

Recap of Bayesian Methods

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Recap: Bayesian Methods

$$\{f(y|\theta) : \theta \in \Theta\}, y_i \sim \text{i.i.d.}$$

Frequentist - $\hat{\theta} = \underset{\theta}{\operatorname{argmax}} L_D(\theta) = \underset{\theta}{\operatorname{argmax}} \prod_{i=1}^n f(y_i|\theta)$

Point Estimators - MLE

Bayesian Decision Theory

- Ingredients:
 - Parameter space Θ . $\theta \in \Theta$
 - Prior: Distribution $p(\theta)$ on Θ .
 - Action space \mathcal{A} .
 - Loss function: $\ell: \mathcal{A} \times \Theta \rightarrow \mathbb{R}$.
- The posterior risk of an action $a \in \mathcal{A}$ is
$$\begin{aligned} r(a) &:= \mathbb{E}[\ell(\theta, a) | \mathcal{D}] \\ &= \int \underbrace{\ell(\theta, a)}_{\text{Loss}} \underbrace{p(\theta | \mathcal{D})}_{\text{Posterior}} d\theta. \end{aligned}$$
- It's the expected loss under the posterior.
- A Bayes action a^* is an action that minimizes posterior risk:

$$r(a^*) = \min_{a \in \mathcal{A}} r(a)$$

The Posterior Predictive Distribution

- Suppose we've already seen data \mathcal{D} .
- The **posterior predictive distribution** is given by

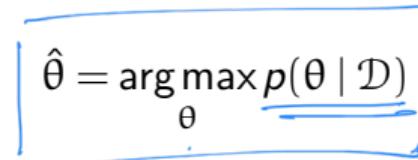
$$x \mapsto p(y | x, \mathcal{D}) = \int \underbrace{p(y | x; \theta)}_{\text{posterior distribution of } \theta} \underbrace{p(\theta | \mathcal{D})}_{d\theta}.$$

- This is an average of all conditional densities in our family, weighted by the posterior.
- May not have closed form.
- Numerical integral may be hard to compute.

MAP Estimator Versus Posterior Predictive Distribution

Maximum a posteriori

- How do we predict by posterior predictive distribution given a new data point x^* ?
- We can use $\hat{y} = \arg \max_y p(y | x, \mathcal{D})$
- What about our MAP estimator for θ ?

$$\hat{\theta} = \arg \max_{\theta} p(\theta | \mathcal{D})$$

$$\frac{\partial p(\theta | \mathcal{D})}{\partial \theta} = 0$$


- We can also predict y by

$$\hat{y} = \arg \max_y p(y | x; \theta = \hat{\theta})$$

- In general, the predictions from two methods are different.

Questions

Question 1

Question 1. (From DeGroot and Schervish) Let θ denote the proportion of registered voters in a large city who are in favor of a certain proposition. Suppose that the value of θ is unknown, and two statisticians A and B assign to θ the following different prior PDFs $\xi_A(\theta)$ and $\xi_B(\theta)$, respectively:

$$\begin{aligned}\xi_A(\theta) &= 2\theta \quad \text{for } 0 < \theta < 1, \\ \xi_B(\theta) &= 4\theta^3 \quad \text{for } 0 < \theta < 1.\end{aligned}$$

In a random sample of 1000 registered voters from the city, it is found that 710 are in favor of the proposition. $D' + 710 = 290$

- ① Find the posterior distribution that each statistician assigns to θ .
- ② Find the Bayes estimate of θ (minimizer of posterior expected loss) for each statistician based on the squared error loss function. — mean
- ③ Show that after the opinions of the 1000 registered voters in the random sample had been obtained, the Bayes estimates for the two statisticians could not possibly differ by more than 0.002, regardless of the number in the sample who were in favor of the proposition.

Question 1: Solution

Note that both prior distributions are from the Beta family.

- ① We have

$$\xi_A(\theta|x) \propto \frac{f(x|\theta)\xi_A(\theta)}{\text{Likelihood}} \propto \theta^{711}(1-\theta)^{290}$$

$\xrightarrow{(2,1)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$

$\xrightarrow{(4,1)} \theta^{\alpha}(1-\theta)^{\beta}$

and

$$\xi_B(\theta|x) \propto f(x|\theta)\xi_B(\theta) \propto \theta^{713}(1-\theta)^{290}$$

Thus the posteriors from A and B are both beta with parameters $(712, 291)$ and $(714, 291)$, respectively.

- ② The respective means are $\frac{712}{1003}$ and $\frac{714}{1005}$.

- ③ In general the two means are given by

$$\frac{a+2}{1003} \quad \text{and} \quad \frac{a+4}{1005}.$$

The difference is less than $2/1000 = .002$.

$$\begin{aligned} \text{Mean} &= \frac{a}{\alpha + \beta} \\ &= \frac{a}{1005a + 2010} \\ &= \frac{-1003a - 4012}{1003 \times 1005} \\ &= \frac{2a - 2002}{1003 \times 1005} \end{aligned}$$

$a=0 \quad \approx 0.002$
 $a=1000 \quad a \approx 6$

Question 2 and 3

- **Question 2.** Two statistics students decide to compute 95% confidence intervals for the distribution parameter θ using an i.i.d. sample X_1, \dots, X_n . Student B uses Bayesian methods to find a 95% credible set $[L_B, R_B]$ for θ . Student F uses frequentist methods to find a 95% confidence interval $[L_F, R_F]$ for θ . Both conclude that parameter θ is in their respective intervals with probability at least .95. Who is correct? Explain.
- **Question 3.** Suppose θ has prior distribution $\text{Beta}(a, b)$ for some $a, b > 0$. Given θ , suppose we make independent coin flips with heads probability θ . Find values of a, b and the coin flips so that the posterior variance is larger than the prior variance. [Hint: Recall that a $\text{Beta}(a, b)$ random variable has variance given by

$$\frac{ab}{(a+b)^2(a+b+1)}.$$

Try $b = 1$.]

Question 2: Solution

$$P\left(\bar{Y} - z_{1-\alpha/2} \frac{s_n}{\sqrt{n}} \leq \theta \leq \bar{Y} + z_{1-\alpha/2} \frac{s_n}{\sqrt{n}}\right) \approx (1-\alpha)$$

- **Question 2.** Two statistics students decide to compute 95% confidence intervals for the distribution parameter θ using an i.i.d. sample X_1, \dots, X_n . Student B uses Bayesian methods to find a 95% credible set $[L_B, R_B]$ for θ . Student F uses frequentist methods to find a 95% confidence interval $[L_F, R_F]$ for θ . Both conclude that parameter θ is in their respective intervals with probability at least .95. Who is correct? Explain.

Solution:

$$95\% \Rightarrow \alpha = 0.05 \rightarrow 1 - \alpha = 0.95$$

- The frequentist student is totally incorrect, since they have misunderstood what a frequentist confidence interval is. Using frequentist methodology, θ is not a random variable, so it doesn't make sense to say it lies in some fixed interval $[L_F, R_F]$. The correct interpretation is that if independent experiments like this were repeated, then at least 95% of the time $[L_F, R_F]$ will contain θ . **That is, the interval is random not θ .**
- We can say that the Bayesian student is consistent. Recall that to compute the credible set, the Bayesian student had to introduce some prior distribution π on θ . What we can say is if someone believes π is correct, then it is rational, given the data, to conclude that θ will lie in the posterior credible set with probability 95%.

$$P(\theta \in [L_B, R_B] | D) \approx .95$$

Question 3: Solution

- **Question 3.** Suppose θ has prior distribution $\text{Beta}(a, b)$ for some $a, b > 0$. Given θ , suppose we make independent coin flips with heads probability θ . Find values of a, b and the coin flips so that the posterior variance is larger than the prior variance. [Hint: Recall that a $\text{Beta}(a, b)$ random variable has variance given by

$$\frac{ab}{(a+b)^2(a+b+1)} \cdot p(\theta) = \theta^{a-1}(1-\theta)^{b-1}$$

Try $b = 1$.]

Solution: As hinted, let's try $a = 10$, $b = 1$ and 9 coin flips all landing tails. The prior variance is given by

$$\frac{10 \cdot 1}{(10+1)^2(10+1+1)} = \frac{5}{726} \approx .0069$$

while the posterior variance is given by

$$\frac{(a+n_h)(b+n_t)}{(a+n_h)(b+n_t)^2(a+n_h+b+n_t+1)} \quad \stackrel{a+n_h, b+n_t}{\overbrace{\qquad\qquad\qquad}} \quad \frac{10 \cdot 10}{(10+10)^2(10+10+1)} = \frac{1}{84} \approx .0119.$$

Question 4

Question 4. What would be the Maximum a Posteriori (MAP) estimator for λ for i.i.d. $\{x_1, x_2, \dots, x_N\}$ where $x_i \sim \exp(\lambda)$ with prior $\lambda \sim \text{Uniform}[u_0, u_1]$?

Question 4: Solution

$$f(x) = \lambda e^{-\lambda x}$$

- Likelihood: $L(x_1, \dots, x_N | \lambda) = \lambda^N e^{-\lambda(x_1 + \dots + x_N)}$
- log-likelihood: $\ell(\lambda | x_1, \dots, x_N) = N \ln \lambda - \lambda(x_1 + \dots + x_N)$

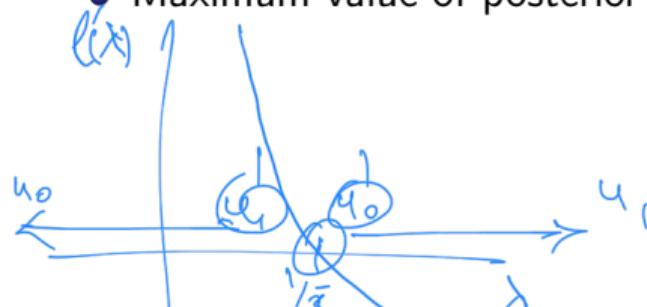
- $\ell'(\lambda) = \frac{N}{\lambda} - (x_1 + \dots + x_N)$

$$\begin{cases} > 0 & \text{if } 0 < \lambda < 1/\bar{x} = N/(x_1 + \dots + x_N), \\ = 0 & \text{if } \lambda = 1/\bar{x} \\ < 0 & \text{if } \lambda > 1/\bar{x} \end{cases}$$

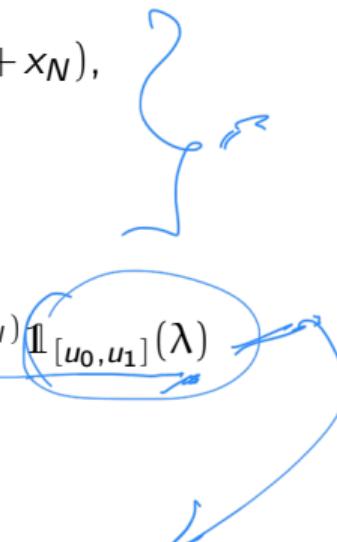
- Prior: $p(\lambda) = \frac{1}{u_1 - u_0} \mathbb{1}_{[u_0, u_1]}(\lambda)$

- Posterior: $p(\lambda | x_1, \dots, x_N) \propto L(x_1, \dots, x_N | \lambda) p(\lambda) = \lambda^N e^{-\lambda(x_1 + \dots + x_N)} \mathbb{1}_{[u_0, u_1]}(\lambda)$

- Maximum value of posterior is attained at



$$\lambda = \begin{cases} u_0 & \text{if } u_0 > 1/\bar{x}, \\ 1/\bar{x} & \text{if } u_0 \leq 1/\bar{x} \leq u_1, \\ u_1 & \text{if } u_1 < 1/\bar{x}. \end{cases}$$



References

$$\varepsilon \sim N(0, \sigma^2) \leftrightarrow N(0, \beta)$$

$$\beta = 1/\sigma^2$$

- DS-GA 1003 Machine Learning Spring 2019