1. The computational cost of the forward substitution algorithm, for a matrix of size $n \times n$, scales as $\mathcal{O}(n)$.

2. The computational cost to calculate the LU decomposition of an invertible matrix scales as $\mathcal{O}(n^2)$.

3. Suppose that A is a symmetric, positive definite matrix. Calculating the Cholesky decomposition of A requires fewer floating point operations than calculating the LU decomposition of A.

4. The condition number $\kappa_2(\mathsf{A})$ of a square invertible matrix A is always strictly larger than 1.

5. The spectral radius of a matrix square A is zero if and only if A is zero.

6. If the condition number of a matrix A is very large ($\gg 1$), then it is possible that rounding errors arising from floating point arithmetic will have a large impact on the accuracy of the numerical solution to the linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ (calculated by LU decomposition followed by forward and backward substitution, for example).

7. Suppose that $\mathsf{A} \in \mathbf{R}^{n \times n}$ is symmetric and positive definite, and let $\boldsymbol{b} \in \mathbf{R}^n$. Consider the following iterative method for solving the linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$:

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \omega \left(\boldsymbol{b} - \mathsf{A}\boldsymbol{x}^{(k)} \right). \tag{1}$$

   This iteration converges to the exact solution of the linear system for all $\omega \in \mathbf{R}$.

8. The convergence speed of the iteration (1), for the optimal value of ω, is independent of the condition number $\kappa_2(\mathsf{A})$.

9. If A is symmetric positive definite, there always exists $\omega \in \mathbf{R}$ such that the iteration (1) converges.

10. If A is a nonzero matrix, then its norm $\|\mathsf{A}\|_2$ is strictly positive.

11. Suppose that we want to solve the linear systems $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ and $\mathsf{A}\boldsymbol{x} = \boldsymbol{c}$ by a direct method. To this end, the LU decomposition of A can be calculated only once.

12. Assuming that the matrix `A` and the vector `b` are already defined, write (on paper) Julia code implementing 100 iterations of (1).

Solutions

1. False. The computational cost scales as $\mathcal{O}(n^2)$.
2. False. The computational cost scales as $\frac{2}{3}n^3 + \mathcal{O}(n^2)$ in general.
3. True. The computational cost of Cholesky decomposition is $\frac{1}{3}n^3 + \mathcal{O}(n^2)$.
4. False. The condition number is always greater than *or equal to* 1. For the identity matrix, the condition number is 1.
5. False. For example, the only eigenvalue of the nonzero matrix `A = [0 1; 0 0]` is 0, so its spectral radius is 0.
6. True.
7. False. For example, for $\omega = 0$, it holds that $\boldsymbol{x}^{(k)} = \boldsymbol{x}^{(0)}$ for all k. Therefore, unless $\boldsymbol{x}^{(0)}$ coincides with the solution of the linear system, the iteration does not converge to the solution.
8. False. We proved in class that the larger the condition number, the slower the convergence.
9. True.
10. True. If this was false, then $\|\cdot\|_2$ would not be called a norm.
11. True. This is the advantage of calculating the LU decomposition, compared to Gaussian elimination with corresponding operations on the right-hand side.
12. For example:

```
A = [3 1; 1 3]
b = [1, 1]
w = .01
x = [0, 0]  # Initial guess
for i in 1:100
    x += w * (b - A*x)
end
```