1. **(T/F)** Let ε denote the machine epsilon for the <MINTED> format. If $x \in \mathbf{R}$ is such that $0 < x < \varepsilon$, then x cannot be represented in the <MINTED> format.

2. **(T/F)** Machine multiplication is commutative, meaning that $a \widehat{*} b = b \widehat{*} a$ for any <MINTED> point numbers a and b.

3. **(T/F)** If `x` is a <MINTED> and `y` is a <MINTED> number, then the result of <MINTED> is a <MINTED> number.

4. **(T/F)** The only polynomial p of degree at most 3 such that $p(-1) = p(0) = p(1) = 1$ is the constant polynomial $p(x) = 1$.

5. **(T/F)** Given $x_0 < x_1 < x_2 < x_3$ and $y_0, y_1, y_2, y_3 \in \mathbb{R}$, the unique polynomial passing through these data points is given by

$$\begin{aligned} p(x) = &\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 \\ &+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3. \end{aligned}$$

6. Given $x_0 < \ldots < x_n$ and $y_0, \ldots, y_n \in \mathbb{R}$, prove that the constant polynomial p that minimizes the expression $\sum_{i=0}^{n} |y_i - p(x_i)|^2$ is given by

$$p(x) = \frac{1}{n+1} \sum_{i=0}^{n} y_i.$$

7. **(T/F)** Let $f(x) = \exp(2x)$, and for any $n \in \mathbb{N}$, let $f_n \in \mathcal{P}_n$ denote the polynomial interpolating f at $n+1$ equidistant points $-1 = x_0 < x_1 < \ldots < x_n = 1$. Then

$$\lim_{n \to \infty} \left(\max_{-1 \leqslant x \leqslant 1} \left| f(x) - f_n(x) \right| \right) = 0.$$

8. **(T/F)** In Julia, if `A` is a 10 by 10 matrix, then <MINTED> gives the matrix obtained by keeping only the rows with odd indices.

9. **(T/F)** The degree of precision of the Gauss–Legendre quadrature rule with $n+1$ points is equal to $2n$.

10. Describe in words what the following code does.

    <MINTED>

In the next questions, we consider the following linear system:

$$\mathsf{A}\boldsymbol{x} = \boldsymbol{b}, \qquad \mathsf{A} \in \mathbf{R}^{n\times n}, \qquad \boldsymbol{b} \in \mathbf{R}^{n}. \tag{1}$$

1. Suppose that `A` is a 10 by 10 upper triangular matrix, and that <MINTED> is a vector of size 10. Complete the following implementation of the backward substitution algorithm:

   <MINTED>

2. How many floating point operations does your implementation require?

3. Suppose that A is symmetric and positive definite. Describe step by step an efficient direct method for solving (1) in this case.

4. **(T/F)** Suppose that A is symmetric and positive definite. Then there exists a unique solution $\boldsymbol{x}_*$ to (1) and, furthermore,

$$\boldsymbol{x}_* \in \operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbf{R}^n} \left(\frac{1}{2}\boldsymbol{x}^T\mathsf{A}\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{x}\right).$$

5. **(T/F)** The Gauss–Seidel basic iterative method is convergent if and only if the matrix A is strictly diagonally dominant.

Until the end of the quiz, we consider the following scalar, nonlinear equation, where the function f is *twice continuously differentiable*:

$$f(x) = 0, \qquad f\colon \mathbf{R} \to \mathbf{R}, \qquad x \in \mathbf{R}. \tag{2}$$

1. **(T/F)** If $f'(x) > 1$ for all $x \in \mathbf{R}$, then there exists a unique solution to (2).

2. **(T/F)** Suppose that $f'(x) > 1$ for all $x \in \mathbf{R}$ and that the following converges to some $x_* \in \mathbf{R}$ when started from $x_0 = 1$. Then $f(x_*) = 0$.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \tag{3}$$

3. **(T/F)** If $f(0)f(1) > 0$, then there cannot exists a solution $x_* \in (0, 1)$ to (2).

4. **(2 marks)** Suppose that iteration (3) converges to some $x_* \in \mathbf{R}$. Prove that

$$\lim_{k\to\infty} \left|\frac{x_{k+1} - x_*}{x_k - x_*}\right| = \left|\frac{f''(x_*)}{2f'(x_*)}\right|.$$