Numerical Analysis: Midterm

(**50 marks**)

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Question 1 (Floating point arithmetic, **10 marks**). True or false?

1. Let $(\bullet)_2$ denote binary representation. It holds that $(0.1011)_2 + (0.0101)_2 = 1$.
2. Let $(\bullet)_3$ denote base 3 representation. It holds that $(1000)_3 \times (0.002)_3 = 2$.
3. A natural number with binary representation $(b_4 b_3 b_2 b_1 b_0)_2$ is even if and only if $b_0 = 0$.
4. In Julia, `Float64(.4) == Float32(.4)` evaluates to `true`.
5. Machine addition $\widehat{+}$ is a commutative operation. More precisely, given any two double-precision floating point numbers $x \in \mathbf{F}_{64}$ and $y \in \mathbf{F}_{64}$, it holds that $x \mathbin{\widehat{+}} y = y \mathbin{\widehat{+}} x$.
6. Let $\mathbf{F}_{32}$ and $\mathbf{F}_{64}$ denote respectively the sets of single and double precision floating point numbers. It holds that $\mathbf{F}_{32} \subset \mathbf{F}_{64}$.
7. In Julia, `eps(Float16)` returns the smallest strictly positive number that can be represented exactly in the `Float16` format.
8. Let $\mathbf{F}_{64}$ denote the set of double precision floating point numbers. For any $x \in \mathbf{R}$ such that $x \in \mathbf{F}_{64}$, it holds that $x + 1 \in \mathbf{F}_{64}$.
9. Let $x \in \mathbf{R}$ and $y \in \mathbf{R}$ be two numbers that are exactly representable in the `Float64` format. Then $x \mathbin{\widehat{+}} y = x + y$: machine addition is exact in this case.
10. It holds that $(0.\overline{2200})_3 = (0.9)_{10}$.

⚙ **Question 2** (Interpolation and approximation, **10 marks**). Are the following assertions true or false? Throughout this exercise, we use the notation $x_i^n = i/n$. The notation $\mathbf{P}(n)$ denotes the set of polynomials of degree less than or equal to n. We proved in class that, for any function $u\colon \mathbf{R} \to \mathbf{R}$ and for all $n \in \mathbf{N}_{>0}$, there exists a unique polynomial $p_n \in \mathbf{P}(n)$ such that

$$\forall i \in \{0, \dots, n\}, \qquad p_n(x_i^n) = u(x_i^n). \tag{1}$$

1. If u is not the zero function, then the degree of p_n is exactly n.

2. If $u(x) = 2x + 1$, then $p_n = u$ for all $n \in \{1, 2, 3, \dots\}$.

3. Fix $u(x) = 1 + \sin(57\pi x)$. Then $p_3(x) = 1$.

4. Fix $u(x) = (2x - 1)^3$. Then $p_2(x) = 2x - 1$.

5. Fix $n \in \mathbf{N}_{>0}$ and suppose that $u\colon \mathbf{R} \to \mathbf{R}$ is a smooth function. There exists a constant $K > 0$ independent of x such that

$$\forall x \in \mathbf{R}, \qquad u(x) - p_n(x) = K \prod_{i=0}^{n} (x - x_i^n).$$

6. It holds that

$$\forall x \in [0, 1], \qquad \Bigl|(x - x_0^n) \dots (x - x_n^n)\Bigr| \leqslant \left(\frac{1}{n}\right)^n.$$

7. In Julia, assuming n and the function u have already been defined, the following code enables to calculate the interpolating polynomial p_n of u:

```
using Polynomials
# Assume `n=5` and `u` have already been defined
x = LinRange(0, 1, n + 1)
p = fit(x, u)
```

8. Let Δ denote the finite difference operator: for a function $f\colon \mathbf{R} \to \mathbf{R}$, the function Δf is defined as

$$\Delta f(x) = f(x + 1) - f(x).$$

Then $f \in \mathbf{P}(n)$ if and only if $\Delta^{n+1} f = 0$. Here Δ^{n+1} denotes the composition of $n + 1$ applications of the operator Δ.

9. In Julia, the following code enables to fit the data `x` and `y` by a straight line.

```
using Polynomials
x = [1, 2, 3, 4]
y = [4, 3, 2, 1]
p = fit(x, y, 1)
```

10. The choice of interpolation nodes may have an impact on the quality of the interpolation.

Question 3 (Numerical integration, **10 marks**). The Gauss–Legendre quadrature formula with n nodes is an approximate integration formula of the form

$$I(u) := \int_{-1}^{1} u(x)\,\mathrm{d}x \approx \sum_{i=1}^{n} w_i\, u(x_i) =: \widehat{I}_n(u), \tag{2}$$

which is exact when u is a polynomial of degree less than or equal to $2n-1$. (Note that the nodes are here numbered starting from 1.)

1. (**5 marks**) Find the nodes and weights of the Gauss–Legendre rule with $n = 3$ nodes, without using any formula (unless you prove it beforehand).

   Hint: Recall that a necessary and sufficient condition in order for (2) to be satisfied for any polynomial $p \in \mathbf{P}(5)$ is that

   $$\int_{-1}^{1} x^d\,\mathrm{d}x = \sum_{i=1}^{n} w_i x_i^d, \qquad \text{for all } d \in \{0,1,2,3,4,5\}.$$

   Furthermore, given the symmetry of the problem, it is reasonable to look for a solution of the following form, which enables to reduce the number of unknowns.

   $$(x_1, x_2, x_3, w_1, w_2, w_3) = (-x, 0, x, w_1, w_2, w_1).$$

2. (**5 marks**) Are the following assertions true of false :

   - The degree of precision of the composite trapezium rule is 2.
   - The composite Simpson rule can be used to integrate exactly a quadratic polynomial.
   - The degree of precision of the following rule is 1

     ```
     function my_integrate(f, a, b)
         x = LinRange(a, b, 100)
         h = x[2] - x[1]
         return h * sum(f, x[1:end-1])
     end
     ```

   - The degree of precision of the following integration rule is 2:

     $$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx 2f(0) + \frac{1}{3}f''(0).$$

   - Suppose that $u\colon \mathbf{R} \to \mathbf{R}$ is a smooth function, and let $\widehat{I}_n(u)$ denote an approximation of the integral $I(u) := \int_{-1}^{1} u(x)\,\mathrm{d}x$ by the composite trapezium approximation with n points. Let

     $$\widehat{J}_n(u) = 2\widehat{I}_{2n}(u) - \widehat{I}_n u.$$

     It holds that

     $$\lim_{n\to\infty} n^2 \left| I(u) - \widehat{J}_n(u) \right| = 0.$$

💻 **Computer exercise 1** (Interpolation, **10 marks**)**.** Consider the following data:

Time (hours)	**Temperature (°C)**
6	10.5
9	15.0
12	20.2
15	25.1
18	22.8
21	17.4

Table 1: Recorded temperatures at different times of the day.

We wish to approximate the temperature as a smooth function of time. To this end, calculate the interpolation polynomial, as well as the best quadratic polynomial approximation (in the sense that the sum of square errors is minimized). You may use the `Polynomials` library. Plot on the same graph:

- The data points using `scatter`;
- The polynomial p_{int} interpolating the data points;
- The quadratic polynomial p_{app} that best approximates the data, in the sense of least squares.

💻 **Computer exercise 2** (Numerical integration, **10 marks**)**.** Boole's integration rule reads

$$\int_{-1}^{1} u(x)\,\mathrm{d}x \approx \frac{7}{45}u(-1) + \frac{32}{45}u\left(-\frac{1}{2}\right) + \frac{12}{45}u\left(0\right) + \frac{32}{45}u\left(\frac{1}{2}\right) + \frac{7}{45}u(1).$$

- Write a function `comp_boole(u, a, b, N)`, which returns an approximation of the integral

  $$I(u) = \int_a^b u(x)\,\mathrm{d}x$$

  obtained by partitioning the integration interval $[a, b]$ into N cells, and applying Boole's rule within each cell.

- Take $u(x) = \cos(x)$, $a = -1$ and $b = 1$. Plot the evolution of the error for $N \in \{1, \dots, 1000$.

- Estimate the order of convergence with respect to N, i.e. find α such that

  $$|\widehat{I}_N - I| \propto CN^{-\alpha},$$

  where I denotes the exact value of the integral and $\widehat{I}_N$ denotes its approximation. In order to find α, use the function `fit` from the `Polynomials` package to find a linear approximation of the form

  $$\log|\widehat{I}_N - I| \approx \log(C) - \alpha\log(N).$$