SERIES

Series Expansion

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Sequences

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In formal terms, a complex sequence is a function whose domain
is the positive integers and whose range is a subset of the complex numbers.
The following are examples of sequences:

$$f(n) = \left(2 - \frac{1}{n}\right) + \left(5 + \frac{1}{n}\right) i \qquad (n = 1, 2, 3, \ldots);$$

$$g(n) = e^{i \frac{\pi n}{4}} \qquad (n = 1, 2, 3, \ldots);$$

$$h(k) = 5 + 3 i + \left(\frac{1}{1+i}\right)^k \qquad (k = 1, 2, 3, \ldots);$$

$$r(n) = \left(\frac{1}{4} + \frac{i}{2}\right)^n \qquad (n = 1, 2, 3, \ldots).$$

Definition (Limit of a Sequence). $\lim_{n\to\infty} z_n = \zeta$ means that for any real number $\epsilon > 0$ there corresponds a positive integer N_ε (which depends on ϵ) such that $z_n \in D_\varepsilon(\zeta)$ whenever $n > N_\varepsilon$. That is $|\zeta - z_n| < \epsilon$ whenever $n > N_\varepsilon$. Figure below illustrates a convergent sequence.

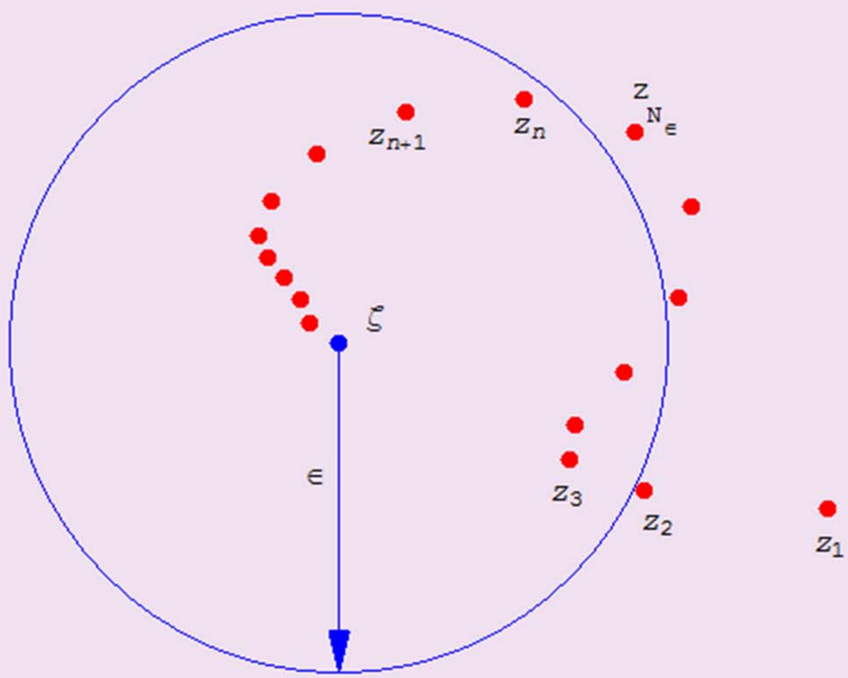

Figure A sequence $\{z_n\}_1^\infty$ that converges to ζ. (If $n > N_\varepsilon$ then $z_n \in D_\varepsilon(\zeta)$.)

Theorem Let $z_n = x_n + i\, y_n$ and $\zeta = u + i\, v$. Then

$$\lim_{n\to\infty} z_n = \zeta, \quad \text{iff}$$

$$\lim_{n\to\infty} x_n = u \quad \text{and} \quad \lim_{n\to\infty} y_n = v.$$

Example

Find the limit of the sequence $\{z_n\} = \left\{\frac{\sqrt{n} + i\,(n+1)}{n}\right\}$.

We write $z_n = x_n + i\, y_n = \frac{1}{\sqrt{n}} + i\,\frac{n+1}{n}$. Using results concerning sequences of real numbers, we find that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \quad \text{and} \quad \lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

Therefore $\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{\sqrt{n} + i\,(n+1)}{n} = 0 + i = i.$

Example

Show that the sequence $\{z_n\} = \{(1+i)^n\}$ diverges.

Solution. We have

$$z_n = (1+i)^n = x_n + i\, y_n$$

$$z_n = \left(\sqrt{2}\right)^n \cos\frac{n\pi}{4} + i\left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}$$

The real sequences $x_n = \left(\sqrt{2}\right)^n \cos\frac{n\pi}{4}$ and $y_n = \left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}$ both exhibit divergent oscillations, so we conclude that $z_n = (1+i)^n$ diverges.

Definition (Cauchy Sequence). The sequence $\{z_n\}$ is said to be a Cauchy sequence if for every $\epsilon > 0$ there exists a positive integer N_ε, such that if $n, m > N_\varepsilon$, then $|z_n - z_m| < \epsilon$, or, equivalently, $z_n - z_m \in D_\varepsilon(0)$.

One of the most important notions in analysis (real or complex) is a theory that allows us to add up infinitely many terms. To make sense of such an idea we begin with a sequence $\{z_n\}$, and form a new sequence $\{S_n\}$, called the sequence of partial sums, as follows.

$$
\begin{aligned}
S_1 &= z_1, \\
S_2 &= z_1 + z_2, \\
S_3 &= z_1 + z_2 + z_3, \\
&\vdots \\
S_n &= z_1 + z_2 + \ldots + z_n = \textstyle\sum_{k=1}^{\infty} z_k, \\
&\vdots
\end{aligned}
$$

Series

Definition (Infinite Series).

The formal expression $\sum_{k=1}^{\infty} z_k = z_1 + z_2 + \ldots + z_n + \ldots$ is called an infinite series, and $z_1, z_2, \ldots, z_n, \ldots,$ are called the terms of the series.

If there is a complex number S for which

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} z_k,$$

we will say that the infinite series $\sum_{k=1}^{\infty} z_k$ converges to S, and that S is the sum of the infinite series. When this occurs, we write

$$S = \sum_{k=1}^{\infty} z_k.$$

The series $\sum_{k=1}^{\infty} z_k$ is said to be absolutely convergent provided that the (real) series of magnitudes $\sum_{k=1}^{\infty} |z_k|$ converges.

Theorem Let $z_n = x_n + i\, y_n$ and $S = U + i\, V$. Then

$$S = \sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} (x_n + i\, y_n) \qquad \text{(converges)}$$

if and only if both

$$U = \sum_{n=1}^{\infty} x_n \quad \text{and} \quad V = \sum_{n=1}^{\infty} y_n \qquad \text{(converge).}$$

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Theorem (Geometric Series).

If $|z| < 1$, the series $\sum_{n=0}^{\infty} z^n$ converges to $f(z) = \frac{1}{1-z}$. That is, if $|z| < 1$ then

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \ldots + z^k + \ldots = \frac{1}{1-z}.$$

If $|z| \geq 1$, the series diverges.

Example Show that $\sum_{n=0}^{\infty} \frac{(1-i)^n}{2^n} = 1 - i$.

Solution. If we set $z = \frac{1-i}{2}$, then $|z| = \left|\frac{1-i}{2}\right| = \frac{\sqrt{2}}{2} < 1$. The sum is

$$1 \Big/ \left(1 - \frac{1-i}{2}\right) = \frac{2}{2-1+i} = \frac{2}{1+i} = 1 - i.$$

Example Evaluate $\sum_{n=3}^{\infty} \frac{i^n}{2^n}$.

Solution. We can put this expression in the form of a geometric series:

$$\sum_{n=3}^{\infty} \frac{i^n}{2^n} = \sum_{n=3}^{\infty} \left(\frac{i}{2}\right)^n = \sum_{n=3}^{\infty} \left(\frac{i}{2}\right)^3 \left(\frac{i}{2}\right)^{n-3}$$

$$= \left(\frac{i}{2}\right)^3 \sum_{n=3}^{\infty} \left(\frac{i}{2}\right)^{n-3} = \left(\frac{i}{2}\right)^3 \frac{1}{1-\frac{i}{2}}$$

$$= -\frac{i}{8} \frac{1}{1-\frac{i}{2}} = \frac{-i}{8-4\,i}$$

$$= \frac{-i}{8-4\,i} \frac{8+4\,i}{8+4\,i} = \frac{4-8\,i}{80}$$

$$= \frac{1}{20} - \frac{i}{10}$$

Taylor series

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We know that analytic functions also have derivatives of all orders.
It seems natural, therefore, that there would be some connection between analytic functions and power series. As you might guess, the connection exists via the Taylor and Maclaurin series of analytic functions.

Definition (Taylor Series). If $f(z)$ is analytic at $z = \alpha$, then the series

$$f(\alpha) + f'(\alpha)(z-\alpha) + \frac{f^{(2)}(\alpha)}{2!}(z-\alpha)^2 + \frac{f^{(2)}(\alpha)}{3!}(z-\alpha)^3 + \ldots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!}(z-\alpha)^k$$

is called the Taylor series for $f(z)$ centered at $z = \alpha$. When the center is $\alpha = 0$, the series is called the Maclaurin series for $f(z)$.

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Theorem (Taylor's Theorem). Suppose $f(z)$ is analytic in a domain G, and that $D_R(\alpha) = \{z : |z - \alpha| < R\}$ is any disk contained in G. Then the Taylor series for $f(z)$ converges to $f(z)$ for all z in $D_R(\alpha)$; that is,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!}(z-\alpha)^k \quad \text{for all} \quad z \in D_R(\alpha).$$

$$\frac{1}{1-z} = \sum_{i=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots \qquad R = 1$$

$$\ln(1+z) = \sum_{i=0}^{\infty} (-1)^{n-1} \frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \qquad R = 1$$

$$(1+z)^k = \sum_{k} \binom{k}{n} x^n = 1 + kz + \frac{k(k-1)}{2!} z^2 + \frac{k(k-1)(k-2)}{3!} z^3 + \cdots \qquad R = 1$$

$$\exp(z) = \sum_{i=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad R = \infty$$

$$sinz = \sum_{i=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots \qquad R = \infty$$

$$cosz = \sum_{i=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \qquad R = \infty$$

Proof

Laurent series

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Suppose $f(z)$ is not analytic in $D_R(\alpha)$, but is analytic in $D_R^*(\alpha) = \{z : 0 < |z - \alpha| < R\}$. For example, the function $f(z) = \frac{1}{z^3} e^z$ is not analytic when $z = 0$ but is analytic for $|z| > 0$. Clearly, this function does not have a Maclaurin series representation.

Definition (Laurent Series).

Let c_n be a complex number for $n = 0, \pm 1, \pm 2, \pm 3, \ldots.$

The doubly infinite series $\sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n$, called a Laurent series, is defined by

$$\sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n = \sum_{n=1}^{\infty} c_{-n} (z-\alpha)^{-n} + \sum_{n=0}^{\infty} c_n (z-\alpha)^n,$$

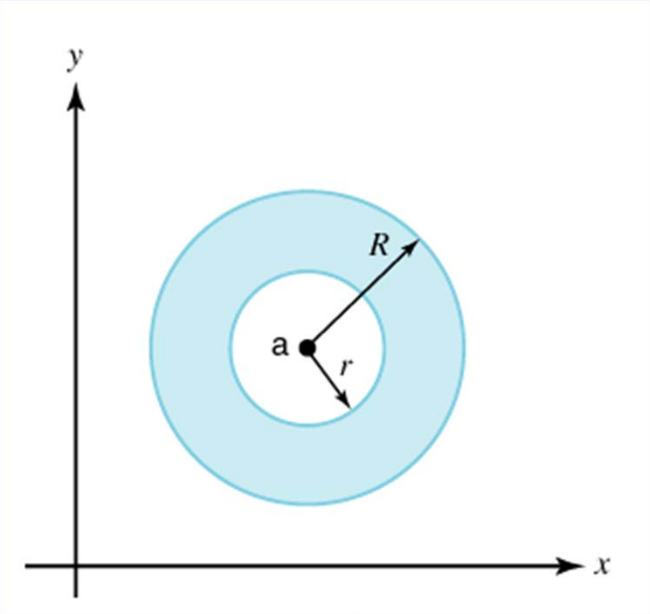
provided the series on the right-hand side of this equation converge.

Theorem (Laurent's Theorem). Suppose $0 \leq r < R$, and that $f(z)$ is analytic in the annulus $A = A(\alpha, r, R) = \{z : r < |z - \alpha| < R\}$ shown in Figure below If ρ is any number such that $r < \rho < R$, then for all $z_0 \in A(\alpha, r, R)$ the function value $f(z_0)$ has the Laurent series representation

$$f(z_0) = \sum_{n=-\infty}^{\infty} c_n (z_0 - \alpha)^n = \sum_{n=1}^{\infty} c_{-n} (z_0 - \alpha)^{-n} + \sum_{n=0}^{\infty} c_n (z_0 - \alpha)^n,$$

where for $n = 0, 1, 2, \ldots$, the coefficients c_{-n} and c_n are given by

$$c_{-n} = \frac{1}{2\pi i} \int_{C_\rho^+(\alpha)} \frac{f(z)}{(z-\alpha)^{-n+1}} dz \quad \text{and} \quad c_n = \frac{1}{2\pi i} \int_{C_\rho^+(\alpha)} \frac{f(z)}{(z-\alpha)^{n+1}} dz.$$



Figure

Example

Find three different Laurent series representations for the function $f(z) = \frac{3}{2 + z - z^2}$ involving powers of z.

Solution. The function $f(z)$ has singularities at $z = -1, 2$ and is analytic in the disk $D: |z| < 1$, in the annulus $A: 1 < |z| < 2$, and in the region $R: |z| > 2$.
We want to find a different Laurent series for $f(z)$ in each of the three domains D, A, and R.
We start by writing $f(z)$ in its partial fraction form:

$$f(z) = \frac{3}{(1+z)(2-z)} = \frac{1}{1+z} + \frac{1}{2-z} = \frac{1}{1+z} + \frac{1}{2}\frac{1}{1-\frac{z}{2}}.$$

We can obtain the following representations for the terms on the right side of Equation:

(1) $\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$ valid for $|z| < 1$,

(2) $\frac{1}{1+z} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n}$ valid for $|z| > 1$,

(3) $\frac{1}{2}\frac{1}{1-\frac{z}{2}} = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ valid for $|z| < 2$,

(4) $\frac{-1}{z}\frac{1}{1-\frac{2}{z}} = \sum_{n=1}^{\infty} \frac{-2^{n-1}}{z^n}$ valid for $|z| > 2$.

Representations (1) and (3) are both valid in the disk $D: |z| < 1$, and thus we have

$$f(z) = \sum_{n=0}^{\infty} (-1)^n z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \left((-1)^n + \frac{1}{2^{n+1}} \right) z^n \quad \text{valid for} \quad |z| < 1,$$

which is a Laurent series that reduces to a Maclaurin series.

In the annulus $A: 1 < |z| < 2$, representations (2) and (3) are valid; hence we get

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \quad \text{valid for} \quad 1 < |z| < 2.$$

Finally, in the region $R: |z| > 2$ we use Representations (2) and (4) to obtain

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n} + \sum_{n=1}^{\infty} \frac{-2^{n-1}}{z^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} - 2^{n-1}}{z^n} \quad \text{valid for} \quad |z| > 2.$$

Example

Find the Laurent series representation for $f(z) = \frac{\cos(z) - 1}{z^4}$ that involves powers of z.

We know that $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$, and hence the Maclaurin series for $\cos z - 1$ is

$$\cos z - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = -\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \ldots,$$

then we can write

$$f(z) = \frac{1}{z^4} \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n-4}}{(2n)!}$$

or in another way we can write

$$f(z) = \frac{\cos z - 1}{z^4} = \frac{-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \ldots}{z^4}$$

We formally divide each term by z^4 to obtain the Laurent series

$$f(z) = -\frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!} - \frac{z^2}{6!} + \frac{z^4}{8!} - \frac{z^6}{10!} + \ldots$$

$$= -\frac{1}{2z^2} + \frac{1}{24} - \frac{z^2}{720} + \frac{z^4}{40\,320} - \frac{z^6}{3\,628\,800} + \ldots$$

Example

Find the Laurent series for $f(z) = \exp\left(\frac{-1}{z^2}\right)$ centered at $\alpha = 0$.

The Maclaurin series for $\exp z$ is $\exp z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$,
which is valid for all z. We let $\frac{-1}{z^2}$ take the role of z in this equation to get

$$\exp\left(\frac{-1}{z^2}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-1}{z^2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n}{(z^2)^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(z^2)^n},$$

which is valid for $|z| > 0$.