Complex Number

Complex Number

- Introduction of complex number
- Algebraic properties
- Geometric properties
- Exponential Form
- Regions in the complex plane

Introduction

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1. In your high-school math class, you probably worked with problem to find square root of a negative number, for example $\sqrt{-1}$ You might begin by using the symbol $i = \sqrt{-1}$. That is mostly like how we start to know complex number.

2. However, most of us just pretend we can and begin by using this symbol, but still very curious about whether this simple expression can really doing magic rather than mathematics.

3. Hopefully we will answer this question in this class.

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Definition

The complex number is an ordered pair of real numbers. It is defined as

Where x and y are both real numbers $z = (x, y)$.

The reason we say ordered pair is because we are thinking of a point in the plane.
The point (3, 4), for example, is not the same as (4, 3). The order in which we write x and y in the equation makes a difference. Clearly, then, two complex numbers are equal if and only if their x coordinates are and their y coordinates are equal. In other words,

$$(x, y) = (u, v) \quad \text{iff} \quad x = u \text{ and } y = v.$$

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Algebraic properties

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A meaningful number system requires a method for combining ordered pairs. The definition of algebraic operations must be consistent so that the sum, difference, product, and quotient of any two ordered pairs will again be an ordered pair.

Then, If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are arbitrary complex numbers, we have

$$
\begin{aligned}
z_1 + z_2 &= (x_1, y_1) + (x_2, y_2) \\
&= (x_1 + i\, y_1) + (x_2 + i\, y_2) \\
&= (x_1 + x_2) + i\,(y_1 + y_2) \\
&= (x_1 + x_2,\ y_1 + y_2)
\end{aligned}
$$

Now, we can have following definitions (addition, subtraction, multiplication and division) for complex number system.

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Definition

Addition: $z_1 + z_2 = (x_1 + i\, y_1) + (x_2 + i\, y_2) = (x_1 + x_2,\ y_1 + y_2)$

Subtraction: $z_1 - z_2 = (x_1 + i\, y_1) - (x_2 + i\, y_2) = (x_1 - x_2,\ y_1 - y_2)$

Example:

Given $z_1 = 3 + 7\,i$ and $z_2 = 5 - 6\,i$. **(a)** Find $z_1 + z_2$ and **(b)** $z_1 - z_2$.

$z_1 + z_2 = (3, 7) + (5, -6) = (3 + 5, 7 - 6) = (8, 1)$ and

$z_1 - z_2 = (3, 7) - (5, -6) = (3 - 5, 7 + 6) = (-2, 13)$.

We can also use the notation $z_1 = 3 + 7\,i$ and $z_2 = 5 - 6\,i$:

$z_1 + z_2 = 3 + 7\,i + 5 - 6\,i = 8 + i$ and

$z_1 - z_2 = 3 + 7\,i - (5 - 6\,i) = -2 + 13\,i$.

We are now looking at the graphical representation of the above examples

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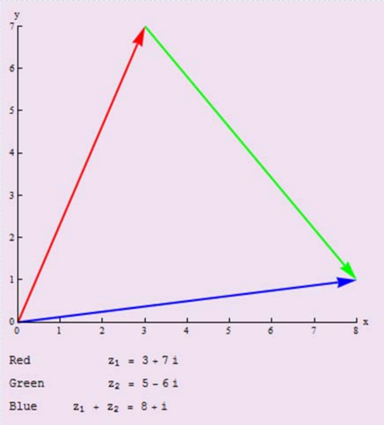
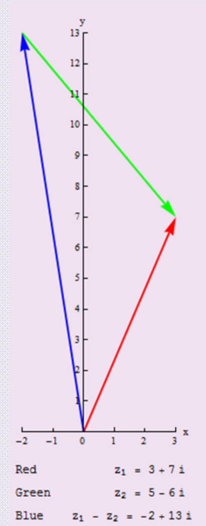
Addition



Subtraction



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Definition

Multiplication: $z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$

Division: $\frac{z_1}{z_2} = \frac{(x_1, y_1)}{(x_2, y_2)} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \frac{-x_1 y_2 + x_2 y_1}{x_2^2 + y_2^2}\right)$, for $z_2 \neq 0$

Example:

Given $z_1 = 3 + 7i$ and $z_2 = 5 - 6i$. Find $z_1 z_2$.

$$z_1 z_2 = (3, 7)(5, -6)$$
$$= (3 * 5 - 7(-6), 3(-6) + 5 * 7)$$
$$= (15 + 42, -18 + 35)$$
$$= (57, 17)$$

We get the same answer by using the notation $z_1 = 3 + 7i$ and $z_2 = 5 - 6i$:

$$z_1 z_2 = (3, 7)(5, -6) = (3 + 7i)(5 - 6i)$$
$$= 15 - 18i + 35i - 42i^2$$
$$= 15 - 42(-1) + (-18 + 35)i$$
$$= 57 + 17i$$
$$= (57, 17)$$

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Example

Given $z_1 = 3 + 7\,i$ and $z_2 = 5 - 6\,i$. Find $\frac{z_1}{z_2}$.

$$\frac{z_1}{z_2} = \frac{(3,7)}{(5,-6)} = \left(\frac{15-42}{25+36}, \frac{18+35}{25+36}\right) = \left(-\frac{27}{61}, \frac{53}{61}\right)$$

As with the example for multiplication, we also get this answer if we use the notation $x + i\,y$:

$$\frac{z_1}{z_2} = \frac{(3,7)}{(5,-6)} = \frac{(3+7\,i)}{(5-6\,i)}$$

$$= \frac{(3+7\,i)}{(5-6\,i)}\,\frac{(5+6\,i)}{(5+6\,i)} = \frac{15+18\,i+35\,i+42\,i^2}{25+30\,i-30\,i-36\,i^2}$$

$$= \frac{15-42+(18+35)\,i}{(25+3)} = -\frac{27}{61} + \frac{53}{61}\,i$$

$$= \left(-\frac{27}{61}, \frac{53}{61}\right)$$

We are now looking at the graphical representation of the above examples

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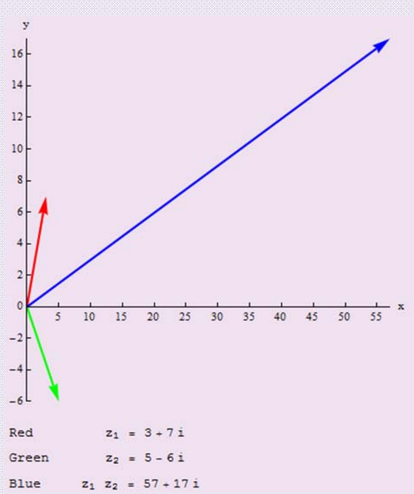
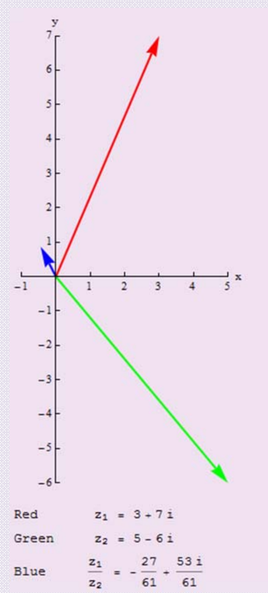
Multiplication



Division



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Definition

Field: In formal terms, a field is a set (in this case, the complex numbers) together with two binary operations (in this case, addition and multiplication) having the following properties.

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(P1) Commutative Law for Addition.

$$z_1 + z_2 = z_2 + z_1.$$

(P2) Associative Law for Addition.

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3.$$

(P3) Additive Identity. There is a complex number ω such that

$z + \omega = z$ for all complex numbers z.

The number ω is obviously the ordered pair $(0, 0)$.

(P4) Additive Inverses. Given any complex number z, there is a complex number η (depending on z) with the property that

$$z + \eta = (0, 0).$$

Obviously, if $z = (x, y) = x + i y$, the number η will be $\eta = (-x, -y) = -x - i y = -z$.

(P5) Commutative Law for Multiplication.

$$z_1 z_2 = z_2 z_1.$$

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(P6) Associative Law for Multiplication.

$$z_1 (z_2 z_3) = (z_1 z_2) z_3.$$

(P7) Multiplicative Identity. There is a complex number ξ such that

$\xi z = z$ for all complex numbers z.

(P8) Multiplicative Inverses.

Given any number z other than the number (0, 0), there is a complex number (depending on z) which we shall denote by z^{-1} with the property that

$$z z^{-1} = (1, 0) = 1.$$

Based on our definition for division, it seems reasonable that the number z^{-1} would be

$$\begin{aligned} z^{-1} &= \frac{(1,0)}{z} = \frac{1}{x+i y} \\ &= \frac{x-i y}{(x+i y)(x-i y)} = \frac{x-i y}{x^2+y^2} \\ &= \frac{x}{x^2+y^2} + i \frac{-y}{x^2+y^2} \\ &= \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) \end{aligned}$$

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(P9) The Distributive Law.

$$z_1 (z_2 + z_3) = z_1 + z_1 z_3.$$

We are now looking at the derivation of properties. None of these properties is difficult to prove. Most of the proofs make use of corresponding facts in the real number system.

To illustrate, we give a proof of property (P1).

Proof of the commutative law for addition: Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ be arbitrary complex numbers. Then,

$$\begin{aligned} z_1 + z_2 &= (x_1, y_1) + (x_2, y_2) \\ &= (x_1 + x_2, y_1 + y_2) && \text{(by definition of addition of complex numbers)} \\ &= (x_2 + x_1, y_2 + y_1) && \text{(by the commutative law for real numbers)} \\ &= (x_2, y_2) + (x_1, y_1) && \text{(by definition of addition of complex numbers)} \\ &= z_2 + z_1 \end{aligned}$$

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Definition

(Real Part of z). The real part of $z = x + i y$, denoted by $\mathrm{Re}(z)$, is the real number x.

(Imaginary Part of z). The imaginary part of $z = x + i y$, denoted by $\mathrm{Im}(z)$, is the real number y.

(Conjugate of z). The conjugate of $z = x + i y$, denoted by $\overline{z}$, is the complex number $(x, -y) = x - i y$.

Examples

Given $z_1 = -3 + 7 i$ and $z_2 = 9 + 4 i$.

We have $\mathrm{Re}[z_1] = \mathrm{Re}[-3 + 7 i] = -3$ and $\mathrm{Re}[z_2] = \mathrm{Re}[9 + 4 i] = 9$.

We have $\mathrm{Im}[z_1] = \mathrm{Im}[-3 + 7 i] = 7$ and $\mathrm{Im}[z_2] = \mathrm{Im}[9 + 4 i] = 4$.

We have $\overline{z_1} = \overline{-3 + 7 i} = -3 - 7 i$ and $\overline{z_2} = \overline{9 + 4 i} = 9 - 4 i$.

We are now looking at the graphical representation of the above examples

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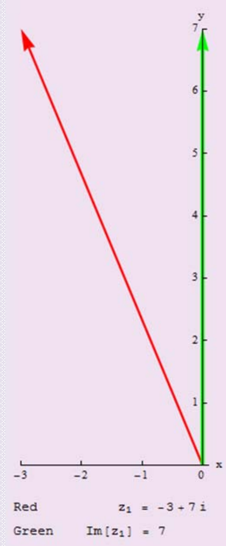
Real Part

Projection on x-axis

Imaginary Part



Projection on y-axis

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Let's take a quick look at $i^2 = (0, 1)^2 = -1$,

It is now time to show specifically how the symbol i relates to the quantity $\sqrt{-1}$. Note that

$$\begin{aligned}(0, 1)^2 &= (0, 1)\,(0, 1) \\ &= (0 - 1,\ 0 + 0) \quad \text{(by definition of multiplication of complex numbers)} \\ &= (-1, 0) \\ &= -1 \quad \text{(by our agreed correspondence)}\end{aligned}$$

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Example

Let z_1 , z_2 be complex numbers, Then $z_1\, z_2 = 0$ implies $z_1 = 0$ or $z_2 = 0$

Suppose that $z_1 \neq 0$, Then z_1 has multiplicative inverse z_1^{-1}

Then we have $z_2 = 1 \cdot z_2 = z_1^{-1}\, z_1\, z_2 = z_1^{-1}\, 0 = 0$

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Remarks

Suppose z_1, z_2 and z_3 are arbitrary complex numbers. Then if $z_2 \neq 0$

$$\overline{\overline{z}} = z$$

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

$$\overline{z_1\, z_2} = \overline{z_1}\ \overline{z_2}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}} \text{ if } z_2 \neq 0$$

$$\mathrm{Re}[z] = \frac{z + \overline{z}}{2}$$

$$\mathrm{Im}[z] = \frac{z - \overline{z}}{2\,i}$$

$$\mathrm{Re}[i\, z] = -\,\mathrm{Im}[z]$$

$$\mathrm{Im}[i\, z] = \mathrm{Re}[z]$$

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Geometric properties

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Complex numbers are ordered pairs of real numbers, so they can be represented by points in the plane. In this subsection, we show the effect that algebraic operations on complex numbers have on their geometric representations.

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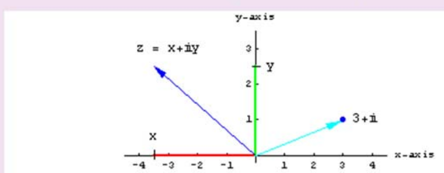
We can represent the number $z = x + i\,y = (x, y)$ by a position vector in the xy plane whose tail is at the origin and whose head is at the point (x, y).

When the xy plane is used for displaying complex numbers, it is called the complex plane, or more simply, the z plane.

Recall that $\mathrm{Re}\,(z) = x$ and $\mathrm{Im}\,(z) = y$.

Geometrically, $\mathrm{Re}\,(z)$ is the projection of $z = (x, y)$ onto the x axis, and $\mathrm{Im}\,(z)$ is the projection of z onto the y axis. It makes sense, then, to call the x axis the real axis and the y axis the imaginary axis.



Complex Plane

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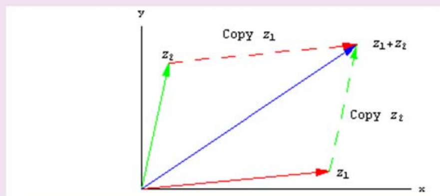
Addition

Addition of complex numbers is analogous to addition of vectors in the plane. As we already know, the sum of

$z_1 = x_1 + i\, y_1 = (x_1,\ y_1)$ and $z_2 = x_2 + i\, y_2 = (x_2,\ y_2)$ is $(x_1 + x_2,\ y_1 + y_2)$.

Hence, $z_1 + z_2$ can be obtained vectorially by using the "parallelogram law," where the vector sum is the vector represented by the diagonal of the parallelogram formed by the two original vectors.



Addition of Two Complex Number

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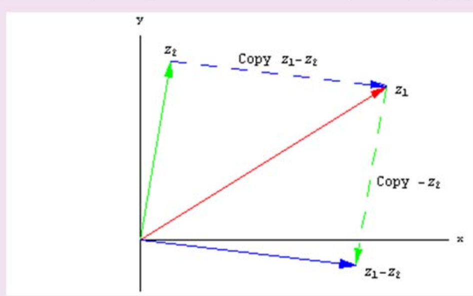
Subtraction

The difference $z_1 - z_2$ can be represented by the displacement vector from the point $z_2 = (x_2,\ y_2)$ to the point $z_1 = (x_1,\ y_1)$



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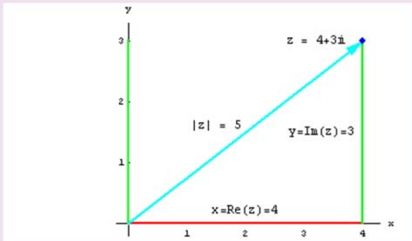
Modulus or absolute value

The modulus, or absolute value, of the complex number $z = x + i\,y$ is a nonnegative real number denoted by $|z|$ and is given by the equation

$$|z| = \sqrt{x^2 + y^2}\,.$$

The number $|z|$ is the distance between the origin and the point $z = (x, y)$. The only complex number with modulus zero is the number 0.
The number $z = 3 + 4\,i$ has modulus $|4 + 3\,i| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$, and is depicted in Figure below.



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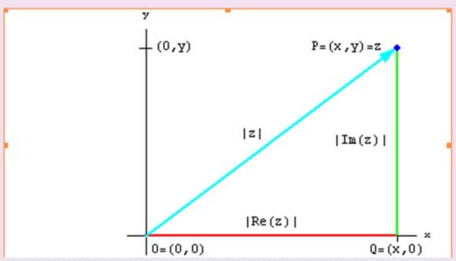
The numbers $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ and $|z|$ are the lengths of the sides of the right triangle OPQ shown in Figure below.



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Remarks:

The inequality $|z_1| < |z_2|$ means that the point z_1 is closer to the origin than the point z_2.
Although obvious from figure above, it is still profitable to work out algebraically the standard results that

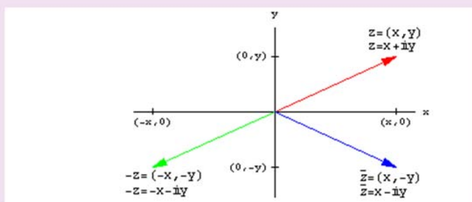
$$|x| = |\mathrm{Re}(z)| \le |z| \text{ and } |y| = |\mathrm{Im}(z)| \le |z|.$$

The difference $z_1 - z_2$ represents the displacement vector from z_2 to z_1, so the distance between z_1 and z_2 is given by $|z_1 - z_2|$. We can obtain this distance

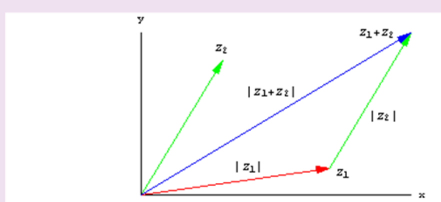
$$\mathrm{dist}(z_1, z_2) = |z_2 - z_1| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Note: A complex number could be viewed as vector and also a point in the plane

If $z = (x, y) = x + i y$, then $-z = (-x, -y) = -x - i y$ is the reflection of z, through the origin, and $\overline{z} = (x, -y) = x - i y$ is the reflection of z through the x axis, as illustrated in Figure below.



A beautiful and important application of the above identity is its use in establishing the triangle inequality, which states that the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side. Figure below illustrates this inequality.



Well, we can also verify it with algebraic proof .

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(Triangle Inequality). If z_1 and z_2 are arbitrary complex numbers, then

$$|z_1 + z_2| \le |z_1| + |z_2|.$$

Proof. We appeal to basic results:

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\,\overline{(z_1 + z_2)} \\
&= (z_1 + z_2)\,(\overline{z_1} + \overline{z_2}) \\
&= z_1\,\overline{z_1} + z_1\,\overline{z_2} + z_2\,\overline{z_1} + z_2\,\overline{z_2} \\
&= |z_1|^2 + z_1\,\overline{z_2} + \overline{z_1}\,z_2 + |z_2|^2 \\
&= |z_1|^2 + z_1\,\overline{z_2} + \overline{z_1\,\overline{z_2}} + |z_2|^2 \\
&= |z_1|^2 + 2\,\mathrm{Re}\,(z_1\,\overline{z_2}) + |z_2|^2 \\
&\le |z_1|^2 + 2\,|z_1\,\overline{z_2}| + |z_2|^2 \\
&= (|z_1| + |z_2|)^2
\end{aligned}$$

Taking square roots yields the desired inequality.

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We can also establish other important identities by means of the triangle inequality. Note that

$$\begin{aligned}
|z_1| &= |(z_1 + z_2) + (-z_2)| \\
&\le |z_1 + z_2| + |-z_2| \\
&\le |z_1 + z_2| + |z_2|.
\end{aligned}$$

Subtracting $|z_2|$ from the left and right sides of this string of inequalities gives an important relationship that will be used in determining lower bounds of sums of complex numbers:

$$|z_1 + z_2| \ge |z_1| - |z_2|.$$

Using the identity $|z|^2 = z\,\overline{z}$ and the commutative and associative laws it follows that

$$\begin{aligned}
|z_1\,z_2|^2 &= (z_1\,z_2)\,\overline{(z_1\,z_2)} \\
&= (z_1\,\overline{z_1})\,(z_2\,\overline{z_2}) \\
&= |z_1|^2\,|z_2|^2
\end{aligned}$$

Taking square roots of the terms on the left and right establishes another important identity

$$|z_1\,z_2| = |z_1|\,|z_2|.$$

As an exercise, we ask you to show

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \text{ provided } z_2 \ne 0.$$

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Exponential Form

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We saw that a complex number $z = x + i\,y$ could be viewed as a vector in the xy-plane whose tail is at the origin and whose head is at the point (x, y). A vector can be uniquely specified by giving its magnitude (i.e., its length) and direction (i.e., the angle it makes with the positive x-axis). In this section, we focus on these two geometric aspects of complex numbers.

Let r be the modulus of z (i.e., $r = |z|$), and let θ be the angle that the line from the origin to the complex number z makes with the positive x-axis. (Note: The number θ is undefined if $z=0$). We make the following definition.

(Polar Representation). The identity

$$z = (r\cos\theta,\ r\sin\theta) = r\,(\cos\theta + i\sin\theta)$$

is known as a polar representation of z, and the values r and θ are called polar coordinates of z,

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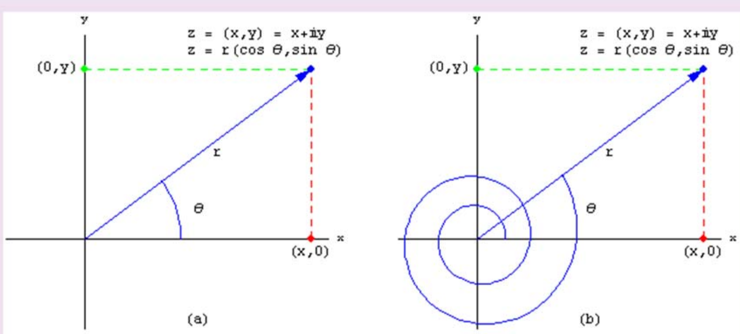

Polar representation of complex numbers.

As Figure shows, the angle θ can be any value for which the identities $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$ hold true. Given $z \neq 0$, the collection of all values of θ for which $z = r\,(\cos\theta + i\sin\theta)$ is denoted by $\arg z$.

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Example

If $z = 1 + i$, then $r = \sqrt{1+1} = \sqrt{2}$ and $z = \left(\sqrt{2}\cos\frac{\pi}{4},\ \sqrt{2}\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ is a polar representation of z. The polar coordinates in this case are $r = \sqrt{2}$, and $\theta = \frac{\pi}{4}$.

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(Argument, arg z). If $z \neq 0$, we denote $\arg z$ by

$$\arg z = \{\theta : z = r(\cos\theta + i\sin\theta)\}.$$

If $\theta \in \arg z$, we say that θ is an argument of z.

Notice that we write $\theta \in \arg z$ as opposed to $\theta = \arg z$. This is because $\arg z$ is a set, and the designation $\theta \in \arg z$ indicates that θ belongs to that set. Notice also that, if $\theta_1 \in \arg z$ and $\theta_2 \in \arg z$, then there exists some integer n such that

$$\theta_2 = \theta_1 + 2n\pi.$$

Example

Since $1 + i = \sqrt{2}\left(\mathrm{Cos}\left[\frac{\pi}{4}\right] + i\,\mathrm{Sin}\left[\frac{\pi}{4}\right]\right)$, we have

$$\begin{aligned}\arg(1+i) &= \left\{\frac{\pi}{4} + 2n\pi : n \text{ an integer}\right\} \\ &= \left\{\ldots, \frac{-15\pi}{4}, \frac{-7\pi}{4}, \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}, \frac{25\pi}{4}, \ldots\right\}\end{aligned}$$

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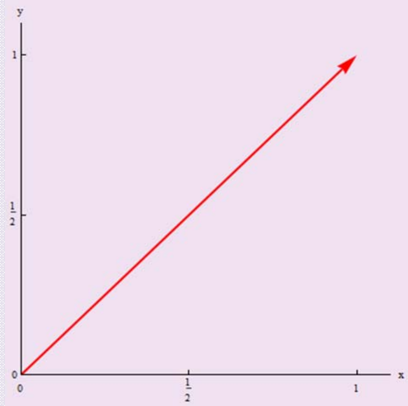
Example



$z_1 = 1 + i$

$r = \sqrt{2}$

$\theta = \frac{\pi}{4}$

$z_2 = \sqrt{2}\ e^{i\pi/4} = \sqrt{2}\ e^{\frac{i\pi}{4}} = 1 + i$

$z_3 = \sqrt{2}\ e^{i\,13\pi/4} = \sqrt{2}\ e^{\frac{i\pi}{4}} = 1 + i$

$z_4 = \sqrt{2}\ e^{-i\,11\pi/4} = \sqrt{2}\ e^{\frac{i\pi}{4}} = 1 + i$

$z_5 = \sqrt{2}\ e^{i\,25\pi/4} = \sqrt{2}\ e^{\frac{i\pi}{4}} = 1 + i$

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Mathematicians have agreed to single out a special choice of $\theta \in \arg z$.
It is that value of θ for which $-\pi < \theta \le \pi$ as the following definition indicates.

Definition (The principal value of argument, Arg z). Let $z \ne 0$, be a complex number. Then

$$\operatorname{Arg} z = \theta, \quad \text{provided } z = r\,(\cos\theta + i\sin\theta) \text{ and } -\pi < \theta \le \pi.$$

If $\theta = \operatorname{Arg} z$, we call θ the argument of z.

Example $\operatorname{Arg}(1 + i) = \frac{\pi}{4}$.

Remark. Clearly if $z = x + i\,y = r\,(\cos\theta + i\sin\theta)$, where $x \ne 0$, then

$$\arg z \subset \arctan\frac{y}{x}.$$

where $\arctan\frac{y}{x} = \left\{\theta : \tan\theta = \frac{y}{x}\right\}$. Note that, as with $\arg$, $\arctan$ is a set (as opposed to Arctan, which is a number). We specifically identify $\arg z$ as a proper subset of $\arctan\frac{y}{x}$ because $\tan\theta$ has period π, whereas $\cos$ and $\sin$ have period 2π. In selecting the proper values for $\arg z$, we must be careful in specifying the choices of $\arctan\frac{y}{x}$ so that the poin z associated with r and θ lies in the appropriate quadrant.

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Example If $z = -\sqrt{3} - i = r(\cos\theta + i\sin\theta)$, then

$$r = |z| = \left|-\sqrt{3} - i\right| = \sqrt{3+1} = 2 \text{ and}$$

$$\theta \in \arctan\frac{y}{x} = \arctan\frac{-1}{-\sqrt{3}} = \left\{\frac{\pi}{6} + n\pi : \ n \text{ is an integer}\right\}.$$

It would be a mistake to use $\frac{\pi}{6}$ as an acceptable value for θ, as the point z associated with $r = 2$ and $\theta = \frac{\pi}{6}$ is in the first quadrant, whereas $z = -\sqrt{3} - i$ is in the third quadrant. A correct choice for θ is $\theta = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$. Thus

$$\sqrt{3} - i = 2\,\mathrm{Cos}\left[-\frac{5\pi}{6}\right] + i\,2\,\mathrm{Sin}\left[-\frac{5\pi}{6}\right], \text{ and}$$

$$-\sqrt{3} - i = 2\,\mathrm{Cos}\left[-\frac{5\pi}{6} + 2n\pi\right] + i\,2\,\mathrm{Sin}\left[-\frac{5\pi}{6} + 2n\pi\right],$$

where n is any integer. In this case,

$$\mathrm{Arg}\left(-\sqrt{3} - i\right) = -\frac{5\pi}{6}, \text{ and}$$

$$\arg\left(-\sqrt{3} - i\right) = \left\{-\frac{5\pi}{6} + 2n\pi : \ n \text{ is an integer}\right\}.$$

Remark. Note that $\arg\left(-\sqrt{3} - i\right) = \left\{-\frac{5\pi}{6} + 2n\pi : \ n \text{ is an integer}\right\}$ is indeed a proper subset of $\arctan\frac{-1}{-\sqrt{3}} = \left\{\frac{\pi}{6} + n\pi : \ n \text{ is an integer}\right\}$.

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Definition $e^{i\theta} = \cos\theta + i\sin\theta = (\cos\theta, \sin\theta)$

Definition

$$\begin{aligned} z &= x + i\,y \\ &= r(\cos\theta + i\sin\theta) \\ &= r\,e^{i\theta} \end{aligned}$$

Consider the product of two complex numbers:

$$\begin{aligned} z_1 z_2 &= r_1 e^{iy_1} r_2 e^{iy_2} \\ &= r_1(\cos y_1 + i\sin y_1) r_2(\cos y_2 + i\sin y_2) \\ &= r_1 r_2[\cos(y_1)\cos(y_2) - \sin(y_1)\sin(y_2) + i\cos(y_1)\sin(y_2) + i\sin(y_1)\cos(y_2)] \\ &= r_1 r_2[\cos(y_1 + y_2) + i\sin(y_1 + y_2)] \\ &= r_1 r_2 e^{i(y_1 + y_2)} \end{aligned}$$

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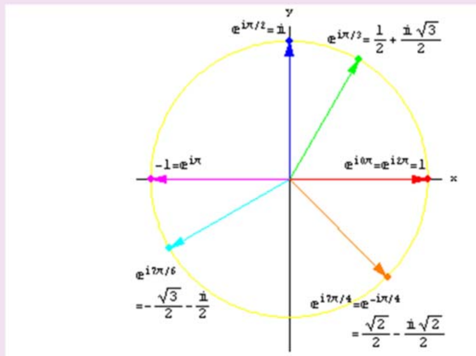
Figure illustrates the location of the points $e^{i\theta}$ for various values of θ.



The location of $e^{i\theta}$ for various values of θ.

Let's look at the rules of computation based on the exponential form of complex number.

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Rules for Computation

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

$$z^{-1} = \frac{1}{r} e^{-i\theta}$$

$$z^m = \left(re^{i\theta}\right)^m = r^m e^{im\theta}$$

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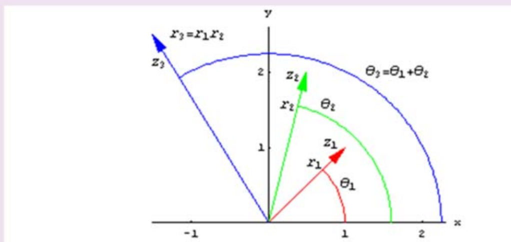
$z_1\ z_2$

If $z_1 = r_1\, e^{i\theta_1}$ and $z_2 = r_2\, e^{i\theta_2}$, then

$$z_1\, z_2 = r_1\, e^{i\theta_1}\, r_2\, e^{i\theta_2} = r_1\, r_2\, e^{i\,(\theta_1+\theta_2)}$$

$$= r_1\, r_2\, (\cos\,(\theta_1+\theta_2) + i\,\sin\,(\theta_1+\theta_2))$$

Figure illustrates the geometric significance of this equation.



The product of two complex numbers $z_3 = z_1\, z_2$

We have already seen that the modulus of the product is the product of the moduli; that is, $|\, z_1\, z_2\, | = |\, z_1\, |\, |\, z_2\, |$.
The above identity establishes that an argument of $z_1\, z_2$ is an argument of z_1 plus an argument of z_2.

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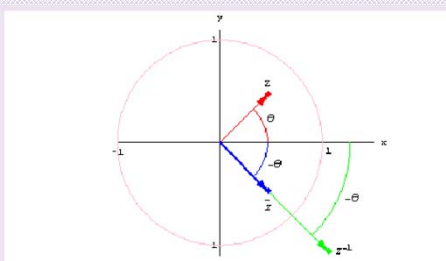
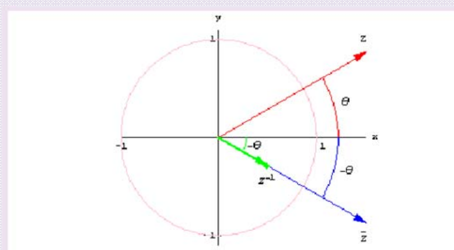
More examples



Relative positions z, $\bar{z}$, and z^{-1}, when $|\, z\, | < 1$.



Relative positions of z, $\bar{z}$, and z^{-1}, when $|\, z\, | > 1$.

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Example

If $z = 1 + i$, then $r = |1 + i| = \sqrt{1+1} = \sqrt{2}$ and $\theta = \text{Arg}(z) = \text{Arg}(1+i) = \frac{\pi}{4}$. Therefore

$$\begin{aligned} z^{-1} &= \frac{1}{r} e^{-i\pi/4} \\ &= \frac{1}{\sqrt{2}}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right] \\ &= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2}\right) \\ &= \frac{1}{2} - \frac{i}{2} \end{aligned}$$

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Example

Given $z_1 = 8\,i$ and $z_2 = 1 + i\sqrt{3}$, compute $\frac{z_1}{z_2}$

If $z_1 = 8\,i$ and $z_2 = 1 + i\sqrt{3}$, then representative polar forms for these numbers are $z_1 = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ and $z_2 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$. Hence

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{8}{2} e^{i\left(\frac{\pi}{2}-\frac{\pi}{3}\right)} \\ &= \frac{8}{2}\left[\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right)\right] \\ &= 4\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right] \\ &= 2\sqrt{3} + 2\,i \end{aligned}$$

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Example

Show that $\left(-\sqrt{3}-i\right)^3 = -8\,i$ in two ways.

Solution. (Method 1): The binomial formula (Exercise 14 of Section 1.2) gives

$$\left(-\sqrt{3}-i\right)^3 = 1\left(-\sqrt{3}\right)^3 + 3\left(-\sqrt{3}\right)^2 (-i) + 3\left(-\sqrt{3}\right)(-i)^2 + (-i)^3$$

$$= -3\sqrt{3} + -9\,i + 3\sqrt{3} + i$$

$$= -8\,i$$

(Method 2): Using ildentity stated above and Example 1.12 yields

$$\left(-\sqrt{3}-i\right)^3 = \left(2\,e^{i\,(-5\pi/6)}\right)^3$$

$$= (2)^3\left(e^{i\,(-5\pi/6)}\right)^3$$

$$= 8\left(e^{i\,(-15\pi/6)}\right)$$

$$= 8\left(\text{Cos}\left[\tfrac{-15\pi}{6}\right] + i\,\text{Sin}\left[\tfrac{-15\pi}{6}\right]\right)$$

$$= -8\,i$$

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Example

Which method would you use if you were asked to compute $\left(-\sqrt{3}-i\right)^{30}$?
Evaluate $\left(-\sqrt{3}-i\right)^{30}$.

Solution. $$\left(-\sqrt{3}-i\right)^{30} = \left(2\,e^{i\,(-5\pi/6)}\right)^{30}$$

$$= 2^{30}\left(e^{i\,(-5\pi/6)}\right)^{30}$$

$$= 2^{30}\left(e^{i\,(-150\pi/6)}\right)$$

$$= 2^{30}\left(\text{Cos}\left[\tfrac{-150\pi}{6}\right] + i\,\text{Sin}\left[\tfrac{-150\pi}{6}\right]\right)$$

$$= 2^{30}\left(\text{Cos}[-25\pi] + i\,\text{Sin}[-25\pi]\right)$$

$$= -\,2^{30}$$

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Rules for Computation Cont.

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n^{th} root of complex number

If $c = \rho e^{i\phi} = \rho(\cos\phi + i\sin\phi)$ and $z = r e^{i\theta}$, then $z^n = c$ iff $r^n e^{in\theta} = \rho e^{i\phi}$. But this last equation is satisfied iff

$$r^n = \rho, \text{ and } n\theta = \phi + 2k\pi, \text{ where } k \text{ is an integer.}$$

As before, we get n distinct solutions given by

$$z_k = \rho^{\frac{1}{n}} e^{i\frac{\phi+2\pi k}{n}} = \rho^{\frac{1}{n}}\left(\cos\frac{\phi+2\pi k}{n} + i\sin\frac{\phi+2\pi k}{n}\right) \quad \text{for } k = 0, 1, \cdots, n-1.$$

Each of the above solutions can be considered an n^{th} root of c. Geometrically, the n^{th} roots of c are equally spaced points that lie on the circle $C_{\rho^{\frac{1}{n}}} = \left\{z : |z| = \rho^{\frac{1}{n}}\right\}$ and form the vertices of a regular polygon with n sides.

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Figure illustrates the case for n=5.

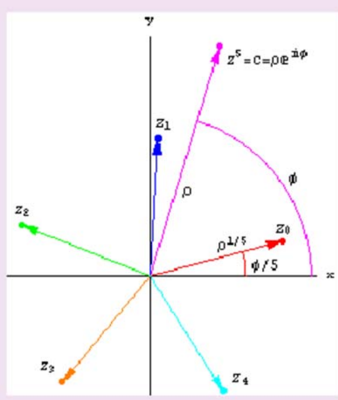

Figure The five solutions to the equation $z^5 = c$.

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Example Find all the cube roots of $8\,i = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$, i.e. find all solutions to the equation $z^3 = 8\,i$.

Solution.

$$z_k = 8^{\frac{1}{3}}\, e^{i\left(\frac{\pi}{2}+2\pi k\right)/3}$$

$$= 2\left(\cos\frac{\frac{\pi}{2}+2\pi k}{3} + i\sin\frac{\frac{\pi}{2}+2\pi k}{3}\right)$$

$$= 2\left(\cos\left(\frac{\pi}{6}+\frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{6}+\frac{2k\pi}{3}\right)\right)$$

for $k = 0, 1, 2, \ldots, n-1$. The Cartesian forms of the solutions are

$$z_0 = \sqrt{3} + i,\quad z_1 = -\sqrt{3} + i,\quad \text{and}\quad z_2 = -2\,i.$$

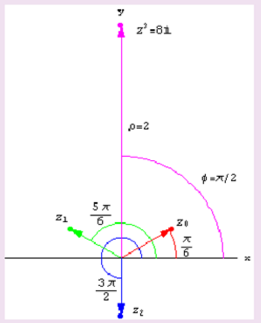

Figure The point $z = 8\,i$ and its three cube roots $z_0,\ z_1,$ and z_2.

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Regions In the Complex Plane

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We need to develop some vocabulary that will help describe sets of points in the plane. One fundamental idea is that of an ϵ neighborhood of the point z_0. It is the open disk of radius $\epsilon > 0$ about z_0 shown in Figure. Formally, it is the set of all points satisfying the inequality $D_\epsilon(z_0) = \{z : |z - z_0| < \epsilon\}$ and is denoted $D_\epsilon(z_0)$. That is,

$$D_\epsilon(z_0) = \{z : |z - z_0| < \epsilon\}.$$

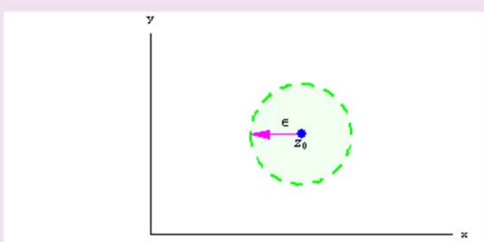

Figure An ϵ neighborhood of the point z_0.

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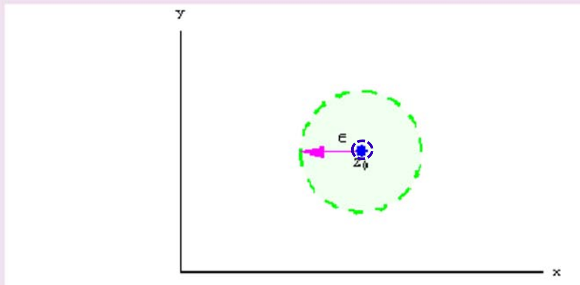
$$D^*_\epsilon(z_0) = \{z : 0 < |z - z_0| < \epsilon\}.$$



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Definitions (Interior Point, Exterior Point, Boundary Point).

The point z_0 is said to be an interior point of the set S provided that there exists an ϵ neighborhood of z_0 that contains only points of S; z_0 is called an exterior point of the set S if there exists an ϵ neighborhood of z_0 that contains no points of S. If z_0 is neither an interior point nor an exterior point of S, then it is called a boundary point of S and has the property that each ϵ neighborhood of z_0 contains both points in S and points not in S. Figure illustrates this situation.

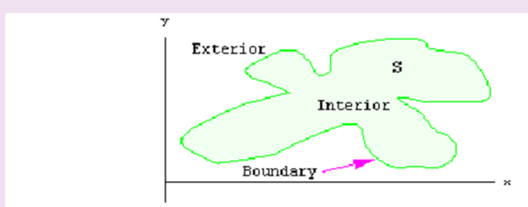

Figure The interior, exterior, and boundary of a set S.

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Example

Let $S = \{z : |z| < 1\}$. **(a)** Find the interior of S. **(b)** Find the exterior of S. **(c)** Find boundary of S.

Find the interior of S.

Let z_0 be a point of S. Then $|z_0| < 1$ so that we can choose $\epsilon = 1 - |z_0| > 0$.
If z lies in the disk $|z - z_0| < \epsilon$, then

$$|z| = |z_0 + z - z_0| \leq |z_0| + |z - z_0| < |z_0| + \epsilon < 1.$$

Hence the ϵ-neighborhood of z_0 is contained in S, and z_0 is an interior point of S.
It follows that the interior of S is the open unit disk.

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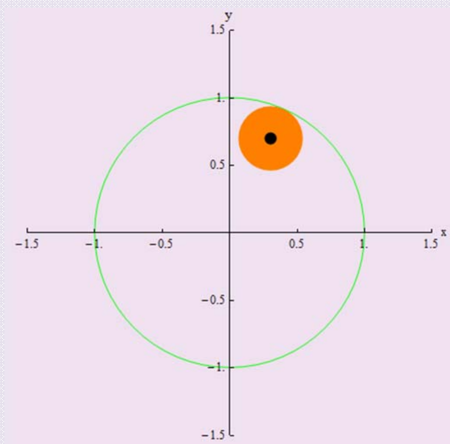

The ϵ-neighborhood of z_0 is contained in $S = \{z : |z| < 1\}$.

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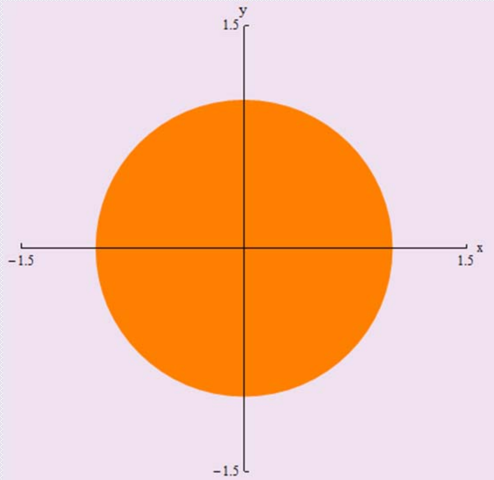


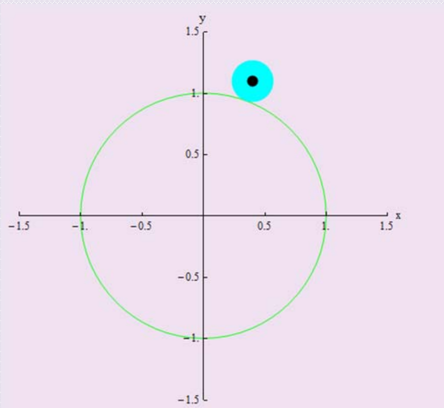
The interior of the set S is the unit disk $\{z: |z| < 1\}$.

Find the exterior of S.

If $|z_0| > 1 + \epsilon$, and $|z - z_0| < \epsilon$ then a similar argument will produce an ϵ-neighborhood of z_0 that lies entirely in the region $|z| > 1$,
Then $|z_0| > 1 + \epsilon > 1 + |z - z_0|$

$$|z_0| - |z - z_0| > 1$$

$$|z| = |z_0 + z - z_0| > |z_0| - |z - z_0| > 1$$



The ϵ-neighborhood of z_0 is contained in the region $|z| > 1$.

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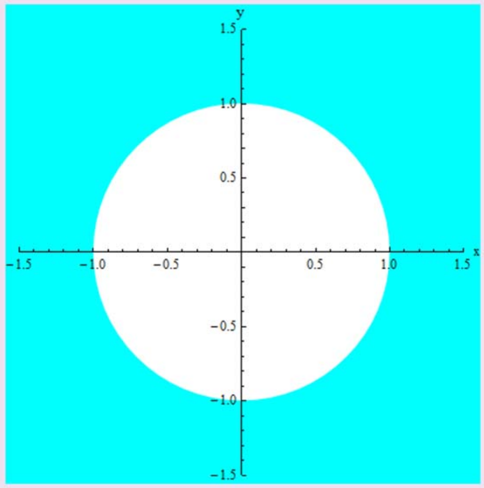

The exterior of the set S is the open region {z: |z| > 1}.

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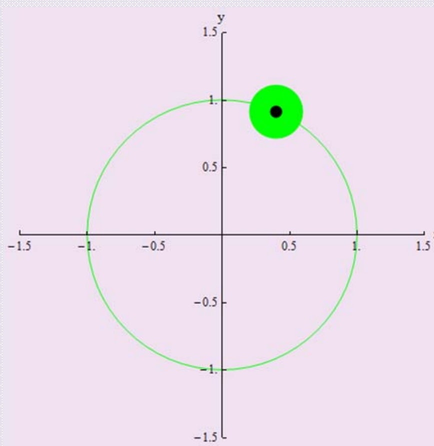
Find the boundary of S.



Every ε-neighborhood of z_0 contains points of S and points that do not belong to S.

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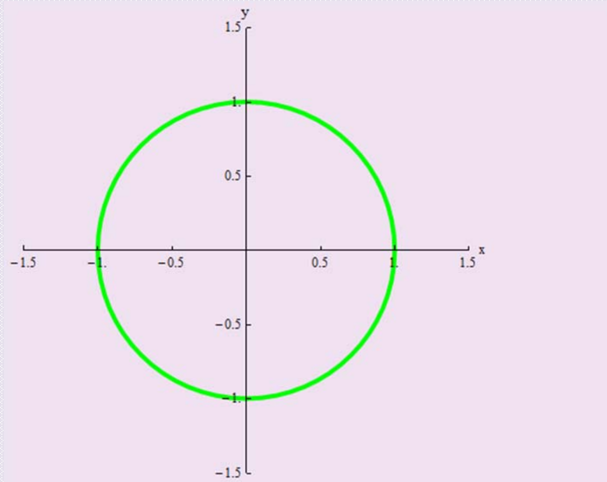

The boundary of the set S = {z:|z|<1 is the unit circle {z: |z| = 1}.

Definition

A set S is called an open set if every point of S is an interior point of S.

A set S s called a closed set if it contains all its boundary points.

A set S is said to be a connected set if every pair of points z_1 and z_2 contained in S can be joined by a curve that lies entirely in S.
Roughly speaking, a connected set consists of a "single piece."
The unit disk $D_1(0) = \{z : |z| < 1\}$ is a connected open set.

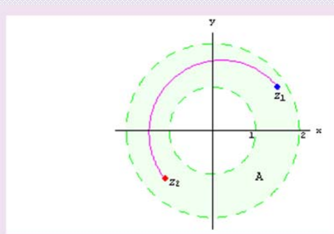

Figure The annulus $A = \{z : 1 < |z| < 2\}$ is a connected set.

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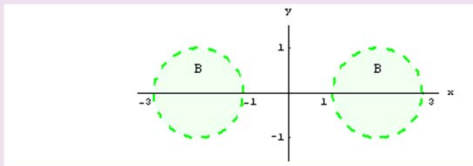

Figure The set $B = \{z : |z+2| < 1 \text{ or } |z-2| < 1\}$ is not a connected set.

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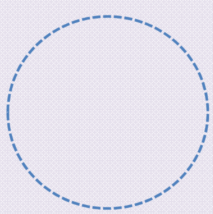
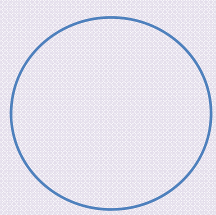
Definition

We call a connected open set a domain. In the exercises we ask you to show that the open unit disk $D_1(0) = \{z : |z| < 1\}$ is a domain and that the closed unit disk $\overline{D_1}(0) = \{z : |z| \le 1\}$ is not a domain.



Domain



Not a domain

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