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Residues and Poles

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Residues

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- Singular Points
- Types of Isolated Singularities
- Zeros and Poles
- Residues
- Residue Theorem
- Residue at Poles

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Singular Points

Recall that the point $z = \alpha$ is called a singular point, or singularity of the complex function $f(z)$ if $f(z)$ is not analytic at $z = \alpha$, but every neighborhood $D_R(\alpha)$ of α contains at least one point at which $f(z)$ is analytic.

For example, the function $f(z) = \frac{1}{1-z}$ is not analytic at $z = 1$, but is analytic for all other values of z. Thus the point $z = 1$ is a singular point of $f(z)$. As another example, consider $g(z) = \mathrm{Log}(z)$. We saw that $g(z)$ is analytic for all z except at the origin and at all points on the negative real-axis. Thus, the origin and each point on the negative real axis is a singularity of $g(z) = \mathrm{Log}(z)$.

Isolated Singularity

The point α is called a isolated singularity of the complex function $f(z)$ if f is not analytic at $z = \alpha$, but there exists a real number $R > 0$ such that $f(z)$ is analytic everywhere in the punctured disk $D_R^*(\alpha)$. For example:

1.The function $f(z) = \frac{1}{1-z}$ has an isolated singularity at $z = 1$.

2. The function $g(z) = \mathrm{Log}(z)$, however, the singularity at $z = 0$ (or at any point of the negative real axis) that is not isolated, because any neighborhood of contains points on the negative real axis, and $g(z) = \mathrm{Log}(z)$ is not analytic at those points.

Functions with isolated singularities have a Laurent series because the punctured disk $D_R^*(\alpha)$ is the same as the annulus $A(\alpha, 0, R)$. The logarithm function $g(z) = \mathrm{Log}(z)$ does not have a Laurent series at any point $z = -a$ on the negative real-axis.

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Types of Isolated Singularities

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- Removable Singularity
- Pole of Order K
- Essential Singularity

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Definition

Let $f(z)$ have an isolated singularity at α with Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n \quad \text{valid for} \quad z \in A(\alpha, 0, R).$$

Then we distinguish the following types of singularities at α.

(i) **Removable Singularity:** If $c_n = 0$ for $n = -1, -2, -3, \ldots,$ then we say that $f(z)$ has a removable singularity at α.

(ii) **Pole of order k:** If k is a positive integer such that $c_{-k} \neq 0$ but $c_n = 0$ for $n = -k-1, -k-2, -k-3, \ldots,$ then we say that $f(z)$ has a pole of order k at α.

(iii) **Essential Singularity:** If $c_n \neq 0$ for infinitely many negative integers n, then we say that $f(z)$ has an essential singularity at $z = \alpha$.

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Let's investigate some examples of these three cases of isolated singularities.

1. If $f(z)$ has a removable singularity at $z = \alpha$, then it has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n \quad \text{valid for} \quad z \in A(\alpha, 0, R).$$

If we use this series to define $f(\alpha) = c_0$, then the function $f(z)$ becomes analytic at $z = \alpha$, removing the singularity.

For example, consider the function $f(z) = \frac{\sin(z)}{z}$. It is undefined at $z = 0$ and has an isolated singularity at $z = 0$, as the Laurent series for $f(z)$ is

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!} + \ldots\right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \frac{z^8}{9!} - \frac{z^{10}}{11!} + \ldots$$

valid for $|z| > 0$.

We can remove this singularity if we define $f(0) = 1$, for then $f(z)$ will be analytic at $z = 0$.

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Another example is

$g(z) = \frac{\cos(z) - 1}{z^2}$, which has an isolated singularity at the point $z = 0$, as the Laurent series for $g(z)$ is

$$g(z) = \frac{1}{z^2}\left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \ldots\right)$$

$$= -\frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \frac{z^6}{8!} - \frac{z^8}{10!} + \ldots$$

valid for $|z| > 0$. If we define $f(0) = -\frac{1}{2}$, then $g(z)$ will be analytic for all z.

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2. If $f(z)$ has a pole of order k at $z = \alpha$, the Laurent series for $f(z)$ is

$$f(z) = \sum_{n=-k}^{\infty} c_n (z-\alpha)^n \quad \text{valid for} \quad z \in A(\alpha, 0, R).$$

where $c_{-k} \neq 0$.

For example,

$$f(z) = \frac{\sin(z)}{z^3}$$

$$= \frac{1}{z^3}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!} + \ldots\right)$$

$$= \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \frac{z^6}{9!} - \frac{z^{10}}{11!} + \ldots$$

has a pole of order $k = 2$ at $z = 0$.

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If $f(z)$ has a pole of order 1 at $z = \alpha$, we say that $f(z)$ has a simple pole at $z = \alpha$.

For example,

$$
\begin{aligned}
g(z) &= \frac{1}{z} e^{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{n!} z^{n} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} z^{n-1} \\
&= \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \frac{z^4}{5!} + \frac{z^5}{6!} + \ldots
\end{aligned}
$$

has a simple pole at $z = 0$.

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3. If infinitely many negative powers of $(z - \alpha)$ occur in the Laurent series, then $f(z)$ has an essential singularity at $z = \alpha$. For example,

$$f(z) = z^2 \sin\left(\frac{1}{z}\right)$$

$$= z^2 \left(\frac{1}{z} - \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^5}{5!} - \frac{\left(\frac{1}{z}\right)^7}{7!} + \frac{\left(\frac{1}{z}\right)^9}{9!} - \frac{\left(\frac{1}{z}\right)^{11}}{11!} + \ldots\right)$$

$$= z - \frac{1}{3!} z^{-1} + \frac{1}{5!} z^{3} - \frac{1}{7!} z^{-5} + \frac{1}{9!} z^{-7} - \frac{1}{11!} z^{-9} + \ldots$$

has an essential singularity at the origin.

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Zeros

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Definition (Zero of order k).
A function $f(z)$ analytic in $D_r(\alpha)$ has a zero of order k at the point $z = \alpha$ if and only if

$$f^{(n)}(\alpha) = 0 \text{ for } n = 0, 1, 2, \ldots, k-1, \text{ and } f^{(k)}(\alpha) \neq 0.$$

A zero of order one is sometimes called a simple zero.

Theorem
A function $f(z)$ analytic in $D_R(\alpha)$ has a zero of order k at the point $z = \alpha$ iff its Taylor series given by $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ has

$$c_0 = c_1 = \ldots = c_{k-1} = 0 \text{ and } c_k \neq 0.$$

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Example

From we see that the function

$$f(z) = z\sin\left(z^2\right) = z^3 - \frac{z^7}{3!} + \frac{z^{11}}{5!} - \frac{z^{15}}{7!} + \ldots$$

has a zero of order $k = 3$ at $z = 0$. Definition confirms this fact because

$$f'(z) = 2z^2 \operatorname{Cos} z^2 + \operatorname{Sin} z^2$$

$$f''(z) = 6z \operatorname{Cos} z^2 - 4z^3 \operatorname{Sin} z^2$$

$$f'''(z) = 6 \operatorname{Cos} z^2 - 8z^4 \operatorname{Cos} z^2 - 24z^2 \operatorname{Sin} z^2$$

Then, $f(0) = f''(0) = f''(0) = 0$, but $f'''(0) = 6 \neq 0$.

Poles

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Theorem A function $f(z)$ analytic in the punctured disk $D_R^*(\alpha)$ has a pole of order k at $z = \alpha$ if and only if it can be expressed in the form

$$f(z) = \frac{h(z)}{(z-\alpha)^k},$$

where the function $h(z)$ is analytic at the point $z = \alpha$ and $h(\alpha) \neq 0$.

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Corollary If $f(z)$ is analytic and has a zero of order k at the point $z = \alpha$, then $g(z) = \frac{1}{f(z)}$ has a pole of order k at $z = \alpha$.

Corollary If $f(z)$ has a pole of order k at the point $z = \alpha$, then $g(z) = \frac{1}{f(z)}$ has a removable singularity at $z = \alpha$. If we define $g(\alpha) = 0$, then $g(z)$ has a zero of order k at $z = \alpha$.

Corollary If $f(z)$ and $g(z)$ have poles of orders m and n, respectively at the point $z = \alpha$, then their product $h(z) = f(z)\, g(z)$ has a pole of order $m + n$ at $z = \alpha$.

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Corollary Let $f(z)$ and $g(z)$ be analytic with zeros of orders m and n, respectively at $z = \alpha$. Then their quotient $h(z) = \frac{f(z)}{g(z)}$ has the following behavior:

(i) If $m > n$, then $h(z)$ has a removable singularity at $z = \alpha$.
If we define $h(\alpha) = 0$, then $h(z)$ has a zero of order $m - n$ at $z = \alpha$.

(ii) If $m < n$, then $h(z)$ has a pole of order $n - m$ at $z = \alpha$.

(iii) If $m = n$, then $h(z)$ has a removable singularity at $z = \alpha$, and can be defined so that $h(z)$ is analytic at $z = \alpha$, by $h(\alpha) = \lim_{z \to \alpha} h(z)$.

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Residues

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The Residue Theorem

We now have the necessary machinery to see some amazing applications of the tools we developed in the last few chapters. You will learn how Laurent expansions can give useful information concerning seemingly unrelated properties of complex functions. You will also learn how the ideas of complex analysis make the solution of very complicated integrals of real-valued functions as easy - literally - as the computation of residues. We begin with a theorem relating residues to the evaluation of complex integrals.

The Cauchy integral formulae are useful in evaluating contour integrals over a simple closed contour C where the integrand has the form $\frac{f(z)}{(z - z_0)^k}$ and f is an analytic function. In this case, the singularity of the integrand is at worst a pole of order k at z_0. We begin this section by extending this result to integrals that have a finite number of isolated singularities inside the contour C.

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Definition (Residue). Let $f(z)$ have a nonremovable isolated singularity at the point z_0. Then $f(z)$ has the Laurent series representation for all z in some disk $D_R^*(z_0)$ given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

The coefficient a_{-1} of $\frac{1}{z - z_0}$ is called the residue of $f(z)$ at z_0 and we use the notation

$$\text{Res}[f, z_0] = a_{-1}.$$

$$\int_C f(z)\, dz = 2 \pi i\, a_{-1} = 2 \pi i\, \text{Res}[f, z_0].$$

Example If $f(z) = e^{\frac{2}{z}}$, then the Laurent series of f about the point $z_0 = 0$ has the form

$$f(z) = 1 + 2 \frac{1}{z} + \frac{2^2}{2!\, z^2} + \frac{2^3}{3!\, z^3} + \frac{2^4}{4!\, z^4} + \frac{2^5}{5!\, z^5} + \ldots,$$ and

$$\text{Res}[f, 0] = a_{-1} = 2.$$

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Example

Find $\text{Res}[g, z_0]$ if $g(z) = \frac{3}{2z + z^2 - z^3}$.

Solution. we find that $g(z)$ has three Laurent series representations involving powers of z. The Laurent series valid in the punctured disk $D_1^*(0) = \{z : 0 < |z| < 1\}$ is

$$g(z) = \sum_{n=0}^{\infty} \left((-1)^n + \frac{1}{2^{n+1}} \right) z^{n-1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} + 1}{2^{n+1}} z^{n-1}.$$

Computing the first few coefficients, we obtain

$$\begin{aligned} g(z) &= \frac{(-1)^0 2^1 + 1}{2} \frac{1}{z} + \frac{(-1)^1 2^2 + 1}{2^2} + \frac{(-1)^2 2^3 + 1}{2^3} z + \frac{(-1)^3 2^4 + 1}{2^4} z^2 \\ &\quad + \frac{(-1)^4 2^5 + 1}{2^5} z^3 + \frac{(-1)^5 2^6 + 1}{2^6} z^4 + \frac{(-1)^n 2^{n+1} + 1}{2^{n+1}} z^{n-1} + \ldots \\ &= \frac{3}{2} \frac{1}{z} - \frac{3}{2^2} + \frac{9}{2^3} z - \frac{15}{2^4} z^2 + \frac{33}{2^5} z^3 - \frac{63}{2^6} z^4 + \ldots \end{aligned}$$

Therefore, $\text{Res}[f, 0] = a_{-1} = \frac{3}{2}$.

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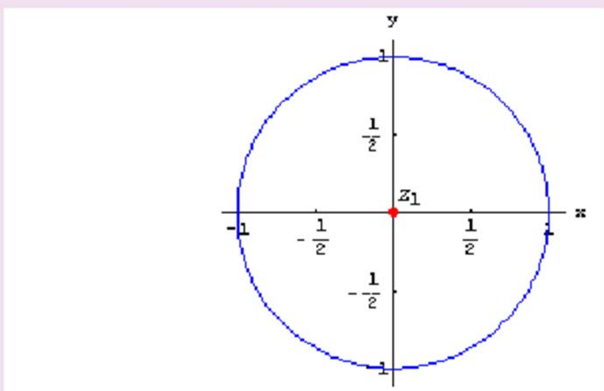
Example

Evaluate $\int_{C_1^+(0)} \exp\left(\frac{2}{z}\right) dz$ where $C_1^+(0)$ denotes the circle $C_1^+(0) = \{z : |z| = 1\}$ with positive orientation.



Solution. we showed that the residue of $f(z) = e^{\frac{2}{z}}$ at $z_0 = 0$ is $\text{Res}[f, 0] = 2$. We get

$$\int_{C_1^+(0)} \exp\left(\frac{2}{z}\right) dz = 2\pi i \,\text{Res}[f, 0] = 4\pi i.$$

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Residues Theorem

Theorem (Cauchy's Residue Theorem). Let D be a simply connected domain, and let C be a simple closed positively oriented contour that lies in D. If $f(z)$ is analytic inside C and on C, except at the points $z_1, z_2, \ldots, z_n$ that lie inside C, then

$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}[f, z_k].$$

The situation is illustrated in Figure below.

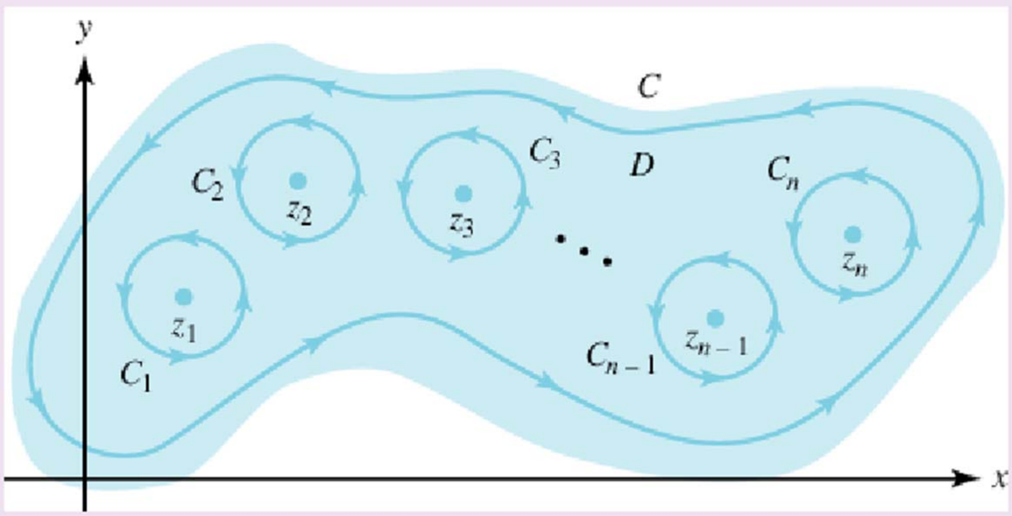

Figure The domain D and contour C and the singular points $z_1, z_2, \ldots, z_n$ in the statement of Cauchy's residue theorem.

Residue at Poles

An isolated singular point of a function f is pole of order m if and only if f(z) can be written in the form

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

where $\phi(z)$ is analytic and nonzero at z_0, Moreover,

$\mathrm{Res}[f, z_0] = \phi(z_0)$ if $m = 1$.

$\mathrm{Res}[f, z_0] = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geqslant 2$.

Theorem (Residues at Poles).

(i) If $f(z)$ has a simple pole at z_0, then $\mathrm{Res}[f, z_0] = \lim_{z\to z_0} (z - z_0) f(z)$.

(ii) If $f(z)$ has a pole of order 2 at z_0, then $\mathrm{Res}[f, z_0] = \lim_{z\to z_0} \frac{d}{dz}\left((z - z_0)^2 f(z)\right)$.

(iii) If $f(z)$ has a pole of order 3 at z_0, then $\mathrm{Res}[f, z_0] = \frac{1}{2!} \lim_{z\to z_0} \frac{d^2}{dz^2}\left((z - z_0)^3 f(z)\right)$.

(v) If $f(z)$ has a pole of order k at z_0, then $\mathrm{Res}[f, z_0] = \frac{1}{(k-1)!} \lim_{z\to z_0} \frac{d^{k-1}}{dz^{k-1}}\left((z - z_0)^k f(z)\right)$.

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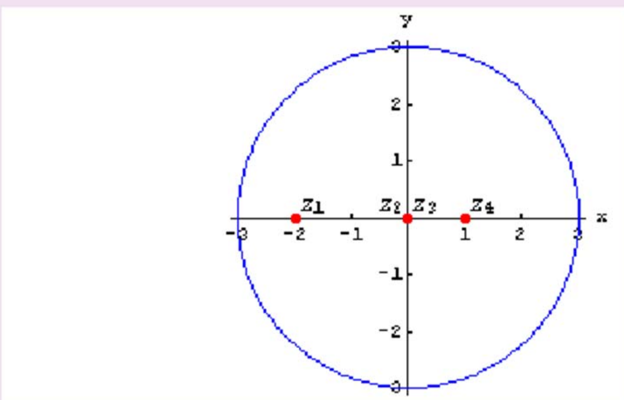
Example Find $\int_{C_3^+(0)} \frac{1}{z^4 + z^3 - 2z^2} \, dz$ where $C_3^+(0)$ denotes the circle $C_3^+(0) = \{z : |z| = 3\}$ with positive orientation.



Solution. We write the integrand as $f(z) = \frac{1}{z^4 + z^3 - 2z^2} = \frac{1}{z^2 (z+2)(z-1)}$.

The singularities of $f(z)$ that lie inside $C_3^+(0)$ are simple poles at the points $z = 1$ and $z = -2$, and a pole of order 2 at the origin.

We compute the residues as follows:

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$$\text{Res}[f, 0] = \lim_{z\to 0} \frac{d}{dz}\left(z^2 f(z)\right) = \lim_{z\to 0} \frac{d}{dz}\left(z^2 \frac{1}{z^2 (z+2)(z-1)}\right)$$

$$= \lim_{z\to 0} \frac{d}{dz}\left(\frac{1}{(z+2)(z-1)}\right) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{1}{z^2+z-2}\right)$$

$$= \lim_{z\to 0} \frac{-2z-1}{(z^2+z-2)^2} = -\frac{1}{4}$$

$$\text{Res}[f, 1] = \lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} (z-1) \frac{1}{z^2 (z+2)(z-1)}$$

$$= \lim_{z\to 1} \frac{1}{z^2 (z+2)} = \frac{1}{3}$$

$$\text{Res}[f, -2] = \lim_{z\to -2} (z-(-2)) f(z) = \lim_{z\to -2} (z+2) \frac{1}{z^2 (z+2)(z-1)}$$

$$= \lim_{z\to -2} \frac{1}{z^2 (z-1)} = -\frac{1}{12}$$

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Finally, the residue theorem yields

$$\int_{C_3^+(0)} \frac{1}{z^4 + z^3 - 2z^2}\, dz = 2\pi i\,(\mathrm{Res}[f, 0] + \mathrm{Res}[f, 1] + \mathrm{Res}[f, -2])$$

$$= 2\pi i \left(-\frac{1}{4} + \frac{1}{3} - \frac{1}{12}\right)$$

$$= 0$$

The answer, $\int_{C_3^+(0)} \frac{1}{z^4 + z^3 - 2z^2}\, dz = 0$, is not at all obvious, and all the preceding calculations are required to get it.

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