Graph traversals

Outline

We will look at traversals of graphs

- Breadth-first or depth-first traversals
- Must avoid cycles
- Depth-first traversals can be recursive or iterative
- Problems that can be solved using traversals

Strategies

Traversals of graphs are also called *searches*

We can use either breadth-first or depth-first traversals
- Breadth-first requires a queue
- Depth-first requires a stack

We each case, we will have to track which vertices have been visited requiring $\Theta(|V|)$ memory
- One option is a hash table
- If we can use a bit array, this requires only $|V|/8$ bytes

The time complexity cannot be better than and should not be worse than $\Theta(|V| + |E|)$
- Connected graphs simplify this to $\Theta(|E|)$
- Worst case: $\Theta(|V|^2)$

Breadth-first traversal

Consider implementing a breadth-first traversal on a graph:

- Choose any vertex, mark it as visited and push it onto queue
- While the queue is not empty:
  - Pop to top vertex v from the queue
  - For each vertex adjacent to v that has not been visited:
    - Mark it visited, and
    - Push it onto the queue

This continues until the queue is empty

- Note: if there are no unvisited vertices, the graph is connected,

Iterative depth-first traversal

An implementation can use a queue

```
void Graph::depth_first_traversal( Vertex *first ) const {
    unordered_map<Vertex *, int> hash;
    hash.insert( first );
    std::queue<Vertex *> queue;
    queue.push( first );

    while ( !queue.empty() ) {
        Vertex *v = queue.front();
        queue.pop();
        // Perform an operation on v

        for ( Vertex *w : v->adjacent_vertices() ) {
            if ( !hash.member( w ) ) {
                hash.insert( w );
                queue.push( w );
            }
        }
    }
}
```

Breadth-first traversal

The size of the queue is $\mathrm{O}(|V|)$

- The size depends both on:
  - The number of edges, and
  - The out-degree of the vertices

Depth-first traversal

Consider implementing a depth-first traversal on a graph:

- Choose any vertex, mark it as visited
- From that vertex:
  - If there is another adjacent vertex not yet visited, go to it
  - Otherwise, go back to the most previous vertex that has not yet had all of its adjacent vertices visited and continue from there
- Continue until no visited vertices have unvisited adjacent vertices

Two implementations:

- Recursive
- Iterative

Recursive depth-first traversal

A recursive implementation uses the call stack for memory:

```
void Graph::depth_first_traversal( Vertex *first ) const {
    std::unordered_map<Vertex *, int> hash;
    hash.insert( first );

     first->depth_first_traversal( hash );
}

void Vertex::depth_first_traversal( unordered_map<Vertex *, int> &hash ) const {
    // Perform an operation on this

    for ( Vertex *v : adjacent_vertices() ) {
        if ( !hash.member( v ) ) {
            hash.insert( v );
            v->depth_first_traversal( hash );
        }
    }
}
```

Iterative depth-first traversal

An iterative implementation can use a stack

```
void Graph::depth_first_traversal( Vertex *first ) const {
    unordered_map<Vertex *, int> hash;
    hash.insert( first );
    std::stack<Vertex *> stack;
    stack.push( first );

    while ( !stack.empty() ) {
        Vertex *v = stack.top();
        stack.pop();
        // Perform an operation on v

        for ( Vertex *w : v->adjacent_vertices() ) {
            if ( !hash.member( w ) ) {
                hash.insert( w );
                stack.push( w );
            }
        }
    }
}
```

Iterative depth-first traversal

If memory is an issue, we can reduce the stack size:

- For the vertex:
  - Mark it as visited
  - Perform an operation on that vertex
  - Place it onto an empty stack
- While the stack is not empty:
  - If the vertex on the top of the stack has an unvisited adjacent vertex,
    - Mark it as visited
    - Perform an operation on that vertex
    - Place it onto the top of the stack
  - Otherwise, pop the top of the stack

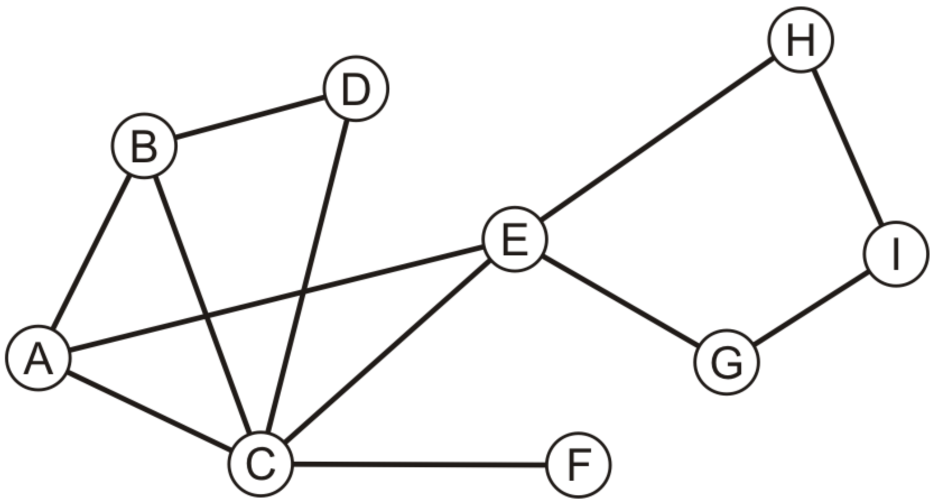
Standard Template Library (STL) approach

An object-oriented STL approach would be create a iterator class:

- The hash table and stack/queue are private member variables created in the constructor
- Internally, it would store the current node
- The auto-increment operator would pop the top of the stack and place any unvisited adjacent vertices onto the stack/queue
- The auto-decrement operator would not be implemented
  - You can't go back…

Example

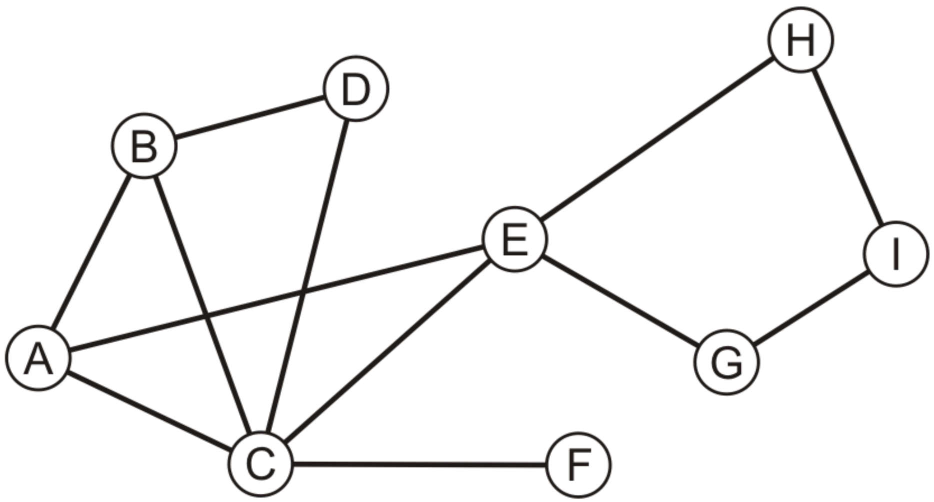
Consider this graph



Example

Performing a breadth-first traversal

- Push the first vertex onto the queue



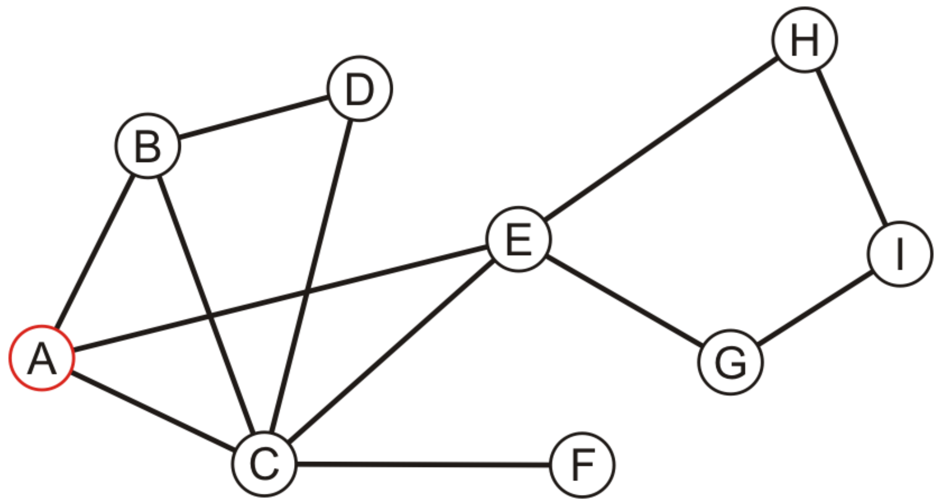
A					

Example

Performing a breadth-first traversal

– Pop A and push B, C and E

A



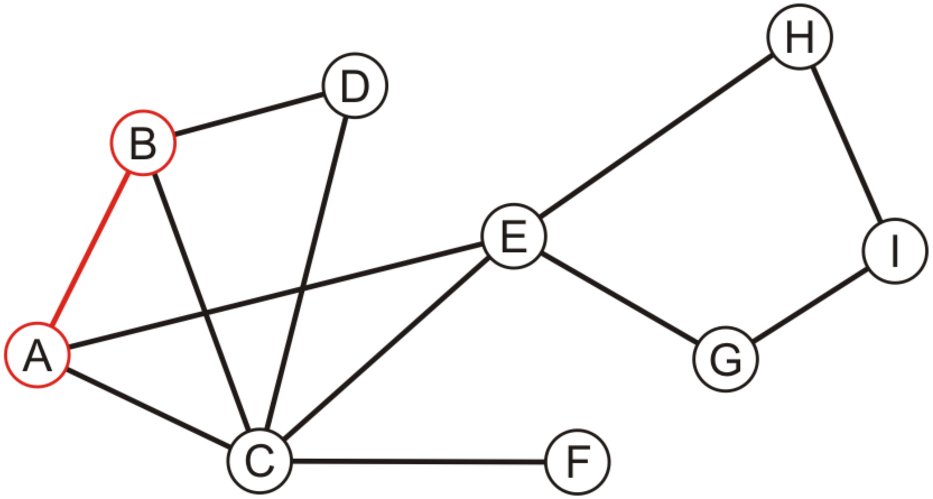
B	C	E			

Example

Performing a breadth-first traversal:

- Pop B and push D

A, B



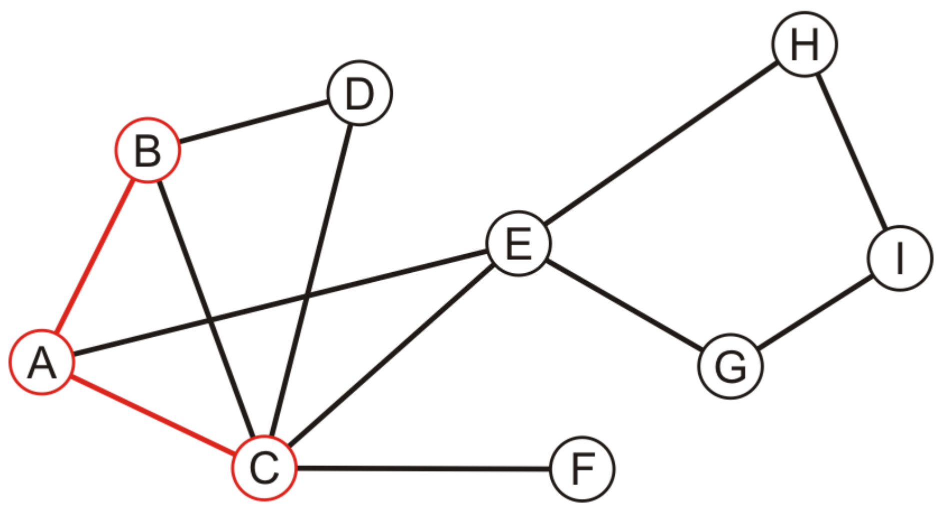
C	E	D			

Example

Performing a breadth-first traversal:

- Pop C and push F

A, B, C



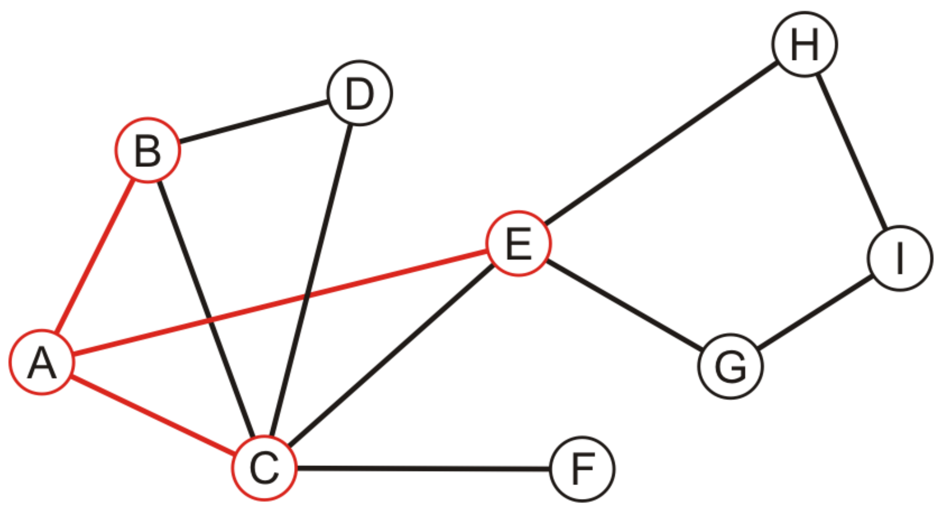
E	D	F			

Example

Performing a breadth-first traversal:

– Pop E and push G and H

A, B, C, E

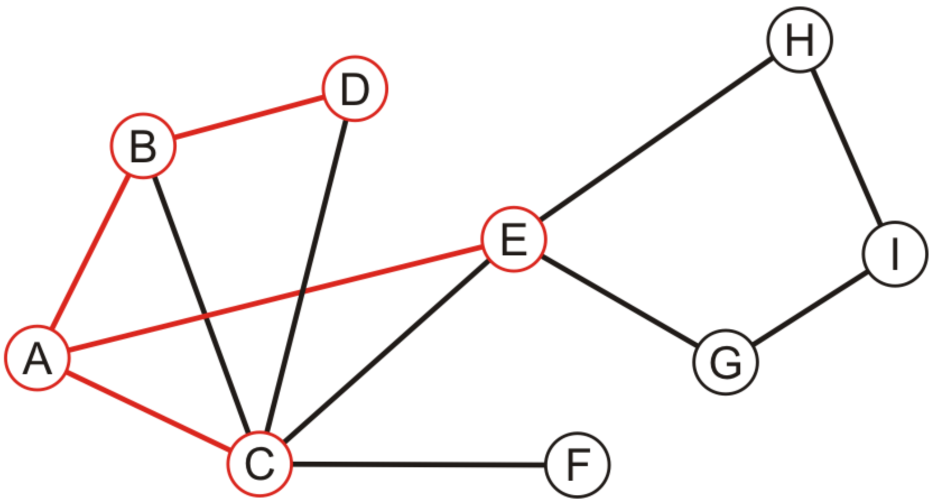


D	F	G	H		

Example

Performing a breadth-first traversal:
- Pop D

A, B, C, E, D



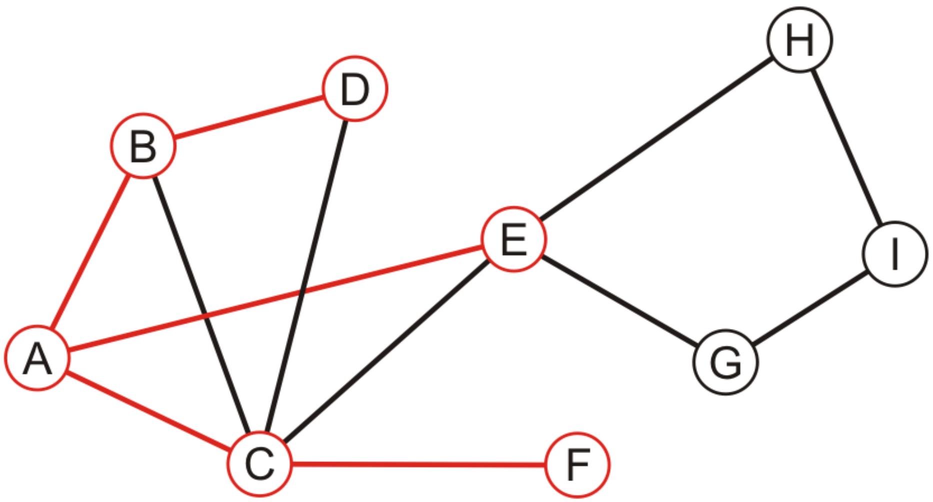
F	G	H			

Example

Performing a breadth-first traversal:

– Pop F

A, B, C, E, D, F



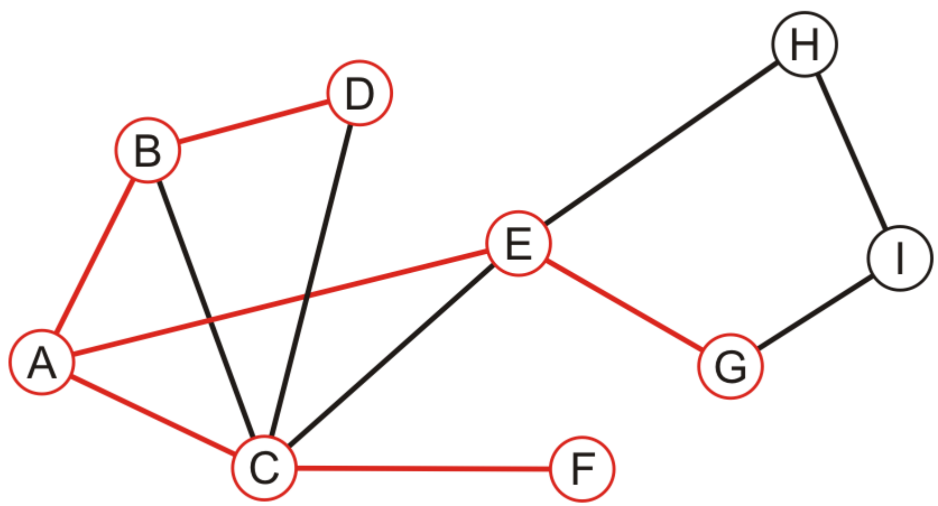
G	H				

Example

Performing a breadth-first traversal:

- Pop G and push I

A, B, C, E, D, F, G



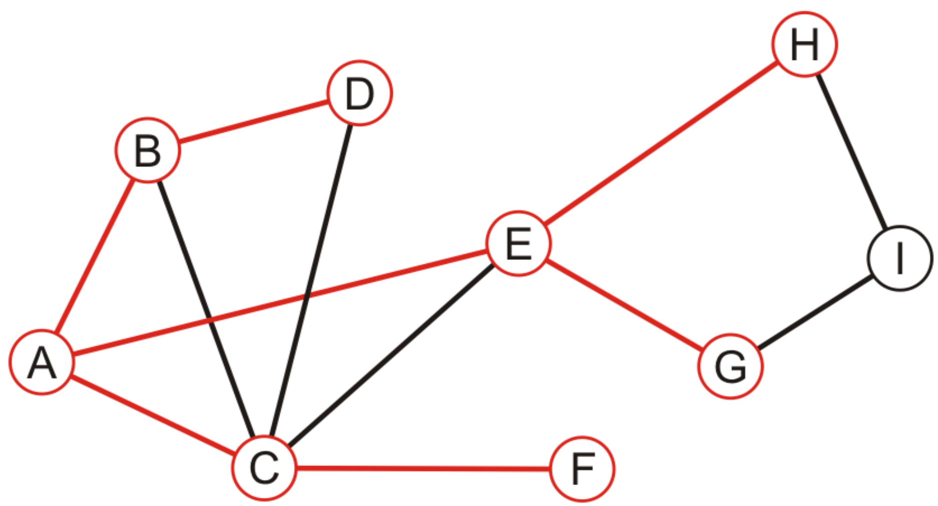
H	I				

Example

Performing a breadth-first traversal:

- Pop H

A, B, C, E, D, F, G, H



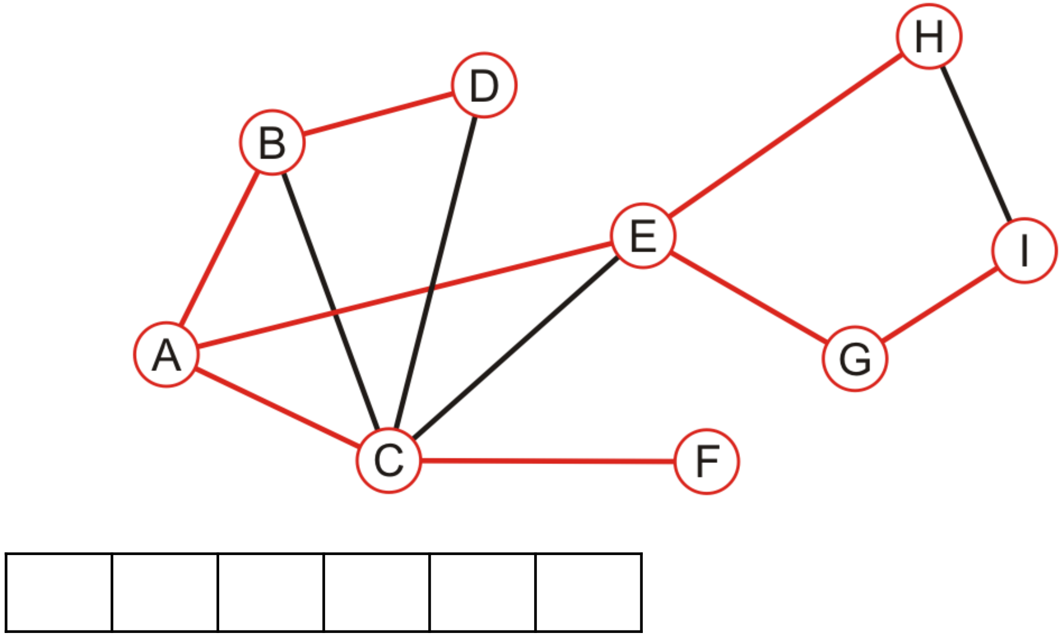
I					

Example

Performing a breadth-first traversal:

– Pop I

A, B, C, E, D, F, G, H, I

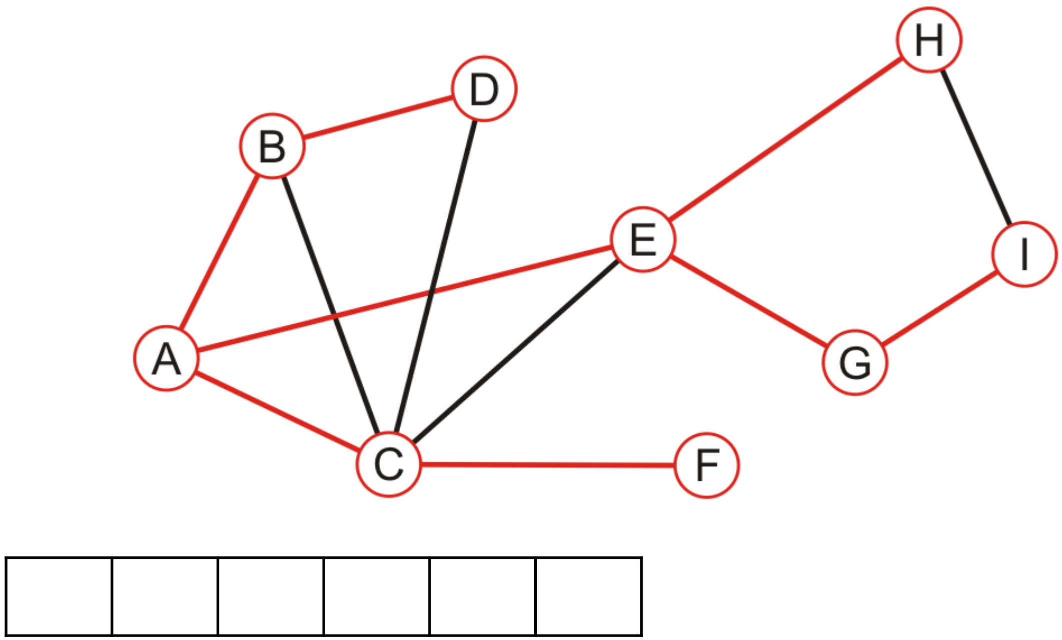


Example

Performing a breadth-first traversal:

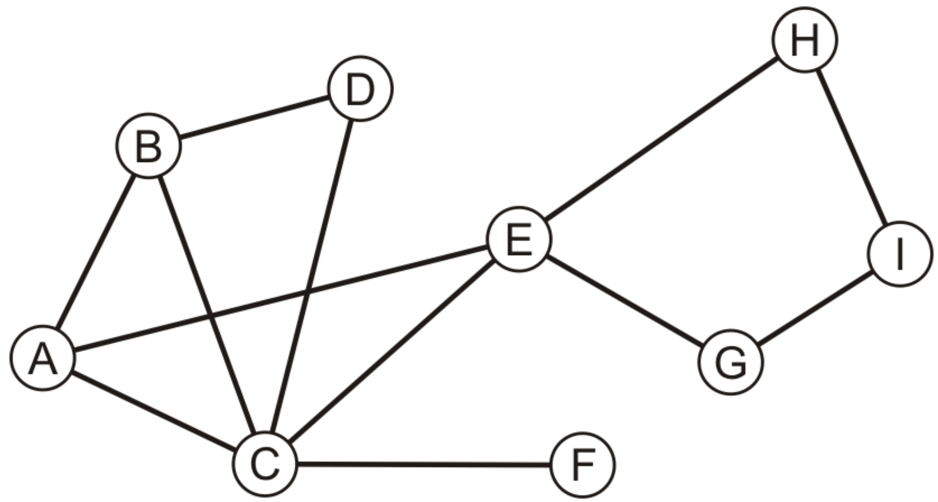
– The queue is empty: we are finished

A, B, C, E, D, F, G, H, I



Example

Perform a recursive depth-first traversal on this same graph

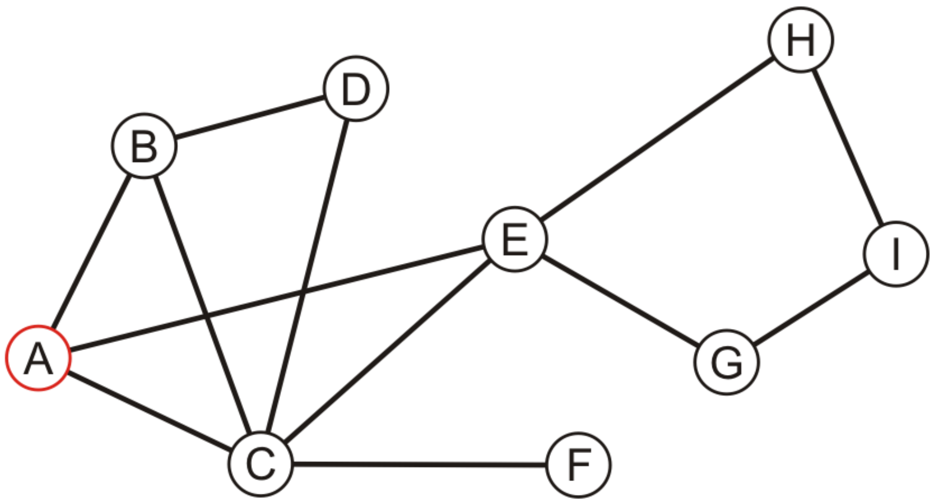


Example

Performing a recursive depth-first traversal:

– Visit the first node

A

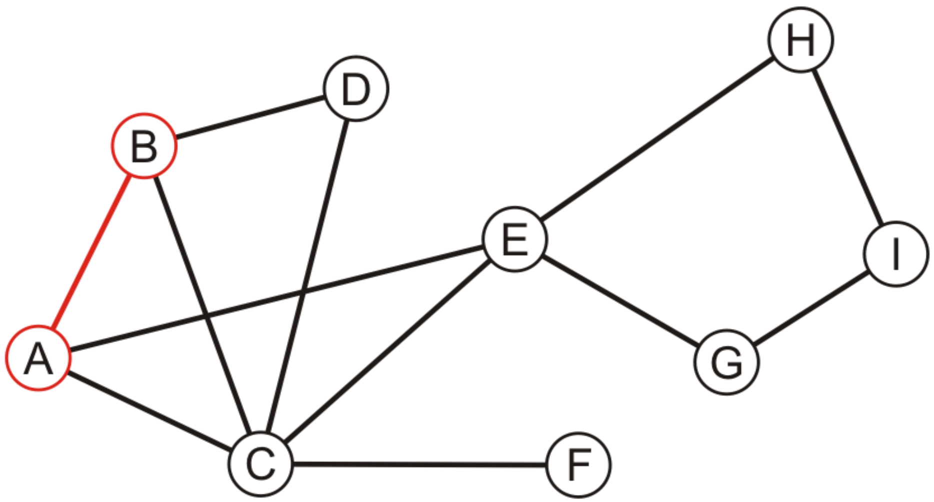


Example

Performing a recursive depth-first traversal:

– A has an unvisited neighbor

A, B

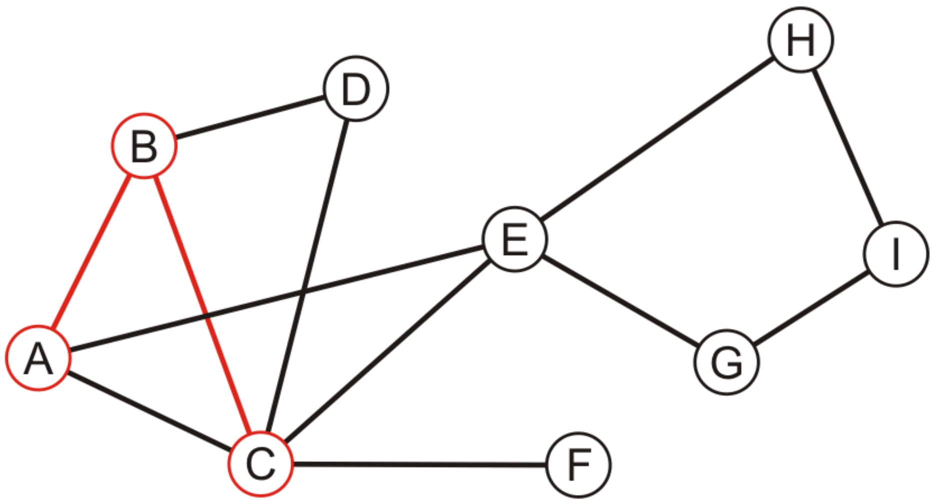


Example

Performing a recursive depth-first traversal:

- B has an unvisited neighbor

A, B, C

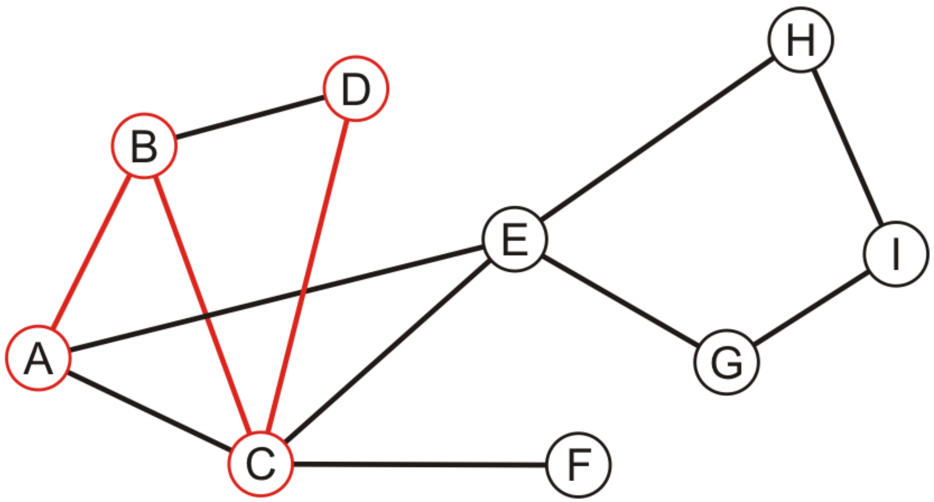


Example

Performing a recursive depth-first traversal:

– C has an unvisited neighbor

A, B, C, D

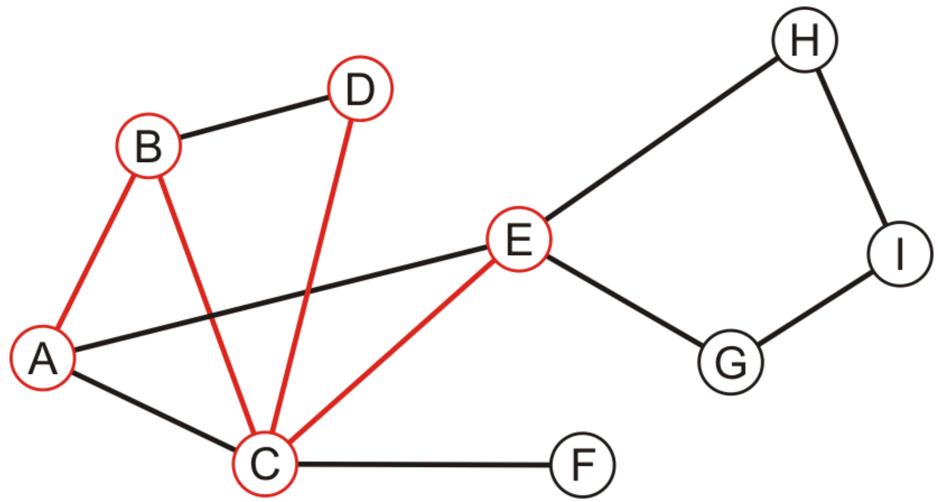


Example

Performing a recursive depth-first traversal:

- D has no unvisited neighbors, so we return to C

A, B, C, D, E

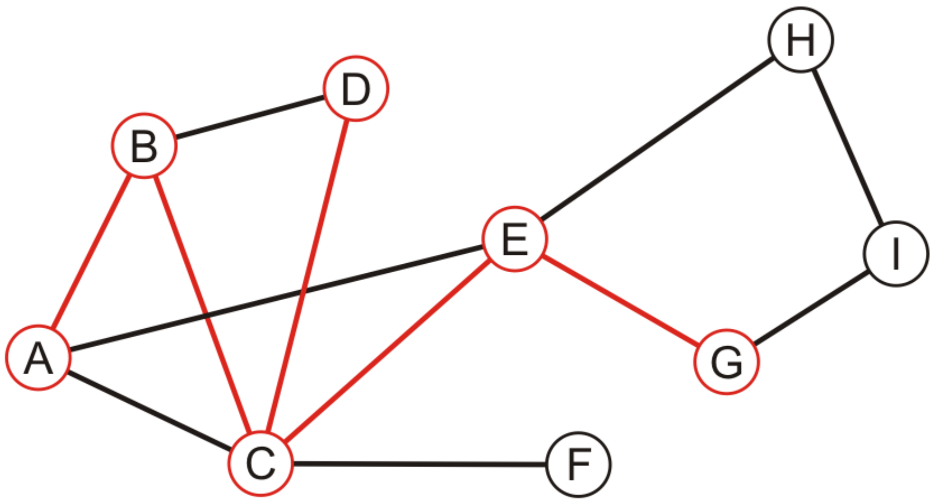


Example

Performing a recursive depth-first traversal:

- E has an unvisited neighbor

A, B, C, D, E, G

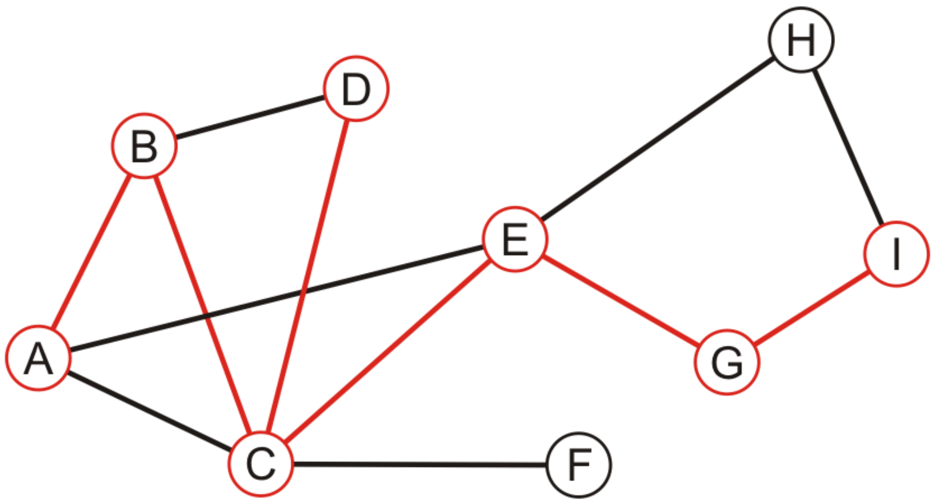


Example

Performing a recursive depth-first traversal:

- F has an unvisited neighbor

A, B, C, D, E, G, I

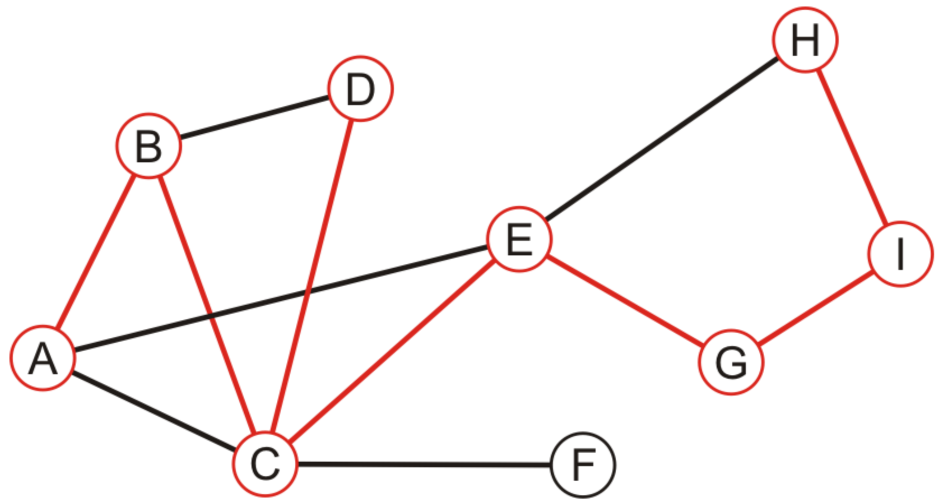


Example

Performing a recursive depth-first traversal:

- H has an unvisited neighbor

A, B, C, D, E, G, I, H

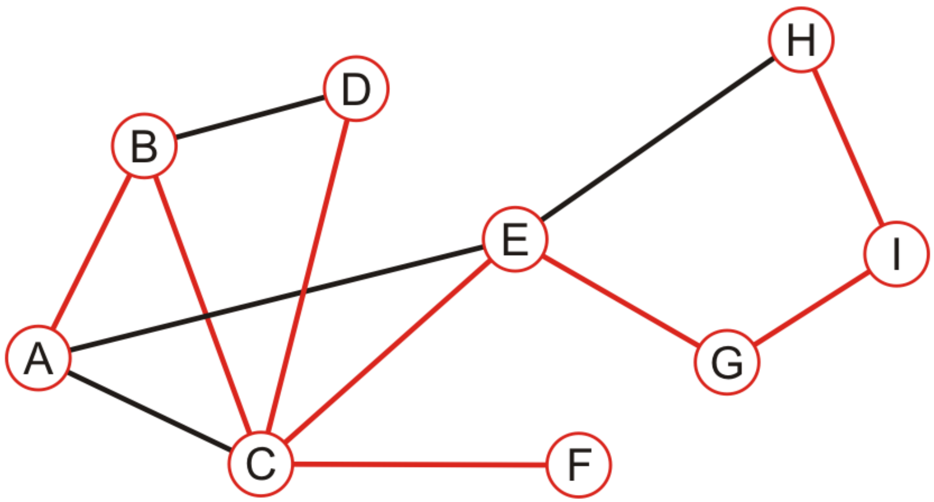


Example

Performing a recursive depth-first traversal:

– We recurse back to C which has an unvisited neighbour

A, B, C, D, E, G, I, H, F

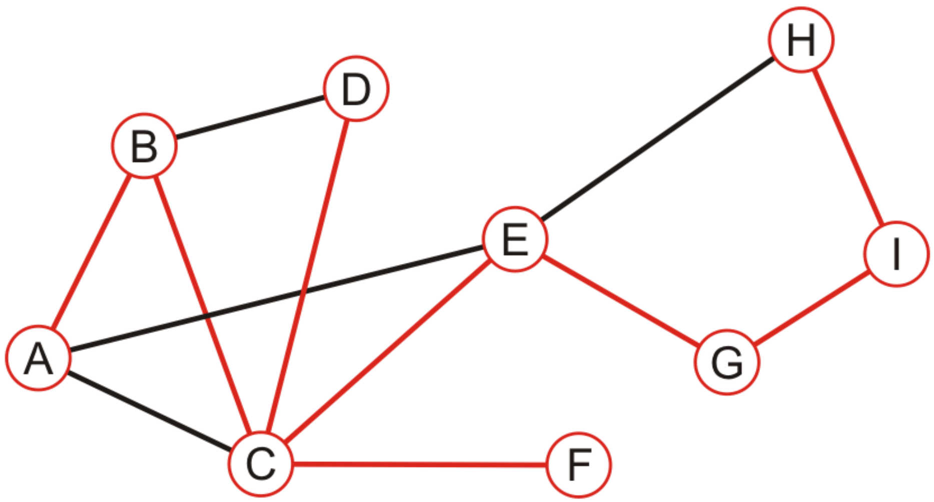


Example

Performing a recursive depth-first traversal:

- We recurse finding that no other nodes have unvisited neighbours

A, B, C, D, E, G, I, H, F

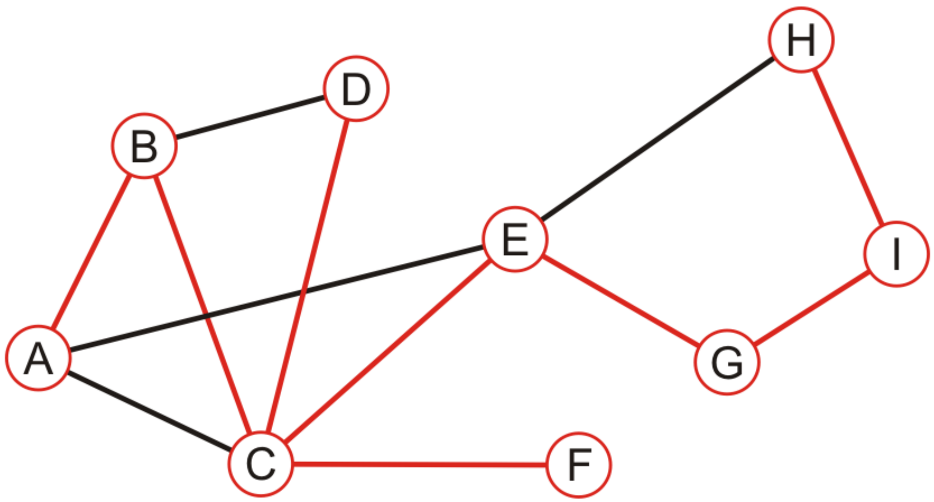


Comparison

Performing a recursive depth-first traversal:

– We recurse finding that no other nodes have unvisited neighbours

A, B, C, D, E, G, I, H, F

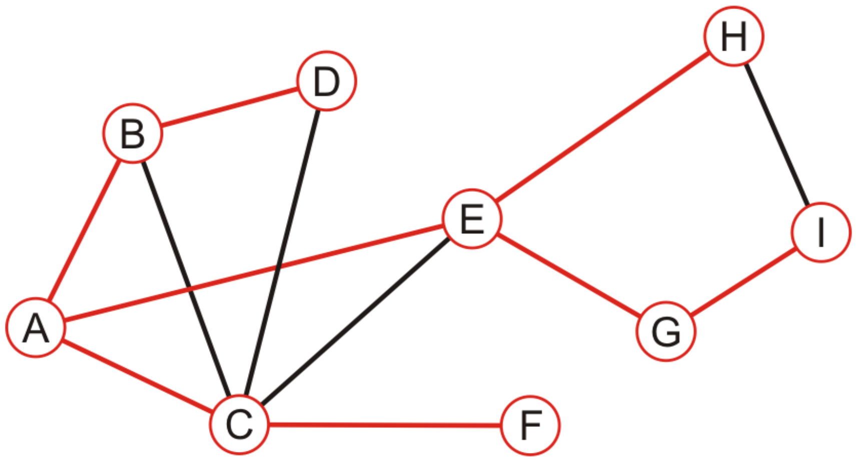


Comparison

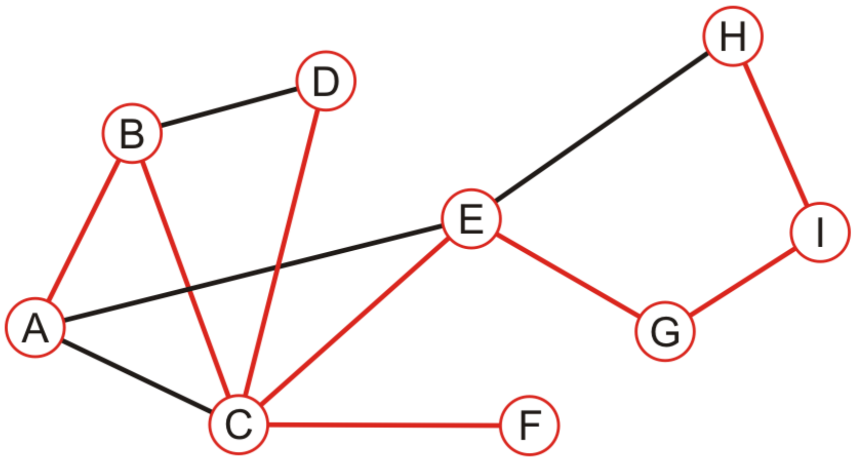
The order in which vertices can differ greatly

- An iterative depth-first traversal may also be different again

A, B, C, E, D, F, G, H, I



A, B, C, D, E, G, I, H, F



Applications

Applications of tree traversals include:

- Determining connectiveness and finding connected sub-graphs
- Determining the path length from one vertex to all others
- Testing if a graph is bipartite
- Determining maximum flow
- Cheney's algorithm for garbage collection

Summary

This topic covered graph traversals

- Considered breadth-first and depth-first traversals
- Depth-first traversals can recursive or iterative
- More overhead than traversals of rooted trees
- Considered a STL approach to the design
- Considered an example with both implementations
- They are also called *searches*

References

Wikipedia, http://en.wikipedia.org/wiki/Graph_traversal
http://en.wikipedia.org/wiki/Depth-first_search
http://en.wikipedia.org/wiki/Breadth-first_search