1. If $\mathsf{A} \in \mathbf{R}^{n\times n}$ is invertible, then for any $\boldsymbol{b} \in \mathbf{R}^n$, there exists a unique solution to the linear system
$$\mathsf{A}\boldsymbol{x} = \boldsymbol{b}\,.$$

2. For any vector $\boldsymbol{x} \in \mathbf{R}^n$, it holds that $\|\boldsymbol{x}\|_1 \leqslant \|\boldsymbol{x}\|_\infty$. Recall that, by definition,
$$\|\boldsymbol{x}\|_1 := |x_1| + \cdots + |x_n|, \qquad \|\boldsymbol{x}\|_\infty = \max\big\{|x_1|, \ldots, |x_n|\big\}\,.$$

3. For a vector norm $\|\bullet\|$ on $\mathbf{R}^n$ (for example a p-norm), the *subordinate* matrix norm is defined by
$$\begin{aligned}\|\mathsf{A}\| &:= \max\big\{\|\mathsf{A}\boldsymbol{x}\| : \|\boldsymbol{x}\| \leqslant 1\big\} \\ &= \max\left\{\frac{\|\mathsf{A}\boldsymbol{x}\|}{\|\boldsymbol{x}\|} : \boldsymbol{x} \neq \mathbf{0}\right\}.\end{aligned}$$
We recall that the two definitions are equivalent. Then it holds that
$$\forall \boldsymbol{x} \in \mathbf{R}^n, \qquad \|\mathsf{A}\boldsymbol{x}\| \leqslant \|\mathsf{A}\|\|\boldsymbol{x}\|\,.$$

4. Let $\|\bullet\|$ denote a matrix p-norm, with $p \in [1, \infty)$. Then for all $\mathsf{A} \in \mathbf{R}^{n\times n}$, it holds that $\|\mathsf{A}^2\|_p < \|\mathsf{A}\|_p^2$, with a strict inequality.

5. Consider the matrix
$$\mathsf{A} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}.$$
Then it holds that $\|\mathsf{A}\|_2 \leqslant 10$. Recall that for a symmetric matrix A, its 2-norm is given by the largest absolute eigenvalue.

6. If all the eigenvalues of a general matrix A (not necessarily symmetric) are positive, then there exists a unique lower triangular matrix $\mathsf{C} \in \mathbf{R}^{n\times n}$ such that $\mathsf{A} = \mathsf{C}\mathsf{C}^T$.

7. The computational cost of solving the linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$, with a diagonal matrix $\mathsf{A} \in \mathbf{R}^{n\times n}$ and a vector $\boldsymbol{b} \in \mathbf{R}^n$, scales as $2n^3 + \mathcal{O}(n^2)$.

8. The only matrix $\mathsf{A} \in \mathbf{R}^{n\times n}$ such that $\kappa_2(\mathsf{A}) = 1$ is the identity matrix.

9. Let $\mathsf{A} \in \mathbf{R}^{n\times n}$ be a symmetric matrix. In this case, the matrix can be decomposed as
$$\mathsf{A} = \mathsf{Q}\Lambda\mathsf{Q}^T\,,$$
for a diagonal matrix Λ containing the eigenvalues and an orthogonal matrix Q. Then $\mathsf{A}^k \to 0$ in the limit $k \to \infty$ if and only if the spectral radius satisfies $\rho(\mathsf{A}) < 1$.

10. Suppose that $\mathsf{A} \in \mathbf{R}^{n\times n}$ is symmetric and positive definite, and let $\boldsymbol{b} \in \mathbf{R}^n$. Consider the following iterative method for solving the linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$:
$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \omega\left(\boldsymbol{b} - \mathsf{A}\boldsymbol{x}^{(k)}\right). \tag{1}$$
If $\omega \neq 0$ and this iteration converges to some vector $\boldsymbol{x}^\infty \in \mathbf{R}$, then $\boldsymbol{x}^\infty$ is the solution of the linear system.

11. **Bonus**: Suppose that $\mathsf{A} \in \mathbf{R}^{n\times n}$ is symmetric and positive definite. We proved in class that the iteration (1) converges to the exact solution if and only if
$$\rho(\mathsf{I} - \omega\mathsf{A}) := \max_{\lambda\in\sigma(\mathsf{A})} |1 - \omega\lambda| < 1.$$
Assuming that all the eigenvalues of A are contained in the interval $[1, 2]$, write a sufficient condition on the real parameter ω to guarantee that the iteration converges.

Your answer: It suffices that $\omega \in$

12. **Bonus:** Suppose again that $\mathsf{A} \in \mathbf{R}^{n\times n}$ is symmetric and positive definite. We proved in class that the error for Richardson's iteration (1) satisfies
$$\forall k \in \mathbf{N}, \qquad \left|\boldsymbol{x}^{(k)} - \boldsymbol{x}_*\right| \leqslant \rho(\mathsf{I} - \omega\mathsf{A})^k \left|\boldsymbol{x}^{(0)} - \boldsymbol{x}_*\right|\,.$$
Assuming that all the eigenvalues of A are contained in the interval $[1, 2]$, what value of *omega* would you choose to optimize this bound, that is to say to minimize the factor $\rho(\mathsf{I} - \omega\mathsf{A})$?

Your answer: