial \alpha f(x) = \alpha \partial f(x)$ for $(\alpha > 0)$
- Summation: $\partial(f_1(x) + f_2(x)) = d_1 + d_2$ for any $d_1 \in \partial f_1$ and $d_2 \in \partial f_2$
- Composing with affine functions: $\partial f(Ax + b) = A^T \partial f(z)$ where $z = Ax + b$
- max: convex combinations of argmax gradients

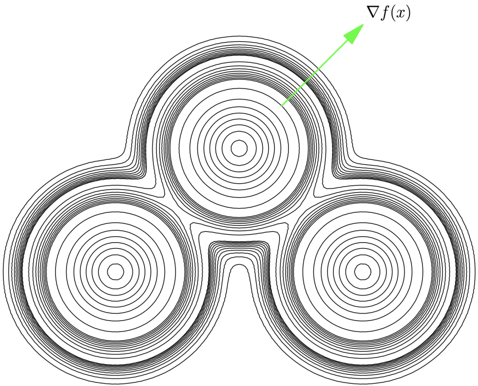
$$\partial \max(f_1(x), f_2(x)) = \begin{cases} \nabla f_1(x) & \text{if } f_1(x) > f_2(x), \\ \nabla f_2(x) & \text{if } f_1(x) < f_2(x), \\ \nabla \theta f_1(x) + (1-\theta) f_2(x) & \text{if } f_1(x) = f_2(x), \end{cases}$$

where $\theta \in [0, 1]$.

Subgradient Descent

Gradient orthogonal to level sets

We know that gradient points to the fastest ascent direction. What about subgradients?



Plot courtesy of Brett Bernstein.

Contour Lines and Subgradients

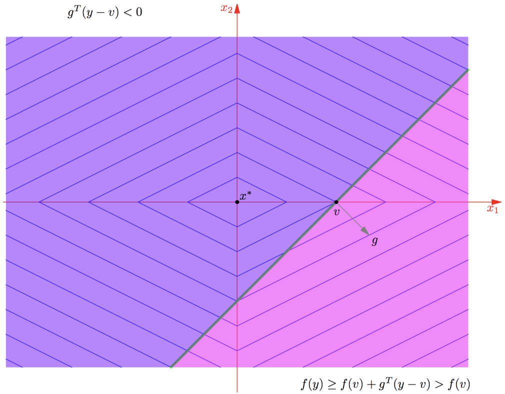
A hyperplane H **supports** a set S if H intersects S and all of S lies one one side of H.

Claim: If $f : \mathrm{R}^d \to \mathrm{R}$ has subgradient g at x_0, then the hyperplane H orthogonal to g at x_0 must **support** the level set $S = \left\{x \in \mathrm{R}^d \mid f(x) = f(x_0)\right\}$.

Proof:

- For any y, we have $f(y) \geqslant f(x_0) + g^T(y - x_0)$. (def of subgradient)
- If y is strictly on side of H that g points in,
  - then $g^T(y - x_0) > 0$.
  - So $f(y) > f(x_0)$.
  - So y is not in the level set S.
- $\therefore$ All elements of S must be on H or on the $-g$ side of H.

Subgradient of $f(x_1, x_2) = |x_1| + 2|x_2|$



- Points on g side of H have larger f-values than $f(x_0)$. (from proof)
- But points on $-g$ side may **not** have smaller f-values.
- So $-g$ may **not** be a descent direction. (shown in figure)

Plot courtesy of Brett Bernstein.

Subgradient Descent

- Move along the negative subgradient:

$$x^{t+1} = x^t - \eta g \quad \text{where } g \in \partial f(x^t) \text{ and } \eta > 0$$

- This can increase the objective but gets us closer to the minimizer if f is convex and η is small enough:

$$\|x^{t+1} - x^*\| < \|x^t - x^*\|$$

- Subgradients don't necessarily converge to zero as we get closer to x^*, so we need decreasing step sizes, e.g. $O(1/t)$ or $O(1/\sqrt{t})$.

- Subgradient methods are slower than gradient descent, e.g. $O(1/\epsilon^2)$ vs $O(1/\epsilon)$ for convex functions.

Based on https://www.cs.ubc.ca/~schmidtm/Courses/5XX-S20/S4.pdf

Subgradient descent for SVM (HW3)

SVM objective function:

$$J(w) = \frac{1}{n}\sum_{i=1}^{n} \max\left(0, 1 - y_i w^T x_i\right) + \lambda \|w\|^2.$$

Pegasos: stochastic subgradient descent with step size $\eta_t = 1/(t\lambda)$

```
Input: λ > 0. Choose w_1 = 0, t = 0
While termination condition not met
  For j = 1, ..., n (assumes data is randomly permuted)
    t = t + 1
    η_t = 1/(tλ);
    If y_j w_t^T x_j < 1
      w_{t+1} = (1 − η_t λ) w_t + η_t y_j x_j
    Else
      w_{t+1} = (1 − η_t λ) w_t
```

Summary

- Subgradient: generalize gradient for non-differentiable convex functions
- Subgradient "descent":
    - General method for non-smooth functions
    - Simple to implement
    - Slow to converge