

1. The degree of precision of the following quadrature rule is 1:

$$\int_{-1}^1 f(x) dx \approx 2f(0) - f(-1) + f(1).$$

2. The closed Newton–Cotes rule with n points is exact for all polynomials of degree up to $n + 1$.
 3. Simpson’s composite integration rule is always exact if f is a polynomial.
 4. The degree of precision of the following quadrature rule is 0:

$$\int_{-1}^1 f(x) dx \approx 2f(1).$$

5. Suppose that $f \in C^\infty[a, b]$ and let $I_n[f]$ denote the approximate integral of f using the composite Simpson rule with n integration points. Then it holds that

$$\lim_{n \rightarrow \infty} n^3 |I[f] - I_n[f]| = 0, \quad I[f] := \int_a^b f(x) dx.$$

6. Suppose that $f \in C^\infty[a, b]$ and let $I_n[f]$ denote the approximate integral of f using the composite trapezium rule with n integration points. Then for all $C \in \mathbf{R}$ it holds that

$$\forall n \in \{2, 3, \dots\}, \quad |I[f] - I_n[f]| \leq \frac{C}{n^2}.$$

7. Suppose that $f \in C^\infty[a, b]$ and let $I_n[f]$ denote the approximate integral of f using the composite trapezium rule with n integration points. Then it holds that

$$\lim_{n \rightarrow \infty} n |I[f] - I_n[f]| = 0.$$

8. There exist w_1 and w_2 such that the degree of precision of the following rule is 4:

$$\int_{-1}^1 f(x) dx = w_1 f(0) + w_2 f(1) + w_3 f(-1) + w_4 f\left(\frac{1}{3}\right).$$

9. In Julia, the following code implements the composite trapezium rule with $n + 1$ points:

```
f(x) = sin(x)
function I_approx(a, b, n)
    x = LinRange(a, b, n + 1)
    h = x[2] - x[1]
    return h/2 * sum(f, x[1:n]) + h/2 * sum(f, x[2:end])
end
```

10. In Julia, the following code implements the composite Simpson rule with $n + 1$ points:

```
f(x) = sin(x)
function I_approx(a, b, n)
    x = LinRange(a, b, n + 1)
    h = x[2] - x[1]
    w = [1; [2 + 7(i%2) for i in 1:n-1]; 1]
    return h/3 * w'f.(x)
end
```

11. **Bonus 1.** There exist integration points x_1, x_2, x_3 and weights w_1, w_2, w_3 such that the following integration rule has a degree of precision equal to 5.

$$\int_{-1}^1 u(x) dx \approx w_1 u(x_1) + w_2 u(x_2) + w_3 u(x_3).$$

12. **Bonus 2.** Assume that $x_1 < x_2 < x_3$ are given points. Write expressions for the weights w_1, w_2, w_3 to ensure that the following integration rule has a degree of precision equal to at least 2.

$$\int_{-1}^1 u(x) dx \approx w_1 u(x_1) + w_2 u(x_2) + w_3 u(x_3).$$

You may use the Lagrange polynomials ℓ_1, ℓ_2, ℓ_3 associated with the points in these expressions