Function of a Complex Variable

Function of a complex variable

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- Introduction of function of a complex variable
- Limits
- Continuity
- Derivatives and Differentiability
- Analytic Functions
- Harmonic Functions
- Elementary Functions

Introduction of function of a complex variable

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We have talked about a basic theory of complex numbers. For the next few lectures we turn our attention to functions of complex numbers. They are defined in a similar way to functions of real numbers that you studied in calculus; the only difference is that they operate on complex numbers rather than real numbers. This section focuses primarily on very basic functions, their representations, and properties associated with functions such as limits, continuity and derivatives.

Functions and Linear Mappings

A complex-valued function f of the complex variable z is a rule that assigns to each complex number z in a set D one and only one complex number w. We write $w = f(z)$ and call w the image of z under f.

A simple example of a complex-valued function is given by the formula $w = f(z) = z^4$. The set D is called the domain of f, and the set of all images $\{w = f(z) : z \in D\}$ is called the range of f.

We can define the domain to be any set that makes sense for a given rule, so for $w = f(z) = z^4$, we could have the entire complex plane for the domain D, or we might artificially restrict the domain to some set such as $D = D_1(0) = \{z : |z| < 1\}$. Determining the range for a function defined by a formula is not always easy, but we will see plenty of examples later on. In some contexts functions are referred to as mappings or transformations.

We used the term domain to indicate a connected open set. When speaking about the domain of a function, however, we mean only the set of points on which the function is defined. This distinction is worth noting, and context will make clear the use intended.

Just as z can be expressed by its real and imaginary parts, $z = x + i\,y$, we write $f(z) = w = v + i\,v$, where u and v are the real and imaginary parts of w, respectively. Doing so gives us the representation

$$w = f(z) = f(x, y) = f(x + i\,y) = u + i\,v.$$

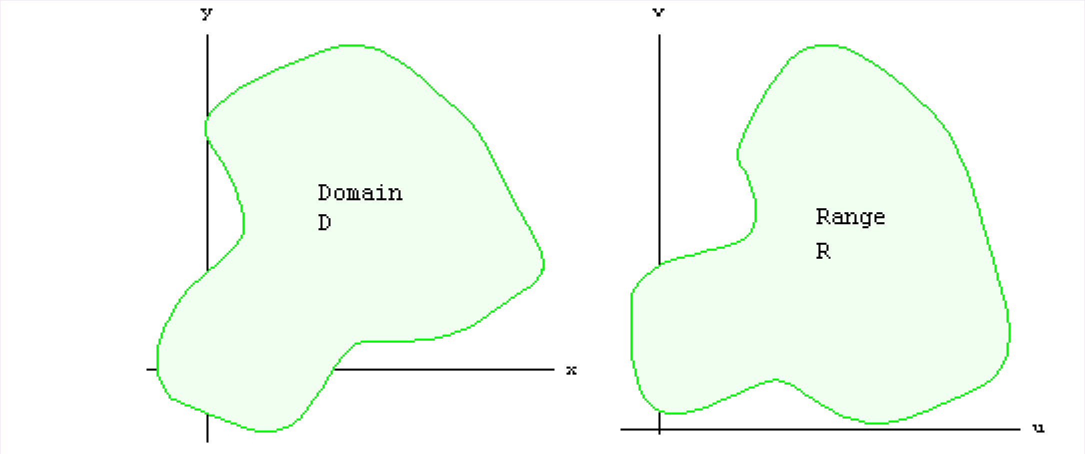
Because u and v depend on x and y, they can be considered to be real-valued functions of the real variables x and y; that is,

$$u = u(x, y) \text{ and } v = v(x, y).$$

Combining these ideas, we often write a complex function f in the form

$$f(z) = f(x + i\,y) = u(x, y) + i\,v(x, y).$$

Illustration of the notion of a function (mapping) using these symbols.



The mapping $w = f(z) = u(x, y) + i\,v(x, y)$.

We now give several examples that illustrate how to express a complex function.

Example

Write $f(z) = z^4$ in the for $f(z) = u(x, y) + i\, v(x, y)$.

Solution. Using the binomial formula, we obtain

$$
\begin{aligned}
f(z) = f(x + i\, y) &= (x + i\, y)^4 \\
&= x^4 + 4\, x^3 (i\, y) + 6\, x^2 (i\, y)^2 + 4\, x\, (i\, y)^3 + (i\, y)^4 \\
&= x^4 + 4\, i\, x^3\, y - 6\, x^2\, y^2 - 4\, i\, x\, y^3 + y^4 \\
&= x^4 - 6\, x^2\, y^2 + y^4 + i\left(4\, x^3\, y - 4\, x\, y^3\right) \\
&= u(x, y) + i\, v(x, y)
\end{aligned}
$$

so that $u(x, y) = x^4 - 6\, x^2\, y^2 + y^4$ and $v(x, y) = 4\, x^3\, y - 4\, x\, y^3$.

Example

Express the function $f(z) = \overline{z}\,\mathrm{Re}[z] + z^2 + \mathrm{Im}[z]$ in the form $f(z) = u(x, y) + i\, v(x, y)$.

Solution. Using the elementary properties of complex numbers, it follows that

$$f(z) = (x - i\,y)\,x + (x + i\,y)^2 + y = \left(2\,x^2 - y^2 + y\right) + i\,(x\,y)$$

so that $u(x, y) = 2\,x^2 - y^2 + y$ and $v(x, y) = x\,y$.

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Examples show how to find $u(x,y)$ and $v(x,y)$ when a rule for computing f is given. Conversely, if $u(x,y)$ and $v(x,y)$ are two real-valued functions of the real variables x and y, they determine a complex-valued function $f(z) = u(x, y) + i\, v(x, y)$, and we can use the formulas

$$x = \frac{z+\bar{z}}{2} \quad \text{and} \quad y = \frac{z-\bar{z}}{2i}$$

to find a formula for f involving the variables z and $\bar{z}$.

Example Express $f(z) = 4x^2 + i\,4y^2$ by a formula involving the variables z and $\bar{z}$.

Solution. Calculation reveals that

$$f(z) = 4\left(\frac{z+\bar{z}}{2}\right)^2 + i\,4\left(\frac{z-\bar{z}}{2i}\right)^2$$

$$= z^2 + 2z\bar{z} + \bar{z}^2 - i\left(z^2 - 2z\bar{z} + \bar{z}^2\right)$$

$$= (1-i)\,z^2 + (2+2i)\,z\bar{z} + (1-i)\,\bar{z}^2$$

Using $z = r e^{i\theta}$ in the expression of a complex function f may be convenient. It gives us the polar representation

$$f(z) = f\left(r e^{i\theta}\right) = U(r, \theta) + i V(r, \theta),$$

where U and V are real functions of the real variables r and θ.

Remark. For a given function f, the functions u and v defined above are different from those used previously in $f(z) = f(x + i y) = u(x, y) + i v(x, y)$ which used Cartesian coordinates instead of polar coordinates.

Example Express $f(z) = z^2$ in both Cartesian and polar form.

Solution. For the Cartesian form, a simple calculation gives

$$f(z) = f(x + i y) = (x + i y)^2$$

$$= x^2 - y^2 + 2 i x y$$

$$= u(x, y) + i v(x, y)$$

so that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2 x y$.

For the polar form, we get v

$$f(z) = f\left(r e^{i\theta}\right) = \left(r e^{i\theta}\right)^2 = r^2 e^{i 2\theta}$$

$$= r^2 (\cos 2\theta + i \sin 2\theta)$$

$$= r^2 \cos 2\theta + i r^2 \sin 2\theta$$

$$= U(r, \theta) + i V(r, \theta)$$

so that $U(r, \theta) = r^2 \cos 2\theta$ and $V(r, \theta) = r^2 \sin 2\theta$.

Example Express $f(z) = z^5 + 4z^2 - 6$ in polar form.

Solution. We obtain

$$f(z) = f\left(r e^{i\theta}\right) = \left(r e^{i\theta}\right)^5 + 4\left(r e^{i\theta}\right)^2 - 6$$

$$= r^5 e^{i5\theta} + 4r^2 e^{i2\theta} - 6$$

$$= r^5 \cos 5\theta + 4r^2 \cos 2\theta - 6 + i\left(r^5 \sin 5\theta + 4r^2 \sin 2\theta\right)$$

$$= U(r, \theta) + i V(r, \theta)$$

so that $U(r, \theta) = r^5 \cos 5\theta + 4r^2 \cos 2\theta - 6$ and $V(r, \theta) = r^5 \sin 5\theta + 4r^2 \sin 2\theta$.

Functions and Linear Mappings Cont.

We now look at the geometric interpretation of a complex function. If D is the domain of real-valued functions $u(x,y)$ and $v(x,y)$, the equations

$$u = u(x, y) \text{ and } v = v(x, y)$$

describe a transformation (or mapping) from D in the xy plane into the uv plane, also called the w plane. Therefore, we can also consider the function

$$w = f(z) = u(x, y) + i v(x, y)$$

to be a transformation (or mapping) from the set D in the z plane onto the range R in the w plane.

If A is a subset of the domain D of f, the set $B = \{w = f(z) : z \in A\}$ is called the image of the set A, and f is said to map A onto B. The image of a single point is a single point, and the image of the entire domain, D, is the range, R. The mapping $w = f(z)$ is said to be from A into S if the image of A is contained in S. Figure illustrates a function f whose domain is D and whose range is R. The shaded areas depict that the function maps A onto B. The function also maps A into R, and, of course, it maps D onto R.

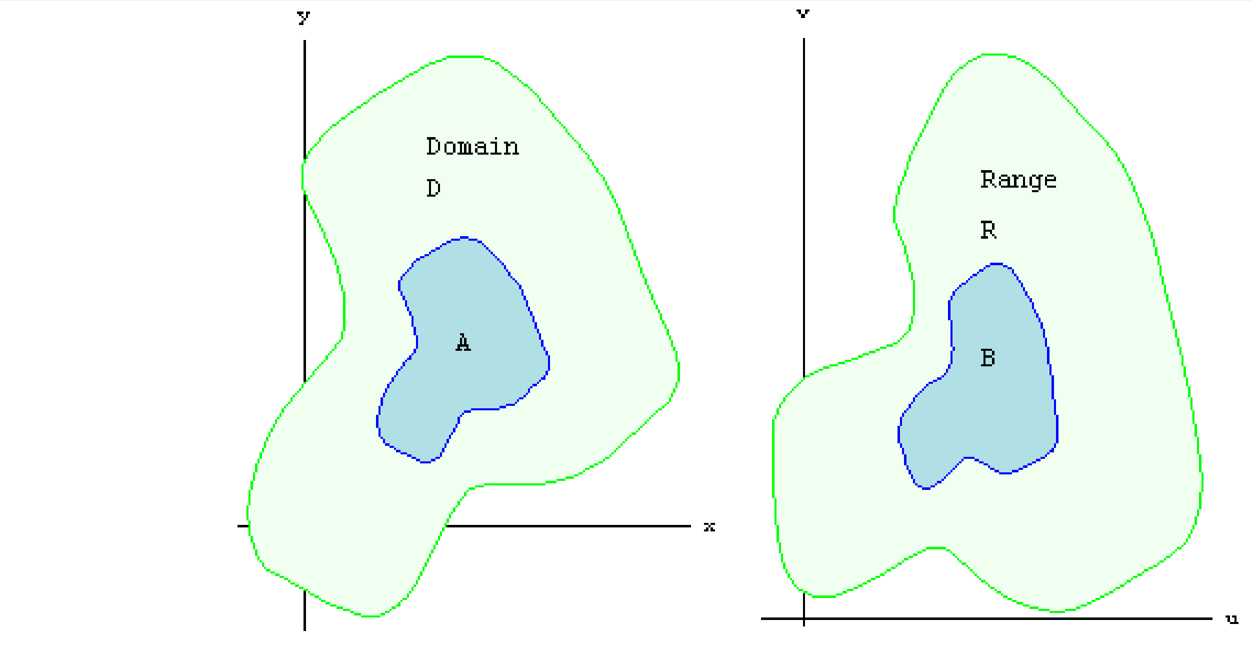

Figure $w = f(z)$ maps A onto B; $w = f(z)$ maps A into R.

The inverse image of a point w is the set of all points z in D such that $w = f(z)$. The inverse image of a point may be one point, several points, or nothing at all. If the last case occurs then the point w is not in the range of f.

For example, if $w = f(z) = i z$, the inverse image of the point -1 is the single point i, because $w = f(i) = i(i) = -1$, and i is the only point that maps to -1.
In the case of $w = f(z) = z^2$, the inverse image of the point -1 is the set $\{i, -i\}$.
if $w = f(z) = e^z$, the inverse image of the point 0 is the empty set---there is no complex number z such that $e^z = 0$.

Figure illustrates the idea of a one-to-one function: distinct points get mapped to distinct points.

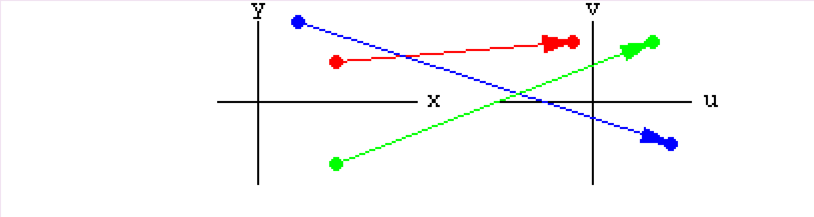

Figure A function $w = f(z)$ that is one-to-one.

The function $f(z) = z^2$ is not one-to-one because $-i \neq i$, but $f(i) = f(-i) = -1$.

Figure depicts this situation: at least two different points get mapped to the same point.

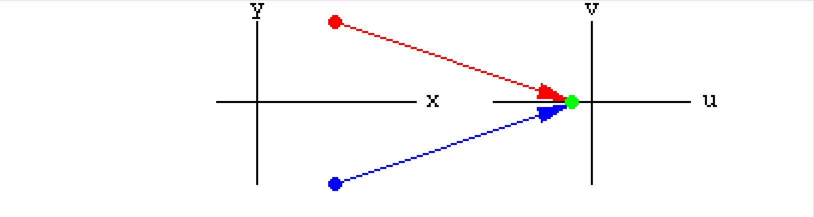

Figure A function that is not one-to-one.

Example If $w = f(z) = i\,z$ for any complex number z, find $f^{-1}(w)$.

Solution. We can easily show f is one-to-one and onto the entire complex plane. We solve for z, given $w = f(z) = i$, to get $z = \frac{w}{i} = -i$
This result implies that $f^{-1}(w) = -i\,w$ for all complex numbers w.

Remark. Once we have specified $f^{-1}(w) = -i\,w$ for all complex numbers w, we note that there is nothing magical about the symbol w. We could just as easily write $f^{-1}(z) = -i\,z$ for all complex numbers z.

Example Show that the function $f[z] = i\,z$ maps the line $y = x + 1$ in the xy plane onto the line $v = -u - 1$ in the w plane.

We write $u + i\,v = w = f(z) = i\,(x + i\,y) = -y + i\,x$ and note that the transformation can be given by the equations $u = -y$ and $v = x$.

Because A is described by $A = \{x + i\,y : y = x + 1\}$, we can substitute $u = -y$ and $v = x$ into the equation $y = x + 1$ to obtain $-u = v + 1$, which we can rewrite as $v = -u - 1$.

Limits and Continuity

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Let $u = u(x, y)$ be a real-valued function of the two real variables x and y. Recall that u has the limit u_0 as (x, y) approaches (x_0, y_0) provided that the value of $u(x, y)$ can be made to get as close as we please to the value u_0 by taking (x, y) to be sufficiently close to (x_0, y_0). When this happens we write

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0.$$

In more technical language, u has the limit u_0 as (x, y) approaches (x_0, y_0) iff $|u(x, y) - u_0|$ can be made arbitrarily small by making both $|x - x_0|$ and $|y - y_0|$ small. This condition is like the definition of a limit for functions of one variable.

Definition (limit of u(x,y)). The expression $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$ means that for each number $\epsilon > 0$, there corresponds a number $\delta > 0$ such that

$$|u(x, y) - u_0| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

Example Show, if $u(x, y) = \frac{2x^3}{x^2 + y^2}$, then $\lim_{(x,y)\to(0,0)} u(x, y) = 0$.

Solution. If $x = r\cos\theta$ and $y = r\sin\theta$, and $z = x + i\,y \neq 0$ then

$$u(x, y) = \frac{2 r^3 \cos^3\theta}{r^2 \cos^2\theta + r^2 \sin^2\theta} = 2 r \cos^3\theta.$$

Because $\sqrt{(x-0)^2 + (y-0)^2} = r$ and because $|\cos^3\theta| < 1$, we have

$$|u(r, \theta)| = 2 r\,|\cos^3\theta| < \epsilon \text{ whenever } \sqrt{(x-0)^2 + (y-0)^2} = r < \frac{\epsilon}{2}.$$

Hence, for any $\epsilon > 0$, Inequality is satisfied for $\delta = \frac{\epsilon}{2}$; that is, $u(x, y)$ has the limit $u_0 = 0$ as (x, y) approaches $(0, 0)$.

Remark

The value u_0 of the limit must not depend on how (x, y) approaches (x_0, y_0), so $u(x, y)$ must approach the value u_0 when (x, y) approaches (x_0, y_0) along any curve that ends at the point (x_0, y_0). Conversely, if we can find two curves C_1 and C_2 that end at (x_0, y_0) along which $u(x, y)$ approaches the two distinct values u_1 and u_2, respectively, then $u(x, y)$ does not have a limit as (x, y) approaches (x_0, y_0).

Example Show that the function $u(x, y) = \frac{x\,y}{x^2 + y^2}$ does not have a limit as (x, y) approaches $(0, 0)$.

Solution. If we let (x, y) approach $(0, 0)$ along the x axis, then

$$\lim_{(x,0)\to(0,0)} u(x, y) = \lim_{(x,0)\to(0,0)} \frac{(x)\,(0)}{x^2 + 0^2} = 0.$$

But if we let (x, y) approach $(0, 0)$ along the line $y = x$, then

$$\lim_{(x,x)\to(0,0)} u(x, y) = \lim_{(x,x)\to(0,0)} \frac{(x)\,(x)}{x^2 + x^2} = \frac{1}{2}.$$

Because the value of the limit differs depending on how (x, y) approaches $(0, 0)$, we conclude that $u(x, y)$ does not have a limit as (x, y) approaches $(0, 0)$.

Limits

Definition (limit of f(z)).

The expression $\lim_{z \to z_0} f(z) = w_0$ means that for each number $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

we can also express the last relationship as

$$f(z) \in D_\epsilon(w_0) \text{ whenever } z \in D_\delta^*(z_0).$$

The formulation of limits in terms of open disks provides a good context for looking at this definition. It says that for each disk of radius $\epsilon > 0$ about the point w_0 (represented by $D_\epsilon(w_0)$) there is a punctured disk of radius $\delta > 0$ about the point z_0 (represented by $D_\delta^*(z_0)$) such that the image of each point in the punctured δ-disk lies in the ϵ-disk. The image of the δ-disk does not have to fill up the entire ϵ-disk; but if z approaches z_0 along a curve that ends at z_0, then $w = f(z)$ approaches w_0. The situation is illustrated in Figure below.

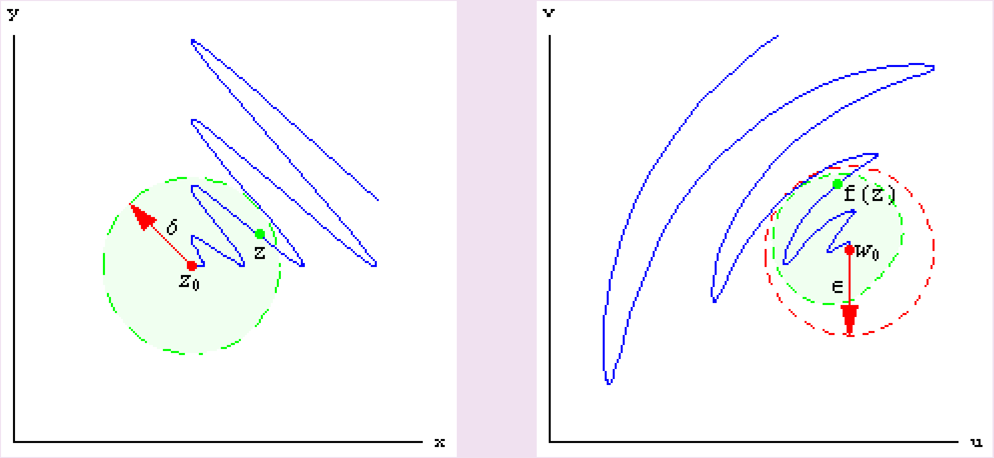

Figure The limit $f(z) \to w_0$ as $z \to z_0$.

Remark

Let $f(z) = u(x, y) + i\,v(x, y)$ be a complex function that is defined in some neighborhood of z_0, except perhaps at $z_0 = x_0 + i\,y_0$. Then

$$\lim_{z \to z_0} f(z) = w_0 = u_0 + i\,v_0$$

iff both

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0.$$

Example. Show that $\lim_{z\to 1+i} (z^2 - 2z + 1) = -1$.

Solution. We have

$$f(z) = 1 - 2z + z^2$$

$$f(x + i\,y) = (x^2 - y^2 - 2x + 1) + i\,(2xy - 2y)$$

$$u(x, y) = x^2 - y^2 - 2x + 1$$

$$v(x, y) = 2xy - 2y$$

Computing the limits for u and v, we obtain

$$\lim_{(x,y)\to(1,1)} u(x, y) = \lim_{(x,y)\to(1,1)} (x^2 - y^2 - 2x + 1) = -1, \text{ and}$$

$$\lim_{(x,y)\to(1,1)} v(x, y) = \lim_{(x,y)\to(1,1)} (2xy - 2y) = 0,$$

so our previous theorem implies that $\lim_{z\to 1+i} f(z) = -1$.

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Theorem

Limits of complex functions are formally the same as those of real functions, and the sum, difference, product, and quotient of functions have limits given by the sum, difference, product, and quotient of the respective limits. We state this result as a theorem and leave the proof as an exercise.

Suppose that $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} g(z) = B$. Then

$\lim_{z\to z_0} f(z) + g(z) = A + B$, and

$\lim_{z\to z_0} f(z) - g(z) = A - B$,

$$\lim_{z\to z_0} f(z)\, g(z) = A\, B,$$

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{A}{B} \quad \text{where } B \neq 0.$$

Theorem

Let $f(z) = u(x, y) + i\, v(x, y)$ be a defined in some neighborhood of z_0. Then $f(z)$ is continuous at $z_0 = x_0 + i\, y_0$ iff $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0).

Example

Show that $\lim_{z \to 1+i} \frac{z^2 - 2i}{z^2 - 2z + 2} = 1 - i$.

Solution. Here P and Q can be factored in the form

$$P(z) = (z - 1 - i)(z + 1 + i),$$

and

$$Q(z) = (z - 1 - i)(z - 1 + i).$$

so that the limit is obtained by the calculation

$$\lim_{z \to 1+i} \frac{z^2 - 2i}{z^2 - 2z + 2} = \lim_{z \to 1+i} \frac{(z-1-i)(z+1+i)}{(z-1-i)(z-1+i)}$$

$$= \lim_{z \to 1+i} \frac{z+1+i}{z-1+i}$$

$$= \frac{(1+i)+1+i}{(1+i)-1+i}$$

$$= \frac{2+2i}{2i}$$

$$= 1 - i$$

Continuity

Definition (continuity of f(z)).

Let $f(z)$ be a complex function of the complex variable z that is defined for all values of z in some neighborhood of z_0. We say that f is continuous at z_0 if three conditions are satisfied:

$$\lim_{z\to z_0} f(z) \text{ exists},$$

$$f(z_0) \text{ exists},$$

$$\lim_{z\to z_0} f(z) = f(z_0).$$

Theorem

Suppose that $f(z)$ and $g(z)$ are continuous at the point z_0. Then the following functions are continuous at z_0.

The sum $f(z) + g(z)$,

The difference $f(z) - g(z)$,

The product $f(z)\, g(z)$,

The quotient $\frac{f(z)}{g(z)}$, provided that $g(z_0) \neq 0$.

The composition $f(g(z))$, provided that $f(z)$ is continuous in a neighborhood of the point $g(z_0)$.

Derivative

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Derivative

Using our imagination, we take our lead from elementary calculus and define the derivative of $f(z)$ at z_0, written $f'(z_0)$, by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

provided that the limit exists. If it does, we say that the function $f(z)$ is differentiable at z_0. If we write $\Delta z = z - z_0$, then we can express Equation above in the form

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

If we let $w = f(z)$ and $\Delta w = f(z) - f(z_0)$, then we can use the notation $\frac{dw}{dz}$ for the derivative:

$$f'(z_0) = \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}.$$

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Example

Use the limit definition to find the derivative of $f(z) = z^3$.

Solution.

$$f'(z_0) = \lim_{z \to z_0} \frac{(z - z_0)^3}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{\left(z^2 + z\, z_0 + z_0^2\right)(z - z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \left(z^2 + z\, z_0 + z_0^2\right)$$

$$= \left(z_0^2 + z_0\, z_0 + z_0^2\right)$$

$$= 3\, z_0^2$$

Remark

Pay careful attention to the complex value Δz ; the value of the limit must be independent of the manner in which $\Delta z \to 0$. If we can find two curves that end at z_0 along which $\frac{\Delta w}{\Delta z}$ approaches distinct values, then $\frac{\Delta w}{\Delta z}$ does not have a limit as $\Delta z \to 0$ and $f(z)$ does not have a derivative at z_0.

Example

Show that the function $f(z) = \overline{z}$ is nowhere differentiable.

Solution. We choose two approaches to the point $z_0 = x_0 + i\, y_0$ and compute limits of the difference quotients. First, we approach $z_0 = x_0 + i\, y_0$ along a line parallel to the x axis by forcing z to be of the form $z = x + i\, y_0$.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(x+i\,y_0) \to (x_0+i\,y_0)} \frac{f(x+i\,y_0) - f(x_0+i\,y_0)}{(x+i\,y_0) - (x_0+i\,y_0)}$$

$$= \lim_{(x+i\,y_0) \to (x_0+i\,y_0)} \frac{(x-i\,y_0) - (x_0-i\,y_0)}{(x+i\,y_0) - (x_0+i\,y_0)}$$

$$= \lim_{(x+i\,y_0) \to (x_0+i\,y_0)} \frac{(x-i\,y_0) - (x_0-i\,y_0)}{(x-x_0) + i\,(y_0-y_0)}$$

$$= \lim_{(x+i\,y_0) \to (x_0+i\,y_0)} \frac{x - x_0}{x - x_0}$$

$$= 1.$$

Next, we approach z_0 along a line parallel to the y axis by forcing z to be of the form $z = x_0 + i\, y$.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(x_0 + i\, y) \to (x_0 + i\, y_0)} \frac{f(x_0 + i\, y) - f(x_0 + i\, y_0)}{(x_0 + i\, y) - (x_0 + i\, y_0)}$$

$$= \lim_{(x_0 + i\, y) \to (x_0 + i\, y_0)} \frac{(x_0 - i\, y) - (x_0 - i\, y_0)}{(x_0 + i\, y) - (x_0 + i\, y_0)}$$

$$= \lim_{(x_0 + i\, y) \to (x_0 + i\, y_0)} \frac{(x_0 - i\, y) - (x_0 - i\, y_0)}{(x_0 - x_0) + i\,(y - y_0)}$$

$$= \lim_{(x_0 + i\, y) \to (x_0 + i\, y_0)} \frac{-i\,(y - y_0)}{i\,(y - y_0)}$$

$$= -1.$$

The limits along the two paths are different, so there is no possible value for the right side of Equation. Therefore $f(z) = \overline{z}$ is not differentiable at the point z_0, and since z_0 was arbitrary, $f(z) = \overline{z}$ is nowhere differentiable.

Analytic Functions

Definition (Analytic Function).
The complex function $f(z)$ is analytic at the point z_0 provided there is some $\epsilon > 0$ such that $f'(z)$ exists for all $z \in D_\epsilon(z_0)$. In other words, $f(z)$ must be differentiable not only at z_0, but also at all points in some ϵ-neighborhood of z_0.

If $f(z)$ is analytic at each point in the region R, then we say that $f(z)$ is analytic on R. Again, we have a special term if $f(z)$ is analytic on the whole complex plane.

Definition (Entire Function). If $f(z)$ is analytic on the whole complex plane then $f(z)$ is said to be entire.

Points of nonanalyticity for a function are called singular points. They are important for certain applications in physics and engineering.

Our definition of the derivative for complex functions is formally the same as for real functions and is the natural extension from real variables to complex variables. The basic differentiation formulas are identical to those for real functions, and we obtain the same rules for differentiating powers, sums, products, quotients, and compositions of functions. We can easily establish the proof of the differentiation formulas by using the limit theorems.

The Rules for Differentiation.

Suppose that $f(z)$ and $g(z)$ are differentiable. We can establish the following rules, which are virtually identical to those for real-valued functions.

$$\frac{d}{dz} C = 0, \text{ where } C \text{ is a constant},$$

$$\frac{d}{dz} z^n = n z^{n-1}, \text{ where } n \text{ is a positive integer},$$

$$\frac{d}{dz}[C f(z)] = C f'(z),$$

$$\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z),$$

$$\frac{d}{dz}[f(z) g(z)] = f(z) g'(z) + g(z) f'(z),$$

$$\frac{d}{dz} \frac{f(z)}{g(z)} = \frac{g(z) f'(z) - f(z) g'(z)}{[g(z)]^2}, \text{ provided that } g(z) \neq 0,$$

□ $$\frac{d}{dz} f(g(z)) = f' f(g(z)) g'(z).$$

Example

If we use rules with $f(z) = z^2 + i\,2\,z + 3$, and $f'(z) = 2\,z + 2\,i$, then we get

$$\frac{d}{dz}\left(z^2 + i\,2\,z + 3\right)^4 = 4\left(z^2 + i\,2\,z + 3\right)^3 (2\,z + 2\,i)$$

$$= 8\left(z^2 + i\,2\,z + 3\right)^3 (z + i).$$

Cauchy-Riemann Equations

We showed that computing the derivative of complex functions written in a form such as $f(z) = z^2$ is a rather simple task. But life isn't always so easy.
Many times we encounter complex functions written as $f(z) = u(x, y) + i\,v(x, y)$. For example, suppose we had

$$f(z) = f(x + i\,y) = u(x, y) + i\,v(x, y) = \left(x^3 - 3\,x\,y^2\right) + i\left(3\,x^2\,y - y^3\right).$$

Is there some criterion - perhaps involving the partial derivatives for $u(x,y)$, and $v(x,y)$ - that we can use to determine whether f is differentiable, and if so, to find the value of $f(z)$?

The answer to this question is yes, thanks to the independent discovery of two important equations by the French mathematician Augustin Louis Cauchy (1789-1857) and the German mathematician Georg Friedrich Bernhard Riemann (1826-1866).

Theorem (Cauchy-Riemann Equations). Suppose that

$$f(z) = f(x + iy) = u(x, y) + i\,v(x, y)$$

is differentiable at the point $z_0 = x_0 + i\,y_0$. Then the partial derivatives of u and v exist at the point (x_0, y_0), and

$$f'(z_0) = u_x(x_0, y_0) + i\,v_x(x_0, y_0),$$ and also

$$f'(z_0) = v_y(x_0, y_0) - i\,u_y(x_0, y_0).$$

Equating the real and imaginary parts of Equations gives

$$u_x(x_0, y_0) = v_y(x_0, y_0) \text{ and } u_y(x_0, y_0) = -v_x(x_0, y_0).$$

Proof

Remark

Note some of the important implications of this theorem.

(i). If f is differentiable at z_0, then the Cauchy-Riemann Equations will be satisfied at z_0, and we can use either either Equation to evaluate $f'(z_0)$.

(ii). Taking the contrapositive, if Cauchy-Riemann Equations are not satisfied at z_0, then we know automatically that $f(z)$ is not differentiable at z_0.

(iii). Even if Cauchy-Riemann Equations are satisfied at z_0, we cannot necessarily conclude that f is differentiable at z_0.

Example

We know that $f(z) = z^2 = x^2 - y^2 + i\,2\,x\,y$ is differentiable and that $f'(z) = 2\,z$. We also have

$$f(z) = z^2 = (x + i\,y)^2 = \left(x^2 - y^2\right) + i\,(2\,x\,y).$$

It is easy to verify that Cauchy-Riemann Equations are indeed satisfied:

$$u_x(x, y) = 2\,x = v_y(x, y) \quad \text{and} \quad u_y(x, y) = -2\,y = -v_x(x, y).$$

Using Cauchy-Riemann Equations, respectively, to compute $f'(z)$ gives

$$f'(z) = u_x(x, y) + i\,v_x(x, y) = 2\,x + i\,2\,y = 2\,z, \quad \text{and}$$

$$f'(z) = v_y(x, y) - i\,u_y(x, y) = 2\,x - i\,(-2\,y) = 2\,z,$$

as expected.

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Example

Show that $f(z) = \overline{z}$ is nowhere differentiable.

Solution. We have $f(z) = f(x + i y) = x - i y = u(x, y) + i v(x, y)$, where $u(x, y) = x$ and $v(x, y) = -y$. Thus, for any point (x, y), $u_x(x, y) = 1$ and $v_y(x, y) = -1$. The Cauchy-Riemann equations are not satisfied at any point $z = (x, y)$, so we conclude that $f(z) = \overline{z}$ is nowhere differentiable.

Example Show that the function defined by

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z} = \dfrac{x^3-3xy^2}{x^2+y^2} + i\,\dfrac{y^3-3x^2y}{x^2+y^2} & \text{when } z \neq 0, \text{ and} \\ 0 & \text{when } z \neq 0. \end{cases}$$

is not differentiable at the point $z_0 = 0$ even though the Cauchy-Riemann equations are satisfied at $(0, 0)$.

Solution. We must use limits to calculate the partial derivatives at $(0,0)$.

$$u_x(0, 0) = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{\frac{x^3-0}{x^2+0}-0}{x} = \lim_{x\to 0} \frac{x}{x} = 1,$$

$$u_y(0, 0) = \lim_{y\to 0} \frac{u(0,y)-u(0,y)}{y} = \lim_{y\to 0} \frac{\frac{0-0}{0+y^2}-0}{y} = \lim_{y\to 0} \frac{0}{y} = 0,$$

$$v_x(0, 0) = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{\frac{0-0}{x^2+0}-0}{x} = \lim_{x\to 0} \frac{0}{x} = 0,$$

$$v_y(0, 0) = \lim_{y\to 0} \frac{v(0,y)-v(0,y)}{y} = \lim_{y\to 0} \frac{\frac{y^3-0}{0+y^2}-0}{y} = \lim_{y\to 0} \frac{y}{y} = 1.$$

Thus we have shown that $u_x(0, 0) = 1$, $u_y(0, 0) = 0$, $v_x(0, 0) = 0$, $v_y(0, 0) = 1$

Hence the Cauchy-Riemann equations hold at the point $(0,0)$.

We now show that f is not differentiable at $z_0 = 0$. Letting z approach 0 along the x axis gives

$$\lim_{(x,0)\to(0,0)} \frac{f(x+0\,i)-f(0)}{x+0\,i-0} = \lim_{(x,0)\to(0,0)} \frac{\frac{x^3-0}{x^2+0}+i\,\frac{0-0}{x^2+0}-0}{x} = \lim_{(x,0)\to(0,0)} \frac{\frac{x^3}{x^2}+0\,i-0}{x}$$

$$= \lim_{(x,0)\to(0,0)} \frac{x}{x} = 1$$

But if we let z approach 0 along the line $y = x$ given by the parametric equations $x = t$ and $y = t$, then

$$\lim_{(t,t)\to(0,0)} \frac{f(t+i\,t)-f(0)}{t+i\,t-0} = \lim_{(t,t)\to(0,0)} \frac{\frac{t^3-3\,t^3}{t^2+t^2}+i\,\frac{t^3-3\,t^3}{t^2+t^2}-0}{t+i\,t} = \lim_{(t,t)\to(0,0)} \frac{\frac{-2\,t^3}{2\,t^2}+i\,\frac{-2\,t^3}{2\,t^2}}{t+i\,t}$$

$$= \lim_{(t,t)\to(0,0)} \frac{-t-i\,t}{t+i\,t} = -1$$

The two limits are distinct, so f is not differentiable at the origin.

Theorem (Cauchy-Riemann conditions for differentiability). Let $f(z) = f(x + i y) = u(x, y) + i v(x, y)$ be a continuous function that is defined in some neighborhood of the point $z_0 = x_0 + i y_0$. If all the partial derivatives u_x, u_y, v_x and v_y are continuous at the point (x_0, y_0) and if the Cauchy-Riemann equations $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$ hold at (x_0, y_0), then $f(z)$ is differentiable at z_0 and the derivative $f'(z_0)$ can be computed with formula

$$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0), \text{ or}$$

$$f'(z_0) = v_y(x_0, y_0) - i u_y(x_0, y_0).$$

Example

$f(z) = u(x, y) + i\, v(x, y) = x^3 - 3xy^2 + i\left(3x^2 y - y^3\right).$

Show that this function is differentiable for all z, and find its derivative.

Solution. We compute $u_x(x, y) = 3x^3 - 3y^2 = v_y(x, y)$ and $u_y(x, y) = -6xy = -v_x(x, y)$, so the Cauchy-Riemann Equations, are satisfied. Moreover, u, v, u_x, u_y, v_x and v_y are continuous everywhere. Therefore, f is differentiable everywhere, and,

$$f'(z) = u_x(x, y) + i\, v_x(x, y) = 3x^3 - 3y^2 + i\, 6xy$$

$$= 3\left(x^2 - y^2 + 2ixy\right) = 3(x + iy)^2 = 3z^2$$

Alternatively,

$$f'(z) = v_y(x, y) - i\, u_y(x, y) = 3x^3 - 3y^2 - i(-6xy)$$

$$= 3\left(x^2 - y^2 + 2ixy\right) = 3(x + iy)^2 = 3z^2$$

This result isn't surprising because $(x + iy)^3 = x^3 - 3xy^2 + i\left(3x^2 y - y^3\right)$ and so the function f is really our old friend $f(z) = z^3$.

Example

Show that the function $f(z) = f(x + i y) = e^{-y} \cos x + i e^{-y} \sin x$ is differentiable for all $z = x + i y$ and find its derivative.

Solution. We first write $u(x, y) = e^{-y} \cos x$ and $v(x, y) = e^{-y} \sin x$ and then compute the partial derivatives.

$$u_x(x, y) = v_y(x, y) = -e^{-y} \sin x, \quad \text{and}$$

$$v_x(x, y) = -u_y(x, y) = e^{-y} \cos x.$$

We note that u, v, u_x, u_y, v_x and v_y are continuous functions and that the Cauchy-Riemann equations hold for all values of (x, y). Hence, we write

$$f'(z) = u_x(x, y) + i v_x(x, y) = -e^{-y} \sin x + i e^{-y} \cos x.$$

Example

Show that the function $f(z) = x^3 + 3xy^2 + i(3x^2y + y^3)$ is differentiable at points that lie on the x and y axes but analytic nowhere.

Solution. Recall (Definition 3.1) that when we say a function is analytic at a point z_0 we mean that the function is differentiable not only at z_0, but also at every point in some neighborhood of z_0. With this in mind, we proceed to determine where the Cauchy-Riemann equations are satisfied. We write $u(x, y) = x^3 + 3xy^2$ and $v(x, y) = 3x^2y + y^3$ and compute the partial derivatives:

$$u_x(x, y) = 3x^3 + 3y^2, \quad v_y(x, y) = 3x^3 + 3y^2, \quad \text{and}$$

$$u_y(x, y) = 6xy, \quad v_x(x, y) = 6xy.$$

Here u, v, u_x, and v_y are continuous, and $u_x(x, y) = v_y(x, y)$ holds for all (x, y). But $u_y(x, y) = -v_x(x, y)$ iff $6xy = -6xy$, which is equivalent to $12xy = 0$. The Cauchy-Riemann equations hold only when $x = 0$ or $y = 0$, and f is differentiable only at points that lie on the coordinate axes.

But this means that f is nowhere analytic because any ϵ-neighborhood about a point on either axis contains points that are not on those axes.

Harmonic Functions

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Definition Harmonic Functions

Let $\phi(x, y)$ be a real-valued function of the two real variables x and y defined on a domain D. (Recall that a domain is a connected open set.) The partial differential equation

$$\phi_{xx}(x, y) + \phi_{yy}(x, y) = 0$$

is known as Laplace's equation and is sometimes referred to as the potential equation. If ϕ, ϕ_x, ϕ_y, ϕ_{xx}, ϕ_{xy}, ϕ_{yx} and ϕ_{yy} are all continuous and if $\phi(x, y)$ satisfies Laplace's equation, then $\phi(x, y)$ is called a harmonic function.

We begin with an important theorem relating analytic and harmonic functions.

Theorem Let $f(z) = f(x + iy) = u(x, y) + i\,v(x, y)$ be an analytic function on a domain D. Then both $u(x, y)$ and $v(x, y)$ are harmonic functions on D. In other words, the real and imaginary parts of an analytic function are harmonic.

Theorem Let $f(z) = f(x+iy) = u(x, y) + i\,v(x, y)$ be an analytic function on a domain D. Then both $u(x, y)$ and $v(x, y)$ are harmonic functions on D. In other words, the real and imaginary parts of an analytic function are harmonic.

Proof. We will show that, if f(z) is analytic, then all partial derivatives of u and v are continuous. Using that result here, we see that, as f is analytic, u and v satisfy the Cauchy-Riemann equations

$$u_x(x, y) = v_y(x, y) \quad \text{and} \quad u_y(x, y) = -v_x(x, y).$$

Taking the partial derivative with respect to x of each side of these equations gives

$$u_{xx}(x, y) = v_{yx}(x, y) \quad \text{and} \quad u_{yx}(x, y) = -v_{xx}(x, y).$$

Similarly, taking the partial derivative of each side with respect to y yields

$$u_{xy}(x, y) = v_{yy}(x, y) \quad \text{and} \quad u_{yy}(x, y) = -v_{xy}(x, y).$$

The partial derivatives u_{xy}, u_{yx}, v_{xy} and v_{yx} are all continuous, so we use a theorem from the calculus of real functions that states that the mixed partial derivatives are equal; that is,

$$u_{xy}(x, y) = u_{yx}(x, y) \quad \text{and} \quad v_{xy}(x, y) = v_{yx}(x, y).$$

Combining all these results finally gives

$$u_{xx}(x, y) + u_{yy}(x, y) = v_{yx}(x, y) - v_{xy}(x, y) = 0, \text{ and}$$

$$v_{xx}(x, y) + v_{yy}(x, y) = -u_{yx}(x, y) + u_{xy}(x, y) = 0.$$

Therefore both u and v are harmonic functions on D.

Example

Show that $u(x, y) = x y^3 - x^3 y$ is a harmonic function and find the harmonic conjugate $v(x, y)$.

Solution. We follow the construction process. The first partial derivatives are

$$u_x(x, y) = y^3 - 3 x^2 y \quad \text{and} \quad u_y(x, y) = 3 x y^2 - x^3.$$

To verify that u is harmonic, we compute the second partial derivatives and note that
$u_{xx}(x, y) + u_{yy}(x, y) = -6 x y + 6 x y = 0$, so u satisfies Laplace's Equation

To construct v, we get

$$v(x, y) = \int \left(y^3 - 3 x^2 y\right) dy + C(x)$$

$$= \frac{1}{4} y^4 - \frac{3}{2} x^2 y^2 + C(x)$$

Differentiating the left and right sides of this equation with respect to x and using
$-u_y(x, y) = v_x(x, y)$ and on the left side yields

$$-3 x y^2 + x^3 = 0 - 3 x y^2 + C'(x)$$

which implies that

$$C'(x) = x^3,$$

then an easy integration yields $C(x) = \frac{1}{4} x^4 + C$, where C is a constant. Now we substitute $C(x)$
and obtain the desired solution

$$v(x, y) = \frac{1}{4} x^4 + \frac{1}{4} y^4 - \frac{3}{2} x^2 y^2 + C.$$