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Acoustic detection potential of single minimum ionising ionising particles in noble liquids

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1 Introduction

There are a lot papers about acoustic signal generated by a particle beam in liquids. There are quite a few about acoustic signals generated by a particle beam in cryo liquids. To our knowledge, there are no papers about acoustic waves generated by single particles through cryogenic fluids. This is our goal. To be the first paper about acoustic signals in cryo liq. We present three things:

2 Acoustic wave generation

We introduce here the mechanism behind the generation of an acoustic signal due to a single particle interaction in a liquid. Let us first consider the effect of an arbitrary heat deposition in the bulk of the liquid. For this purpose, we add a source term and a damping term to the well-known acoustic wave equation in an isothermal fluid:

$$\Delta p(\boldsymbol{x}, t) = \rho_0 \kappa \frac{\partial^2}{\partial t^2} p(\boldsymbol{x}, t), \tag{2.1}$$

where $p(\boldsymbol{x}, t)$ is the pressure difference at a some point $\boldsymbol{x}$ in the liquid at time t, ρ_0 is the rest density and it is taken as constant, and κ_T is the isothermal compressibility of the liquid[1]

2.1 Damping term derivation

Due to the small energy deposition by a single particle in the fluid, it is reasonable to assume that the damping effect caused by the viscosity of the liquid will significantly affect the decay time of the acoustic wave. To derive the viscous wave equation (also known as the strongly damped wave equation ref) we exploit the principle of conservation of mass and momentum and include the damping term $\mu \Delta \boldsymbol{v}$:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0 \tag{2.2}$$

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{v} = -\boldsymbol{\nabla} P + \mu \Delta \boldsymbol{v}, \tag{2.3}$$

where $\rho(\boldsymbol{x}, t)$, $P(\boldsymbol{x}, t)$, and $\boldsymbol{v}(\boldsymbol{x}, t)$ are the density, pressure, and velocity, respectively, while μ is the coefficient of bulk viscosity. As done in in other related literature ref, we assume a fluid with no vorticity ($\boldsymbol{\nabla} \times \boldsymbol{v} = \boldsymbol{0}$). Thus, Navier-Stokes equation (2.3) takes the form:

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \boldsymbol{\nabla} \cdot \boldsymbol{v} = -\boldsymbol{\nabla} P + \mu \Delta \boldsymbol{v}. \tag{2.4}$$

Let us consider a small perturbation to each of the variables:

$$\rho = \rho_0 + \delta\rho \tag{2.5}$$

$$P = p_0 + \delta p \tag{2.6}$$

$$\boldsymbol{v} = \boldsymbol{v}_0 + \delta \boldsymbol{v} = \delta \boldsymbol{v}, \tag{2.7}$$

where ρ_0, p_0, and $\boldsymbol{v}_0$ are the pressure, density, and velocity of the fluid at equilibrium and we have assumed the fluid to be initially at rest. From now on we will exclusively deal with $\delta p, \delta\rho$ and $\delta \boldsymbol{v}$, as such, we will be dropping the delta to keep notation simple. By plugging the perturbed variables into (2.2) and (2.4) and neglecting higher order terms, i.e. $\mathcal{O}((\delta\rho/\rho_0)^2)$, $\mathcal{O}((\delta P/P_0)^2)$, and $\mathcal{O}((\delta v/v_0)^2)$, we obtain the following equations:

[1] Note that in liquids the isothermal compressibilty is greater than the isentropic compressibility, which can therefore be neglected.

$$\frac{\partial \rho}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \tag{2.8}$$

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla} p = \mu \Delta \boldsymbol{v}. \tag{2.9}$$

Taking the divergence of (2.9) leads to

$$\rho_0 \frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \boldsymbol{v} + \Delta p = \mu \Delta \left(\boldsymbol{\nabla} \cdot \boldsymbol{v} \right), \tag{2.10}$$

where $\boldsymbol{\nabla} \cdot \boldsymbol{v}$ can be replaced using (2.10) to obtain

$$\frac{\partial^2 \rho}{\partial t^2} = \Delta \left(p + \frac{\mu}{\rho_0} \frac{\partial \rho}{\partial t} \right) \tag{2.11}$$

For an isentropic process $\rho = \rho_0 \kappa_T p$. Hence, (2.11) can be expressed only as a function of pressure:

$$\Delta \left(p + \mu \, \kappa_T \frac{\partial p}{\partial t} \right) = \rho_0 \, \kappa_T \, \frac{\partial^2 p}{\partial t^2} \tag{2.12}$$

We define the speed of the wave, c, as $c = 1/\sqrt{\rho_0 \kappa_T}$ and the attenuation frequency, ω_0, as $\omega_0 = 1/\mu \kappa_T$. Thus, (2.12) becomes:

$$\Delta \left(p + \frac{1}{\omega_0} p_t \right) = \frac{1}{c^2} p_{tt} \tag{2.13}$$

where the partial derivatives are denoted by the corresponding subscript.

2.2 Frequency effect on damping

Using (2.13), we can study the effect of the viscosity (damping term) on a plane acoustic wave. To do so, we first derive the dispersion relation. Let us assume an oscillating solution of the form

$$p(\boldsymbol{x}, t) = e^{i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)}. \tag{2.14}$$

Plugging (2.14) in (2.13) leads to:

$$\left(1 - i \frac{\omega}{\omega_0} \right) \Delta p = -\frac{\omega^2}{c^2} p,$$

from which we can obtain the following dispersion relation for $|\boldsymbol{k}| = k(\omega)$:

$$k(\omega) = \pm \frac{\omega}{c} \left(1 - i \frac{\omega}{\omega_0} \right)^{-\frac{1}{2}} = \alpha(\omega) + i\beta(\omega) \tag{2.15}$$

Equation (2.15) shows the dispersion relation for the fluid and an expansion. Replacing (2.15) in (2.14) results in an exponential decay factor of the form $e^{-\beta(\omega)x}$, which describes the damping in amplitude of the wave over time (as shown in figure **??**).

2.3 Source term derivation

Let us now address the mechanism by which the wave is generated, i.e. the source term in (2.1). It is well documented ref that a particle passing through a liquid deposits energy such that, locally, the temperature sharply increases. The (almost) instantaneous change in temperature leads to a rapid volume expansion and the subsequent change in density propagates through the liquid. Here we assume that this effect, referred to as “local heating”, is the biggest contributor to the generation of the sound wave. This is consistent with past literature where acoustic signals due to particle beams were studied ref. Therefore the theoretical estimates presented in section 5 constitute a lower bound for the signal amplitude.

Let us consider the effect of some local temperature fluctuation $\tau(\boldsymbol{x}, t)$ such that the total temperature is given by $T(\boldsymbol{x}, t) = T_0 + \tau(\boldsymbol{x}, t)$, where T_0 is the equilibrium temperature. With a variation in temperature, density will change as a function of both pressure and temperature. Specifically, at first order (using ρ, p, and τ to denote the changes in density, pressure, and temperature respectively) we can express the change in density by:

$$\rho = \left.\frac{\partial \rho}{\partial P}\right|_T p + \left.\frac{\partial \rho}{\partial T}\right|_P \tau \tag{2.16}$$

where P and T represent total pressure and temperature respectively, while V represents a small volume of the liquid around the particle interaction point. Let us recall the definitions of isothermal compressibility κ_T and coefficient of thermal expansion:

$$\kappa_T = -\frac{1}{V}\left.\frac{\partial V}{\partial P}\right|_T = \frac{1}{\rho_0}\left.\frac{\partial \rho}{\partial P}\right|_T \tag{2.17}$$

$$\beta = \frac{1}{V}\left.\frac{\partial V}{\partial T}\right|_P = -\frac{1}{\rho_0}\left.\frac{\partial \rho}{\partial T}\right|_P . \tag{2.18}$$

We can then rewrite (2.16) as follows:

$$\rho = \rho_0(\kappa_T p - \beta\tau).$$

From here we can proceed just as in section 2.1 to derive (2.11). First, we use (2.16) to write the density in terms of the pressure difference p and temperature fluctuation τ. Then, we assume that, to first order, the functions $\kappa_T(P, T, t)$ and $\beta(P, T, t)$ vary slowly with time, such that:

$$\rho_0\kappa_T\frac{\partial^2 p}{\partial t^2} - \rho_0\beta\frac{\partial^2 \tau}{\partial t^2} = \Delta\left(p - \mu\kappa_T\frac{\partial p}{\partial t} + \mu\beta\frac{\partial \tau}{\partial t}\right). \tag{2.19}$$

Equation (2.19) shows the presence of an extra damping term. However, since the spatial temperature fluctuation over time ($\frac{\partial}{\partial t}\Delta\tau$) is approximately zero, we may neglect that term. Using the second law of thermodynamics, we can determine the heat per unit volume added to the liquid $\epsilon(\boldsymbol{x}, t)$.

$$\epsilon = \frac{\delta Q}{\delta V} = \rho C_p \tau, \tag{2.20}$$

where C_p is the specific heat capacity of the liquid at constant pressure. We may now introduce the complete wave equation by substituting τ with (2.20):

$$\Delta\left(p - \mu\kappa_T \frac{\partial p}{\partial t}\right) - \rho_0 \kappa_T \frac{\partial^2 p}{\partial t^2} = -\frac{\beta}{C_p}\frac{\partial^2 \epsilon}{\partial t^2} \tag{2.21}$$

which may be more compactly written as:

$$\Delta\left(p - \frac{1}{\omega_0} p_t\right) - \frac{1}{c^2} p_{tt} = -\frac{\beta}{C_p}\epsilon_{tt}. \tag{2.22}$$

We have successfully derived the correction terms to the acoustic wave equation (2.1) to model the damped wave that is generated by a heat source inside a liquid. In Section 2.4, we will show how to estimate the heat deposition $\epsilon(\boldsymbol{x}, t)$ for single, charged particles through liquids using the Bethe-Bloch formula. At this stage it is important to note that our assumption that κ_T and β do not vary in time is only valid away from the center of heating. Therefore our model is accurate to predicting the signal away from the particle interaction point.

2.4 Modelling Single Particle Energy deposition

To model the effect of a particle going through a fluid, we will ignore any nonlinear effects, such as direct collisions between the particle and the fluid molecules. That way we will be able to analytically describe the average energy deposition of the particle. Multiple attempts to describe the energy deposition profile of a single charged particles can be found in literature [1, 2].

For the purposes of our analysis we are using the Bethe-Bloch formula [3] to obtain an accurate estimate of the energy lost by the particle in the medium. However, according to Amsler at. al. [4] we recognize that the average $\langle dE/dx \rangle$ is not a correct estimate of the energy deposited, hence why the most probable value from the Bethe distribution was used described by:

$$\left.\frac{dE}{dx}\right|_M = \text{SOMETHING}, \tag{2.23}$$

where.... The primary assumptions we made were that the particles travel at a straight line through the fluid (along $\hat{\boldsymbol{x}}$) and that they have high enough energy that the deposition in the medium does not change their speed appreciably. In more rigorous terms we assume that:

$$\frac{d}{dt}\frac{dE}{dx} = 0. \tag{2.24}$$

As a result, the rate at which energy is deposited in the medium can be given by:

$$\frac{dE}{dt} = \frac{dx}{dt}\left.\frac{dE}{dx}\right|_M = v\left.\frac{dE}{dx}\right|_M, \tag{2.25}$$

where v is the speed of the particle through the fluid. Furthermore we can set up a cylindrical coordinate system around $\hat{\boldsymbol{x}}$ where the particle would always be at position $(\rho, \phi, x) = (0, 0, vt)$ at time t. As a result, what we now need to derive, is the rate of change energy density $\epsilon_t(\boldsymbol{x}, t) = \epsilon_t(\rho, x, t) = dE/dt\, d\Omega$ in order to plug in to the wave equation (2.22).

To do so, consider the energy deposition in some volume Ω. The rate of energy deposition over the volume can be written as (using (2.25)):

$$\frac{dE}{dt} = \int_\Omega d\Omega\, v\, \frac{dE}{dx}\bigg|_M G(\boldsymbol{x}), \tag{2.26}$$

where $G(\boldsymbol{x})$ is the spatial distribution of the energy deposited by the particle. From (2.26) we can derive that the rate of change of energy density, in cylindrical coordinates, can be given by:

$$\epsilon_t(\boldsymbol{x}, t) = \epsilon_t(\rho, \phi, x, t) = \frac{dE}{dt} G(\rho, \phi, x - vt). \tag{2.27}$$

At this point, inspired by ref, we choose a Gaussian distribution for G. Since our distribution is cylindrically symmetric we can express ϵ_t as:

$$\epsilon_t(\rho, x, t) = \frac{dE}{dt} G(\rho, x - vt) = \frac{dE}{dx}\bigg|_M \frac{v}{(2\pi\sigma^2)^{3/2}} \exp\left\{\frac{1}{2\sigma^2}\left[(\rho^2 + (x - vt)^2)\right]\right\}, \tag{2.28}$$

where σ is the standard deviation of the spatial spread of the energy deposition, for which we choose a crude estimate of the mean distance of the fluid particles given by $\sigma = \langle r \rangle \approx n^{-1/3}$ where n is the number density of the fluid particles. As a result, by taking one time derivative we can get the source term ϵ_{tt} to be:

$$\epsilon_{tt}(\rho, x, t) = \frac{dE}{dx}\bigg|_M \frac{v^2(x - vt)}{(2\pi\sigma^4)^{3/2}} \exp\left\{\frac{1}{2\sigma^2}\left[(\rho^2 + (x - vt)^2)\right]\right\}. \tag{2.29}$$

3 Mechanism of Propagation and Decay

In this section we focus on obtaining qualitative and quantitative aspects of the wave motion by solving the equations derived in Section 2. To do so we approach the subject from 3 main points of view. We analytically solve for the spherical harmonics to study the damping behaviour of the waves, then proceed in determining a complete analytical solution for the problem by obtaining the fundamental solution. Finally we computationally solve the equation for our specific source term using a Finite Element Method to gain some more rigorous qualitative understanding of the signal generated. Alas, we also provide a perturbation theory approximation method for obtaining accurate enough estimates of the signal generated. The full numerical estimates, along with commentary on detection are presented in Sections 5 and 6 respectively.

3.1 Spherical Harmonics

To understand how the sound waves propagate in the liquid we consider the homogeneous wave equation shown in (2.13). Specifically, we formulate the following initial value problem:

$$\begin{cases} \Delta\left(p + \frac{1}{\omega_0}p_t\right) - \frac{1}{c^2}p_{tt} = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}^+ \\ p(\boldsymbol{x}, 0) = p_0(\boldsymbol{x}) & \text{in } \mathbb{R}^3 \\ p_t(\boldsymbol{x}, 0) = p_0'(\boldsymbol{x}) & \text{in } \mathbb{R}^3. \end{cases} \tag{3.1}$$

We search for separable solutions of the form:

$$p(\boldsymbol{x}, t) = P(r)T(t) \equiv \frac{\Psi(r)}{r}T(t), \tag{3.2}$$

where $|\boldsymbol{x}| = r$. For now we are only interested in studying the behaviour of the spherical harmonics for this situation. As a result, we are going to assume arbitrary non singular functions to be possible initial conditions (i.e. $p_0, p_t \in L^2(\mathbb{R}^3)$). Applying (3.2) to (3.1) we obtain the following equations for the time dependent component ($T(t)$) and the spatially dependent component ($P(r)$):

$$\Delta P(r) = \frac{\Psi''(r)}{r} = \lambda\frac{\Psi(r)}{r} \tag{3.3}$$

$$T''(t) - \frac{\lambda c^2}{\omega_0}T'(t) - \lambda c^2 T = 0, \tag{3.4}$$

where $\lambda \in \mathbb{R}$ is some arbitrary constant obtained from separation of variables. Solving for (3.3) we find that the only non singular, Lebesgue square integrable solutions in $\mathbb{R}^3$ require $\lambda < 0$. Hence, it is with foresight that we define $\lambda = -k^2$, as k is the wavenumber. Substituting for lambda in (3.4) we obtain:

$$T''(t) + \frac{k^2c^2}{i\omega_0}T'(t) + k^2c^2T = 0. \tag{3.5}$$

By using the ansatz $T(t) = \exp(i\omega t)$ we obtain the following dispersion relation:

$$i\omega = -\frac{(kc)^2}{2\omega_0} \pm kc\sqrt{\left(\frac{kc}{2\omega_0}\right)^2 - 1}. \tag{3.6}$$

Equation 3.6 is nothing more than the inverse of (2.15). However, it is significantly more helpful in understanding the behaviour of the wave. Specifically, we can now define a characteristic constant for our system α like so:

$$\alpha = \frac{kc}{2\omega_0}. \tag{3.7}$$

Hence, we can rewrite the dispersion relation like so:

$$i\omega = -kc\left(\alpha \pm \sqrt{\alpha^2 - 1}\right). \tag{3.8}$$

As a sanity check we can see that at the limit where $\omega_0 \to \infty$, $\alpha \to 0$, recovering the undamped dispersion relation for a sound wave (i.e. $\omega = kc$). More interestingly, we have managed to quantify that independently of the value of α the spherical harmonic is going to decay by, at least, an $\exp(-kc\,\alpha\,t)$ term. Furthermore, in the case where $\alpha^2 > 1$ we will have an overdamped behaviour, with an additional exponential envelope. Finally, for $\alpha^2 < 1$, which is the most likely case (i.e. undamped frequency less than attenuation frequency), we see that the second term in (3.8) becomes imaginary, hence we obtain oscillatory solutions.

3.2 Solution for Stationary Source

To construct a particular solution to (2.22) with the source term derived in Section 2.3, we can fist write the solution in terms of the fundamental solution $F(\boldsymbol{x}, t)$ and the source term like so:

$$p(\boldsymbol{x}, t) = -\frac{\beta}{C_p}[F * \epsilon_{tt}](\boldsymbol{x}, t) = -\frac{\beta}{C_p}\int_{\mathbb{R}} d\tau \int_{\mathbb{R}^3} d\boldsymbol{y}\, F(\boldsymbol{x} - \boldsymbol{y}, t - \tau)\, \epsilon_{tt}(\boldsymbol{y}, \tau), \tag{3.9}$$

where β is the coefficient of isothermal expansion of the liquid medium, and C_p is its specific heat capacity at constant pressure. Hence, we first set out to obtain the fundamental solution by solving the following Global Cauchy Problem:

$$\begin{cases} \Delta\left(F + \frac{1}{\omega_0}F_t\right) - \frac{1}{c^2}F_{tt} = \delta^3(\boldsymbol{x})\delta(t) & \text{in } \mathbb{R}^3 \times \mathbb{R}^+ \\ F(\boldsymbol{x}, 0) = 0 & \text{in } \mathbb{R}^3 \\ F_t(\boldsymbol{x}, 0) = 0 & \text{in } \mathbb{R}^3 \\ F \in L^2(\mathbb{R}^3 \times \mathbb{R}). \end{cases} \tag{3.10}$$

The last condition in (3.10) states that the pressure function (F) is a Lebesgue square integrable function, physically implying an outgoing wave.

To obtain the fundamental solution ($F(\boldsymbol{x}, t)$) we will first define the following Fourier transform:

$$\tilde{F}(\boldsymbol{x}, \omega) = \int_{\mathbb{R}} dt\, F(\boldsymbol{x}, t)e^{-i\omega t} \tag{3.11}$$

$$F(\boldsymbol{x}, t) = \frac{1}{2\pi}\int_{\mathbb{R}} d\omega\, F(\boldsymbol{x}, \omega)e^{i\omega t}. \tag{3.12}$$

Since $F \in L^2$ it can be written as an inverse Fourier transform (3.12). Hence, we can solve the equation of problem (3.10) for the Fourier transformed $\tilde{F}$ at each ω like so:

$$(\Delta + k(\omega)^2)\,\tilde{F} = \frac{1}{1 - i\frac{\omega}{\omega_0}}\delta^3(\boldsymbol{x}), \tag{3.13}$$

where $k(\omega) = \omega/c\sqrt{1 - i\omega/\omega_0}$ is once again the wavenumber and has the beefit of recovering the famillar dispersion relation that was explored in 2.15.

Equation (3.11) is a well defined inhomogeneous Helmholtz equation. This is rigorously solved in A. Here, we use the result to obtain the following expression for the transform of the fundamental solution:

$$\tilde{F}(r,\omega) = -\frac{1}{4\pi r}\frac{1}{1 - i\omega/\omega_0}\exp\left\{\frac{i\omega r}{c\sqrt{1 - i\omega/\omega_0}}\right\} = \frac{1}{4\pi r}\frac{k^2}{\omega^2}e^{ikr}. \tag{3.14}$$

Here, to emphasize that the solution is spherically symmetric we have used $r = |\boldsymbol{x}|$ for the spatial variable. Finally, we can express the fundamental solution as the inverse transform of $\tilde{F}$:

$$F(r,t) = -\frac{1}{8\pi^2 r}\int_{\mathbb{R}} d\omega\,\frac{1}{1 - i\omega/\omega_0}\exp\left\{\frac{i\omega r}{c\sqrt{1 - i\omega/\omega_0}} + i\omega t\right\}. \tag{3.15}$$

Now we can use (3.9) along with (2.29) and (3.15) to solve for the true pressure wave generated by any massive particle passing through noble fluids.

3.3 Perturbative Approximation

Since the integrals presented in Section 3.2 are non standard and hard to compute, we offer here a perturbation method of approximating the effect of damping based on solving the standard inhomogeneous wave equation. Here we describe the procedure using the Kirchhoff's formula to obtain solutions to the wave equation, however other standard solving methods, analytical or numerical may be used.

To begin we rewrite the wavelike equation (2.22) in terms of the d'Alembert operator ($\Box := \Delta - \partial_{tt}$), an arbitrary source function $f(\boldsymbol{x},t)$ and the damping term as a perturbation like so:

$$\Box u - f = \lambda\, Lu, \tag{3.16}$$

where λ is as small value of order $1/\omega_0$, and L is a linear differential operator defined by $L := \Delta\partial_t$. We can now express our solution u in terms of λ:

$$u(\boldsymbol{x},t) = \sum_{n=0}^{\infty}\lambda^n\, u_n(\boldsymbol{x},t), \tag{3.17}$$

where u_0 satisfies $\Box u_0 = f$. As a result we obtain the following to first order in λ:

$$\Box u_0 - f + \lambda\Box u_1 = \lambda\, Lu_0 + \mathcal{O}(\lambda^2) \tag{3.18}$$

$$\Box u_1 = Lu_0 + \mathcal{O}(\lambda). \tag{3.19}$$

Now we realize that $Lu_0 = \partial_t(u_{0tt} + f)$ and by combining with (3.19) we can create the following recursive formula for the perturbation terms:

$$\Box u_n = \partial_t^n f + \sum_{m=0}^{n} \partial_t^{n-m+2} u_m. \tag{3.20}$$

Now we can obtain the terms using any numerical or analytical technique that solves the inhomogeneous wave equation, essentially transforming problem (3.10) to a series of wave equations. It is important to know, however, that in most cases λ is sufficiently small so that the first order approximation would suffice.

4 Numerical Estimation

Unfortunately the integral derived in Section 3.2 is very cumbersome to solve analytically. Hence we opt for a numerical integration method to calculate the waveform produced by a single massive particle in a noble fluid. Here we describe how we derived the Finite Element Method (FEM) scheme, as well as provide results for the qualitative behaviour of the pressure wave.

4.1 Finite Element Method (FEM) Scheme Derivation

4.1.1 Nondimensionalization

To convert the global Cauchy problem described in (3.10) in a version that is easy to attack numerically we need to first nondimensionalize the wavelike equation (2.1) by scaling the variables like so:

$$t \to Tt \qquad \boldsymbol{x} \to L\boldsymbol{x} \qquad p(\boldsymbol{x}, t) \to Pu(\boldsymbol{x}, t), \tag{4.1}$$

where T, L, and P are time, length, and pressure constants respectively to nondimensionalize the relevant variables. This way we can rewrite (2.22) like so:

$$\Delta u + \frac{1}{T\omega_0}\Delta u_t - \frac{L^2}{T^2c^2}u_{tt} = -\frac{\beta}{C_p}\frac{L^2}{P}\epsilon_{tt}. \tag{4.2}$$

This way we can set $T = 1/\omega_0$, $L = Tc$, and $P = \beta L^2/C_p$ to obtain:

$$\Delta u + \Delta u_t - u_{tt} = -\epsilon_{tt}, \tag{4.3}$$

which is our nondimensionalized wave equation.

4.1.2 From a Global Problem to a weakly formulated, bounded IVP

We further need to constrain the problem in (3.10) to a finite volume $\Omega \subset \mathbb{R}^3$. Since the source term we have derived in (2.29) is cylindrically symmetric as time goes on, we can use a cylinder for our bounded space $\Omega = S^1 \times [0, 1]$. Hence we can reformulate (3.10) as the bounded Initial Value Problem (IVP) P1:

$$P_1 = \begin{cases} \Delta u(\boldsymbol{x}, t) + \Delta u_t(\boldsymbol{x}, t) - u_{tt}(\boldsymbol{x}, t) = -\epsilon_{tt} & \forall (\boldsymbol{x}, t) \in \Omega \times \mathbb{R}^+ \\ u(\boldsymbol{x}, 0) = 0 & \forall \boldsymbol{x} \in \Omega \\ u(\boldsymbol{x}, t) = 0 & \forall \boldsymbol{x} \in \partial\Omega \times \mathbb{R}^+ \end{cases}. \tag{4.4}$$

We also impose that the solution $u(\boldsymbol{x}, t)$ is cylindrically symmetric specifically:

$$u(\rho, \phi, x, t) = u(\rho, x, t)\, v(\phi). \tag{4.5}$$

Plugging this to (4.4) we obtain:

$$v\Delta_{\rho x}(u + u_t) + (u + u_t)\Delta_\phi v - vu_{tt} = -\epsilon_{tt}, \tag{4.6}$$

Where it is apparent that we can choose a solution to be $v(\phi) \equiv 1$, effectively removing one degree of freedom from our problem, allowing for some reduction of computation time in the numerical scheme.

To create a FEM scheme as well as prove well posedness of our new IVP we will express it in its weak formulation. Consider the space $H^3(\Omega)$ of all locally summable functions in Ω with derivatives that belong to $L^3(\Omega)$. Then assuming a solution $u \in H^3(\Omega)$ of problem P_1 (4.4) then $\forall v \in H^3(\Omega)$ the following must be true $\forall t \in \mathbb{R}^+$:

$$\int_\Omega \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} u \; dx \int_\Omega \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} u_t \; dx + \int_\Omega vu \; dx - \int_\Omega v\epsilon_{tt} \; dx = 0. \tag{4.7}$$

Here we have just multiplied (4.3) with v, integrated over Ω and used Green's Identities to simplify the integrals involving Laplace's operator. We can now construct an inner product on $H^3(\Omega)$ making it a Sobolev Space (The proof is omitted here, but it is shown in full in Appendix B):

$$\begin{aligned} (\cdot,\cdot) : H^3(\Omega) \times H^3(\Omega) &\to \mathbb{R} \\ (u, v) &\mapsto \int_\Omega d\Omega \; vu \in \mathbb{R}. \end{aligned} \tag{4.8}$$

By slightly abusing notation we can rewrite (4.7) like so:

$$(\boldsymbol{\nabla} v, \boldsymbol{\nabla} u) + (\boldsymbol{\nabla} v, \boldsymbol{\nabla} u_t) + (v, u_{tt}) = (v, \epsilon_{tt}). \tag{4.9}$$

Finally, it possible at this point to define a multilinear functional $\alpha(v, u)$ and a source term $F(v)$ so that we can later restate our problem in terms of them:

$$\begin{aligned} \alpha(v, u) &= (\boldsymbol{\nabla} v, \boldsymbol{\nabla} u) + (\boldsymbol{\nabla} v, \boldsymbol{\nabla} u_t) + (v, u_{tt}) \\ F(v) &= (v, \epsilon_{tt}). \end{aligned} \tag{4.10}$$

Hence restating (4.3) like so:

$$\alpha(v, u) = F(v), \tag{4.11}$$

which through the Lax-Milligram Theorem ref proves the existence of a unique solution for the weak formulation of P_1 (4.4).

4.1.3 Discretization

So far P_1 in. (4.4) has been describing a continuous problem. We will now convert it to a discrete P_2 in (4.12) by finding a suitable subspace $W \subset H^3(\Omega)$ such that any function $u \in H^3(\Omega)$ can be approximated by $\tilde{u} \in W$. To do this we will discretize Ω like so. Take a half plane along the axis of the cylinder Ω and splice it into triangles. Then revolve the triangles about the central axis in order to create triangular volume elements T_i such that the set $T = \{T_i\}$ of all the elements covers Ω. We associate a basis function ϕ_i to each edge of our triangulation T and set $W = \text{span}(\Phi)$ where $\Phi = \{\phi_i\}$. The choice of basis functions ϕ_i is piecewise linear functions commonly used in FEM ref.

Now assuming $u \approx \tilde{u}$ we can write the new discrete problem P_2:

$$P_2 = \begin{cases} \alpha(v,u) = F(v) & \forall v \in W \\ u(\boldsymbol{x},0) = 0 & \forall \boldsymbol{x} \in U \\ u(\boldsymbol{x},t) = 0 & \forall \boldsymbol{x} \in \partial U \times \mathbb{R}^+ st\ z \neq 0 \\ \frac{\partial u}{\partial \boldsymbol{\nu}}\Big|_{(\boldsymbol{x},t)} = 0 & \forall \boldsymbol{x} \in \partial U \times \mathbb{R}^+ st\ z = 0. \end{cases} \tag{4.12}$$

As stated before, we do not need to solve for the entire cylinder Ω since we know that the solution u is cylindrically symmetric. Hence this is a, spatially, a 2D problem that evolves in time. With that in mind we proceed in deriving a numerical scheme for P_2. We first express $u \in H^3(\Omega)$ and $v \in H^3(\Omega)$ as:

$$\begin{aligned} u(\boldsymbol{x},t) \approx \tilde{u} = \sum_n c_n(t)\phi_n(\boldsymbol{x}) \\ v(\boldsymbol{x}) \approx \sum_n v_n \phi_n(\boldsymbol{x}). \end{aligned} \tag{4.13}$$

By substituting into (4.12) we obtain:

$$\sum_{i,j} v_j(c_i + c_{j\,t})(\boldsymbol{\nabla}\phi_i, \boldsymbol{\nabla}\phi_j) + (v_j c_{itt} - v_j \epsilon_{tt})(\phi_i, \phi_j) = 0. \tag{4.14}$$

A relation that can be simplified by realizing that it must hold true for all v_j. Hence by constructing the matrices $A_{ij} = (\boldsymbol{\nabla}\phi_i, \boldsymbol{\nabla}\phi_j)$ and $B_{ij} = (\phi_i, \phi_j)$ we can write a linear system to solve for the vector $\boldsymbol{u} = (c_i)$:

$$A(\boldsymbol{u} + \boldsymbol{u}_t) + B(\boldsymbol{u} - \boldsymbol{\epsilon}_{tt}), \tag{4.15}$$

where $\boldsymbol{\epsilon}_{tt} = (\epsilon_{tt}(\boldsymbol{x}_i))$ where $\boldsymbol{x}_i$ is the coordinate of the edge associated with the basis function ϕ_i. Now we can use any numerical scheme for the time evolution. For this case we chose forward Euler to obtain:

$$A\left[\frac{1}{\Delta t}((1+\Delta t)\,\boldsymbol{u}^{n+1} - \boldsymbol{u}^n)\right] + B\left[\frac{1}{\Delta t^2}(\boldsymbol{u}^{n+1} - 2\boldsymbol{u}^n + \boldsymbol{u}^{n-1}) + \boldsymbol{\epsilon}_{tt}\right] = 0. \tag{4.16}$$

A program was written to numerically solve the following system that can be found here ref.

4.2 Qualitative Simulation Results

5 Theoretical Signal Estimates

6 Potential for Detection

6.1 Signal to noise Ratio

6.2 Potential Detectors

A Solution of Helmholtz Equation

Appendix A is very useful and helpful in understanding the universe as we know it.

B Proving H is a Sobolev space

Acknowledgments

This is the most common positions for acknowledgments. A macro is available to maintain the same layout and spelling of the heading.

Note added. This is also a good position for notes added after the paper has been written.

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