1. Is $(U, \subseteq)$ a lattice?

A **lattice** is a partially ordered set in which every pair of elements has both a **least upper bound** (join) and a **greatest lower bound** (meet). In the context of sets ordered by inclusion, the join is the **union** and the meet is the **intersection**.

Let $A, B \in U$. There are three cases to consider:

1. **Both A and B are finite.**
    - **Union:** $A \cup B$ is finite.
    - **Intersection:** $A \cap B$ is finite.
    - **Conclusion:** Both $A \cup B$ and $A \cap B$ belong to U.
2. **One is finite and the other is co-finite.**
    - Without loss of generality, suppose A is finite and $B = \mathbb{N} \setminus F$ is co-finite (with F finite).
    - **Union:** $A \cup B = (\mathbb{N} \setminus F) \cup A$. Since A is finite, adding it to a co-finite set still gives a co-finite set.
    - **Intersection:** $A \cap B$ is a subset of A (hence finite).
    - **Conclusion:** Both $A \cup B$ and $A \cap B$ are in U.
3. **Both A and B are co-finite.**
    - Write $A = \mathbb{N} \setminus F$ and $B = \mathbb{N} \setminus G$ for finite sets F and G.
    - **Intersection:**

      $$A \cap B = (\mathbb{N} \setminus F) \cap (\mathbb{N} \setminus G) = \mathbb{N} \setminus (F \cup G),$$

      and since $F \cup G$ is finite, $A \cap B$ is co-finite.
    - **Union:**

      $$A \cup B = (\mathbb{N} \setminus F) \cup (\mathbb{N} \setminus G) = \mathbb{N} \setminus (F \cap G),$$

      and $F \cap G$ is finite, so $A \cup B$ is co-finite.
    - **Conclusion:** Both $A \cup B$ and $A \cap B$ belong to U.

Since in every case the union and intersection of any two elements in U remain in U, every pair of elements in U has a join and a meet. Therefore, $(U, \subseteq)$ **is a lattice.**

2. Is $(U, \subseteq)$ bounded?

A poset is **bounded** if it has both a least element (often denoted $\perp$) and a greatest element (often denoted $\top$).

- **Least Element:** The empty set $\varnothing$ is finite. Thus, $\varnothing \in U$ and for every $X \in U$, we have $\varnothing \subseteq X$.
- **Greatest Element:** The whole set $\mathbb{N}$ is co-finite because

$$\mathbb{N} = \mathbb{N} \setminus \varnothing,$$

  and $\varnothing$ is finite. Thus, $\mathbb{N} \in U$ and for every $X \in U$, we have $X \subseteq \mathbb{N}$.

Since both $\varnothing$ and $\mathbb{N}$ are in U and serve as the minimum and maximum elements respectively, **$(U, \subseteq)$ is bounded.**

3. Is $(U, \subseteq)$ complete?

A poset is **complete** if every subset (not just every pair of elements) has a supremum (least upper bound) and an infimum (greatest lower bound) within the poset.

While we have shown that every pair of elements in U has a join and meet, completeness is a much stronger property. Let's examine a counterexample.

Counterexample: Consider the collection

$$\mathcal{S} = \{\, \{2n\} \mid n \in \mathbb{N} \,\},$$

i.e. the collection of all singleton sets containing even numbers. Note that each $\{2n\}$ is finite, so $\{2n\} \in U$ for every n.

- **Supremum (Union):**

  $$\bigcup_{n \in \mathbb{N}} \{2n\} = \{2, 4, 6, \ldots\},$$

  which is the set of all even numbers.

- **Analysis of the union:**
  - The set of even numbers is infinite.
  - Its complement, the set of odd numbers, is also infinite.
  - Therefore, the set of even numbers is **neither finite nor co-finite**.

Since the union $\{2, 4, 6, \ldots\}$ does **not** belong to U, the supremum of $\mathcal{S}$ does not exist in U.

Because we have found a subset of U that does not have a supremum (and a similar issue could arise for the infimum of other subsets), $(U, \subseteq)$ **is not complete.**

A **co-finite set** (in a given universal set, say $\mathbb{N}$ in our context) is a set whose complement is finite. In other words, if $A \subseteq \mathbb{N}$ is co-finite, then the set $\mathbb{N} \setminus A$ contains only finitely many elements.

For example:

- In $\mathbb{N}$, the set $\mathbb{N} \setminus \{1, 2, 3\}$ is co-finite because its complement $\{1, 2, 3\}$ is finite.
- Conversely, a set like the set of even numbers is not co-finite in $\mathbb{N}$ because its complement (the set of odd numbers) is infinite.