Conformal Mapping

Conformal Mapping

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- Properties of Conformal Mapping
- Bilinear Transformations
- Mapping of a region

Properties of Conformal Mapping

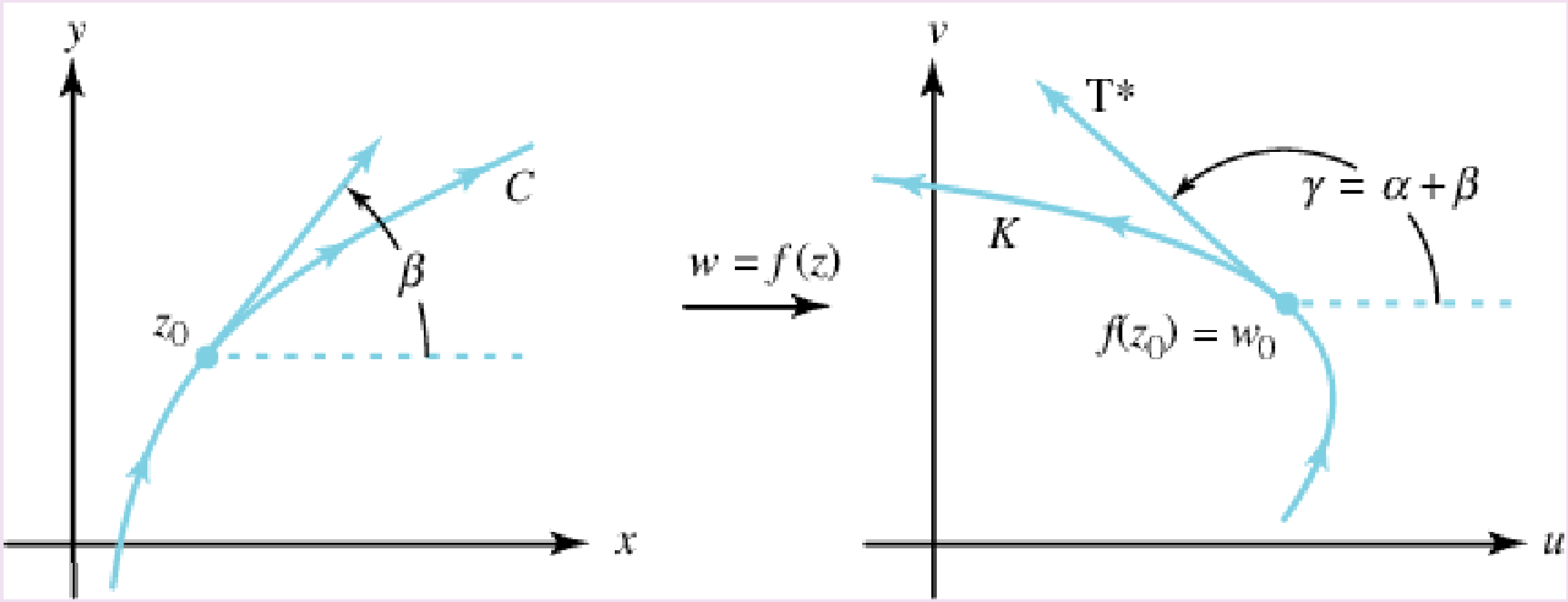
Let $f(z)$ be an analytic function in the domain D, and let z_0 be a point in D. If $f'(z_0) \neq 0$, then we can express $f(z)$ in the form

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0),$$

where $\eta(z) \to 0$ as $z \to z_0$. If z is near z_0, then the transformation $w = f(z)$ has the linear approximation

$$S(z) = A + B(z - z_0),$$

where $A = f(z_0)$ and $B = f'(z_0)$. Because $\eta(z) \to 0$ when $z \to z_0$, for points near z_0 the transformation $w = f(z)$ has an effect much like the linear mapping $w = S(z)$. The effect of the linear mapping S is a rotation of the plane through the angle $\alpha = \mathrm{Arg}(f'(z_0))$, followed by a magnification by the factor $|f'(z_0)|$, followed by a rigid translation by the vector $A + Bz_0$. Consequently, the mapping $w = S(z)$ preserves angles at the point z_0. We now show that the mapping $w = f(z)$ also preserves angles at z_0.


y
C
β
z_0
x
$w = f(z)$
v
T*
$\gamma = \alpha + \beta$
K
$f(z_0) = w_0$
u

(Conformal Mapping).

Let $f(z)$ be an analytic function in the domain D, and let z_0 be a point in D. If $f'(z_0) \neq 0$, then $f(z)$ is conformal at z_0.

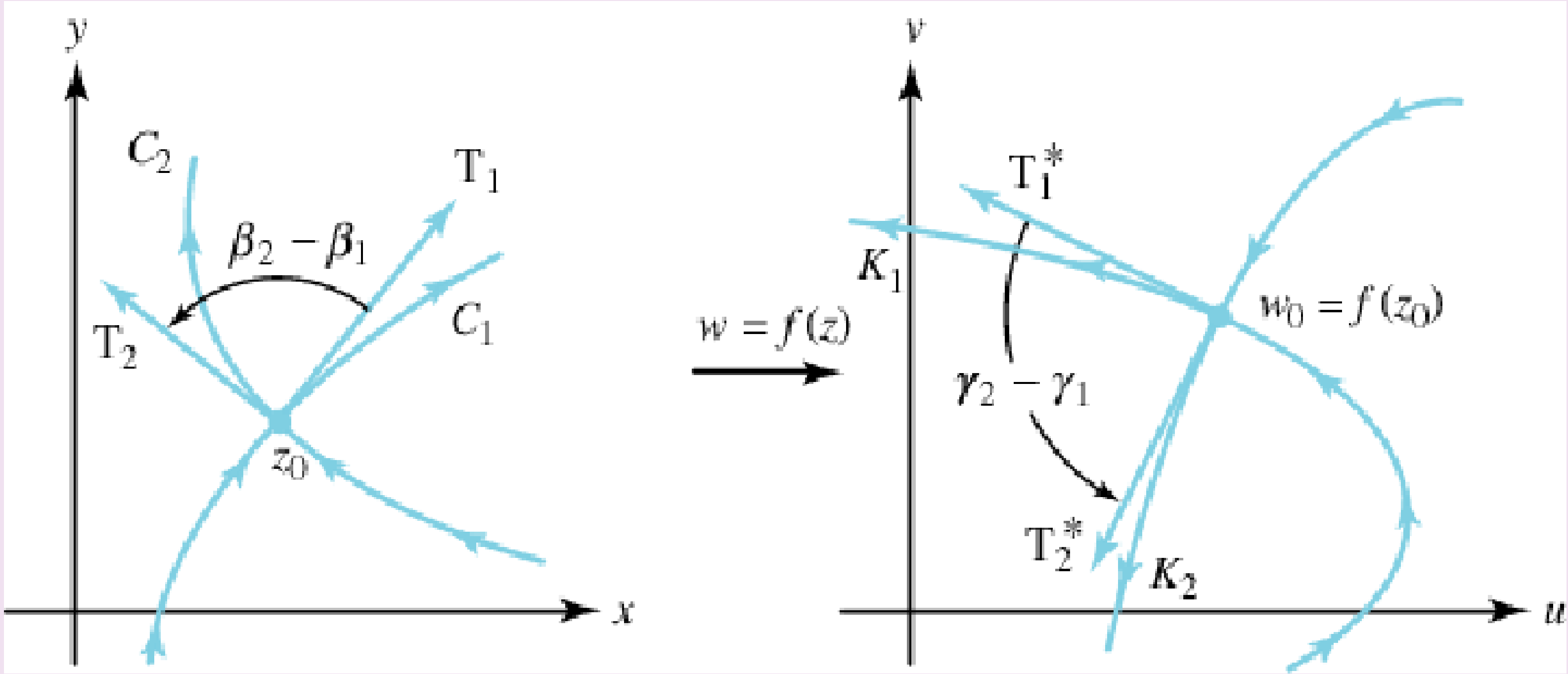


Figure The analytic mapping $w = f(z)$ is conformal at the point z_0, where $f'(z_0) \neq 0$.

Bilinear Transformations

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Another important class of elementary mappings was studied by August Ferdinand Möbius (1790-1868). These mappings are conveniently expressed as the quotient of two linear expressions and are commonly known as linear fractional or bilinear transformations.

Let a, b, c, and d denote four complex constants with the restriction that $ad \neq bc$. Then the function

$$w = S(z) = \frac{az + b}{cz + d}$$

is called a bilinear transformation, a Möbius transformation, or a linear fractional transformation.

If the expression for $S(z)$ is multiplied through by the quantity $cz + d$, then the resulting expression has the bilinear form $cwz - az + dw - b = 0$. We collect terms involving z and write $z(cw - a) = -dw + b$. Then, for values of $w \neq \frac{a}{c}$ the inverse transformation is given by

$$z = S^{-1}(w) = \frac{-dw + b}{cw - a}.$$

We can extend $S(z)$ and $S^{-1}(w)$ to mappings in the extended complex plane. The value $S(\infty)$ should be chosen to equal the limit of $S(z)$ as $z \to \infty$. Therefore we define

$$S(\infty) = \lim_{z \to \infty} S(z) = \lim_{z \to \infty} \frac{a + \frac{b}{z}}{c + \frac{d}{z}} = \frac{a}{c},$$

and the inverse is $S^{-1}\left(\frac{a}{c}\right) = \infty$. Similarly, the value $S^{-1}(\infty)$ is obtained by

$$S^{-1}(\infty) = \lim_{w\to\infty} S^{-1}(w) = \lim_{w\to\infty} \frac{-d+\frac{b}{w}}{c-\frac{a}{w}} == \frac{-d}{c},$$

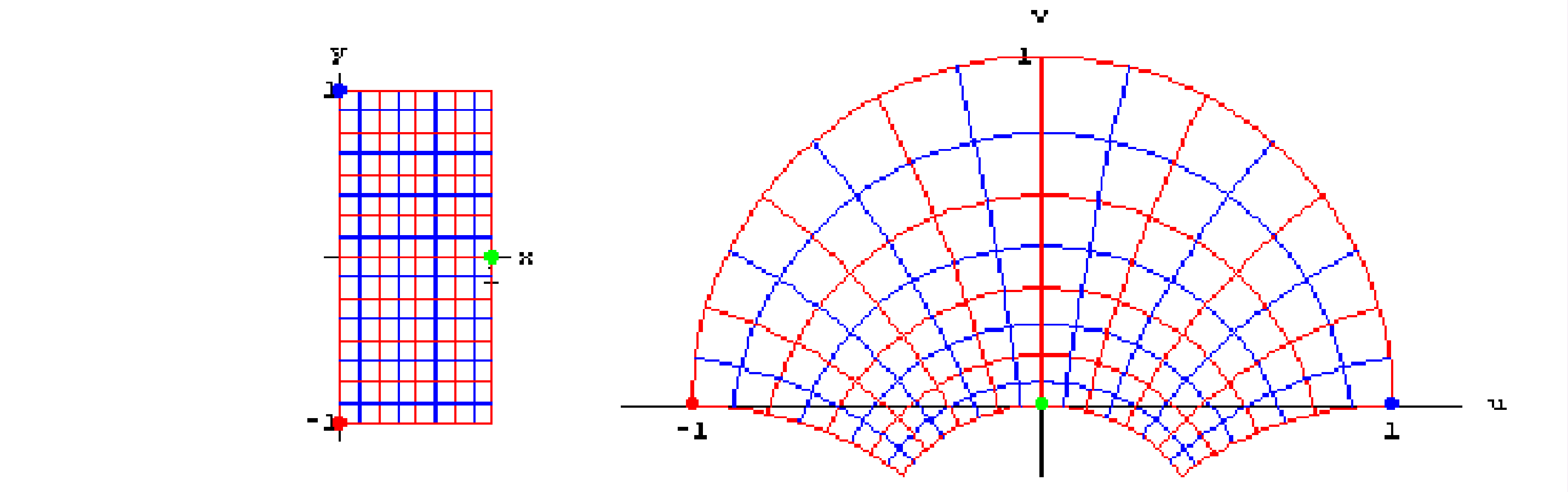
and the inverse is $S\left(\frac{-d}{c}\right) = \infty$. With these extensions we conclude that the transformation $w = S(z)$ is a one-to-one mapping of the extended complex z-plane onto the extended complex w-plane.

(The Implicit Formula). There exists a unique bilinear transformation that maps three distinct points z_1, z_2, and z_3 onto three distinct points w_1, w_2, and w_3, respectively. An implicit formula for the mapping is given by the equation

$$\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}.$$

Example

Construct the bilinear transformation $w = S(z)$ that maps the points $z_1 = -i$, $z_2 = 1$, $z_3 = i$ onto the points $w_1 = -1$, $w_2 = 0$, $w_3 = 1$, respectively.



Solution. We use the implicit formula,

$$\frac{(z-(-i))(1-i)}{(z-i)(1-(-i))} = \frac{(w-(-1))(0-1)}{(w-1)(0-(-1))}$$

$$\frac{(z+i)(1-i)}{(z-i)(1+i)} = \frac{(w+1)(0-1)}{(w-1)(0+1)}$$

$$\frac{(z+i)(1-i)}{(z-i)(1+i)} = \frac{w+1}{-w+1}.$$

Expanding this equation, collecting terms involving w and zw on the left and then simplify.

$$(z - i)(1 + i)(w + 1) = (z + i)(1 - i)(-w + 1)$$

$$(1 + i) z w + (1 - i) w + (1 + i) z + (1 - i)$$
$$=$$
$$(-1 + i) z w + (-1 - i) w + (1 - i) z + (1 + i)$$

$$z w + i z w + w - i w + z + i z + 1 - i$$
$$=$$
$$-z w + i z w - w - i w + z - i z + 1 + i$$

$$2 z w + 2 w = -2 i z + 2 i$$

$$z w + w = - i z + i$$

$$w (1 + z) = i (1 - z)$$

Therefore the desired bilinear transformation is

$$w = S(z) = \frac{i(1-z)}{1+z}.$$

(The Implicit Formula with a point at Infinity)

The point at infinity can be introduced as one of the prescribed points in either the z plane or the w plane.

Case 1. If $z_3 = \infty$, then we can write $\frac{(z_2 - z_3)}{(z - z_3)} = \frac{(z_2 - \infty)}{(z - \infty)} = 1$ and substitute this expression into implicait formula to obtain $\frac{(z - z_1)(z_2 - \infty)}{(z - \infty)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}$ which can be rewritten as $\frac{(z - z_1)(z_2 - \infty)}{(z_2 - z_1)(z - \infty)} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}$ and simplifies to obtain

$$\frac{z - z_1}{z_2 - z_1} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}.$$

Case 2. If $w_3 = \infty$, then we can write $\frac{(w_2 - w_3)}{(w - w_3)} = \frac{(w_2 - \infty)}{(w - \infty)} = 1$ and substitute this expression into implicait formula to obtain $\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - \infty)}{(w - \infty)(w_2 - w_1)}$ which can be rewritten as $\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - \infty)}{(w_2 - w_1)(w - \infty)}$ and simplifies to obtain

$$\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{w - w_1}{w_2 - w_1}.$$

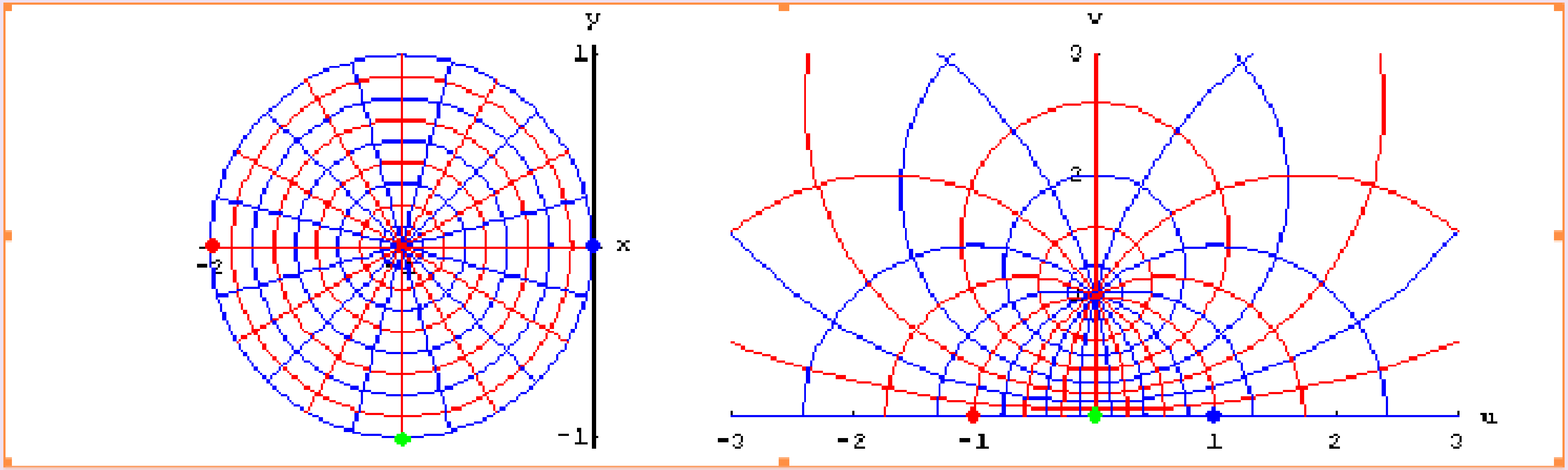
Mapping of a region

We let D be a region in the z plane that is bounded by either a circle or a straight line C. We further let z_1, z_2, and z_3 be three distinct points that lie on C and have the property that an observer moving along C from z_1 to z_3 through z_2 finds the region D to be on the left. If C is a circle and D is the interior of C, then we say that C is positively oriented. Conversely, the ordered triple (z_1, z_2, z_3) uniquely determines a region that lies to the left of C.

We let G be a region in the w plane that is bounded by either a circle of a straight line K. We further let w_1, w_2, and w_3 be three distinct points that lie on K such that an observer moving along K from w_1 to w_3 through w_2 finds the region G to be on the left. Because a bilinear transformation is a conformal mapping that maps the class of circles and straight lines onto itself, we can use the implicit formula to construct a bilinear transformation $w = S(z)$ that is a one-to-one mapping of D onto G.

Example

Show that the mapping $w = S(z) = \frac{(1-i)z+2}{(1+i)z+2}$ maps the disk $D: \ |z+1| < 1$ one-to-one and onto the upper half plane $\mathrm{Im}(w) > 0$.



Solution. For convenience, we choose the ordered triple $z_1 = -2,\ z_2 = -1 - i,\ z_3 = 0$, which gives the circle $C:\ |z+1| = 1$ a positive orientation and the disk D a left orientation. The corresponding image points are

$$w_1 = S(z_1) = S(-2) = -1,$$

$$w_2 = S(z_2) = S(-1-i) = 0,$$

$$w_3 = S(z_3) = S(0) = 1.$$

Because the ordered triple of points $w_1 = -1,\ w_2 = 0,\ w_3 = 1$, lie on the u axis, it follows that the image of circle C is the u axis. The points $w_1 = -1,\ w_2 = 0,\ w_3 = 1$ give the upper half-plane $G:\ \mathrm{Im}(w) > 0$ a left orientation. Therefore $w = S(z) = \frac{(1-i)z + 2}{(1+i)z + 2}$ maps the disk D onto the upper half-plane G. To check our work, we choose a point z_0 that lies in D and find the half-plane in which its image, w_0 lies. The choice $z_0 = -1$ yields $w_0 = S(z_0) = i$. Hence the upper half-plane is the correct image.

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This situation is illustrated in Figure below

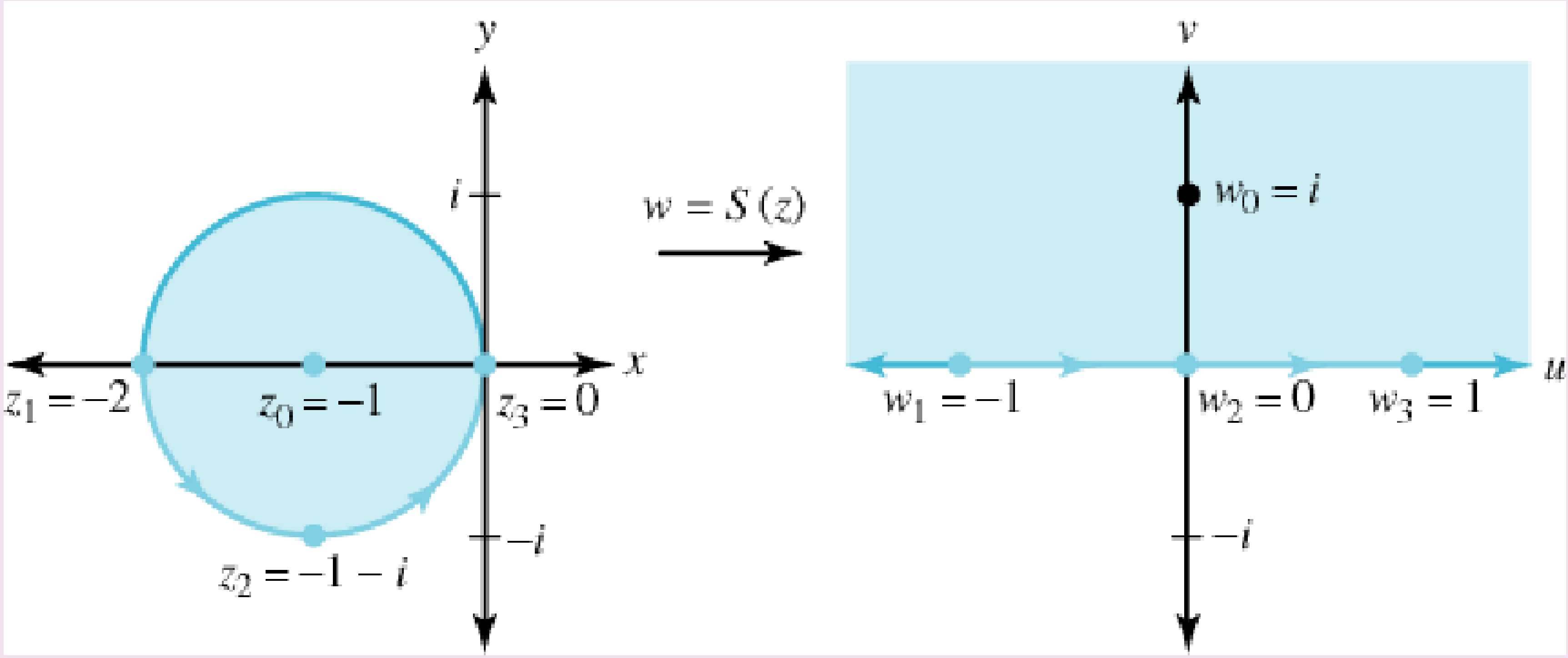
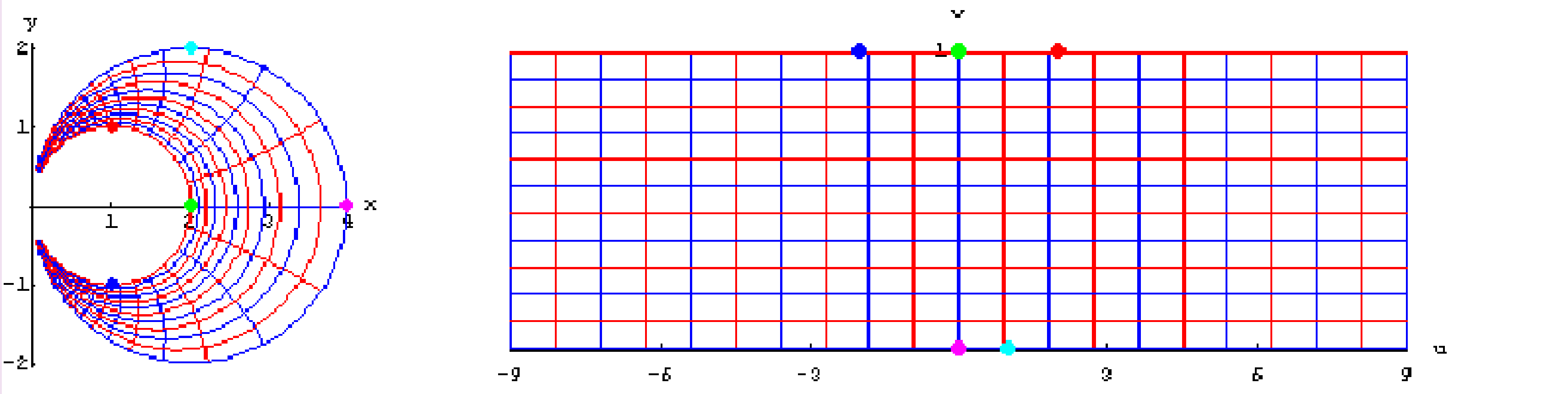


Figure The bilinear mapping $w = S(z) = \frac{(1-i)z + 2}{(1+i)z + 2}$.

Example

Find the bilinear transformation $w = S(z)$ that maps the crescent-shaped region that lies inside the disk $D : |z - 2| < 2$ and outside the circle $|z - 1| = 1$ onto a horizontal strip.



Solution. For convenience we choose $z_1 = 4$, $z_2 = 2 + 2i$, $z_3 = 0$ and the image values $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$, respectively. The ordered triple $z_1 = -4$, $z_2 = 2 + 2i$, $z_3 = 0$ gives the circle $C : |z - 2| = 2$ a positive orientation and the disk $D : |z - 2| < 2$ has a left orientation. The image points $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$ all lie on the extended u axis, and they determine a left orientation for the upper half-plane $\mathrm{Im}(w) > 0$. Therefore we can use the second implicit formula to write

$$\frac{(z-4)\,(2+2\,i-0)}{(z-0)\,(2+2\,i-4)} = \frac{w-0}{1-0},$$

which determines a mapping of the disk $D: |z-2| < 2$ onto the upper half-plane $\mathrm{Im}(w) > 0$. Use the fact that $\frac{2+2\,i}{-2+2\,i} = -i$ to simplify the preceding equation and get

$$\frac{z-4}{z}\,\frac{2+2\,i}{-2+2\,i} = \frac{z-4}{z}(-i) = \frac{w}{1}$$

which can be written in the form

$$w = S(z) = \frac{-iz+i4}{z}.$$

A straightforward calculation shows that the points $z_4 = 1 - i$, $z_5 = 2$, $z_6 = 1 + i$ are mapped onto the points

$$w_4 = S(z_4) = S(1 - i) = -2 + i,$$

$$w_5 = S(z_5) = S(2) = i,$$

$$w_6 = S(z_6) = S(1 + i) = 2 + i,$$

respectively. The points $w_4 = -2 + i$, $w_5 = i$, $w_6 = 2 + i$ lie on the horizontal line $\mathrm{Im}(w) > 1$ in the upper half-plane. Therefore the crescent-shaped region is mapped onto the horizontal strip $0 < \mathrm{Im}(w) < 1$, as shown in Figure below.

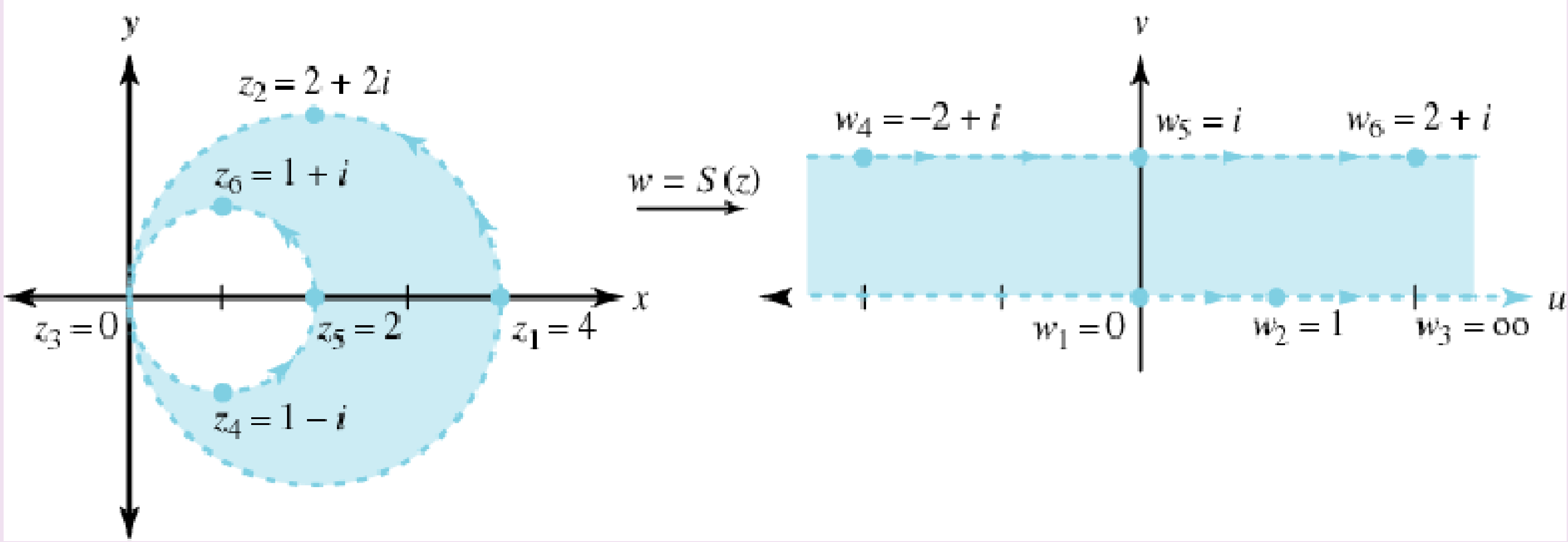


Figure The mapping $w = S(z) = \frac{-iz + i4}{z}$.